

# Addressing Estimation Errors on Expected Asset Returns through Robust Portfolio Optimization

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It is well known that the performance of the classical Markowitz model for portfolio optimization is extremely sensitive to estimation errors on the expected asset returns. Robust optimization mitigates this issue. We focus on ellipsoidal uncertainty sets around a point estimate of the expected asset returns. An important issue is the choice of the parameters that specify this ellipsoid, namely the point estimate and the estimation-error matrix. We show that there exist diagonal estimation-error matrices that achieve an arbitrarily small loss in the expected portfolio return as compared to the optimum. We empirically investigate the sample size needed to compute the point estimate. We also conduct an empirical study of different estimation-error matrices and give a heuristic to choose the size of the uncertainty set. The results of our experiments show that robust portfolio models featuring a family of diagonal estimation-error matrices outperform the classical Markowitz model.

*Key words:* robust optimization, portfolio optimization, optimization under uncertainty

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## 1. Introduction

Consider a portfolio optimization problem where we allocate capital across  $n$  assets to maximize the return on investment. If the return vector  $\mathbf{r} \in \mathbb{R}^n$  is known, the problem can be formulated as  $\max_{\mathbf{x} \in \mathbb{R}_+^n} \{\mathbf{r}^\top \mathbf{x} : \mathbf{1}^\top \mathbf{x} = 1\}$ , where  $\mathbf{x}$  denotes the fraction of investment in each asset assuming long positions only. In this case, the problem has a trivial optimal solution: invest only in the asset with the largest return.

In practice, however, investors must consider that the assets are risky and that the return vector  $\mathbf{r} \sim \mathbb{D}$  belongs to some probability distribution. The classical mean-variance portfolio optimization problem introduced by Markowitz (1952) addresses this uncertainty by maximizing the *expected* return of the portfolio subject to a constraint on the risk modeled as the variance of the portfolio return. Let  $\boldsymbol{\mu} \in \mathbb{R}^n$  and  $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$  denote the expectation vector and covariance matrix of the asset returns, respectively. Then the Markowitz model is formulated as

$$\max_{\mathbf{x} \in \mathbb{R}_+^n} \boldsymbol{\mu}^\top \mathbf{x} \tag{1}$$

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$$\text{subject to } \mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} \leq v \quad (2)$$

$$\mathbf{1}^\top \mathbf{x} = 1. \quad (3)$$

Despite the theoretical promise of this mean-variance model, practitioners face an overwhelming challenge. The *true* expectation vector  $\boldsymbol{\mu}$  and covariance matrix  $\boldsymbol{\Sigma}$  of the random asset returns are unknown. Therefore, one can only optimize (1)-(3) with *estimated* parameters.

### 1.1. Literature Review

It has been observed that even small errors in estimating  $\boldsymbol{\mu}$  produce large changes in the returns of portfolio holdings (see, for example, Best and Grauer 1991, Chopra and Ziemba 1993, Michaud and Michaud 2008). The issue of obtaining reliable estimates for  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  has been studied extensively, leading to a vast literature. Several papers advocate the use of shrinkage estimators, such as the James-Stein (James and Stein 1992), Jobson (Jobson 1979), Jorion (Jorion 1986), Frost-Savarino (Frost and Savarino 1986), and Ledoit-Wolf (Ledoit and Wolf 2003, 2004a,b, 2020) estimators. The James-Stein, Jobson, and Jorion procedures are estimators for the mean asset returns, the Frost-Savarino procedure is a joint estimator of the means and covariances, and the Ledoit-Wolf procedure is an estimator of the covariance. Shrinkage is often interpreted as a form of empirical Bayesian procedure, which assumes a prior to establish an exogenous structure on potential estimates. We refer the reader to Avramov and Zhou (2010) for a survey of Bayesian procedures used in portfolio selection. An interesting approach to addressing estimation errors is the Black-Litterman model (Black and Litterman 1990, 1992), which combines market information and investor views into the mean-variance optimization problem. Other works have incorporated diversification across market and estimation risk into the model (Jagannathan and Ma 2003, ter Horst et al. 2006, Kan and Zhou 2007). ter Horst et al. (2006) also suggest portfolio weight adjustments. DeMiguel et al. (2009), Brodie et al. (2009), Gotoh and Takeda (2011) additionally study the imposition of norm constraints to regularize the optimal portfolio against large errors. There have also been studies on the intersection of machine learning and portfolio optimization in the context of error mitigation (Lim et al. 2012, Ban et al. 2018). Finally, we name robust portfolio optimization. Robust optimization has been well-studied in the portfolio management literature Goldfarb and Iyengar (2003), Tütüncü and Koenig (2004), Natarajan et al. (2008), Calafiore and Monastero (2012), Bertsimas et al. (2018). This paper contributes to the stream of robust portfolio optimization literature, focusing on errors on the estimates for  $\boldsymbol{\mu}$ .

### 1.2. Background on Robust Optimization

We assume that the *true* expected return vector  $\boldsymbol{\mu}$  that parametrizes the real-world returns distribution is unknown and belongs to an ellipsoidal uncertainty set given by

$$\mathcal{U} := \{\boldsymbol{\mu} \in \mathbb{R}^n : (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\top \boldsymbol{\Xi}^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \leq \kappa^2\}$$

where  $\hat{\boldsymbol{\mu}}$  is an *estimated* expected return vector and  $\boldsymbol{\Xi}$  is a positive definite matrix referred to as the estimation-error matrix. The robust optimization problem hopes to find a portfolio that maximizes the expected returns of the investment in a worst-case scenario  $\boldsymbol{\mu} \in \mathcal{U}$ . That is, the robust optimization problem is formulated as

$$\max_{\mathbf{x} \in \mathcal{X}} \min_{\boldsymbol{\mu} \in \mathcal{U}} \{ \boldsymbol{\mu}^\top \mathbf{x} \}$$

where  $\mathcal{X}$  is the feasible region characterized by constraints (2) and (3) over the non-negative orthant. Estimation errors in the mean  $\boldsymbol{\mu}$  have a greater impact on portfolio performance than those in the covariance matrix  $\boldsymbol{\Sigma}$  or other parameters (Jagannathan and Ma 2003, DeMiguel et al. 2009, Gotoh and Takeda 2011). Indeed, Chopra and Ziemba (1993) show that errors in  $\boldsymbol{\mu}$  are around twenty times as significant as errors in the covariances (see also Ulf and Raimond 2006). Additionally, there is a common perception that return variances and covariances are much easier to estimate from historical data, and that a few factors can capture the general covariance structure, making it more manageable to estimate compared to expected returns (Merton 1980, Nelson 1992, Chan et al. 1999). This paper focuses on the estimation of  $\boldsymbol{\mu}$  and on how to deal with estimation errors in  $\boldsymbol{\mu}$ . For the remainder of the paper, we assume that the *true* covariance matrix  $\boldsymbol{\Sigma}$  is known.

Following Ben-Tal and Nemirovski (1999), the robust optimization problem can be reformulated as a quadratically constrained convex programming problem

$$\max_{\mathbf{x} \in \mathcal{X}} \{ \hat{\boldsymbol{\mu}}^\top \mathbf{x} - \kappa \sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}} \}. \quad (4)$$

The term  $\sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}}$  can be interpreted as an estimation risk that must be considered in addition to the market risk  $\mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x}$  (see Fabozzi et al. 2007, p. 371). While  $\kappa$  can be absorbed into  $\boldsymbol{\Xi}$  in the reformulation, we maintain a distinction between the two terms as  $\kappa$  may be interpreted as the weight on the estimation risk relative to the expected return term in the objective.

In this paper we assume that we have access to historical data on asset returns over an extended period. A fascinating question in data science is to estimate  $\boldsymbol{\mu}$  and  $\boldsymbol{\Xi}$  from this data, especially when one can expect some degree of obsolescence in the older data.

### 1.3. Contributions

A common critique of robust portfolio models is that they often lack guidance on how to define the uncertainty set. Indeed, the literature on constructing  $\mathcal{U}$  in portfolio management is scarce. Stubbs and Vance (2005) provide a comprehensive overview of the practical impacts of computing suitable estimation-error matrices. Additionally, there have been studies in which a scalar multiple of  $\boldsymbol{\Sigma}$  is used as the estimation-error matrix (Scherer 2007, Garlappi et al. 2007). Among these, Scherer (2007) has a skeptical take on robust optimization and shows that such a choice for  $\boldsymbol{\Xi}$  is equivalent to some other well known shrinkage approaches.

The primary contribution of this paper is a practical framework for constructing the uncertainty set  $\mathcal{U}$  for the expected return vector  $\boldsymbol{\mu}$ . We focus on constructing  $\mathcal{U}$  based on observed historical asset returns. We conduct theoretical and empirical analyses on the choice of the estimation-error matrix  $\Xi$  and the parameter  $\kappa$ , and provide empirically strong choices for the sample size of historical data used to estimate  $\boldsymbol{\mu}$ . We first give theoretical results on the class of *diagonal* estimation-error matrices for the choice of  $\Xi$  in Section 2. In particular, we show that one can achieve an arbitrarily small loss in the expected portfolio return as compared to the optimal portfolio. We accomplish this by showing that there exists a choice of parameters for the diagonal estimation-error matrix  $\Xi$  such that the resulting solution to (4) is arbitrarily close to the optimal portfolio. This is an existential result. In the following sections, we address constructive aspects of the uncertainty set. In Section 3, we evaluate different sample sizes for constructing the estimate  $\hat{\boldsymbol{\mu}}$  of  $\boldsymbol{\mu}$  from historical returns. Our results highlight the existence of a gap between the expected returns of the Markowitz mean-variance portfolio, constructed using the estimated  $\hat{\boldsymbol{\mu}}$ , and the optimal portfolio, constructed using the true  $\boldsymbol{\mu}$ , across all sample sizes. We observe that larger sample sizes do not produce superior portfolios; rather, an “intermediate” sample size achieves the best results. We explore choices for the estimation-error matrix  $\Xi$  through empirical experiments and present a heuristic to calibrate  $\kappa$  in Section 4. We perform computational experiments on simulated, synthetic data drawn from distributions with parameters constructed from historical asset returns and devise choices for  $\Xi$  and  $\kappa$  that lead to statistically significant improvements. Given that it is infeasible to meaningfully obtain valid and statistically significant results on real-world returns, we validate our findings on additional synthetic data with added temporal uncertainty in the asset returns. Our results demonstrate that a robust portfolio model featuring a family of diagonal estimation-error matrices and an appropriate choice for  $\kappa$  found using our proposed heuristic can significantly improve the performance of the Markowitz model. Section 5 contains some concluding remarks.

## 2. Diagonal Estimation-Error Matrices

In this section, we analyze the class of diagonal estimation-error matrices. Such matrices were first studied by Stubbs and Vance (2005), where the authors argue that a simple diagonal estimation-error matrix is easier to generate than a dense estimation-error matrix. However, a natural question arises: does there exist a trade-off in the performance of the robust portfolio if constructed with the simpler diagonal estimation-error matrix? We address this question by presenting our main theoretical result, which states that for any estimate  $\hat{\boldsymbol{\mu}}$  of  $\boldsymbol{\mu}$ , one can always choose a diagonal, positive definite matrix  $\Xi$  and a positive parameter  $\kappa$  such that the resulting robust portfolio has an expected return arbitrarily close to that of the optimal portfolio. This is an existential theorem, which shows that one can focus on diagonal estimation-error matrices without compromising on the quality of the portfolio. We consider constructive aspects of such a matrix in later sections.

Given a feasible portfolio  $\hat{\mathbf{x}}$  and an optimal portfolio  $\mathbf{x}^*$ , we define the *loss* in expected returns as

$$\text{loss}(\hat{\mathbf{x}}) = \boldsymbol{\mu}^\top \mathbf{x}^* - \boldsymbol{\mu}^\top \hat{\mathbf{x}},$$

where  $\boldsymbol{\mu}$  is the true expected return. A solution  $\hat{\mathbf{x}}^R$  to the robust portfolio problem

$$\max_{\mathbf{x}} \hat{\boldsymbol{\mu}}^\top \mathbf{x} - \kappa \sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}} \quad (5)$$

$$\text{subject to } \mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} \leq v \quad (6)$$

$$\mathbf{1}^\top \mathbf{x} = 1 \quad (7)$$

$$\mathbf{x} \geq \mathbf{0} \quad (8)$$

depends on the expected return estimate  $\hat{\boldsymbol{\mu}}$ , the estimation-error matrix  $\boldsymbol{\Xi}$ , and the parameter  $\kappa$ . In this case, assuming feasibility, we write

$$\text{loss}(\hat{\boldsymbol{\mu}}, \boldsymbol{\Xi}, \kappa) = \boldsymbol{\mu}^\top \mathbf{x}^* - \boldsymbol{\mu}^\top \hat{\mathbf{x}}^R.$$

**THEOREM 1.** *Given  $\epsilon > 0$ , for every  $\hat{\boldsymbol{\mu}}$  there exists a diagonal, positive definite matrix  $\boldsymbol{\Xi}$  and  $\kappa > 0$  such that  $\text{loss}(\hat{\boldsymbol{\mu}}, \boldsymbol{\Xi}, \kappa) < \epsilon$ .*

*Proof.* Let  $\mathcal{X} := \{\mathbf{x} \in \mathbb{R}_+^n : 2\mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} \leq v, \mathbf{1}^\top \mathbf{x} = 1\}$ , let  $I^0 := \{i \in \{1, 2, \dots, n\} : x_i = 0 \text{ for all } \mathbf{x} \in \mathcal{X}\}$  and  $I^+ := \{1, 2, \dots, n\} \setminus I^0$ . Note that  $I^0$  may be nonempty when  $v$  is the minimum variance of a portfolio  $\mathbf{x} \in \mathbb{R}_+^n$  such that  $\mathbf{1}^\top \mathbf{x} = 1$ . Pick  $\tilde{\mathbf{x}} \in \text{rint}(\mathcal{X})$ , where  $\text{rint}(\mathcal{X})$  denotes the relative interior of  $\mathcal{X}$ . Note that  $\tilde{x}_i = 0$  for  $i \in I^0$  and  $\tilde{x}_i > 0$  for  $i \in I^+$ . Let  $\tilde{\mathbf{x}}^* = (1 - \epsilon)\mathbf{x}^* + \epsilon\tilde{\mathbf{x}}$  for  $\epsilon \in (0, 1)$ . Then  $\tilde{\mathbf{x}}^* \in \text{rint}(\mathcal{X})$ .

Observe that  $\tilde{\mathbf{x}}^* \rightarrow \mathbf{x}^*$  as  $\epsilon \rightarrow 0$ . This implies  $\boldsymbol{\mu}^\top \tilde{\mathbf{x}}^* \rightarrow \boldsymbol{\mu}^\top \mathbf{x}^*$  as  $\epsilon \rightarrow 0$ . Therefore, the loss corresponding to  $\tilde{\mathbf{x}}^*$  is arbitrarily small. We now show that there exists a choice of  $\boldsymbol{\Xi}$  and  $\kappa$  such that the solution to the robust optimization problem is arbitrarily close to  $\tilde{\mathbf{x}}^*$ .

First, suppose  $\hat{\boldsymbol{\mu}} = \mathbf{0}$ . By choosing  $\kappa > 0$ ,  $\xi_i = \frac{1}{\tilde{x}_i^*}$  for  $i \in I^+$  and  $\xi_i = 1$  for  $i \in I^0$ , it is easy to check using the Kuhn-Tucker conditions that  $\tilde{\mathbf{x}}^*$  solves the problem  $\max_{\mathbf{x} \in \mathcal{X}'} -\kappa \sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}} \equiv \min_{\mathbf{x} \in \mathcal{X}'} \mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}$ , where  $\mathcal{X}' := \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} \leq v, \sum_{i \in I^+} x_i = 1\}$ .

For any given  $\hat{\boldsymbol{\mu}}$ , again let  $\xi_i = \frac{1}{\tilde{x}_i^*}$  for  $i \in I^+$  and  $\xi_i = 1$  for  $i \in I^0$ . Consider the problem  $\max_{\mathbf{x} \in \mathcal{X}} \hat{\boldsymbol{\mu}}^\top \mathbf{x} - \kappa \sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}} \equiv \min_{\mathbf{x} \in \mathcal{X}} \sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}} - \eta \hat{\boldsymbol{\mu}}^\top \mathbf{x} =: g(\eta)$  with an optimal solution  $\mathbf{x}^*(\eta)$  where  $\eta = \frac{1}{\kappa}$ . Since the objective function is strictly convex and  $\mathcal{X}$  is a convex set,  $\mathbf{x}^*(\eta)$  is unique. Define  $M := \max_{\mathbf{x} \in \mathcal{X}} \hat{\boldsymbol{\mu}}^\top \mathbf{x}$ . The existence of the maximum follows from the boundedness of  $\mathcal{X}$ . Let  $h(\eta) := \min_{\mathbf{x} \in \mathcal{X}} \sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}} - \eta M = \sqrt{\tilde{\mathbf{x}}^{*\top} \boldsymbol{\Xi} \tilde{\mathbf{x}}^*} - \eta M$ . Clearly,  $h(\eta) \leq g(\eta) \leq \sqrt{\tilde{\mathbf{x}}^{*\top} \boldsymbol{\Xi} \tilde{\mathbf{x}}^*} - \eta \hat{\boldsymbol{\mu}}^\top \tilde{\mathbf{x}}^*$ . Taking the limit as  $\eta \rightarrow 0$ , we get  $\sqrt{\tilde{\mathbf{x}}^{*\top} \boldsymbol{\Xi} \tilde{\mathbf{x}}^*} \leq \lim_{\eta \rightarrow 0} g(\eta) \leq \sqrt{\tilde{\mathbf{x}}^{*\top} \boldsymbol{\Xi} \tilde{\mathbf{x}}^*}$ . Therefore  $\lim_{\eta \rightarrow 0} g(\eta) = \sqrt{\tilde{\mathbf{x}}^{*\top} \boldsymbol{\Xi} \tilde{\mathbf{x}}^*} = g(0)$ . Thus, the solution  $\mathbf{x}^*(\eta)$  to the robust optimization problem is arbitrarily close to  $\tilde{\mathbf{x}}^*$ .  $\square$

We now give an example to show that it is not always possible to choose a diagonal positive definite matrix  $\Xi$  and a  $\kappa$  such that  $\text{loss}(\hat{\boldsymbol{\mu}}, \Xi, \kappa) = 0$ .

EXAMPLE 1. Consider a two asset portfolio optimization problem. Let  $\sigma_{11} < v$ ,  $\sigma_{22} > v$ , and  $\mu_1 > \mu_2$ . Suppose  $\hat{\mu}_1 < \hat{\mu}_2$ . Then there does not exist a diagonal positive definite matrix  $\Xi$  and a parameter  $\kappa$  such that  $\text{loss}(\hat{\boldsymbol{\mu}}, \Xi, \kappa) = 0$ .

It is easy to see that the optimal portfolio to the problem (1)-(3) is  $\mathbf{x}^* = (x_1 = 1, x_2 = 0)$  and that it is unique. Moreover, we observe that both constraint (6) and the constraint  $x_1 \geq 0$  are not tight for  $\mathbf{x}^*$ . Then the Lagrangian dual of the robust portfolio problem can be written as

$$\min_{\lambda \in \mathbb{R}, \tau \in \mathbb{R}_+} \max_{x_1, x_2} \hat{\mu}_1 x_1 + \hat{\mu}_2 x_2 - \kappa \sqrt{x_1^2 \xi_1 + x_2^2 \xi_2} + \lambda(1 - (x_1 + x_2)) + \tau x_2$$

with the optimality conditions

$$x_1 + x_2 = 1 \tag{9}$$

$$\hat{\mu}_1 - \frac{\kappa x_1 \xi_1}{\sqrt{x_1^2 \xi_1 + x_2^2 \xi_2}} - \lambda = 0 \tag{10}$$

$$\hat{\mu}_2 - \frac{\kappa x_2 \xi_2}{\sqrt{x_1^2 \xi_1 + x_2^2 \xi_2}} - \lambda + \tau = 0. \tag{11}$$

Since  $\kappa \geq 0$ ,  $\Xi \succ 0$  and  $\tau \geq 0$ , the optimality conditions imply that  $\hat{\mu}_1 = \lambda + \kappa \sqrt{\xi_1} \geq \lambda$  and  $\hat{\mu}_2 = \lambda - \tau \leq \lambda$  for  $\mathbf{x}^*$  for any choice of  $\xi_1, \xi_2$ . However, this is a contradiction to since  $\hat{\mu}_1 < \hat{\mu}_2$ .

Despite it not being possible to achieve zero loss generally, our next result states a sufficient condition for obtaining zero loss.

THEOREM 2. *If all the assets are active in the true optimal solution, then for every  $\hat{\boldsymbol{\mu}}$  there exists a diagonal, positive definite matrix  $\Xi$  and  $\kappa > 0$  such that  $\text{loss}(\hat{\boldsymbol{\mu}}, \Xi, \kappa) = 0$ .*

*Proof.* The result follows by observing  $\mathbf{x}^* \in \text{rint}(\mathcal{X})$  and by using the fact that  $\mathbf{x}^*$  solves the problem  $\max_{\mathbf{x} \in \mathcal{X}} -\kappa \sqrt{\mathbf{x}^\top \Xi \mathbf{x}} \equiv \min_{\mathbf{x} \in \mathcal{X}} \mathbf{x}^\top \Xi \mathbf{x}$  when  $\boldsymbol{\xi} = \frac{1}{\mathbf{x}^*}$  in the proof of Theorem 1.  $\square$

### 3. Estimating the Expected Return Vector Using Observed Historical Asset Returns

In this section, we address the estimation of the vector  $\boldsymbol{\mu}$  of expected asset returns when we have access to historical data. If the historical returns are independent and identically distributed (i.i.d.) random variables, then certainly, by the law of large numbers, we obtain the best estimate for  $\boldsymbol{\mu}$  by using all of the data. However, in reality, the historical returns may become obsolete over time. Therefore it may not be appropriate to assume time independence over long periods. Indeed, Ulf and Raimond (2006) show that even the simple equal weight portfolio outperform portfolios constructed under the assumption that the returns are i.i.d. Our focus is on determining an appropriate sample

size for estimating  $\boldsymbol{\mu}$  from historical returns. Suppose  $r^1, r^2, \dots, r^H$  are the vectors of historical asset returns observed in the real world in some time horizon  $1, 2, \dots, H$ , possibly over several decades. It is natural to estimate the expected asset returns as the average of the  $N$  most recent observations. But then we should ask: what choice of  $N$  gives the best estimates for  $\boldsymbol{\mu}$ ? To this end, we investigate the performance of portfolios constructed using the Markowitz model based on estimates of  $\boldsymbol{\mu}$  obtained using different sample sizes. To be able to repeat the experiments and obtain statistically significant results, we generate a set of synthetic, simulated asset returns.

We generate simulated returns  $\hat{r}^t \sim \mathcal{N}(\boldsymbol{\mu}^t, \boldsymbol{\Sigma})$ , following a multivariate normal distribution with a “true” expected return vector  $\boldsymbol{\mu}^t$  and covariance matrix  $\boldsymbol{\Sigma}$ . That is, the simulated returns  $\hat{r}^t$  are independent random variables, drawn from different distributions for each  $t = 1, \dots, H$ . We refer the reader to Appendix A for our construction of the parameters  $\boldsymbol{\mu}^t$  and  $\boldsymbol{\Sigma}$ . To summarize,  $\boldsymbol{\mu}^t$  is computed from the real data  $r^i$  by averaging over  $T$  consecutive time periods, centered at  $t$ . Asset returns  $\hat{r}^i$  are then generated using the distribution  $\mathcal{N}(\boldsymbol{\mu}^i, \boldsymbol{\Sigma})$ . A Markowitz portfolio  $\hat{\mathbf{x}}^t$  is constructed at time  $t$  using the estimate  $\hat{\boldsymbol{\mu}}^t = \frac{1}{N} \sum_{i=t-N+1}^t \hat{r}^i$ . This portfolio is then held until period  $t+T$  and evaluated using  $\boldsymbol{\mu}^{t+T}$ , which is derived from future returns not used in the portfolio’s construction at time  $t$ . Clearly, if the data  $r^i$  were independent over time, the vector  $\boldsymbol{\mu}^{t+T}$  would be independent of the values  $\boldsymbol{\mu}^i$  for  $i \leq t$ . Therefore, estimates  $\hat{\boldsymbol{\mu}}^t$  that are averaged over larger  $N$  would provide better estimates of  $\boldsymbol{\mu}^{t+T}$  and therefore better Markowitz portfolios  $\hat{\mathbf{x}}^t$ . Our experiments below show that this is not the case.

In addition to the portfolio  $\hat{\mathbf{x}}^t := \max_{\mathbf{x} \in \mathcal{X}} \hat{\boldsymbol{\mu}}^{t\top} \mathbf{x}$ , we also construct the optimal portfolio at time  $t$ , namely  $\mathbf{x}^{t*} := \max_{\mathbf{x} \in \mathcal{X}} \boldsymbol{\mu}^{t\top} \mathbf{x}$ . We compute its *true expected return*  $\boldsymbol{\mu}^{t+T\top} \mathbf{x}^{t*}$ , assuming we hold it for the next  $T$  periods, and, for each choice of  $N$ , we compute the *estimated expected return*  $\hat{\boldsymbol{\mu}}^{t+T\top} \hat{\mathbf{x}}^t$  of the Markowitz portfolio  $\hat{\mathbf{x}}^t$ , and its *actual expected return*  $\boldsymbol{\mu}^{t+T\top} \hat{\mathbf{x}}^t$ , all averaged over the simulated runs and over the time periods  $t$ . We evaluate the average performance of these portfolios over several simulated runs using *efficient frontiers* that plot the expected returns against different risk thresholds  $v$ . We achieve a standard error less than 0.05 for each reported value. A comprehensive overview of our experimental setup can be found in Appendix A.

Figures 1 and 2 present the experimental results for an 11-asset dataset with monthly returns and a 12-asset dataset with daily returns. The average expected returns for the equal-weight portfolios were 1.21% per month and 0.049% per day, respectively, while the minimum variance portfolios yielded 1.23% and 0.034% on average, respectively. The corresponding minimum variances were 11.46 and 0.84. Figure 1 plots efficient frontiers for Markowitz portfolios obtained for different choices of  $N$  in the estimation of  $\hat{\boldsymbol{\mu}}^t$ . Note that for any  $N > 1$ , the Markowitz optimal portfolios outperform the equal-weight portfolio when the risk threshold is  $v \geq 21.25$ . This trend is also observed in Figure 2 for  $v \geq 1.75$ . Both figures reveal a significant gap between true and actual

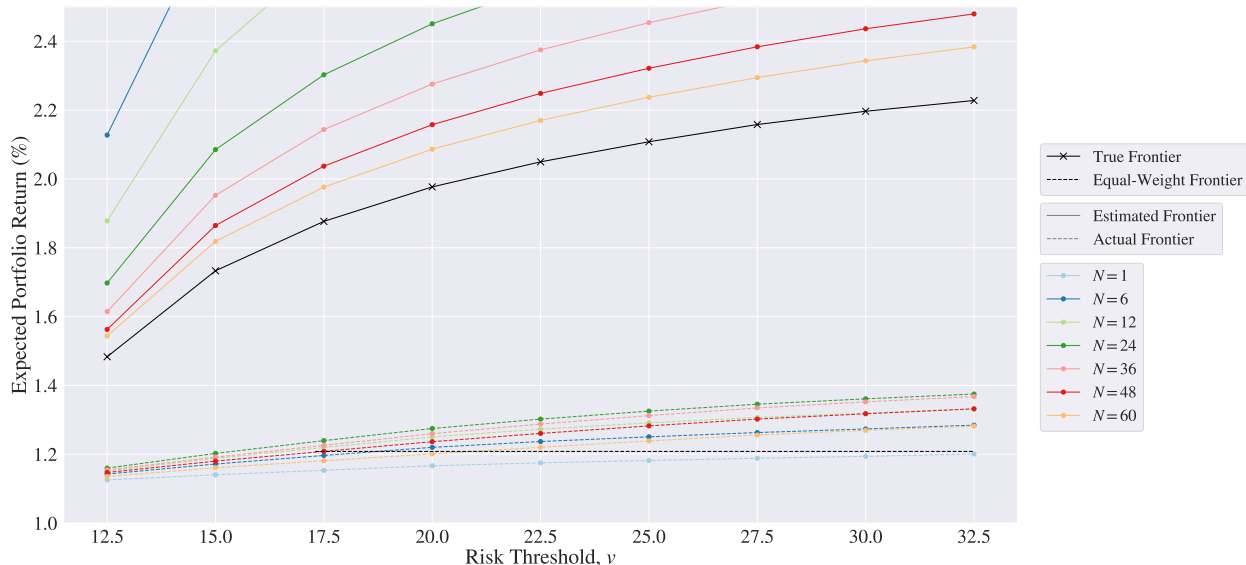


Figure 1 Efficient frontiers of portfolios constructed on the 11-asset GICS monthly returns dataset.

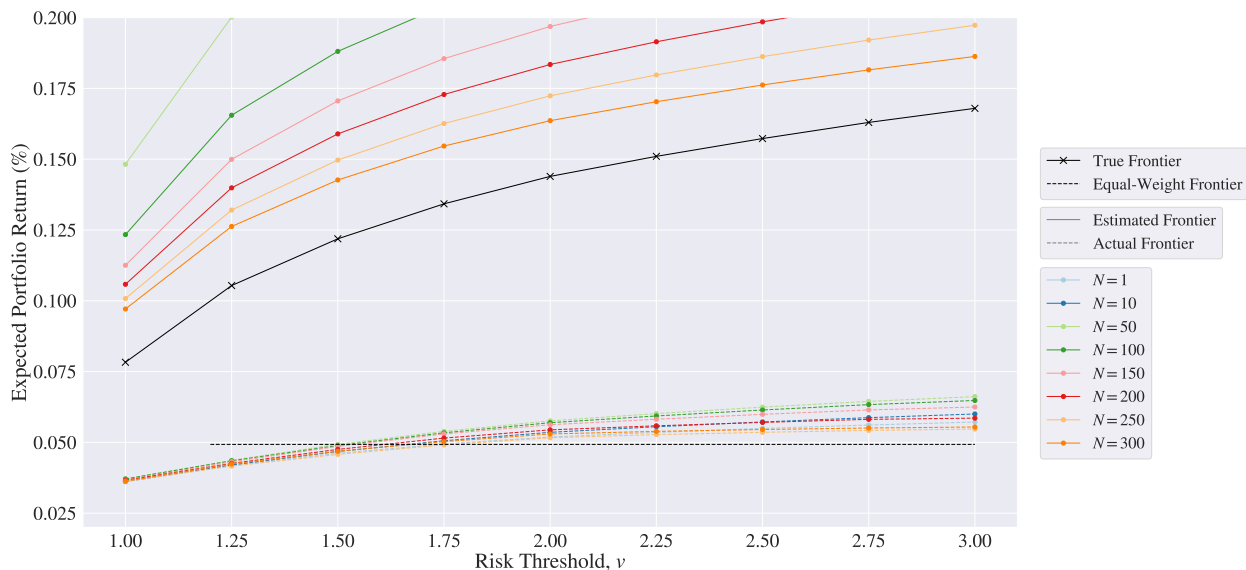


Figure 2 Efficient frontiers of portfolios constructed on the 12-asset Fama-French daily returns dataset.

expected returns for all values of  $N$ . This gap quantifies the value of information: Knowing  $\mu$  improves the expected portfolio return from around 1.3% per month to about 2% per month for a risk level  $v = 20$ . Interestingly, the actual expected returns initially improve with increasing  $N$ , but then deteriorate for larger  $N$ . The best results are achieved by choosing  $N = 24$  months (Figure 1) and  $N = 50$  days (Figure 2). Large values of  $N$  produce poor portfolios. Perhaps this is not surprising as the returns are not identically distributed over time. We repeated the experiments on other data sets summarized in Appendix C. The results are similar. The gap between true and actual expected portfolio returns is very significant. The quality of the Markowitz portfolios improves as  $N$  increases up to some value  $N^*$ , and then it deteriorates as  $N$  increases further. For



monthly data, averaging over 5 data sets and 3 risk levels, we find  $N^* \approx 28$  albeit with substantial variation; for daily data, averaging over 4 data sets and 3 risk levels,  $N^* \approx 100$  again with much variation. In Section 4 we use  $N^* = 24$  for monthly datasets and  $N^* = 100$  for daily datasets. These values for  $N^*$  are not intended to be prescriptive across all datasets, but their existence demonstrates that one cannot mitigate the effects of estimation errors on  $\boldsymbol{\mu}$  by using a large  $N$ . This lends further support to the usefulness of a robust optimization framework.

While there exist several other methods for estimating the expected return vector, Jagannathan and Ma (2003), DeMiguel et al. (2009) show that estimation errors in the sample mean persist regardless of the estimation technique used. In our study, we limit our analyses to estimators obtained with observed historical asset returns.

The significant gap between the true and actual frontiers observed in Figures 1 and 2 highlights the potential for better portfolios. We show in the next section that robust optimization can indeed deliver superior portfolios.

#### 4. Empirical Study of the Estimation-Error Matrix

In this section, we investigate choices for the estimation-error matrix  $\boldsymbol{\Xi}$  in (4) and the associated parameter  $\kappa$ . While robust optimization techniques have been previously employed in the context of portfolio management, there is little guidance on constructing  $\boldsymbol{\Xi}$ . Following our theoretical results from Section 2, we explore the class of diagonal estimation-error matrices. Such a choice for  $\boldsymbol{\Xi}$  requires practitioners to only calibrate  $n + 1$  parameters. We first conduct experiments on simulated streams of i.i.d. returns. In later parts of the section, we validate the results of these experiments with the setting introduced in Section 3, for which we consider returns that are not necessarily independent ally distributed.

Following the works of Scherer (2007) and Garlappi et al. (2007), we are initially motivated to examine candidate choices for a diagonal  $\boldsymbol{\Xi}$  that involve the covariance matrix  $\boldsymbol{\Sigma}$ . Consider the family of matrices  $\boldsymbol{\Xi}(k) = \text{diag}\left(\frac{1}{\sigma_i^k}\right)$ , where  $k \in \mathbb{R}$ . We consider the following choices for  $\boldsymbol{\Xi}(k)$ .

1.  $\boldsymbol{\Xi}(0) = I$ : The identity matrix may be appealing to decision-makers because it requires calibrating only one parameter,  $\kappa$ . A large value of  $\kappa$  corresponds to choosing a solution close to the equally weighted portfolio. As  $\kappa$  decreases more emphasis is put on the estimates  $\hat{\boldsymbol{\mu}}$ . This trade off makes the tuning of the parameter  $\kappa$  fairly intuitive.
2.  $\boldsymbol{\Xi}(-2) = \text{diag}(\sigma_i^2)$ : This choice of  $\boldsymbol{\Xi}$  appeals to a fundamental intuition: assets with a higher variance tend to have a poorer estimate of their expected returns.
3.  $\boldsymbol{\Xi}(2) = \text{diag}\left(\frac{1}{\sigma_i^2}\right)$ : This choice of  $\boldsymbol{\Xi}$  incorporates the notion of a ‘‘risk premium’’. This approach is particularly relevant when investors believe in the Capital Asset Pricing Model (CAPM), suggesting that assets with a greater variance in their return also have a greater expected return.

Selecting an appropriate value for  $\kappa$  is critical. Given the objective function  $\hat{\boldsymbol{\mu}}^\top \mathbf{x} - \kappa \sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}}$  in (4), we may consider  $\kappa$  to be the weight imposed on the penalty term  $\sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}}$ , relative to the expected return term  $\hat{\boldsymbol{\mu}}^\top \mathbf{x}$ . Therefore, we design a heuristic to calibrate  $\kappa$  such that a target ratio  $r$  between these two terms  $\frac{\hat{\boldsymbol{\mu}}^\top \mathbf{x}}{\sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}}}$  is achieved. We present the heuristic as Algorithm 1 in Appendix B. We evaluate three potential choices for the target  $r$ , namely 2, 3, 4.

Our experimental framework is based on real-world stocks in the S&P 500 index and the Fama-French industry portfolio data library. We assume that the returns follow a multivariate normal distribution with mean  $\boldsymbol{\mu}$  and variance  $\boldsymbol{\Sigma}$  computed as follows. Using the observed historical asset returns  $r^i$  for  $i = 1, \dots, H$ , we set  $\boldsymbol{\mu} = \frac{1}{H} \sum_{t=1}^H r^i$ , and  $\boldsymbol{\Sigma}$  to be the covariance matrix of the returns. The estimated vector  $\hat{\boldsymbol{\mu}}$  is then the sample average of  $N$  random samples generated from the distribution  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . Based on the results previously presented in Section 3, we let  $N = 24$  for our data on monthly returns, and  $N = 100$  for our data on daily returns. Our goal is to investigate the various choices for  $\boldsymbol{\Xi}$  and  $\kappa$  mentioned above. We construct portfolios over four evenly-spaced risk thresholds  $v$  for each dataset, denoted by Very Low, Low, Medium, and High.

We present the results of the experiments as the percentage gap closed by the robust portfolio compared to the Markowitz portfolio. More specifically, we report  $(\bar{R} - \bar{M}) / (T - \bar{M})$  where  $\bar{R}$  denotes the expected returns of the robust portfolio,  $\bar{M}$  denotes the expected returns of the Markowitz portfolio, and  $T$  denotes the true optimal expected return obtained by solving the problem in (1)-(3). The values for  $\bar{R}$  and  $\bar{M}$  are averages of 10000 simulated runs. Consequently, the standard error of each reported percentage gap closed is less than 0.1. The optimal target ratio  $r$  was chosen by conducting a grid search for each  $\boldsymbol{\Xi}(k)$ , with the returns for the optimal parameters presented. We further consolidate the results by presenting the average percentage gap closed across all datasets at four different risk thresholds in Table 1. The optimal target ratio was found to be  $r = 4$  for the three rows corresponding to monthly data. For daily data, the ratio was  $r = 4$  for  $\boldsymbol{\Xi}(-2)$ , and  $r = 2$  for  $\boldsymbol{\Xi}(0)$  and  $\boldsymbol{\Xi}(2)$ .

The largest percentage gaps closed for each risk threshold are highlighted in Table 1. Our simulations demonstrate that significant gains can be achieved over the Markowitz model with an appropriately chosen estimation-error matrix and parameter  $\kappa$ . Notably, even the simple choice of the identity matrix,  $\boldsymbol{\Xi}(0)$ , yields portfolios that outperform the Markowitz portfolio for Very Low, Low, and Medium risk levels. On the other hand, we find that  $\boldsymbol{\Xi}(-2)$  is not a strong choice for the estimation-error matrix, suggesting that errors in estimating the expected returns cannot be corrected based directly on the variance of the returns. Interestingly, robust portfolios constructed using  $\boldsymbol{\Xi}(2)$  consistently outperform the Markowitz portfolio across all risk thresholds in datasets with both monthly and daily returns, closing the largest gaps.

|         |  | Risk Threshold, $v$           | Very Low    | Low        | Medium     | High       |
|---------|--|-------------------------------|-------------|------------|------------|------------|
| Monthly |  | Gap Closed (%) with $\Xi(-2)$ | 0.5         | -1.7       | -4.6       | -7.1       |
|         |  | Gap Closed (%) with $\Xi(0)$  | 2.8         | 2.7        | 1.0        | -1.0       |
|         |  | Gap Closed (%) with $\Xi(2)$  | <b>3.5</b>  | <b>5.2</b> | <b>4.9</b> | <b>3.6</b> |
|         |  | Risk Threshold, $v$           | Very Low    | Low        | Medium     | High       |
| Daily   |  | Gap Closed (%) with $\Xi(-2)$ | 6.4         | 3.1        | -0.1       | -1.0       |
|         |  | Gap Closed (%) with $\Xi(0)$  | 10.8        | 6.7        | 2.5        | 1.0        |
|         |  | Gap Closed (%) with $\Xi(2)$  | <b>11.7</b> | <b>8.6</b> | <b>5.6</b> | <b>5.2</b> |

**Table 1** Average percentage gap closed by the robust portfolio compared to the Markowitz portfolio for different choices of estimation-error matrices:  $\Xi(-2) = \text{diag}(\sigma_i^2)$ ,  $\Xi(0) = I$ , and  $\Xi(2) = \text{diag}\left(\frac{1}{\sigma_i^2}\right)$ . The percentages are averages across 5 datasets with varying number of assets.

Overall, for the best choices of  $\Xi$  and  $\kappa$ , the gap between the expected returns of the Markowitz portfolios and the optimal portfolios is reduced by an average of 6.1% across all datasets and risk thresholds in our experiments. Based on these preliminary results, we propose that the family of diagonal estimation-error matrices  $\Xi(k) = \text{diag}\left(\frac{1}{\sigma_i^k}\right)$  for *positive*  $k \in \mathbb{R}_+$  yields robust portfolios with improved expected returns over Markowitz portfolios. The intuition is that, for a desired level of portfolio risk in the constraint, the objective should penalize assets with low volatility, which tend to have lower returns, and favor meeting the risk constraint through diversification using assets with greater volatility, which tend to have better returns.

We conclude the section by validating our choice through additional experiments by incorporating a temporal component to the data. We utilize the same experimental setup as in Section 3 and described in Appendix A. We consider  $k = 2, 4, 10$  for the estimation-error matrices  $\Xi(k)$ . We let  $N = 24$  for monthly returns and  $N = 100$  for our daily returns, and consider an array of risk thresholds  $v$  for each dataset, with  $\kappa$  calibrated with the optimal target ratios found with a grid search. We present our results in Table 2 as the percentage gap closed by the robust portfolio compared to the Markowitz portfolio for each dataset, with a standard error less than 0.05 for each reported value.

Our findings once again show that robust portfolio with the appropriate estimation-error matrix can improve upon the Markowitz portfolio. In particular, the choice of  $\Xi(4)$  results in robust portfolios that consistently outperform the Markowitz portfolios across all datasets on average for all risk thresholds, with  $\Xi(10)$  portfolios covering the largest gaps. For the best choices of  $\Xi$  and  $\kappa$ , we observe that the gap between the expected returns of the Markowitz portfolios and the optimal portfolios is closed by an average of 8% across all datasets and risk thresholds in our experiments. Indeed, these experiments suggests that our recommendations for  $\Xi$ , and our heuristic for choosing  $\kappa$ , can be extended and applied to settings where the historical returns are not identically distributed random variables.

| Dataset | Units       | Number of Sectors | Risk Threshold, $v$           |                               |             |             |             |             |
|---------|-------------|-------------------|-------------------------------|-------------------------------|-------------|-------------|-------------|-------------|
|         |             |                   | Very Low                      | Low                           | Medium      | High        |             |             |
| GICS    | Monthly     | 11                | Gap Closed (%) with $\Xi(2)$  | <b>4.1</b>                    | <b>3.6</b>  | 3.5         | 2.4         |             |
|         |             |                   | Gap Closed (%) with $\Xi(4)$  | 2.1                           | 2.4         | <b>3.7</b>  | 3.7         |             |
|         |             |                   | Gap Closed (%) with $\Xi(10)$ | -2.1                          | -2.4        | 2.7         | <b>6.3</b>  |             |
|         |             | 5                 | Gap Closed (%) with $\Xi(2)$  | 0.7                           | 0.3         | 0.4         | 0.9         |             |
|         |             |                   | Gap Closed (%) with $\Xi(4)$  | <b>1.9</b>                    | 1.8         | 1.8         | 2.1         |             |
|         |             |                   | Gap Closed (%) with $\Xi(10)$ | 1.7                           | <b>2.1</b>  | <b>2.4</b>  | <b>3.8</b>  |             |
|         |             | 10                | Gap Closed (%) with $\Xi(2)$  | <b>4.0</b>                    | <b>5.8</b>  | 5.9         | 5.3         |             |
|         |             |                   | Gap Closed (%) with $\Xi(4)$  | 3.7                           | 5.7         | 6.3         | 6.2         |             |
|         |             |                   | Gap Closed (%) with $\Xi(10)$ | 0.9                           | 5.1         | <b>9.9</b>  | <b>11.1</b> |             |
|         | Monthly     | 12                | Gap Closed (%) with $\Xi(2)$  | 3.3                           | 4.2         | 3.8         | 3.0         |             |
|         |             |                   | Gap Closed (%) with $\Xi(4)$  | <b>3.9</b>                    | 5.8         | 6.2         | 5.8         |             |
|         |             |                   | Gap Closed (%) with $\Xi(10)$ | 1.6                           | <b>6.3</b>  | <b>10.2</b> | <b>10.7</b> |             |
|         |             | 17                | Gap Closed (%) with $\Xi(2)$  | 3.6                           | 2.8         | 2.3         | 1.8         |             |
|         |             |                   | Gap Closed (%) with $\Xi(4)$  | 4.2                           | 4.4         | 3.7         | 3.1         |             |
|         |             |                   | Gap Closed (%) with $\Xi(10)$ | <b>4.5</b>                    | <b>5.0</b>  | <b>7.7</b>  | <b>6.5</b>  |             |
|         | Fama-French | Monthly           | 5                             | Gap Closed (%) with $\Xi(2)$  | 5.0         | 6.5         | 6.4         | 5.9         |
|         |             |                   |                               | Gap Closed (%) with $\Xi(4)$  | 6.4         | 8.5         | 8.7         | 7.9         |
|         |             |                   |                               | Gap Closed (%) with $\Xi(10)$ | <b>6.7</b>  | <b>10.0</b> | <b>11.9</b> | <b>12.7</b> |
| 10      |             |                   | Gap Closed (%) with $\Xi(2)$  | 4.9                           | 3.3         | 2.1         | 1.8         |             |
|         |             |                   | Gap Closed (%) with $\Xi(4)$  | 10.3                          | 8.9         | 7.2         | 6.7         |             |
|         |             |                   | Gap Closed (%) with $\Xi(10)$ | <b>12.7</b>                   | <b>14.6</b> | <b>14.3</b> | <b>13.6</b> |             |
| Daily   |             |                   | 12                            | Gap Closed (%) with $\Xi(2)$  | 7.3         | 6.2         | 4.5         | 3.9         |
|         |             |                   |                               | Gap Closed (%) with $\Xi(4)$  | 7.1         | 6.0         | 4.9         | 4.7         |
|         |             |                   |                               | Gap Closed (%) with $\Xi(10)$ | <b>9.6</b>  | <b>11.5</b> | <b>12.2</b> | <b>12.2</b> |
|         |             | 17                | Gap Closed (%) with $\Xi(2)$  | <b>3.4</b>                    | 5.2         | 5.0         | 4.2         |             |
|         |             |                   | Gap Closed (%) with $\Xi(4)$  | 3.2                           | <b>6.2</b>  | 6.9         | 6.7         |             |
|         |             |                   | Gap Closed (%) with $\Xi(10)$ | 1.4                           | 5.7         | <b>9.8</b>  | <b>11.5</b> |             |

**Table 2** Percentage gap closed by the robust portfolio with estimation-error matrices  $\Xi(2) = \text{diag}\left(\frac{1}{\sigma_i^2}\right)$ ,  $\Xi(4) = \text{diag}\left(\frac{1}{\sigma_i^4}\right)$ , and  $\Xi(10) = \text{diag}\left(\frac{1}{\sigma_i^{10}}\right)$  compared to the Markowitz portfolio, with temporal uncertainty in the data. Columns correspond to the portfolios constructed along various risk thresholds.

## 5. Conclusion

In this paper, we offer a framework for constructing an ellipsoidal uncertainty set of the expected asset returns; this is needed in robust portfolio optimization. Our work addresses a gap in the literature by providing both theoretical and empirical insights into the selection of the estimation-error matrix  $\Xi$  and the weight  $\kappa$  assigned to the estimation risk. These choices are critical to the performance of robust portfolio optimization models. We prove an existential theorem showing that diagonal estimation-error matrices can yield robust portfolios with arbitrarily small loss in expected return compared to an optimum portfolio. Additionally, we challenge a conventional assumption often made for data-driven optimization methods that larger sample sizes from historical data invariably reduce estimation error. Instead, we demonstrate demonstrating through empirical analysis that this is not always the case. Our empirical findings, based on synthetic data reflecting historical asset returns, support the practical usefulness of robust portfolio model as an alternative to the traditional Markowitz model. We find that the penalty term in the objective function can help construct better portfolios by discouraging assets with low volatility that

also tend to have low returns. Instead, it encourages diversification by incentivizing assets with greater volatility and expected returns. Thus the ellipsoid used in the robust formulation does more than just model uncertainty. Our work paves the way for further investigations in constructing estimation-error matrices for robust portfolio optimization, with implications for both academic research and practical financial decision-making.

## Acknowledgements

This research was supported by the U.S. Office of Naval Research under award number N00014-22-1-2528. The authors would like to thank Matthias Köppe for his valuable contributions to an earlier version of this manuscript. We also thank Javier Peña and Karan Singh for their helpful comments.

## Disclosure Statement

The authors report there are no competing interests to declare.

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## Appendix A: Detailed Experimental Setup

We conduct our experiments using simulated data modeled on stocks in the U.S. equity market. We use data on the historical returns of stocks in the S&P 500 index as consolidated by Kocuk and Cornuéjols (2020), classified into 11 sectors according to the Global Industrial Classification Standard (GICS). Table 3 gives the market-weighted averages and variances of their monthly returns over a 30-year period between January 1987 and December 2016. Additionally, we consider the 5-, 10-, 12-, and 17-sector datasets from the Fama-French data library. These datasets include monthly returns, for which we consider a 30-year period between March 1994 and February 2024, and daily returns, for which we consider a 10-year period between March 2014 and February 2024. The sectors are outlined in Tables 4-7 along with the market-weighted average and variance of their returns.

Observations of the financial market suggest that assumptions about structural stationarity are weak. Systemic changes to the market in recent history, such as the 1997 Asian financial crisis, the collapse of the dot-com bubble, the 2008 financial crisis, and the 2020 COVID-19 pandemic substantiate this claim. These events significantly impacted key industries across numerous sectors. Therefore, it seems more reasonable to consider multiple distributions that evolves over time rather than assuming one static distribution that

describes the asset returns as a whole. Consequently, it is unlikely that the asset returns are i.i.d. random variables. Instead, we model the asset returns for each time period as random variables drawn from possibly different multivariate normal distributions.

A fundamental challenge is extracting the *true* expected returns that parameterize the distributions the real-world observed historical data are drawn from. To address this concern and to obtain statistically significant results, we use simulated data in our experiments. We assume that the returns are normally distributed over time with varying mean returns  $\boldsymbol{\mu}^t$  and a fixed variance  $\boldsymbol{\Sigma}$  across all time periods,  $\text{Normal}(\boldsymbol{\mu}^t, \boldsymbol{\Sigma})$ . Consider planning horizon for  $t = 1, \dots, H$ . We compute the true expected returns as  $\boldsymbol{\mu}^t = \frac{1}{T} \sum_{i=t-\frac{T}{2}+1}^{t+\frac{T}{2}} r^i$  for  $t = \frac{T}{2}, \dots, H - \frac{T}{2}$ , where  $r^i$  is the realization of returns observed in the  $i^{\text{th}}$  time unit in the data and  $T$  is a prescribed parameter. We chose  $T = 30$  and  $T = 260$  for our experiments on the monthly and daily datasets, respectively. The parameter  $T$  was tuned in accordance with systemic changes in the market observed in the data. We let the true covariance matrix  $\boldsymbol{\Sigma}$  be the covariance of the asset returns in the stock market data. We then sample  $\hat{r}^t \sim \mathcal{N}(\boldsymbol{\mu}^t, \boldsymbol{\Sigma})$  to simulate additional streams of “observed” returns, and compute the estimated expected return vectors as  $\hat{\boldsymbol{\mu}}^t = \frac{1}{N} \sum_{i=t-N+1}^t \hat{r}^i$  for  $t = \frac{T}{2} + N - 1, \dots, H - \frac{T}{2} - T$ , where  $N$  is a chosen parameter for the sample size of historical time-units to consider for the estimation. We simulated 50 runs for experiments on monthly returns where  $H = 360$ , and 10 runs for experiments on daily returns where  $H = 2600$ .

We define the optimal portfolios as  $\mathbf{x}^{t*} := \max_{\mathbf{x} \in \mathcal{X}} \boldsymbol{\mu}^{t\top} \mathbf{x}$  for each  $t = \frac{T}{2}, \dots, H - \frac{T}{2}$ . That is, for each time period, we find the portfolio that maximizes the true expected return  $\boldsymbol{\mu}^t$ . Similarly, we define the estimated Markowitz portfolios as  $\hat{\mathbf{x}}^t := \max_{\mathbf{x} \in \mathcal{X}} \hat{\boldsymbol{\mu}}^{t\top} \mathbf{x}$  for each  $t = \frac{T}{2} + N - 1, \dots, H - \frac{T}{2} - T$ . We utilize *efficient frontiers* to quantify the performance of the portfolios. An efficient frontier plots the maximum expected return of a portfolio of assets as a function of the risk thresholds (Markowitz 1952). The true frontier is computed using the true expected returns of the assets and the optimal portfolios. The estimated frontier is computed by using the estimated expected returns and the estimated Markowitz portfolios, which describes the expected return of the Markowitz portfolios should the estimated parameters be realized. The actual frontier plots the expected return one actually observes on the true expected returns when one invests in the portfolios constructed with the estimated expected returns. That is, for each choice of  $N$ , we compute the *true expected return*  $\boldsymbol{\mu}^{t+T\top} \mathbf{x}^{t*}$ , the *estimated expected return*  $\hat{\boldsymbol{\mu}}^{t\top} \hat{\mathbf{x}}^t$ , and the *actual expected return*  $\boldsymbol{\mu}^{t+T\top} \hat{\mathbf{x}}^t$ , all averaged over the simulated runs and over the time periods  $t$ . We evaluate the portfolios at time  $t + T$  to negate any sampling bias in our set up.

All experiments were run on an Apple M3 processor with 8GB of memory, running Python version 3.12.4 and Gurobi Optimizer version 11.0.1 (Gurobi Optimization 2024).

## Appendix B: A Heuristic to Calibrate $\kappa$

Algorithm 1 first initializes the portfolios  $\bar{\mathbf{x}}$  and  $\bar{\bar{\mathbf{x}}}$ . Here,  $\bar{\mathbf{x}}$  is the equal-weight portfolio,  $\bar{\bar{\mathbf{x}}}$  is a portfolio that is normalized relative to the inverse of  $\xi_i$  across all assets. The heuristic also initializes the target  $r$  for the ratio  $\frac{\hat{\boldsymbol{\mu}}\bar{\mathbf{x}}}{\sqrt{\bar{\mathbf{x}}\boldsymbol{\Sigma}\bar{\mathbf{x}}}}$  to the middle point in a target range  $[l, u]$ . It then chooses the corresponding  $\kappa$  and solves for the associated robust portfolio  $\mathbf{x}^R$ . The heuristic returns  $\kappa$  if the ratio  $\frac{\hat{\boldsymbol{\mu}}\mathbf{x}^R}{\kappa\sqrt{\mathbf{x}^R\boldsymbol{\Sigma}\mathbf{x}^R}}$  falls in the target range  $[l, u]$ . If not,  $\kappa$  is re-calibrated and the ratio is checked again. We evaluate three potential choices for the range  $[l, u]$ , namely  $[1, 3]$ ,  $[2, 4]$ ,  $[3, 5]$ , conducting a grid search to determine the optimal values.



**Algorithm 1** Heuristic to calibrate  $\kappa$ 

**Input:** Estimates of the expected returns  $\hat{\boldsymbol{\mu}} \in \mathbb{R}^n$ , a diagonal positive definite estimation-error matrix  $\boldsymbol{\Xi} \in \mathbb{R}^{n \times n}$ , a lower bound  $l \in \mathbb{R}$  and an upper bound  $u \in \mathbb{R}$  for the ratio  $\frac{\hat{\boldsymbol{\mu}}^\top \mathbf{x}}{\kappa \sqrt{\mathbf{x}^\top \boldsymbol{\Xi} \mathbf{x}}}$ .

**Output:**  $\kappa \in \mathbb{R}$ .

- 1: Initialize  $\bar{\mathbf{x}} = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$ ,  $\bar{\boldsymbol{\xi}} = \left(\frac{1}{\sum_{i=1}^n \frac{1}{\xi_i}}, \dots, \frac{1}{\sum_{i=1}^n \frac{1}{\xi_i}}\right)$ ,  $r = \frac{u-l}{2}$ .
- 2: Compute  $\hat{\boldsymbol{\mu}} \bar{\mathbf{x}}$ ,  $\sqrt{\bar{\boldsymbol{\xi}}^\top \boldsymbol{\Xi} \bar{\boldsymbol{\xi}}}$ .
- 3: Choose  $\kappa = \frac{\hat{\boldsymbol{\mu}} \bar{\mathbf{x}}}{r \sqrt{\bar{\boldsymbol{\xi}}^\top \boldsymbol{\Xi} \bar{\boldsymbol{\xi}}}}$ .
- 4: **while** Stopping condition not met **do**
- 5:   Solve for  $\mathbf{x}^R$  the solution to (4).
- 6:   **if**  $l \leq \frac{\hat{\boldsymbol{\mu}} \mathbf{x}^R}{\kappa \sqrt{\mathbf{x}^R \boldsymbol{\Xi} \mathbf{x}^R}} \leq u$  **then**
- 7:     **return**  $\kappa$ .
- 8:   **else**
- 9:      $\kappa \leftarrow \frac{\hat{\boldsymbol{\mu}} \mathbf{x}^R}{r \sqrt{\mathbf{x}^R \boldsymbol{\Xi} \mathbf{x}^R}}$ .
- 10:   **end if**
- 11: **end while**

**Appendix C: Data Summary**

|            | Energy | Consumer<br>Discretionary | Consumer<br>Staples | Real<br>Estate | Industrials | Financials | Telecomm-<br>unications<br>Services | Information<br>Technology | Materials | Health<br>Care | Utilities |
|------------|--------|---------------------------|---------------------|----------------|-------------|------------|-------------------------------------|---------------------------|-----------|----------------|-----------|
| $\mu$      | 1.18   | 1.51                      | 1.39                | 1.15           | 1.29        | 1.33       | 1.03                                | 1.73                      | 1.39      | 1.42           | 1.01      |
| $\sigma^2$ | 39.5   | 28.3                      | 17.2                | 52.5           | 26.5        | 39.5       | 30.0                                | 50.5                      | 32.4      | 21.6           | 18.3      |

**Table 3** Sample averages and variances of historical returns of the GICS 11-sector dataset.

|         |            | Cnsmr  | Manuf  | HiTec  | Hlth   | Other  |
|---------|------------|--------|--------|--------|--------|--------|
| Daily   | $\mu$      | 0.0453 | 0.0457 | 0.0553 | 0.0500 | 0.0449 |
|         | $\sigma^2$ | 1.11   | 1.40   | 2.21   | 1.31   | 1.97   |
| Monthly | $\mu$      | 0.939  | 0.907  | 1.12   | 1.03   | 0.876  |
|         | $\sigma^2$ | 17.7   | 20.5   | 38.6   | 18.3   | 28.1   |

**Table 4** Sample averages and variances of historical returns of the Fama-French 5-sector dataset.

|         |            | NoDur  | Durbl  | Manuf  | Enrgy  | HiTec  | Telcm  | Shops  | Hlth   | Utils  | Other  |
|---------|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Daily   | $\mu$      | 0.0410 | 0.0480 | 0.0494 | 0.0516 | 0.0630 | 0.0316 | 0.0489 | 0.0500 | 0.0384 | 0.0449 |
|         | $\sigma^2$ | 0.885  | 3.17   | 1.45   | 2.83   | 2.69   | 1.63   | 1.42   | 1.31   | 1.23   | 1.97   |
| Monthly | $\mu$      | 0.887  | 1.01   | 1.02   | 1.04   | 1.29   | 0.646  | 1.02   | 1.05   | 0.813  | 0.900  |
|         | $\sigma^2$ | 14.1   | 75.2   | 25.0   | 46.2   | 48.3   | 27.4   | 21.6   | 18.1   | 17.3   | 28.1   |

**Table 5** Sample averages and variances of historical returns of the Fama-French 10-sector dataset.

|         |            | NoDur  | Durbl  | Manuf  | Enrgy  | Chems  | BusEq  | Telcm  | Utils  | Shops  | Hlth   | Money  | Other  |
|---------|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Daily   | $\mu$      | 0.0410 | 0.0480 | 0.0527 | 0.0516 | 0.0433 | 0.0632 | 0.0316 | 0.0384 | 0.0489 | 0.0500 | 0.0493 | 0.0364 |
|         | $\sigma^2$ | 0.885  | 3.17   | 1.75   | 2.83   | 1.25   | 2.70   | 1.63   | 1.23   | 1.42   | 1.31   | 2.45   | 1.59   |
| Monthly | $\mu$      | 0.887  | 1.01   | 1.09   | 1.04   | 0.877  | 1.29   | 0.646  | 0.813  | 1.02   | 1.05   | 0.969  | 0.739  |
|         | $\sigma^2$ | 14.1   | 75.2   | 31.6   | 46.2   | 18.5   | 48.4   | 27.4   | 17.3   | 21.6   | 18.1   | 32.5   | 25.8   |

**Table 6** Sample averages and variances of historical returns of the Fama-French 12-sector dataset.

|         |            | Food   | Mines  | Oil    | Clths  | Durbl  | Chems  | Cnsum  | Cnstr  | Steel  | FabPr  | Machn  | Cars   | Trans  | Utils  | Rtail  | Finan  | Other  |
|---------|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Daily   | $\mu$      | 0.0405 | 0.0534 | 0.0507 | 0.0419 | 0.0306 | 0.0445 | 0.0504 | 0.0574 | 0.0473 | 0.0504 | 0.0638 | 0.0528 | 0.0500 | 0.0384 | 0.0492 | 0.0493 | 0.0453 |
|         | $\sigma^2$ | 0.951  | 3.79   | 2.83   | 2.20   | 1.83   | 2.21   | 1.13   | 2.24   | 4.27   | 1.75   | 2.84   | 2.94   | 1.66   | 1.23   | 1.53   | 2.46   | 1.57   |
| Monthly | $\mu$      | 0.880  | 1.01   | 1.03   | 0.897  | 0.633  | 0.874  | 1.08   | 1.19   | 0.959  | 1.05   | 1.31   | 1.08   | 1.01   | 0.813  | 1.03   | 0.969  | 0.935  |
|         | $\sigma^2$ | 14.8   | 67.5   | 46.8   | 39.6   | 37.2   | 36.7   | 15.8   | 37.2   | 83.7   | 32.8   | 52.3   | 65.2   | 29.6   | 17.3   | 23.3   | 32.5   | 25.6   |

**Table 7** Sample averages and variances of historical returns of the Fama-French 17-sector dataset.