Fulkerson100

Blocking and Antiblocking Theory

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Ray Fulkerson in 1964

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BLOCKING POLYHEDRA

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

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Anti-blocking Polyhedra

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Communicated by W. T. Tutte

Received May 1, 1970

A theory parallel to that for blocking pairs of polyhedra is developed for anti-blocking pairs of polyhedra, and certain combinatorial results and problems are discussed in this framework.

Blocking pairs of polyhedra are intimately related to maximum packing problems, anti-blocking pairs to minimum covering problems.

Mathematical Programming Study 1 (1974) 120 132 North-Holland Publishing Company

ON BALANCED MATRICES

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Dedicated to A.W. Tucker, as a token of our gratitude for over 40 years of friendship and inspiration

1. Introduction

Balanced Matrices

A $0,1$ matrix A is *balanced* if it does not contain a $k \times k$ submatrix, k odd, with two 1's per row and per column.
This notion was introduced by Berge as a generalization of the notion of bipartite graph to hypergraphs.

THEOREM: Berge 1972   blocking theory
If A is balanced, all the extreme points of the polyhedron

$$Ax \geq \mathbf{1}$$
$$x \geq 0$$

are $0,1$ vectors.

THEOREM: Fulkerson, Hoffman, and Oppenheim 1974
If A is balanced, the linear system

$$Ax \geq \mathbf{1}$$
$$x \geq 0$$

is totally dual integral.

Balanced Matrices; Antiblocking Theory

THEOREM: Berge 1972

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$$\begin{aligned} Ax &\leq \mathbf{1} \\ x &\geq 0 \end{aligned}$$

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is totally dual integral.

Decomposition and Recognition of Balanced Matrices

THEOREM: Conforti, Cornuéjols and Rao 1999
Balanced matrices can be recognized in polynomial time.

The proof is over 100 pages long and relies on a structural theorem.

A key step in proving the theorem is the 2-join decomposition, which is also critical in proving the Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour, Thomas 2006), also a 100-page paper.

Both papers received the Fulkerson prize, in 2000 and 2009 respectively.

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Blocking pairs of polyhedra are intimately related to maximum packing problems, anti-blocking pairs to minimum covering problems.

Antiblocking Theory and Perfect Graphs

An $m \times n$ 0,1 matrix A is *perfect* if, for every $c \in \{0,1\}^n$

$$\begin{array}{ccc} \text{Max } cx & = & \text{Min } \sum_i y_i \\ Ax \leq \mathbf{1} & & yA \geq c \\ x \in \{0,1\}^n & & y \in \{0,1\}^m \end{array}$$

OBSERVATION If A is the StableSet-Node incidence matrix of a graph G, the above equality says: Max clique = chromatic number, for G and all its vertex-induced subgraphs, i.e. G is a perfect graph.
In fact, if a 0,1 matrix is not a StableSet-Node incidence matrix of a graph, it is not perfect (Chvátal 1975). So checking perfection of a 0,1 matrix amounts to checking whether a graph is perfect.

Fulkerson 1970 says that a 0,1 matrix A is *pluperfect* if equality holds above for every $c \in \mathbb{Z}^n_+$.

Antiblocking Theory and Perfect Graphs

PERFECT GRAPH CONJECTURE Berge 1960 A graph is perfect if and only if its complement is perfect.

THEOREM Fulkerson 1970 A graph is pluperfect if and only if its complement is pluperfect.

Recall that G pluperfect means that the linear system $Ax \leq \mathbf{1}$, $x \geq 0$ is totally dual integral, where A is the StableSet-Node incidence matrix of G.

Fulkerson 1970 "Thus, to prove the perfect graph conjecture, it suffices to show that, if G is perfect, and if we replace a vertex v in G by two vertices v', v'', where v' and v'' are joined by an edge and each is joined by an edge to every neighbor of v, the new graph G' is again perfect."

Antiblocking Theory and Perfect Graphs

Lovász proved the "Replication Conjecture".
As a consequence,

THEOREM Fulkerson 1972 and Lovász 1972
The following properties are equivalent for a 0,1 matrix A.
(i) The matrix A is perfect
(ii) the linear system $Ax \leq \mathbf{1}$, $x \geq 0$ is totally dual integral
(iii) the polytope $Ax \leq \mathbf{1}$, $x \geq 0$ only has 0,1 extreme points.

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BLOCKING POLYHEDRA

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Blocking Theory and Ideal Matrices

DEFINITION: A 0,1 matrix A is *ideal* if all the extreme points of the polyhedron $Ax \geq \mathbf{1}$, $x \geq 0$ are 0,1 vectors.

EXAMPLE 1: Balanced matrices

EXAMPLE 2: Consider a graph with source s and sink t.
Let A be a 0,1 matrix whose columns are indexed by the edges of the graph and whose rows are the characteristic vectors of the st-paths. Fulkerson 1970 proved that A is ideal: the extreme points of $Ax \geq \mathbf{1}$, $x \geq 0$ are the characteristic vectors of the st-cuts.

NOTE: It is standard to only keep the "minimal" rows of A (if rows a, a' satisfy $a' \geq a$, we can drop a' because $a'x \geq 1$ is implied by $ax \geq 1$.)

DEFINITION: The blocker $b(A)$ is the 0,1 matrix whose rows are the minimal solutions of $Ax \geq \mathbf{1}$, $x \geq 0$.

THEOREM: (Lehman 1965) A is ideal if and only if $b(A)$ is ideal.

Primal AND dual integrality

Let A be an $m \times n$ $0,1$ matrix. The matrix A *packs* if

$$\begin{array}{ccc} \tau := \text{Min } \sum_j x_j & = & \nu := \text{Max } \sum_i y_i \\ Ax \geq \mathbf{1} & & yA \leq \mathbf{1} \\ x \in \{0,1\}^n & & y \in \{0,1\}^m \end{array}$$

THEOREM Menger 1927: In a graph, the minimum size of an st-cut equals the maximum number of edge-disjoint st-paths.

THEOREM Lucchesi-Younger 1978: In a digraph, the minimum size of a dijoin equals the maximum number of edge-disjoint dicuts.

CONJECTURE Woodall 1978: The minimum size of a dicut equals the maximum number of edge-disjoint dijoins.

Minors

Let A be a $0,1$ matrix.

In the set covering formulation $Ax \geq \mathbf{1}$, $x \geq 0$,
setting $x_j = 0$ corresponds to removing column j of matrix A;
setting $x_j = 1$ corresponds to removing column j and all the rows with a 1 in column j.

EXAMPLE: For the matrix of st-paths, these operations correspond to contracting and deleting edges.

Any matrix obtained from A by a sequence of these two operations is called a *minor* of A.

REMARK If a $0,1$ matrix is ideal, then also all its minors are ideal.

The packing property

A 0,1 matrix A has the *packing property* if it packs and all its minors pack.

OBSERVATION The packing property is to blocking theory what perfection is to antiblocking theory.

EXAMPLE: For the matrix of st-paths, taking minors corresponds to contracting and deleting edges. It follows from Menger's theorem that the matrix of st-paths has the packing property.

THEOREM Lehman 1990
If a 0,1 matrix has the packing property, it is ideal.

The converse is not true, as shown by the triangles of K_4.

Blocking Theory and The Replication Conjecture

CONJECTURE Conforti and Cornuéjols (1993)

A 0,1 matrix A has the packing property if and only if the system $Ax \geq \mathbf{1}$, $x \geq 0$ is totally dual integral, namely

$$\begin{array}{ccc} \text{Min } cx & = & \text{Max } \sum_i y_i \\ Ax \geq \mathbf{1} & & yA \leq c \\ x \in \{0,1\}^n & & y \in \mathbb{Z}_+^m \end{array}$$

for all $c \in \mathbb{Z}_+^n$.

REMARK From the definition, A has the packing property if and only if equality holds above for all $c \in \{0, 1, n\}^n$.

The above conjecture is called the *Replication Conjecture*:
If equality holds for all $c \in \{0, 1, n\}^n$, it also holds for $c = \mathbf{1} + e^i$ for $i = 1, \ldots, n$.

Differences between perfection and idealness

THEOREM Chudnovsky, Cornuéjols, Liu, Seymour, Vuskovič 2005

Given a 0,1 matrix A, there is a polynomial algorithm to decide whether A is perfect.

THEOREM Ding, Feng, Zang 2008

Given a 0,1 matrix A, it is co-NP-complete to decide whether A is ideal.

The difference boils down to: there is a polynomial algorithm to decide whether a graph contains an odd hole Chudnovsky, Scott, Seymour, Spirkl 2020, whereas it is co-NP-complete to decide whether a 0,1 matrix contains an odd hole minor (Ding, Feng, Zang 2008).

ANOTHER DIFFERENCE

For a 0,1 perfect matrix A, the system $Ax \leq \mathbf{1}$, $x \geq 0$ is TDI.
For a 0,1 ideal matrix A, the system $Ax \geq \mathbf{1}$, $x \geq 0$ is not always TDI.

Minimally imperfect matrices

A $0,1$ matrix A is *minimally imperfect* if it is not perfect but all its proper column submatrices are perfect.

THEOREM Chvátal, Fulkerson, Lovász 1972
Chudnovsky, Robertson, Seymour, Thomas 2006
A $0,1$ matrix A is minimally imperfect if and only if it is the matrix

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & & 1 \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & 1 & & 0 \end{pmatrix}$$

or it is the clique-node matrix of an odd hole or of the complement of an odd hole.

Minimally nonideal 0,1 matrices

A 0,1 matrix A is *minimally nonideal* if it is not ideal but all its minors are ideal.

THEOREM Lehman 1990
An $m \times n$ 0,1 matrix A is minimally nonideal if and only if it is

$$\Delta_n := \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 1 & 0 & 0 & & 1 \end{pmatrix}$$

or its rows of minimum cardinality form a square submatrix $\bar{A}$ satisfying $\bar{A}B = E + kI$ for some square 0,1 matrix B and $k \geq 1$.

Ideal minimally nonpacking 0,1 matrices

CONJECTURE (the $\tau = 2$ Conjecture)
Cornuéjols, Guenin, Margot 2000
Every ideal mnp matrix has covering number $\tau = 2$.

THEOREM Cornuéjols, Guenin, Margot 2000
This conjecture implies the Replication Conjecture.

EXAMPLES
- The triangles of K_4 Lovász 1972
- Another example was found by Schrijver in 1980.
- Cornuéjols, Guenin and Margot 2000 found an infinite class and a few other small examples.
- Younger 2004 found another infinite class
- Abdi, Cornuéjols, Guričanová and Lee 2018 found over 700 small examples ($n \leq 14$).
- Abdi, Cornuéjols, Lee and Superdock 2019 found yet another infinite class.

Intersecting minors

A 0,1 matrix is intersecting if its packing number is 1 (every two rows intersect) and its covering number is at least 2 (there is no column of all 1s).

The following is a restatement of the $\tau = 2$ conjecture.

CONJECTURE Abdi, Cornuéjols, Lee 2020
Let A be a 0,1 matrix with no intersecting minor. The matrix A is ideal if and only if the linear system $Ax \geq 1, x \geq 0$ is totally dual integral.

THEOREM Abdi, Cornuéjols, Lee 2020
One can check in polynomial time that a 0,1 matrix has no intersecting minor.

Insights from the clutter of dijoins

Consider a digraph D with arc weights $w_a \in \mathbb{Z}_+$.

We would like to pack dijoins of D so that every arc a is contained in at most w_a dijoins in the packing.

Can we pack τ dijoins where τ is the smallest weight of a dicut?

NOTE: WMA the arc weights are 0,1 as we can replace an arc with weight w_a by w_a parallel arcs of weight 1.

If every dicut of D has weight at least 2, can we pack 2 disjoint dijoins?

THEOREM Abdi and Neuwohner 2024 If the weight-1 arcs form a connected graph, the answer is YES.

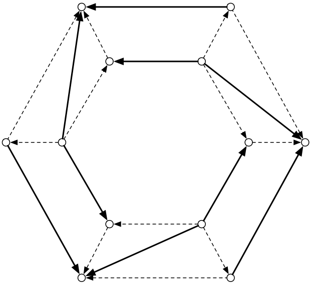
This can be proved using a result of Abdi, Cornuéjols, Zambelli 2024 about the intersection of two submodular flow systems.

This theorem settles a conjecture of Chudnovsky, Edwards, Kim, Scott, and Seymour 2016.

The above theorem fails when the weight-1 arcs form a disconnected graph with 3 or more components.

Insights from minimally nonpacking dijoins

Schrijver's example



minimal dijoin - arc incidence matrix

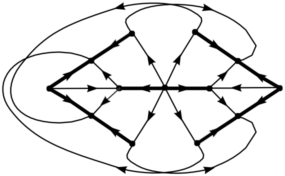
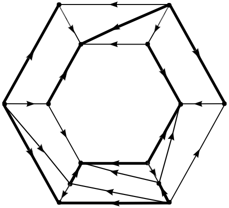
1	1	1	0	0	0	0	0	0
1	0	0	0	1	1	0	1	1
0	1	0	1	0	1	1	0	1
0	0	1	1	1	0	1	1	0
1	1	0	1	0	1	0	0	1
1	1	0	0	0	1	0	1	1
1	0	1	0	1	0	1	1	0
1	0	1	0	1	1	0	1	0
0	1	1	1	1	0	1	0	0
0	1	1	1	0	0	1	0	1

For an ideal mnp clutter $\mathcal{C}$ with covering number $\tau = 2$, the core of $\mathcal{C}$ consists of the members that occur in at least one fractional packing of value 2. WMA duplicate elements are removed.

THEOREM Abdi and Cornuéjols 2022 If $\mathcal{C}$ is an ideal clutter, then its core is also an ideal clutter.

Examples of Cornuéjols and Guenin 2002

Two other examples of dijoin clutters whose core is Q_6 (with column duplications):

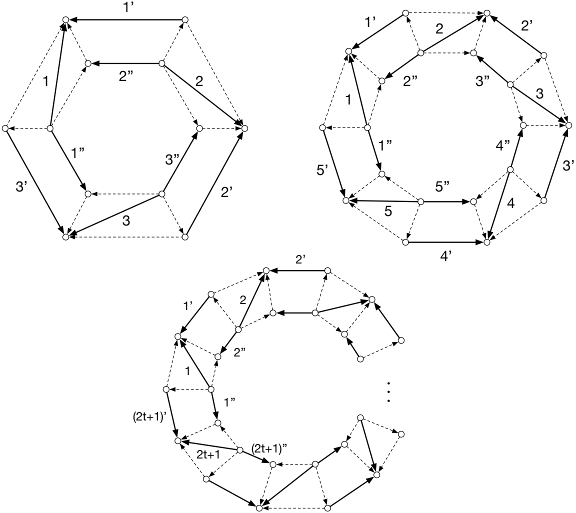



THEOREM Williams 2004

Dijoin clutters whose core is Q_6:

These two examples and Schrijver's example are the only three such digraphs with 0,1 weights (up to some digraph operations such as "folding". See Williams' dissertation).

Younger’s family of minimally nonpacking dijoins



THEOREM Abdi, Cornuéjols, Dalirrooyfard, Liu 2024
The dijoin clutters in Younger’s family are ideal minimally nonpacking.

The core of Younger's family Q_{2k}, $k \geq 3$ odd

$$Q_6 := \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$Q_{10} := \left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{array}\right)$$

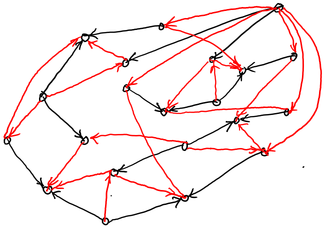
THEOREM Abdi, Cornuéjols, Dalirrooyfard, Liu 2024
Q_{2k} is ideal minimally nonpacking for $k \geq 3$ odd.

Are there other cores arising from mnp dijoins?

Are the cores Q_{2k} arising from Younger's family the only possible cores for mnp dijoins?

The answer is NO.

We recently found two other mnp cuboids that give rise to dijoin clutters. Here is a digraph with 0,1 weights and 5 weight-1 paths whose core is not Q_{10}:



1	1	1	1	1	0	0	0	0	0
1	0	0	0	0	0	1	1	1	1
0	1	0	0	0	1	0	1	1	1
0	0	1	0	0	1	1	0	1	1
0	0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1	0
1	0	1	0	0	0	1	0	1	1
0	1	0	1	0	1	0	1	0	1
0	0	1	0	1	1	1	0	1	0
1	0	0	1	0	0	1	1	0	1
0	1	0	0	1	1	0	1	1	0
0	0	1	1	1	1	1	0	0	0

But its core is Q_{10} plus one row! The core of the other mnp dijoin example is Q_{10} plus two rows.