The Chvátal Rank of 2-Dimensional Integer-Free Polyhedra

Gérard Cornuéjols[1][0000-0002-3976-1021] and
Vrishabh Patil[1][0000-0002-2911-2888]

Tepper School of Business, Carnegie Mellon University, Pittsburgh PA 15213, USA
gc0v@andrew.cmu.edu, vmpatil@andrew.cmu.edu

Abstract. In a seminal paper, Chvátal introduced a rounding procedure to go iteratively from a rational polytope $P \subseteq \mathbb{R}^n$ to its integer hull. He showed that the number of iterations needed to reach the integer hull is finite. This number is known as the Chvátal rank of P. Chvátal also gave a family of examples showing that this rank can be arbitrarily large, even in dimension $n = 2$. In this paper we consider *integer-free* rational polyhedra $P \subseteq \mathbb{R}^n$, that is $P \cap \mathbb{Z}^n = \emptyset$. We show that the Chvátal rank of *integer-free* rational polyhedra is at most 3 when $n = 2$, and we give an example showing that this bound is tight. More generally, for a 2-dimensional integer-free rational polyhedron $P \subseteq \mathbb{R}^n$, the Chvátal rank is at most 3, and this bound is tight.

Keywords: Integer Programming · Chvátal Rank.

1 Introduction

Cutting plane procedures are among the most fundamental methods for strengthening linear relaxations of integer programs. Given a polyhedron, these procedures iteratively generate valid inequalities, or cuts, that eliminate fractional solutions while preserving all integer ones. One of the earliest and most extensively studied cutting plane methods is the *Chvátal procedure* [2].

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron. A linear inequality $\alpha x \leq \beta$ is said to be *valid* for P if it holds for all $x \in P$. When the vector α has integer components, the corresponding *Chvátal cut* is obtained by rounding down the right-hand side: $\alpha x \leq \lfloor \beta \rfloor$. Every such inequality is valid for the integer hull $P_I := \text{conv}(P \cap \mathbb{Z}^n)$. The *Chvátal closure* P' of P is the set of points in P that satisfy all Chvátal cuts derived from valid inequalities $\alpha x \leq \beta$ for P where $\alpha \in \mathbb{Z}^n$.

Chvátal showed that if P is a rational polytope, then its closure P' is also a rational polytope [2]. Iterating the closure operation yields a sequence of polytopes $P^{(0)} = P$ and $P^{(i)} = (P^{(i-1)})'$ for $i \geq 1$. This iterative process is referred to as the *Chvátal procedure*. The *Chvátal rank* of P is the smallest integer t such that $P^{(t)} = P_I$. Chvátal also showed that the rank of rational polytopes is always finite [2]. These two results of Chvátal were later extended to rational polyhedra by Schrijver [14]. Note that the rationality of P is essential for a finite Chvátal rank: for the polyhedron $P = \left\{x \in \mathbb{R}^2 : x_1 + \sqrt{2}\, x_2 = \frac{1}{2}\right\}$ the Chvátal procedure does not produce any nontrivial cut and yet $P_I = \emptyset$.

The Chvátal rank can be exponential in the input size required to describe P, and [2] gave an example illustrating that this may happen even in two dimensions. For a positive integer θ, let

$$P := \{x \in \mathbb{R}^2_+ : \ \theta x_1 + x_2 \le \theta, \ -\theta x_1 + x_2 \le 0\}.$$

This polytope has Chvátal rank θ, which can be made arbitrarily large. In terms of the encoding size, $L \approx \log_2 \theta$, the Chvátal rank is 2^L.

By contrast, positive results on bounding the Chvátal rank are known when one restricts attention to rational polytopes contained in the unit hypercube $[0,1]^n$. [1] showed that the rank of any such polytope is bounded above by $O(n^3 \log n)$ and that, for integer-free polytopes ($P \cap \mathbb{Z}^n = \emptyset$) within the hypercube, the rank is at most n. [12] established that this bound n is tight when the polytope P intersects all the edges of the hypercube. Subsequently, [9] improved the general upper bound for unit hypercubes to $O(n^2 \log n)$, while a lower bound of order $O(n^2)$ was proved by [13].

More recent work has refined and generalized the Chvátal procedure. [8] introduced a class of strengthened cuts for bounded variables and proved that the resulting closure remains polyhedral. Building on this, [5], [6] extended these ideas to incorporate variable bounds and general linear constraints, again showing polyhedrality of the closure. [4] demonstrated that for any rational polyhedron P, one can introduce binary variables to construct a compact extended formulation of P whose Chvátal rank is polynomial in the encoding size of P. These developments collectively illustrate the structural depth and algorithmic richness of the Chvátal closure.

In this paper, we study the Chvátal rank of *integer-free, rational polyhedra* that are not necessarily contained in the unit hypercube $[0,1]^n$. We prove that every 2-dimensional integer-free rational polyhedron has Chvátal rank at most 3, and we construct an example showing that this bound is tight. Section 2 contains our proof of the upper bound in the plane $\mathbb{R}^2$. Section 3 contains the proof of the lower bound. In Section 4, we generalize the result to 2-dimensional integer-free rational polyhedra in $\mathbb{R}^n$.

2 An Upper Bound on the Chvátal Rank of Integer-Free Rational Polyhedra in $\mathbb{R}^2$

We show an upper bound of 3 on the Chvátal rank of *integer-free* rational polyhedra in $\mathbb{R}^2$.

Theorem 1. *Let $P \subseteq \mathbb{R}^2$ be a rational polyhedron with $P \cap \mathbb{Z}^2 = \emptyset$. Then the Chvátal rank of P is at most 3.*

We will prove this theorem by analyzing *maximal lattice-free* convex sets. A convex set $S \subseteq \mathbb{R}^n$ is said to be *lattice-free* if it contains no integer point in its relative interior. Note however that integer points are allowed on the boundary. A lattice-free convex set is *maximal* if it is not properly contained in any other lattice-free convex set. [10] showed that any maximal lattice-free convex set in $\mathbb{R}^n$ with a rational affine hull is a full-dimensional polyhedron with at most 2^n facets, and every facet contains an integer point in its relative interior.

Our proof of Theorem 1 will need two general results. The first result shows that every integer-free rational polyhedron is contained in the interior of a maximal lattice-free convex set. Let $\text{int}(\cdot)$ denote the interior and $\text{conv}(\cdot)$ denote the convex hull.

Theorem 2. *For any nonempty rational polyhedron $P \subset \mathbb{R}^n$ such that $P \cap \mathbb{Z}^n = \emptyset$, there exists a maximal lattice-free convex set $M \subset \mathbb{R}^n$ containing P in its interior.*

Proof. We may assume that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Define $C := \{x \in \mathbb{R}^n : Ax \leq b + \frac{1}{2}\mathbf{1}\}$, where $\mathbf{1}$ denotes the m-dimensional vector of all ones.

We claim that $C \cap \mathbb{Z}^n = \emptyset$. Indeed, take any $z \in \mathbb{Z}^n$. Because $P \cap \mathbb{Z}^n = \emptyset$, there exists an index $i \in \{1, \dots, m\}$ such that $\sum_{j=1}^n a_{ij} z_j > b_i$. Because $\sum_{j=1}^n a_{ij} z_j \in \mathbb{Z}$ and $b_i \in \mathbb{Z}$, we have

$$\sum_{j=1}^{n} a_{ij} z_j \geq b_i + 1 > b_i + \frac{1}{2},$$

and so $z \notin C$, proving the claim.

For $x \in P$, $Ax \leq b < b + \frac{1}{2}\mathbf{1}$, so all inequalities defining the polyhedron C are satisfied strictly, and thus $x \in \text{int}(C)$. Since C is convex and lattice-free, there exists a maximal lattice-free convex set M such that $C \subseteq M$. We have $P \subset \text{int}(C) \subseteq \text{int}(M)$. □

Let $M \subset \mathbb{R}^n$ be a maximal lattice-free convex set. Given $n + 1$ points $\bar{y}, y^1, y^2, \dots, y^n \in \mathbb{R}^n$ that are affinely independent, let

$$C(\bar{y}, y^1 y^2 \dots y^n) := \{\bar{y}\} + \text{cone}(\bar{y} - y^1, \bar{y} - y^2, \dots, \bar{y} - y^n).$$

Lemma 1. *Let $\bar{y} \in \mathbb{Z}^n$, $y^1, y^2, \dots, y^n \in M$ be $n + 1$ points that are affinely independent. Then $M \cap \text{int}(C(\bar{y}, y^1 y^2 \dots y^n)) = \emptyset$.*

Proof. Let $x \in \text{int}(C(\bar{y}, y^1 y^2 \dots y^n))$. Then $x = \bar{y} + \sum_{i=1}^n \alpha_i (\bar{y} - y^i)$ where $\alpha_i > 0$. This can be rewritten as

$$\bar{y} = \frac{1}{1 + \sum_{i=1}^n \alpha_i} x + \sum_{i=1}^{n} \frac{\alpha_i}{1 + \sum_{i=1}^n \alpha_i} y^i.$$

Since all $n+1$ coefficients in the right-hand-side of this equation are strictly positive, the point $\bar{y}$ is in the interior of the polytope with vertices $x, y^1, y^2, \dots, y^n$. By assumption $\bar{y} \in \mathbb{Z}^n$ and $y^1, y^2, \dots, y^n \in M$. Since M contains no integer point in its interior, it follows that $x \notin M$. □

We will now focus on the case $n = 2$. Let $P \subset \mathbb{R}^2$ be a nonempty rational polyhedron such that $P \cap \mathbb{Z}^2 = \emptyset$. By Theorem 2, there is a maximal lattice-free convex set $M \subset \mathbb{R}^2$ containing P in its interior.

If M is a split, i.e. $M = \{x \in \mathbb{R}^2 : \ \pi_0 \leq \pi x \leq \pi_0 + 1\}$ for some $(\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$, it is easy to see that the Chvátal rank of P is 1, since there exists a constant $\epsilon > 0$ such that the inequalities $\pi_0 + \epsilon \leq \pi x$ and $\pi x \leq \pi_0 + 1 - \epsilon$ are both valid for P by Theorem 2 and the assumption that P is a rational polyhedron. So Theorem 1 holds in this case.

Otherwise, M has three or four facets [10]. Furthermore, there exist points $y^1, y^2, y^3 \in \mathbb{Z}^2$ in the relative interior of three distinct facets of M such that any two of the vectors $v^1 := y^2 - y^1$, $v^2 := y^3 - y^1$, $v^3 := y^3 - y^2$ generate the whole integer lattice $\mathbb{Z}^2$ [10], [7]. In other words, v^1, v^2 form a basis for the lattice $\mathbb{Z}^2$. By a unimodular transformation and an integer translation, we may assume

$$y^1 = (0,0), \qquad y^2 = (1,0), \qquad y^3 = (0,1).$$

Set $y^4 := (1,1)$, $y^5 := (1,-1)$, $y^6 := (-1,1)$.

We will use the following three "width–3" strips: $R_1 := \{x \in \mathbb{R}^2 : \ -1 \leq x_1 \leq 2\}$, $R_2 := \{x \in \mathbb{R}^2 : \ -1 \leq x_2 \leq 2\}$, $R_3 := \{x \in \mathbb{R}^2 : \ -1 \leq x_1 + x_2 \leq 2\}$. We also consider the three splits $S_1 := \{x \in \mathbb{R}^2 : \ 0 \leq x_1 \leq 1\}$, $S_2 := \{x \in \mathbb{R}^2 : \ 0 \leq x_2 \leq 1\}$, $S_3 := \{x \in \mathbb{R}^2 : \ 0 \leq x_1 + x_2 \leq 1\}$.

Lemma 2. $M \subseteq S_1 \cup S_2 \cup S_3$.

Proof. The set $\mathbb{R}^2 \setminus (S_1 \cup S_2 \cup S_3)$ is the union of 6 regions, namely the interiors of $C(y^1, y^2y^3)$, $C(y^2, y^1y^3)$, $C(y^3, y^1y^2)$, $C(y^4, y^2y^3)$, $C(y^5, y^1y^2)$, $C(y^6, y^1y^3)$. By Lemma 1, M does not intersect any of these regions. Thus $M \subseteq S_1 \cup S_2 \cup S_3$. □

Lemma 3. *There exists* $i \in \{1,2,3\}$ *such that* $M \subseteq R_i$.

Proof. For the sake of contradiction, suppose that for each $i \in \{1,2,3\}$ there exists a point $p^i = (p^i_1, p^i_2) \in M \setminus R_i$. By Lemma 2, we have $p^1 \in (S_2 \cup S_3) \setminus R_1$, $p^2 \in (S_1 \cup S_3) \setminus R_2$, $p^3 \in (S_1 \cup S_2) \setminus R_3$. Let $c := (\frac{1}{3}, \frac{1}{3})$. Since $(0,0), (1,0), (0,1) \in M$ and M is a convex set, we have $c \in M$.

Claim 1. $p^1 \in S_3$ *or* $p^2 \in S_3$.

Proof. Suppose for the sake of contradiction that $p^1 \in S_2 \setminus (R_1 \cup S_3)$ and $p^2 \in S_1 \setminus (R_2 \cup S_3)$. To prove the claim, we will show that $\mathrm{conv}(p^1, p^2, c)$ contains an integer point in its interior, namely $(0,0)$, $(1,0)$, $(0,1)$, or $(1,1)$. There are 4 cases depending on the signs of p^i_j for $i,j = 1,2$. We give an explicit algebraic argument in the case $p^1_1 < -1$ and $p^2_2 < -1$. The remaining cases follow by symmetry between coordinates 1 and 2.

So assume $p^1_1 < -1$ and $p^2_2 < -1$. From $p^1 \in S_2$ and $p^2 \in S_1$ we also have $0 \leq p^1_2 \leq 1$ and $0 \leq p^2_1 \leq 1$. Define $t := \frac{p^2_1 - p^2_2}{(p^2_1 - p^2_2) + (p^1_2 - p^1_1)}$. Since $p^2_1 - p^2_2 > 0$ and $p^1_2 - p^1_1 > 0$, we have $t \in (0,1)$. Let

$$q := tp^1 + (1-t)p^2.$$

Then $q \in \text{int}(\text{conv}(p^1, p^2))$. Moreover, $q_1 = q_2$ because $q_2 - q_1 = t(p^1_2 - p^1_1) + (1 - t)(p^2_2 - p^2_1) = 0$ by the definition of t. We may write $q_1 = q_2 = \frac{p^2_1 p^1_2 - p^1_1 p^2_2}{(p^2_1 - p^2_2) + (p^1_2 - p^1_1)}$. The denominator is positive. For the numerator, note that $0 \leq p^2_1 p^1_2 \leq 1$, while $p^1_1 p^2_2 > 1$ because $p^1_1 < -1$ and $p^2_2 < -1$. Hence $p^2_1 p^1_2 - p^1_1 p^2_2 < 0$, so $q_1, q_2 < 0$. Since $q = (q_1, q_2)$ lies on the diagonal $\{x_1 = x_2\}$ strictly below the origin and $c = (\frac{1}{3}, \frac{1}{3})$ lies on the same diagonal strictly above the origin, the origin lies in the open segment (q, c). Therefore $(0, 0) \in \text{int}(\text{conv}(p^1, p^2, c))$. Since $p^1, p^2, c \in M$ and M is convex, $\text{conv}(p^1, p^2, c) \subseteq M$, and thus $(0, 0) \in \text{int}(M)$, contradicting that M is lattice-free. □

By Claim 1 and symmetry between coordinates 1 and 2, we may assume $p^1 \in S_3 \setminus R_1$.

Claim 2. $p^1 \in S_3 \cap \{x_1 < -1\}$ *and* $p^3 \in S_2 \cap \{x_1 + x_2 > 2\}$.

Proof. Since $p^1 \in S_3 \setminus R_1$, we have $p^1_1 \notin [-1, 2]$. If $p^1_1 > 2$, then $\text{int}(\text{conv}(p^1, p^3, c))$ contains one of $(0, 0), (1, 0), (1, -1)$, contradicting lattice-freeness of M. Hence $p^1_1 < -1$.

Next, $p^3 \in (S_1 \cup S_2) \setminus R_3$ implies $p^3_1 + p^3_2 \notin [-1, 2]$. Suppose $p^3 \in S_1 \setminus R_3$. Then $\text{int}(\text{conv}(p^1, p^3, c))$ contains $(0, 0)$ if $p^3_1 + p^3_2 < -1$, and it contains $(0, 1)$ if $p^3_1 + p^3_2 > 2$. Therefore $p^3 \in S_2 \setminus R_3$. Now, if $p^3 \in S_2 \cap \{x_1 + x_2 < -1\}$, then $\text{int}(\text{conv}(p^1, p^3, c))$ contains $(-1, 1)$, again a contradiction. Therefore $p^3 \in S_2 \cap \{x_1 + x_2 > 2\}$. □

Claim 3. $p^2 \in S_1 \cap \{x_2 < -1\}$.

Proof. Recall that we have $p^2 \in (S_1 \cup S_3) \setminus R_2$. First suppose $p^2 \in S_3 \setminus R_2$. If $p^2_1 > 2$, the triangle $\text{conv}(p^2, p^3, c)$ contains $(1, 0)$ in its interior, a contradiction. If $p^2_1 < -1$, the fact that $p^2 \notin R_2$ implies $p^2 \in S_3 \cap \{x_2 > 2\}$. But now the triangle $\text{conv}(p^2, p^3, c)$ contains the integer point $(0, 1)$ in its interior, a contradiction. Therefore $p^2 \in S_1 \setminus R_2$. If $p^2_2 > 2$, the triangle $\text{conv}(p^1, p^2, c)$ contains $(0, 1)$ in its interior. Therefore we must have $p^2 \in S_1 \cap \{x_2 < -1\}$. □

The triangle $\text{conv}(p^1, p^2, p^3)$ is contained in M and, by construction, M has three or four facets. Consider the three facets going through $y^1 = (0, 0)$, $y^2 = (1, 0)$, $y^3 = (0, 1)$, and drop the fourth if any. The three lines associated with these facets of M define a triangle T containing M, and they have the following form:

$$x_1 + \alpha x_2 = 0,\ x_2 = \beta(x_1 - 1),\ \gamma x_1 + (x_2 - 1) = 0.$$

Because we have points p^1, p^2, p^3 satisfying Claims 2 and 3, we must have $0 < \alpha < 1$, $\beta > 0$, $\gamma > 0$. Let v^1, v^2, v^3 be the vertices of T, which lie in the regions $S_3 \cap \{x_1 < -1\}$, $S_1 \cap \{x_2 < -1\}$, and $S_2 \cap \{x_1 + x_2 > 2\}$ respectively because T contains M. Then, $v^1_1 < -1$ implies $\gamma > \frac{1-\alpha}{\alpha}$, $v^2_2 < -1$ implies $\beta > \frac{1}{1-\alpha}$, and $v^3_1 + v^3_2 > 2$ implies $\gamma < \frac{1}{2+\beta}$.

Putting these 3 inequalities together, we get

$$\frac{1-\alpha}{\alpha} < \gamma < \frac{1}{2+\beta} < \frac{1-\alpha}{3-2\alpha},$$

which implies $\alpha > 1$, a contradiction.

This completes the proof that there exists $i \in \{1,2,3\}$ such that $M \subseteq R_i$. □

We will need two more lemmas for our proof of Theorem 1. Denote the dimension of a polyhedron P by $\dim(P)$.

Lemma 4. *Let $P \subseteq \mathbb{R}^n$ be a nonempty, rational, integer-free polyhedron with dimension 0 or 1. Then the Chvátal rank of P is 1.*

Proof. If $\dim(P) = 0$, let $\bar{x}$ be the unique point in P and let $\bar{x}_j$ be a fractional coordinate of $\bar{x}$. Then $x_j \leq \lfloor \bar{x}_j \rfloor$ is a Chvátal cut. Hence $P' = \emptyset$.

Assume $\dim(P) = 1$ and let $L := \text{aff}(P)$. Since P is a rational polyhedron, L is a rational line, so we can write $L = \bar{x} + \{\lambda v : \lambda \in \mathbb{R}\}$, where $\bar{x} \in \mathbb{Q}^n$ and $v \in \mathbb{Z}^n \setminus \{0\}$ with relatively prime entries. We may choose a unimodular matrix $U \in \mathbb{Z}^{n \times n}$ such that $Uv = e^1$ [11, Corollary II.1], where e^1 is the first unit vector. Set $Q := UP$. Note that $Q \cap \mathbb{Z}^n = \emptyset$, and $Q' = UP'$ since Q is a unimodular mapping of P. Hence it suffices to prove $Q' = \emptyset$.

Now $\text{aff}(Q) = UL$ is a line parallel to e^1, so there exists $c \in \mathbb{R}^{n-1}$ with $\text{aff}(Q) = \{(t, c) \in \mathbb{R}^n : t \in \mathbb{R}\}$.

If $c \notin \mathbb{Z}^{n-1}$, there exists $j \in \{2, \ldots, n\}$ with $c_j \notin \mathbb{Z}$. Since $x_j \leq c_j$ is valid for Q, $x_j \leq \lfloor c_j \rfloor$ is valid for Q'. But now $Q \subset \{x : x_j = c_j\}$ implies that $Q' = \emptyset$.

If instead $c \in \mathbb{Z}^{n-1}$, then any integer k gives $(k, c) \in \mathbb{Z}^n$. Since Q is an integer-free, convex set and is contained in the line $\text{aff}(Q)$, Q is a line segment lying between two consecutive integer points on $\text{aff}(Q)$. So $Q = \{(t, c) \in \mathbb{R}^n : \alpha \leq t \leq \beta\}$ for some $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Z}$ with $\lceil \alpha \rceil = \lfloor \beta \rfloor + 1$. Then the inequalities $x_1 \leq \lfloor \beta \rfloor$ and $x_1 \geq \lceil \alpha \rceil$ are rank-1 Chvátal cuts. Therefore $Q' = \emptyset$. □

Lemma 5 ([3], Lemma 2). *Let P be a rational polyhedron and F a nonempty face of P. Then $F' = P' \cap F$.*

Proof of Theorem 1. By Theorem 2, there is a maximal lattice-free convex set $M \subset \mathbb{R}^2$ containing P in its interior. We already dealt with the case where M is a split, so we may assume that M has 3 or 4 facets. By Lemma 3, there exist $(\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$ such that $\pi_0 - 1 < \pi x < \pi_0 + 2$ for every $x \in P$. It follows that $\pi x \geq \pi_0$ and $\pi x \leq \pi_0 + 1$ are rank-1 Chvátal inequalities. Consider the integer-free faces $F_1 := P' \cap \{\pi x = \pi_0\}$ and $F_2 := P' \cap \{\pi x = \pi_0 + 1\}$. Observe that $\dim(F_1)$ and $\dim(F_2)$ are at most 1. By Lemma 4, $F_1' = F_2' = \emptyset$. Moreover, by Lemma 5, $P^{(2)} \cap \{\pi x = \pi_0\} = \emptyset$ and $P^{(2)} \cap \{\pi x = \pi_0 + 1\} = \emptyset$, which implies that $\pi x > \pi_0$ and $\pi x < \pi_0 + 1$ hold for $x \in P^{(2)}$. Finally, the rank-3 inequalities $\pi x \geq \pi_0 + 1$ and $\pi x \leq \pi_0$ prove that $P^{(3)} = \emptyset$. □

3 A Lower Bound on the Chvátal Rank of Integer-Free Polyhedra in $\mathbb{R}^2$

We use an example due to [3] to show that there exists a 2-dimensional, integer-free, rational polytope $P \subset \mathbb{R}^2$ with Chvátal rank equal to 3. Let

$$P := \left\{ x \in \mathbb{R}^2 : \begin{array}{cc} 6x_1 + 8x_2 \leq 13, & 6x_1 - 8x_2 \leq 5, \\ -6x_1 + 8x_2 \leq 7, & -6x_1 - 8x_2 \leq -1 \end{array} \right\}.$$

Note that the polytope P is the convex hull of the points

$$\left\{ \left(\tfrac{1}{2}, \tfrac{5}{4}\right), \left(\tfrac{1}{2}, -\tfrac{1}{4}\right), \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{3}{2}, \tfrac{1}{2}\right) \right\}.$$

As observed by [3],

Lemma 6. *The Chvátal closure P' is given by the convex hull of*

$$\left\{ \left(\tfrac{1}{2}, 0\right), \left(\tfrac{1}{2}, 1\right), \left(0, \tfrac{1}{3}\right), \left(0, \tfrac{2}{3}\right), \left(1, \tfrac{1}{3}\right), \left(1, \tfrac{2}{3}\right) \right\}.$$

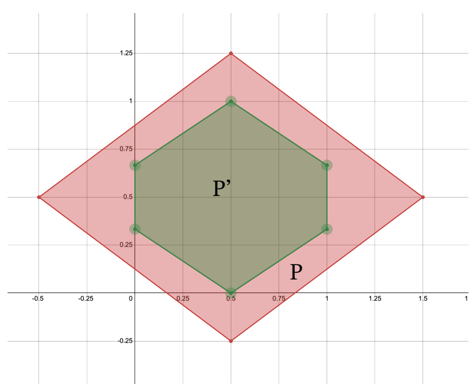

Fig. 1. A 2-dimensional integer-free polytope with Chvátal rank 3.

We provide a proof for completeness.

Proof. The convex hull of the six points above is

$$Q = \left\{ x \in \mathbb{R}^2 : \begin{array}{c} 2x_1 + 3x_2 \leq 4, \;\; 2x_1 - 3x_2 \leq 1 \\ -2x_1 + 3x_2 \leq 2, \;\; -2x_1 - 3x_2 \leq -1 \\ 0 \leq x_1 \leq 1 \end{array} \right\}.$$

Observe that the following inequalities are valid for P

$$2x_1 + 3x_2 \le \frac{19}{4}, \qquad 2x_1 - 3x_2 \le \frac{7}{4},$$
$$-2x_1 + 3x_2 \le \frac{11}{4}, \qquad -2x_1 - 3x_2 \le -\frac{1}{4},$$
$$-x_1 \le \frac{1}{2}, \qquad x_1 \le \frac{3}{2},$$

and that the corresponding rank-1 Chvátal inequalities are precisely those that describe Q above. It remains to verify that each of the six points stated in the lemma belongs to the first Chvátal closure of P.

Recall that a Chvátal inequality of P is obtained from a valid inequality $\alpha x \le \beta$, $\alpha \in \mathbb{Z}^2$, by rounding the right-hand side down to $\lfloor \beta \rfloor$.

We claim that $\left(\frac{1}{2}, 0\right) \in P'$. Consider any rank-1 Chvátal inequality $\alpha x \le \lfloor \beta \rfloor$. If the inequality is valid for both integer points $z^1 = (0,0)$ and $z^2 = (1,0)$, then by convexity it is also valid for their midpoint $\left(\frac{1}{2}, 0\right)$. Hence $\left(\frac{1}{2}, 0\right)$ can only be cut off by a Chvátal inequality that cuts off at least one of $(0,0)$ or $(1,0)$.

Consider any valid inequality for P that cuts off $(0,0)$, i.e., $\alpha z^1 > \beta$. Observe that the ray from z^1 through the vertex $\left(\frac{1}{2}, -\frac{1}{4}\right) \in P$ passes through $(2,-1)$, and the ray from z^1 through $\left(-\frac{1}{2}, \frac{1}{2}\right) \in P$ passes through $(-1,1)$. Both $(2,-1)$ and $(-1,1)$ lie on lines supporting P along the corresponding edges incident to those vertices. Thus, any inequality $\alpha x \le \beta$ that is valid for P and cuts off z^1 must also be valid for $(2,-1)$ and $(-1,1)$. Moreover, since $(2,-1)$ and $(-1,1)$ are integer, they must also satisfy the corresponding Chvátal inequality $\alpha x \le \lfloor \beta \rfloor$. Finally, since the point $\left(\frac{1}{2}, 0\right) \in \mathrm{conv}((2,-1),(-1,1))$, it cannot be cut off by such a Chvátal inequality.

An entirely symmetric argument applies if the inequality cuts off $(1,0)$, i.e., $\alpha z^2 > \beta$. In this case, we may use the two integer points $(-1,-1)$ and $(2,1)$, which lie on the rays from z^2 through the adjacent vertices $\left(\frac{1}{2}, -\frac{1}{4}\right)$ and $\left(\frac{3}{2}, \frac{1}{2}\right)$, respectively. Again, since $(-1,-1)$ and $(2,1)$ are integer points, they must satisfy the corresponding Chvátal inequality $\alpha x \le \lfloor \beta \rfloor$, which implies that the point $\left(\frac{1}{2}, 0\right) \in \mathrm{conv}((-1,-1),(2,1))$ is not cut off.

Therefore, $\left(\frac{1}{2}, 0\right)$ satisfies every rank-1 Chvátal inequality of P, and so, $\left(\frac{1}{2}, 0\right) \in P'$.

The remaining five vertices of P' are handled analogously. Each is in the convex combination of two integer points that lie on lines supporting one of the corners of $[0,1]^2$ and an adjacent vertex of P, and is therefore satisfied by all rank-1 Chvátal inequalities. □

To conclude the proof of the lower bound, we show that P' still requires two further rounds of the Chvátal procedure to obtain the empty set. Indeed, P' is integer-free, contained in $[0,1]^2$, and intersects every edge of the unit square. By [12], it follows that P' has Chvátal rank 2, and therefore P has Chvátal rank 3.

4 Chvátal Rank of 2-Dimensional Integer-Free Polyhedra

In this section we extend our main result beyond $\mathbb{R}^2$ and provide an upper bound for integer-free rational polyhedra $P \subseteq \mathbb{R}^n$ with dimension 2. Our proof will use the following lemma.

Lemma 7. *Let $L \subseteq \mathbb{R}^n$ be a nonempty rational affine subspace such that $L \cap \mathbb{Z}^n = \emptyset$. Then there exist $\pi \in \mathbb{Z}^n \setminus \{0\}$ and $\beta \in \mathbb{Q} \setminus \mathbb{Z}$ such that $L \subseteq \{x \in \mathbb{R}^n : \pi x = \beta\}$.*

Proof. Write $L = \bar{x} + V$, where $\bar{x} \in \mathbb{Q}^n$ and V is a rational linear subspace. Let $\Lambda := V \cap \mathbb{Z}^n$ be an integral sublattice of V with rank $k := \dim(V)$. We may choose basis $v^1, \dots, v^k$ of Λ, and extend it to a unimodular matrix $U \in \mathbb{Z}^{n \times n}$ whose first k columns are $v^1, \dots, v^k$ [11, Corollary II.1].

Under the unimodular transformation $z := U^{-1}x$, we have $U^{-1}V = \mathbb{R}^k \times \{0\}^{n-k}$, and hence $U^{-1}L = \{(t, c) : t \in \mathbb{R}^k\}$ where $c \in \mathbb{Q}^{n-k}$ is the vector of the last $n - k$ coordinates of $\bar{z} = U^{-1}\bar{x}$.

If $c \in \mathbb{Z}^{n-k}$, then $(0, c) \in U^{-1}L \cap \mathbb{Z}^n$, which contradicts $L \cap \mathbb{Z}^n = \emptyset$. Therefore, there exists an index $j \in \{k + 1, \dots, n\}$ with $c_j \notin \mathbb{Z}$. Fix such j and define

$$\pi := e^j U^{-1} \in \mathbb{Z}^n, \qquad \beta := e^j \bar{z} = c_j \in \mathbb{Q} \setminus \mathbb{Z}.$$

Observe $L \subseteq \{x \in \mathbb{R}^n : \pi x = \beta\}$ since for every $x \in L$, $\pi x = e^j U^{-1} x = e^j z = e^j \bar{z} = c_j = \beta$. □

Theorem 3. *The Chvátal rank of any 2-dimensional, integer-free, rational polyhedron $P \subset \mathbb{R}^n$ is at most 3.*

Proof. Let $L := \operatorname{aff}(P)$ define the affine hull of P.

Suppose $L \cap \mathbb{Z}^n = \emptyset$. Then by Lemma 7, there exist $\pi \in \mathbb{Z}^n$ with relatively prime entries and $\beta \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\pi x = \beta$ for all $x \in L$. In particular, by letting $\pi_0 := \lfloor \beta \rfloor$, we have $L \subseteq \{x : \pi_0 < \pi x < \pi_0 + 1\}$, i.e., L is contained in the interior of a split. Then the valid inequality $\pi x \le \beta$ yields, after one Chvátal round, the inequality $\pi x \le \pi_0$ valid for P'. Similarly, from $-\pi x \le -\beta$ we obtain $\pi x \ge \pi_0 + 1$ valid for P'. Therefore $P' = \emptyset$, and the Chvátal rank of P is 1.

Now suppose $L \cap \mathbb{Z}^n \ne \emptyset$. Pick $y \in L \cap \mathbb{Z}^n$ and replace P by $P - y$. Then $0 \in \operatorname{aff}(P)$ and L is a 2-dimensional rational linear subspace. Set $\Lambda := L \cap \mathbb{Z}^n$, which is an integer sublattice of dimension 2. We may then choose a basis $\{v_1, v_2\}$ of Λ and extend it to a unimodular matrix $U \in \mathbb{Z}^{n \times n}$ whose first two columns are v_1 and v_2 [11, Corollary II.1]. By the unimodular transformation $z = U^{-1}x$, we may assume that $L = \mathbb{R}^2 \times \{0\}^{n-2}$, and thus $P \subseteq \mathbb{R}^2 \times \{0\}^{n-2}$.

Now work in the first two coordinates and define

$$\bar{P} := \{(x_1, x_2) \in \mathbb{R}^2 : (x_1, x_2, 0, \dots, 0) \in P\}.$$

Then $\bar{P}$ is a nonempty integer-free polyhedron in $\mathbb{R}^2$. Thus it follows directly from Theorem 1 that $\bar{P}^{(3)} = \emptyset$. □

5 Conclusions

While the Chvátal rank of polyhedra can be arbitrarily large in general, we show the surprising result that, for 2-dimensional *integer-free* polyhedra, the Chvátal rank is bounded above by 3 and that this bound is tight. A natural open direction is to give upper and lower bounds on the Chvátal rank of k-dimensional *integer-free* polyhedra for $k \geq 3$. Matching upper and lower bounds for $k = 3$ is already an interesting challenge.

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