Reduction the Chvátal Rank through Binarization

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1. Introduction

In a seminal paper, Chvátal [1] introduced the following rounding procedure for going from a rational polyhedron $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ to the convex hull $P_1$ of the integer points in $P$: Generate valid inequalities $ax \leq \beta$ for $P$ such that $a \in \mathbb{Z}^n$ and $\beta \in \mathbb{R}$, and round $\beta$ to its integer part $\lfloor \beta \rfloor$. The intersection of all such inequalities $ax \leq \lfloor \beta \rfloor$ is called the Chvátal closure. It is a polyhedron [1]. Chvátal [1] showed for the bounded case, and Schrijver [2] in the general case, that repeating this process a finite number of times produces the integer hull $P_1$. The Chvátal rank of a polyhedron $P$ is the smallest number of iterations of the Chvátal procedure needed to obtain $P_1$. Chvátal [1] observed that the Chvátal rank can be exponential in the input size needed to describe $P$, and he gave an example illustrating that this may happen even in two dimensions. However, for a polytope $P$ contained in the 0-1 hypercube, Eisenbrand and Schulz [3] proved that the Chvátal rank is no more than $O(n^2 \log n)$. Additionally, Rothvös and Sanita [4] showed that there exist polytopes contained in the 0-1 hypercube with rank $O(n^2)$. Other measures of the complexity of integer programs have been considered in the literature, such as the length of cutting plane proofs [5, 6, 7].

This note focuses on the Chvátal rank. We show that for a rational polyhedron $P$ whose variables have integral entries, such that the Chvátal rank is polynomial in $\|P\|$, while also keeping general variables to handle the possible unboundedness of $P$. In Section 2, we focus on the case where $P$ is a polytope and provide a motivating example. We then present our main result in Section 3 where we construct a compact extended formulation of the polyhedron $P$ and show that the Chvátal rank is bounded above by a polynomial function of $P$’s input size. In Section 4, we briefly discuss the Chvátal rank of integer-free polyhedra.

2. The Logarithmic Binarization Scheme and a Motivating Example

In this section, we consider the case where $P$ is a polytope. Therefore we can represent each integer variable using a set of binary variables. These so-called “binary extended formulations” have been previously studied by Glover [8], Roy [9], Sherali and Adams [10], Dash et al. [11], Aprile et al. [12]. In particular, we consider the logarithmic binarization scheme, wherein we replace each bounded integer variable $0 \leq x_i \leq u$ with $[\log_2(u+1)]$ new 0-1 variables $z_{it}$, where $x_i = \sum_{t=1}^{[\log_2(u+1)]} 2^{t-1}z_{it}$. More precisely, if $P = \{x \in \mathbb{R}^n : Ax \leq b, 0 \leq x_i \leq u \ \forall \ i = 1, \ldots, n\}$, then its binary extended formulation is

$$P_B = \{(x, z) \in \mathbb{R}^n \times [0, 1]^{[\log_2(u+1)]} : Ax \leq b,$$

$$x_i = \sum_{t=1}^{[\log_2(u+1)]} 2^{t-1}z_{it} \ \forall \ i = 1, \ldots, n\}.$$ 

We will now present an example that illustrates the potential of binarization in reducing the Chvátal rank.

We refer to the following example with an arbitrarily large Chvátal rank $\theta$, provided by Chvátal [1].

$$P := \{x \in \mathbb{R}^2_+ : \theta x_1 + x_2 \leq \theta, -\theta x_1 + x_2 \leq 0\}.$$ 

For $\theta = 5$ (see Figure 1(a)), this polytope has a Chvátal rank of 5. Observe that conv($P \cap \mathbb{Z}^2$) = $\{x \in \mathbb{R}^2_+ : 0 \leq x_1 \leq 1, x_2 = 0\}$. Prior to defining $P_B$, we note that the variable $x_1$ is already contained in [0, 1]. Thus,

$$P_B = \{(x, z) \in \mathbb{R}^2_+ \times [0, 1]^2 : 5x_1 + x_2 \leq 5,$$

$$-5x_1 + x_2 \leq 0, x_2 = z_1 + 2z_2\}.$$ 

Let $R = \text{proj}_{x_1, z_1, z_2} P_B$. We have $R = \{(x_1, z) \in \mathbb{R}_+ \times [0, 1]^2 : 5x_1 + z_1 + 2z_2 \leq 5, -5x_1 + z_1 + 2z_2 \leq 0\}$ (see Figure 1(b)).
derive these inequalities by applying the Chvátal procedure to \(z_1 \leq \frac{1}{5}\) and \(z_2 \leq \frac{1}{2}\), which are valid inequalities for the Chvátal closure of \(R\).

3. A New Bound on the Chvátal Rank using Binarization

In this section, we show that one can achieve a reduction in the Chvátal rank of a (possibly unbounded) rational polyhedron by using an extended formulation.

Consider a rational, nonempty polyhedron \(P\). We may assume without loss of generality that \(P \subseteq \mathbb{R}^n_+\) as one can replace every unrestricted variable \(x_i \in \mathbb{R}\) by \(x^+_i - x^-_i\), where \(x^+_i, x^-_i \in \mathbb{R}_+\). In the formulation \(P : = \{x \in \mathbb{R}^n_+ : Ax \leq b\}\), we may also assume that \((A, b)\) has integer entries.

We will need the following variation of a classical result from Meyer [13].

**Theorem 1.** Let \(P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}\), where \(A\) is an integral \(m \times n\)-matrix and \(b\) is an integral vector. Let \(C : = \{x \in \mathbb{R}^n_+ : Ax \leq 0\}\). Let \(\Delta\) be the maximum absolute value of the subdeterminants of the matrix \([A, b]\). Then

\[P_I := \text{conv}(P \cap \mathbb{Z}^n) = Q_I + C\]

where \(Q_I\) is the integer hull of the polytope \(Q := P \cap [0, (n+1)\Delta]^n\).

**Proof.** The theorem holds when \(P = \emptyset\), so assume now that \(P\) is nonempty. By the Minkowski-Weyl theorem for polyhedra [14, 15], there exist rational vectors \(v^1, \ldots, v^p\) and integral vectors \(r^1, \ldots, r^Q\) such that \(P = \text{conv}(v^1, \ldots, v^p) + \text{cone}(r^1, \ldots, r^Q)\). We may assume that \(v^1, \ldots, v^p\) are extreme points of \(P\), and therefore each component of \(v^1, \ldots, v^p\) is at most \(\Delta\) in absolute value since they are computed as the quotients of subdeterminants of the matrix \([A, b]\). Additionally, by Cramer’s rule, each component of \(r^1, \ldots, r^Q\) is at most \(\Delta\) in absolute value. Consider the truncation of \(P\)

\[Q := P \cap [0, (n+1)\Delta]^n\]

The set \(T := Q \cap \mathbb{Z}^n\) is finite. Let \(Q_I := \text{conv}(T)\). We claim that \(P_I = Q_I + C\).

Clearly any point in \(Q_I + C\) belongs to \(P_I\). Conversely, consider an integer point \(\tilde{x} \in P_I\). We will show that \(\tilde{x} \in T + C\). Because \(P_I \subseteq P\), we have \(\tilde{x} \in P\). Thus, we can write \(\tilde{x} = \sum_{i=1}^s \lambda_i v^i + \sum_{j=1}^q \mu_j r^j\) with \(\lambda \geq 0, \sum_{i=1}^s \lambda_i = 1\), and \(\mu \geq 0\). By Caratheodory’s theorem, we may assume that there are at most \(n\) nonzero terms in the vector \(\mu\). Let \(\tilde{x}' = \sum_{i=1}^s \lambda_i v^i + \sum_{j=1}^q (\mu_j - [\mu_j]) r^j\), and \(r := \sum_{j=1}^q [\mu_j] r^j\). Then \(\tilde{x} = \tilde{x}' + r\). Note that \(\tilde{x}'\) and \(r\) are integral and so is \(\tilde{x}'\). Furthermore, \(\tilde{x}' \in [0, (n+1)\Delta]^n\) because in the convex combination \(\sum_{i=1}^s \lambda_i v^i\), all points \(v^i\) have components at most \(\Delta\), and in the conic combination \(\sum_{j=1}^q (\mu_j - [\mu_j]) r^j\), there are at most \(n\) vectors \(r^j\) with a nonzero coefficient \(\mu_j - [\mu_j]\), and each of these \(n\) vectors \(r^j\) have components at most \(\Delta\). Therefore \(\tilde{x}' \in T\) and \(\tilde{x} = \tilde{x}' + r \in T + C\).

Any point \(x \in P_I\) is a convex combination of integer points \(\tilde{x} \in P_I\). Since \(\tilde{x} \in T + C\), it follows that any \(x \in P_I\) belongs to \(Q_I + C\). \(\square\)
Theorem 3. Let $P := \{ x \in \mathbb{R}^n_+ : Ax \leq b \}$ be a rational, nonempty polyhedron. Assume that $(A, b)$ has integer entries. Let $\theta = \max(\max_{j \in I} |a_{ij}|, \max_{j \in J} |b_j|)$, and let $\Delta$ be the maximum absolute value of the subdeterminants of the matrix $[A \ b]$. Then

$$R = \left\{ \left( x, (y, z), w \right) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, 1]^{nN} \times \mathbb{R}^n : \right.$n-2

$$x = y + w, \quad Ay \leq b, \quad Aw \leq 0,$n-2

$$y_i = \sum_{t=1}^N 2^{t-1} z_{it}, \forall i = 1, \ldots, n,$n-2

$$y \in [0, (n+1)\Delta]^{nN}\right\}$$n-2

is a compact extended formulation of $P$, where $N = 1 + \log_2(n+1) + n \log_2(n\theta)$.

Proof. By the Minkowski-Weyl theorem for polyhedra, $P = \text{conv}(v^1, \ldots, v^d) + C$, where $v^1, \ldots, v^d$ are the extreme points of $P$ and $C := \{ x \in \mathbb{R}^n_+ : Ax \leq 0 \}$. Each component of the vectors $v^1, \ldots, v^d$ is at most $\Delta$ in absolute value. It follows that $P = Q + C$, where $Q = P \cap [0, (n+1)\Delta]^{nN}$. Then we can apply the logarithmic binarization scheme to each variable $x_i$ that defines $Q$, replacing it by $N$ variables in $[0, 1]$ where $N = 1 + \log_2(n+1)\Delta$. The subdeterminants of $[A \ b]$ can be written as the sum of at most $n!$ products each upper bounded by $\theta^n$. Therefore $\Delta \leq n!\theta^n \leq (n\theta)^n$, and we can choose $N = 1 + \log_2(n+1) + n \log_2(n\theta)$. Thus, we have the desired extended formulation of $P$. Compactness of the extended formulation follows from the fact that this system only has a number of variables and constraints that is polynomial in the size of the input used to describe $P$. \qed

We will also use the following result of Eisenbrand and Schulz [3].

Theorem 2 ([3], Theorem 3.3). The Chvátal rank of a polytope in the $n$-dimensional 0-1 cube is at most $n^2(1 + \log n)$.

Theorem 3. Let $P := \{ x \in \mathbb{R}^n_+ : Ax \leq b \}$ be a rational, nonempty polyhedron. Assume that $(A, b)$ has integer entries. Let $\theta = \max(\max_{j \in I} |a_{ij}|, \max_{j \in J} |b_j|)$. Then there exists a compact extended formulation of $P$ such that the Chvátal rank is at most $O(n^4 \log^2(\theta n))$.

Proof. Consider the extended formulation $R$ of $P$ defined in Lemma 1. Let $R^\Delta = \{ z \in [0, 1]^{nN} : A(\sum_{t=1}^N 2^{t-1} z_i) \leq b \}$, where $\sum_{t=1}^N 2^{t-1} z_i$ is an $n$-dimensional vector such that the $i$th entry corresponds to $\sum_{t=1}^N 2^{t-1} z_i$. Then by Theorem 2, we can obtain $R^\Delta = \text{conv}(R^\Delta \cap [0, 1]^{nN})$ in at most $(nN)^2(1 + \log nN) \approx O(n^4 \log^2(\theta n))$ iterations of the Chvátal procedure. Observe that Chvátal inequalities valid for $R^\Delta$ are also valid for $R_I = \text{conv}(R \cap \mathbb{Z}^n \times \mathbb{Z}^n \times \{0, 1\}^{nN} \times \mathbb{Z}^n)$ as the inequalities that define $R^\Delta$ are a subset of those that define $R$. Consider a vector $(x, (y, z), w) \in R$ in the closure obtained from applying $O(n^4 \log^2(\theta n))$ iterations of the Chvátal procedure on $R$. Since $y \in R, y \in P \cap [0, (n+1)\Delta]^{nN}$. Furthermore, we claim that $y \in Q_I := \text{conv}(P \cap [0, (n+1)\Delta]^{nN} \cap \mathbb{Z}^n)$. Indeed, since $y \in R^\Delta$, we can write $z = \sum_{k \in K} \lambda_k x^k$, where $\sum_{k \in K} \lambda_k = 1$, $\lambda \geq 0$, $x^k \in R^\Delta \cap [0, (n+1)\Delta]^{nN}$ are the integral vertices of $R^\Delta$ and $K$ is the index set. Then $y = \sum_{k \in K} \lambda_k y^k$, where $y^k = \sum_{t=1}^N 2^{t-1} z_i^k$, and thus $y \in Q_I$. By Theorem 1, for all $(x, (y, z), w) \in R_I$, $x \in Q_I + C = P_I$, and therefore, $R_I$ is an extended formulation of $P_I$. \qed

4. The Chvátal Rank of Integer-Free Polyhedra

Cook et al. [5] showed that the length of the cutting plane proof for a rational, integer-free polyhedron $P$ in dimension $d$ is bounded above by a function $g(d)$, which also serves as a bound for the Chvátal rank of $P$. Using results of Reis and Rothvoss [16] on the flatness constant, one can give an explicit bound $g(d) = d^{1+o(d)}e$ for some $\epsilon > 0$. While this result shows that the rank depends only on the dimension of the polyhedron, the bound is exponential in $d$. We present a variation of Theorem 3 for integer-free polyhedra. We will use the following lemma of Bockmayr et al. [17].

Lemma 2 ([17], Lemma 3). Let $P \subseteq [0, 1]^{nN}$ be a $d$-dimensional rational polytope in the 0-1 cube with $P_I = \emptyset$. If $d = 0$, then $P' = \emptyset$; if $d > 0$, then $P(d) = \emptyset$.

The notation $P'$ denotes the elementary Chvátal closure of $P$, and $P(d)$ denotes the $d$th Chvátal closure.

Theorem 4. Let $P := \{ x \in \mathbb{R}^n_+ : Ax \leq b \}$ be a rational, nonempty polyhedron such that its integer hull $P_I = \emptyset$. Assume that $(A, b)$ has integer entries. Let $\theta = \max(\max_{j \in I} |a_{ij}|, \max_{j \in J} |b_j|)$. Then there exists a compact extended formulation of $P$ such that the Chvátal rank is at most $O(n^2 \log^2(\theta n))$.

Proof. Observe that the dimension $d < n$. The proof is identical to that of Theorem 3, where we now invoke Lemma 2 to obtain $R_I = \text{conv}(R^\Delta \cap [0, 1]^{nN}) = \emptyset$ in at most $nN \approx O(n^2 \log^2(\theta n))$ iterations of the Chvátal procedure. \qed

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CRediT authorship contribution statement

Gérard Cornuèjols: Conceptualization, Formal analysis, Funding acquisition, Investigation, Writing – original draft, Writing – review & editing.
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