Reducing the Chvátal Rank through Binarization

Gérard Cornuéjols, Vrishabh Patil
Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213, USA
gc0v@andrew.cmu.edu, vmpatil@andrew.cmu.edu

In a classical paper, Chvátal introduced a rounding procedure for strengthening the polyhedral relaxation P of an integer program; applied recursively, the number of iterations needed to obtain the convex hull of the integer solutions in P is known as the Chvátal rank. Chvátal showed that this rank can be exponential in the input size L needed to describe P. We give a compact extended formulation of P, described by introducing binary variables, whose rank is polynomial in L.

Key words: Chvátal rank, extended formulation, binarization

---

1. Introduction

In a seminal paper, Chvátal (1973) introduced the following rounding procedure for going from a rational polyhedron $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ to the convex hull P_I of the integer points in P: Generate valid inequalities $\alpha x \leq \beta$ for P such that $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{R}$, and round β to its integer part $\lfloor \beta \rfloor$. The intersection of all such inequalities $\alpha x \leq \lfloor \beta \rfloor$ is called the *Chvátal closure*. It is a polyhedron (Chvátal 1973). Chvátal (1973) showed for the bounded case, and Schrijver (1980) in the general case, that repeating this process a finite number of times produces the integer hull P_I. The *Chvátal rank* of a polyhedron P is the smallest number of iterations of the Chvátal procedure needed to obtain P_I. Chvátal (1973) observed that the Chvátal rank can be exponential in the input size needed to describe P, and he gave an example illustrating that this may happen even in two dimensions. However, for a polytope P contained in the 0-1 hypercube, Eisenbrand and Schulz (2003) proved that the Chvátal rank is no more than $O(n^2 \log n)$. Additionally, Rothvoß and Sanita (2013) showed that there exist polytopes contained in the 0-1 hypercube with rank $O(n^2)$. Other measures of the complexity of integer programs have been considered in the literature, such as the length of cutting plane proofs (Cook et al. 1987, Chvátal et al. 1989, Pokutta and Schulz 2010). In this note we focus on the Chvátal rank. We show that for any rational polyhedron P, there exists a *compact extended formulation* such that the Chvátal rank is polynomial in the input size L needed to represent the polyhedron P. By *extended formulation*, we mean a polyhedron $Q \subset \mathbb{R}^{n+q}$ such that $P = \text{proj}_x Q$. By *compact* we mean that Q has encoding size that is polynomial in L. Assuming A and b have integral entries, L can be expressed in terms of the number of variables n, the number of constraints m, and $\log_2 \theta$ where $\theta = \max(\max_{j,i} |a_{ji}|, \max_j |b_j|)$.

The key idea behind this result is to use a binary extended formulation on a bounded polytope that contains a truncation of the polyhedron P, while also keeping general variables to handle the possible unboundedness of P. In Section 2, we focus on the case where P is a polytope and provide a motivating example. We then present our main result in Section 3 where we construct a compact extended formulation of the polyhedron P and show that the Chvátal rank is bounded above by a polynomial function of P's input size. In Section 4, we briefly discuss the Chvátal rank of integer-free polyhedra.

2. The Logarithmic Binarization Scheme and a Motivating Example

In this section, we consider the case where P is a polytope. Therefore we can represent each integer variable using a set of binary variables. These so called "binary extended formulations" have been previously studied by Glover (1975), Roy (2007), Sherali and Adams (1990), Dash et al. (2018), Aprile et al. (2023). In particular, we consider the logarithmic binarization scheme, wherein we replace each bounded integer variable $0 \le x_i \le u$ with $\lceil \log_2(u+1) \rceil$ new 0-1 variables z_{it}, where $x_i = \sum_{t=1}^{\lceil \log_2(u+1)\rceil} 2^{t-1} z_{it}$. More precisely, if $P = \{x \in \mathbb{R}^n : Ax \le b,\ 0 \le x_i \le u \ \forall\ i = 1, \ldots, n\}$, then its binary extended formulation is

$$P_{\mathcal{B}} = \Big\{(x,z) \in \mathbb{R}^n \times [0,1]^{n\lceil \log_2(u+1)\rceil} : Ax \le b,\ x_i = \sum_{t=1}^{\lceil \log_2(u+1)\rceil} 2^{i-1} z_{it} \ \forall\ i = 1, \ldots, n\Big\}.$$

We will now present an example that illustrates the potential of binarization.

We refer to the following example with an arbitrarily large Chvátal rank θ, provided by Chvátal (1973).

$$P := \{x \in \mathbb{R}^2_+ : \theta x_1 + x_2 \le \theta, -\theta x_1 + x_2 \le 0\}.$$

For $\theta = 5$ (see Figure 1a), this polytope has a Chvátal rank of 5. Observe that $\operatorname{conv}(P \cap \mathbb{Z}^2) = \{x \in \mathbb{R}^2 : 0 \le x_1 \le 1, x_2 = 0\}$. Prior to defining $P_{\mathcal{B}}$, we note that the variable x_1 is already contained in $[0,1]$. Thus,

$$P_{\mathcal{B}} = \{(x,z) \in \mathbb{R}^2_+ \times [0,1]^2 : 5x_1 + x_2 \le 5, -5x_1 + x_2 \le 0, x_2 = z_1 + 2z_2\}.$$

Let $R = \operatorname{proj}_{(x_1,z_1,z_2)} P_{\mathcal{B}}$. We have $R = \{(x_1,z) \in \mathbb{R}_+ \times [0,1]^2 : 5x_1 + z_1 + 2z_2 \le 5, -5x_1 + z_1 + 2z_2 \le 0\}$ (see Figure 1b).

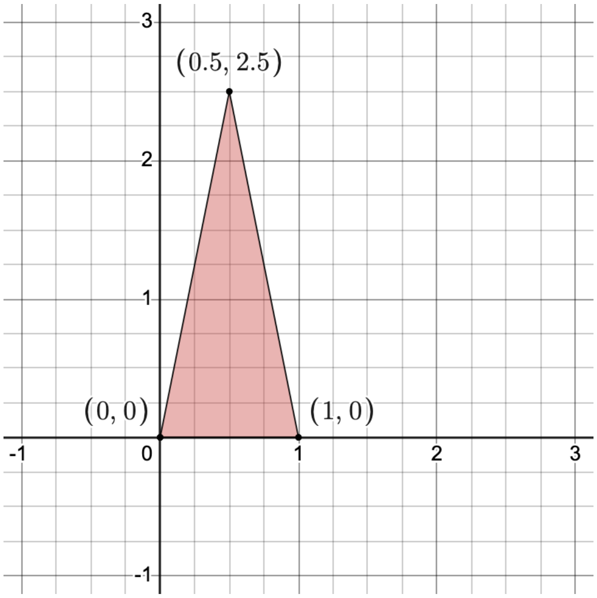
We will show that the Chvátal rank of R is 2. First, observe that the following inequalities are valid for R: $x_1 + z_1 \le 9/5$, $x_1 + z_2 \le 8/5$, $-x_1 + z_1 \le 4/5$, $-x_1 + z_2 \le 3/5$. Therefore the following inequalities are rank 1 Chvátal inequalities for R.

$$x_1 + z_1 \le 1 \tag{1}$$

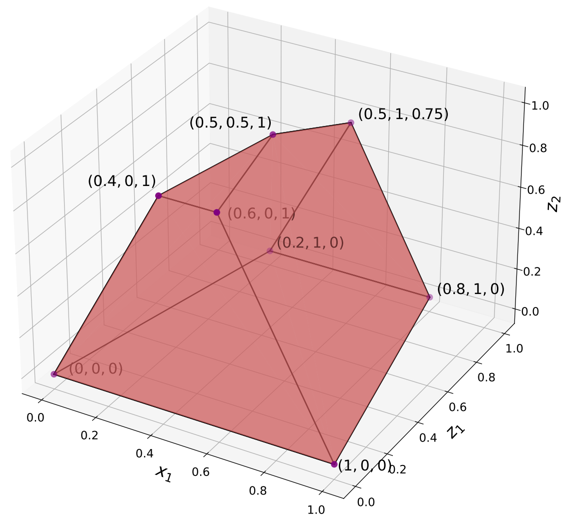
$$x_1 + z_2 \le 1 \tag{2}$$

$$-x_1 + z_1 \le 0 \tag{3}$$

$$-x_1 + z_2 \le 0 \tag{4}$$



(a) Chvátal's polytope P where $\theta = 5$.



(b) The polytope $R = \text{proj}_{(x_1, z_1, z_2)} P_{\mathcal{B}}$.

Figure 1 **Chvátal's polytope and its extended formulation.**

It suffices to show that $z_1 \leq 0$ and $z_2 \leq 0$ are rank 2 Chvátal inequalities for R since $\text{conv}(R \cap \mathbb{Z}^3) = \{(x_1, z_1, z_2) : 0 \leq x_1 \leq 1, z_1 = z_2 = 0\}$. We can derive these inequalities by applying the Chvátal procedure to $z_1 \leq \frac{1}{2}$ and $z_2 \leq \frac{1}{2}$, which are valid inequalities for the Chvátal closure of R.

3. A New Bound on the Chvátal Rank using Binarization

In this section, we show that one can achieve a reduction in the Chvátal rank of a (possibly unbounded) rational polyhedron by using an extended formulation.

Consider a rational, nonempty polyhedron P. We may assume without loss of generality that $P \subset \mathbb{R}^n_+$ as one can replace every unrestricted variable $x_i \in \mathbb{R}$ by $x_i^+ - x_i^-$ where $x_i^+, x_i^- \in \mathbb{R}_+$. In the formulation $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$, we may also assume that (A, b) has integer entries.

We will need the following variation of a classical result of Meyer (1974).

Theorem 1. *Let $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$, where A is an integral $m \times n$-matrix and b is an integral vector. Let $C := \{x \in \mathbb{R}^n_+ : Ax \leq 0\}$. Let Δ be the maximum absolute value of the subdeterminants of the matrix $[A\ b]$. Then*

$$P_I := \text{conv}(P \cap \mathbb{Z}^n) = Q_I + C$$

where Q_I is the integer hull of the polytope $Q := P \cap [0, (n+1)\Delta]^n$.

Proof. The theorem holds when $P = \emptyset$, so assume now that P is nonempty. By the Minkowski-Weyl theorem for polyhedra (Minkowski 1910, Weyl 1950), there exist rational vectors $v^1, \ldots, v^s$ and integral vectors $r^1, \ldots, r^q$ such that $P = \text{conv}(v^1, \ldots, v^s) + \text{cone}(r^1, \ldots, r^q)$. We may assume that $v^1, \ldots, v^s$ are extreme points of P, and therefore each component of $v^1, \ldots, v^s$ is at most

Δ in absolute value since they are computed as the quotients of subdeterminants of the matrix $[A\ b]$. Additionally, by Cramer's rule, each component of $r^1, \ldots, r^q$ is at most Δ in absolute value. Consider the truncation of P

$$Q := P \cap [0, (n+1)\Delta]^n.$$

The set $T := Q \cap \mathbb{Z}^n$ is finite. Let $Q_I := \mathrm{conv}(T)$. We claim that $P_I = Q_I + C$.

Clearly any point in $Q_I + C$ belongs to P_I. Conversely, consider an integer point $\bar{x} \in P_I$. We will show that $\bar{x} \in T + C$. Because $P_I \subseteq P$, we have $\bar{x} \in P$. Thus, we can write $\bar{x} = \sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} \mu_j r^j$ with $\lambda \geq 0$, $\sum_{i=1}^{s} \lambda_i = 1$, and $\mu \geq 0$. By Caratheodory's theorem, we may assume that there are at most n nonzero terms in the vector μ. Let $\bar{x}' = \sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} (\mu_j - \lfloor \mu_j \rfloor) r^j$ and $r := \sum_{j=1}^{q} \lfloor \mu_j \rfloor r^j$. Then $\bar{x} = \bar{x}' + r$. Note that $\bar{x}$ and r are integral and so is $\bar{x}'$. Furthermore, $\bar{x}' \in [0, (n+1)\Delta]^n$ because in the convex combination $\sum_{i=1}^{s} \lambda_i v^i$, all points v^i have components at most Δ, and in the conic combination $\sum_{j=1}^{q} (\mu_j - \lfloor \mu_j \rfloor) r^j$, there are at most n vectors r^j with a nonzero coefficient $\mu_j - \lfloor \mu_j \rfloor$, and each of these n vectors r^j have components at most Δ. Therefore $\bar{x}' \in T$ and $\bar{x} = \bar{x}' + r \in T + C$. Any point $x \in P_I$ is a convex combination of integer points $\bar{x} \in P_I$. Since $\bar{x} \in T + C$, it follows that x belongs to $Q_I + C$. $\square$

The following lemma describes a compact extended formulation of P.

Lemma 1. *Let $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ be a rational, nonempty polyhedron. Assume that (A, b) has integer entries. Let $\theta = \max(\max_{j,i} |a_{ji}|, \max_j |b_j|)$, and let Δ be the maximum absolute value of the subdeterminants of the matrix $[A\ b]$. Then*

$$R = \Big\{ (x, (y, z), w) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,1]^{nN} \times \mathbb{R}^n : \begin{aligned}[t] & x = y + w, \\ & Ay \leq b, \\ & Aw \leq 0, \\ & y_i = \sum_{t=1}^{N} 2^{t-1} z_{it}, \ \forall i = 1, \ldots, n, \\ & y \in [0, (n+1)\Delta]^n \Big\} \end{aligned}$$

is a compact extended formulation of P, where $N = 1 + \log_2(n+1) + n \log_2(n\theta)$.

Proof. By the Minkowski-Weyl theorem, $P = \mathrm{conv}(v^1, \ldots, v^s) + C$, where $v^1, \ldots, v^s$ are the extreme points of P and $C := \{x \in \mathbb{R}^n_+ : Ax \leq 0\}$. Each component of the vectors $v^1, \ldots, v^s$ is at most Δ in absolute value. It follows that $P = Q + C$, where $Q = P \cap [0, (n+1)\Delta]^n$. Then we can apply the logarithmic binarization scheme to each variable y_i that defines Q, replacing it by N variables in $[0, 1]$ where $N = 1 + \lceil \log_2(n+1)\Delta \rceil$. The subdeterminants of $[A\ b]$ can be written as the sum of at most $n!$ products each upper bounded by θ^n. Therefore $\Delta \leq n!\theta^n \leq (n\theta)^n$, we can choose $N = 1 + \log_2(n+1) + n\log_2(n\theta)$. Thus, we have the desired extended formulation of P. Compactness of the extended formulation follows from the fact that this system only has a number of variables and constraints that is polynomial in the size of the input used to describe P. $\square$

We will also use the following result of Eisenbrand and Schulz (2003).

THEOREM 2 **(Eisenbrand and Schulz (2003), Theorem 3.3)**. *The Chvátal rank of a polytope in the n-dimensional 0-1 cube is at most $n^2(1+\log n)$.*

THEOREM 3. *Let $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ be a rational, nonempty polyhedron. Assume that (A, b) has integer entries. Let $\theta = \max(\max_{j,i}|a_{ji}|, \max_j |b_j|)$. Then there exists a compact extended formulation of P such that the Chvátal rank is at most $O(n^4 \log_2^2(n\theta))$.*

Proof. Consider the extended formulation R of P defined in Lemma 1. Let $R^z = \{z \in [0,1]^{nN} : A(\sum_{t=1}^{N} \mathbf{2^{t-1}} \boldsymbol{z_t}) \leq b\}$, where $\sum_{t=1}^{N} \mathbf{2^{t-1}} \boldsymbol{z_t}$ is an n-dimensional vector such that the i^{th} entry corresponds to $\sum_{t=1}^{N} 2^{t-1} z_{it}$. Then by Theorem 2, we can obtain $R^z_I = \text{conv}(R^z \cap \{0,1\}^{nN}\})$ in at most $(nN)^2(1+\log nN) \approx O(n^4 \log_2^2(n\theta))$ iterations of the Chvátal procedure. Observe that Chvátal inequalities valid for R^z_I are also valid for $R_I = \text{conv}\big(R \cap (\mathbb{Z}^n \times \mathbb{Z}^n \times \{0,1\}^{nN} \times \mathbb{Z}^n)\big)$ as the inequalities that define R^z are a subset of those that define R. Consider a vector $(x,(y,z),w)$ in the closure obtained from applying $O(n^4 \log_2^2(n\theta))$ iterations of the Chvátal procedure on R. Since $y \in R$, $y \in P \cap [0,(n+1)\Delta]^n$. Furthermore, we claim that $y \in Q_I := \text{conv}\big((P \cap [0,(n+1)\Delta]^n]) \cap \mathbb{Z}^n\big)$. Indeed, since $z \in R^z_I$, we can write $z = \sum_{k\in K} \lambda_k z^k$ where $\sum_{k \in K} \lambda_k = 1$, $\lambda \geq 0$, $z^k \in R^z \cap \{0,1\}^{nN}$ are the integral vertices of R^z_I and K is the index set. Then $y = \sum_{k\in K} \lambda_k y^k$, where $y^k = \sum_{t=1}^{N} \mathbf{2^{t-1}} \boldsymbol{z^k_t}$, and thus $y \in Q_I$. By Theorem 1, for all $(x,(y,z),w) \in R_I$, $x \in Q_I + C = P_I$, and therefore, R_I is an extended formulation of P_I. $\square$

4. The Chvátal Rank of Integer-Free Polyhedra

Cook et al. (1987) showed that the length of the cutting plane proof for a rational, integer-free polyhedron P in dimension d is bounded above by a function $g(d)$, which also serves as a bound for the Chvátal rank of P. Using results of Reis and Rothvoss (2023) on the flatness constant, one can give an explicit bound $g(d) = d^{(1+\epsilon)d}$ for some $\epsilon > 0$. While this result shows that the rank depends only on the dimension of the polyhedron, the bound is exponential in d. We present a variation of Theorem 3 for integer-free polyehdra. We will use the following lemma of Bockmayr et al. (1999).

LEMMA 2 **(Bockmayr et al. (1999), Lemma 3)**. *Let $P \subseteq [0,1]^n$ be a d-dimensional rational polytope in the 0-1 cube with $P_I = \emptyset$. If $d = 0$, then $P' = \emptyset$; if $d > 0$, then $P^{(d)} = \emptyset$.*

The notation P' denotes the elementary Chvátal closure of P, and $P^{(d)}$ denotes the d^{th} Chvátal closure.

THEOREM 4. *Let $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ be a rational, nonempty polyhedron such that its integer hull $P_I = \emptyset$. Assume that (A, b) has integer entries. Let $\theta = \max(\max_{j,i}|a_{ji}|, \max_j |b_j|)$. Then no more than $O(n^2 \log_2(n\theta))$ iterations of the Chvátal procedure are needed to show that $P_I = \emptyset$.*

Proof. Observe that the dimension $d < n$. The proof is identical to that of Theorem 3, where we now invoke Lemma 2 to obtain $R^z_I = \mathrm{conv}(R^z \cap \{0,1\}^{nN}\}) = \emptyset$ in at most $nN \approx O(n^2 \log_2(n\theta))$ iterations of the Chvátal procedure. □

Acknowledgements: This research was supported by the U.S. Office of Naval Research under award number N00014-22-1-2528.

References

Aprile M, Conforti M, Di Summa M (2023) Binary extended formulations and sequential convexification. *Mathematics of Operations Research* .

Bockmayr A, Eisenbrand F, Hartmann M, Schulz AS (1999) On the chvátal rank of polytopes in the 0/1 cube. *Discrete Applied Mathematics* 98(1-2):21–27.

Chvátal V (1973) Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete mathematics* 4(4):305–337.

Chvátal V, Cook W, Hartmann M (1989) On cutting-plane proofs in combinatorial optimization. *Linear algebra and its applications* 114:455–499.

Cook W, Coullard CR, Turán G (1987) On the complexity of cutting-plane proofs. *Discrete Applied Mathematics* 18(1):25–38.

Dash S, Günlük O, Hildebrand R (2018) Binary extended formulations of polyhedral mixed-integer sets. *Mathematical Programming* 170(1):207–236.

Eisenbrand F, Schulz AS (2003) Bounds on the chvátal rank of polytopes in the 0/1-cube. *Combinatorica* 23:245–261.

Glover F (1975) Improved linear integer programming formulations of nonlinear integer problems. *Management science* 22(4):455–460.

Meyer RR (1974) On the existence of optimal solutions to integer and mixed-integer programming problems. *Mathematical Programming* 7:223–235.

Minkowski H (1910) *Geometrie der zahlen* (BG Teubner).

Pokutta S, Schulz AS (2010) On the rank of cutting-plane proof systems. *Integer Programming and Combinatorial Optimization: 14th International Conference, IPCO 2010, Lausanne, Switzerland, June 9-11, 2010. Proceedings 14*, 450–463 (Springer).

Reis V, Rothvoss T (2023) The subspace flatness conjecture and faster integer programming. *arXiv preprint arXiv:2303.14605* .

Rothvoß T, Sanita L (2013) 0/1 polytopes with quadratic chvátal rank. *Integer Programming and Combinatorial Optimization: 16th International Conference, IPCO 2013, Valparaíso, Chile, March 18-20, 2013. Proceedings 16*, 349–361 (Springer).

Roy JS (2007) "binarize and project" to generate cuts for general mixed-integer programs. *Algorithmic Operations Research* 2(1):37–51.

Schrijver A (1980) On cutting planes. *Combinatorics* 79:291–296.

Sherali HD, Adams WP (1990) A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM Journal on Discrete Mathematics* 3(3):411–430.

Weyl H (1950) The elementary theory of convex polyhedra. *Contributions to the Theory of Games I* 3–18.