Kent Andersen · Gérard Cornuéjols · Yanjun Li

Split closure and intersection cuts⋆

Abstract. In the seventies, Balas introduced intersection cuts for a Mixed Integer Linear Program (MILP), and showed that these cuts can be obtained by a closed form formula from a basis of the standard linear programming relaxation. In the early nineties, Cook, Kannan and Schrijver introduced the split closure of a MILP, and showed that the split closure is a polyhedron. In this paper, we show that the split closure can be obtained using only intersection cuts. We give two different proofs of this result, one geometric and one algebraic. The result is then used to provide a new proof of the fact that the split closure of a MILP is a polyhedron. Finally, we extend the result to more general disjunctions.

1. Introduction

In the seventies, Balas [2] showed that cuts for a Mixed Integer Linear Program (MILP) can be derived from disjunctions applied to the bases of its linear programming relaxation. In that paper, the basis was an optimal basis to the linear programming relaxation. The cut was obtained by a closed form formula and was called the intersection cut.

Later in the seventies, Balas [3] generalized his results to polyhedra. He proved that, given a polyhedron P and a disjunction, a cut for a point $\bar{x} \in P$ that violates the disjunction can be obtained by solving a linear program. The idea was further expanded in the early nineties when Cook, Kannan and Schrijver [6] studied split cuts obtained from disjunctions with two terms called split disjunctions. The intersection of all split cuts is called the split closure of a MILP. Cook, Kannan and Schrijver proved that the split closure of a MILP is a polyhedron. Split cuts are equivalent to Gomory mixed integer cuts [9] and to mixed integer rounding cuts generated from linear combinations of constraints [10, 8]. Relations between these cuts and other families of cuts are surveyed in [7].

Intersection cuts are a special type of split cuts. The main contribution of this paper is to show that, conversely, the split closure of a MILP can be obtained using only intersection cuts, provided infeasible and non-optimal bases are considered. This result was first shown by Balas and Perregaard [4] for mixed 0–1 linear programs. For such programs, it suffices to consider 0–1 disjunctions of the form $x_j \leq 0$ or $x_j \geq 1$. We generalize their

K. Andersen: Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213, USA.
e-mail: `kha@andrew.cmu.edu`

G. Cornuéjols: Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213, USA; and LIF, University of Marseille at Luminy, 13288 Marseille, France. e-mail: `gc0v@andrew.cmu.edu`

Y. Li: Krannert Graduate School of Management, Purdue University, West Lafayette, IN 47907, USA.
e-mail: `li14@mgmt.purdue.edu`

result from 0–1 disjunctions to more general two-term disjunctions. We also consider the more general setting of MILPs. Two different proofs are provided, one geometric and one algebraic. A consequence is a new proof of the fact that the split closure of a MILP is a polyhedron.

Our main result shows that the only interesting split cuts are intersection cuts. This result has algorithmic implications since intersection cuts can be obtained by a closed form formula and many of the cutting planes used in practice are split cuts. In particular, Gomory mixed integer cuts generated from an optimal basis are currently the most effective in commercial codes [5]. In [1], we exploit the main result of this paper by fixing the optimal basis and modifying the split disjunctions that generate the Gomory mixed integer cuts. On several test problems, these cuts drastically reduce the number of nodes in a branch-and-cut framework.

1.1. Notation

We consider the Mixed Integer Linear Program

$$(MILP) \qquad min\{c^T x : Ax \leq b, \ x_j \text{ integer}, \ j \in N_I\},$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $N_I \subseteq N := \{1, 2, \ldots, n\}$. Let LP be the Linear Programming problem obtained from MILP by dropping the integrality conditions on x_j for $j \in N_I$. P_I and P denote the sets of feasible solutions to MILP and LP respectively. The set $M := \{1, 2, \ldots, m\}$ is used to index the rows of A. Given $i \in M$, the ith row of A is denoted $a_{i.}$. We assume $a_{i.} \neq 0_n$ for all $i \in M$. A *basis* of A is an n-subset B of M, such that the vectors $\{a_{i.}\}_{i \in B}$ are linearly independent. Observe that, if the rank of A is less than n, A does not have bases.

The relaxations of P obtained by dropping constraints of P play a major role in this paper. Given $S \subseteq M$, define

$$P(S) := \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i, \forall i \in S\}$$

to be the relaxation of P obtained by keeping the constraints in S only. The rank of the sub-matrix of A induced by the rows in S is denoted $r(S)$. The rank of A is denoted simply by $r := r(M)$.

The case where the vectors $\{a_{i.}\}_{i \in S}$ are linearly independent is especially important. Given a positive integer k, let

$$\mathcal{B}_k^* := \{S \subseteq M : |S| = k \text{ and } r(S) = k\}$$

denote the set of k-subsets S of M, such that the vectors $\{a_{i.}\}_{i \in S}$ are linearly independent ($\mathcal{B}_n^*$ denotes the set of bases of A). Note that $\mathcal{B}_k^*$ is empty if $k > r$. Given $B \in \mathcal{B}_k^*$ where $k \leq r$, the set $P(B)$ is a translate of a polyhedral cone. Specifically, $P(B)$ can be written as $P(B) = C + \bar{x}(B)$, where $\bar{x}(B)$ solves the system $a_{i.}^T x = b_i, \forall i \in B$, and C is the polyhedral cone $C := \{x \in \mathbb{R}^n : a_{i.}^T x \leq 0, \forall i \in B\}$. Translates of polyhedral cones have many properties in common with polyhedral cones. We call them *conic polyhedra* in the remainder of this paper.

The most general two-term disjunction considered in this paper is an expression D of the form $D^1 x \leq d^1 \vee D^2 x \leq d^2$, where $D^1 \in \mathbb{R}^{m_1 \times n}$, $D^2 \in \mathbb{R}^{m_2 \times n}$, $d^1 \in \mathbb{R}^{m_1}$ and $d^2 \in \mathbb{R}^{m_2}$. The set of points in $\mathbb{R}^n$ satisfying the disjunction D is denoted by F_D. The convex hull of $P \cap F_D$, denoted by $Conv\,(P \cap F_D)$, is called the *disjunctive hull* defined by P and D in the remainder of this paper. In addition, given a subset S of the constraints, the set $Conv\,(P(S) \cap F_D)$ is called the disjunctive hull defined by S and D. In particular, given a set $B \in \mathcal{B}_r^*$, the set $Conv\,(P(B) \cap F_D)$ is called a *basic* disjunctive hull. An important two-term disjunction is the *split disjunction* $D(\pi, \pi_0)$ of the form $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$, where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $\pi_j = 0$ for all $j \notin N_I$.

Let $\Pi^n(N_I) := \{(\pi, \pi_0) \in \mathbb{Z}^{n+1} : \pi_j = 0, j \notin N_I\}$. The *split closure* of a MILP, denoted by SC, is defined to be the intersection of the disjunctive hulls defined by P and $D(\pi, \pi_0)$ over all the disjunctions (π, π_0) in $\Pi^n(N_I)$, *i.e.*,

$$\text{SC} := \cap_{(\pi, \pi_0) \in \Pi^n(N_I)} Conv\,(P \cap F_{D(\pi, \pi_0)}).$$

Similarly, given $S \subseteq M$, $\text{SC}(S)$ is defined to be the intersection of the disjunctive hulls defined by $P(S)$ and $D(\pi, \pi_0)$ over all disjunctions (π, π_0) in $\Pi^n(N_I)$. A *split cut* is a valid inequality for SC.

Several results presented in this paper about split disjunctions generalize to disjunctions $D(\pi, \pi_0^1, \pi_0^2)$ of the form $\pi^T x \leq \pi_0^1 \vee \pi^T x \geq \pi_0^2$, where $\pi \in \mathbb{R}^n$ and $\pi_0^1 < \pi_0^2$. These disjunctions are called *general* split disjunctions in the remainder of the paper. Given π_0^1 and π_0^2 satisfying $\pi_0^1 < \pi_0^2$, not all general split disjunctions are valid for MILP, *i.e.*, MILP may have feasible solutions that violate a disjunction $D(\pi, \pi_0^1, \pi_0^2)$ for some $\pi \in \mathbb{R}^n$. However, for the sake of generality, we state and prove some results in terms of general split disjunctions. The results obtained for general split disjunctions imply the corresponding results for split disjunctions.

1.2. Main Contributions

The first contribution of this paper is a result (Corollary 1) stating that the split closure of MILP can be written as the intersection of the split closures of the sets $P(B)$ over all sets $B \in \mathcal{B}_r^*$, *i.e.*, $\text{SC} = \cap_{B \in \mathcal{B}_r^*} \text{SC}(B)$. We prove this result by proving that the disjunctive hull defined by P and a general split disjunction $D(\pi, \pi_0^1, \pi_0^2)$ can be written as the intersection of the basic disjunctive hulls, *i.e.*, $Conv\,(P \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = \cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$. We provide both a geometric and an algebraic proof of this result. The result implies that the disjunctive hull defined by P and a split disjunction $D(\pi, \pi_0)$ can be obtained using only intersection cuts (A precise definition of intersection cut is given in Sect. 2.1). In turn, this implies that the split closure of a MILP can be obtained using only intersection cuts. Furthermore, the result leads to a new proof of the fact that the split closure is a polyhedron (Theorem 2).

The second contribution of this paper is Theorem 3 that describes the disjunctive hull defined by P and a general two-term disjunction D.

The rest of this paper is organized as follows. In Sect. 2, we give an outline of the main results in the paper. We also give examples showing that some, seemingly natural, extensions of these results are incorrect. In Sect. 3, we consider conic polyhedra. In

(a) A polytope   (b) A polytope and a split disjunction   (c) The disjunctive hull
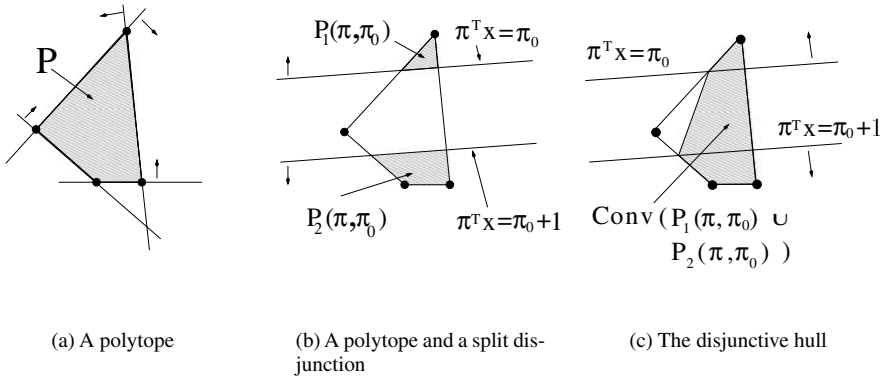
Fig. 1. Effect of applying a split disjunction to a polytope

particular, given $B \in \mathcal{B}_r^*$, we show that the split closure of the conic polyhedron $P(B)$ is a polyhedron. In Sect. 4, a characterization of the split closure in terms of intersection cuts is presented, and a geometric argument for the validity of the result is presented. In Sect. 5, we generalize the results of Sect. 4 to more general two-term disjunctions. The arguments used in Sect. 5 are mostly algebraic. In fact, Sect. 4 and Sect. 5 can be read independently.

2. Outline

We start by giving an outline and by presenting examples of the results derived in the paper. Furthermore, we give counterexamples of some seemingly natural extensions of these results.

Figure 1(a) gives an example of a polyhedron P (in this case a polytope). In Figure 1(b), a split disjunction $D(\pi, \pi_0)$ is given with P. As can be seen from Figure 1(b), $P \cap D(\pi, \pi_0)$ contains two disjoint parts $P_1(\pi, \pi_0)$ and $P_2(\pi, \pi_0)$ (either might be empty). The main reason that split disjunctions are interesting is that the valid inequalities for $P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0)$ are valid inequalities for MILP. Since an inequality is valid for $P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0)$ if and only if it is valid for $Conv\,(P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0))$, this leads naturally to a study of the facial structure of the polyhedron $Conv\,(P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0))$. The shaded area in Figure 1(c) describes $Conv\,(P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0))$.

2.1. Split disjunctions, conic polyhedra and intersection cuts

It seems natural to start with the simplest polyhedra. Consider a conic polyhedron $P(B)$ for $B \in \mathcal{B}_r^*$. Due to the linear independence of the vectors $\{a_{i.}\}_{i \in B}$, this polyhedron is sufficiently simple. Observe that, in the case where $r = n$, B defines a basis of A. Figure 2(a) shows a polyhedron in the two-dimensional plane and gives an example of such a set B. As illustrated in Figure 2(b), $P(B)$ is the set obtained from P by ignoring the constraints in $M \setminus B$.
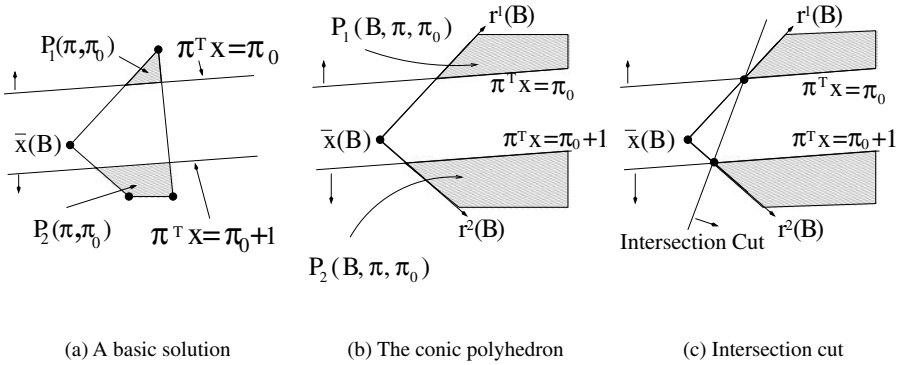
(a) A basic solution (b) The conic polyhedron (c) Intersection cut

Fig. 2. Deriving a conic polyhedron and an intersection cut

In Sect. 1, we gave an inequality description of the polyhedron $P(B)$. However, $P(B)$ can also be described in terms of its extreme points and extreme rays as follows. Let $\bar{x}(B)$ satisfy $a_{i.}^T \bar{x}(B) = b_i$ for all $i \in B$. Furthermore, let $L(B) := \{x \in \mathbb{R}^n : a_{i.}^T x = 0, \forall i \in B\}$. $L(B)$ is the null space of the matrix $[a_{i.}]_{i \in B}$, and $\bar{x}(B) + L(B)$ is the intersection of the hyperplanes $a_{i.}^T x = b_i$ for $i \in B$. In the example of Figure 2, there are two linearly independent vectors in two dimensions, so $L(B)$ reduces to $\{0_n\}$ and $\bar{x}(B)$ is uniquely defined. If $r < n$, this is not the case. The extreme rays $r^i(B)$ of $P(B)$ can be obtained as follows. Since the vectors $\{a_{i.}\}_{i \in B}$ are linearly independent, there exists a solution $r^i(B)$ to the system $a_{k.}^T r^i(B) = 0, \forall k \in B \setminus \{i\}$, and $a_{i.}^T r^i(B) = -1$. Now $P(B)$ can be written as

$$P(B) = \bar{x}(B) + L(B) + Cone\left(\{r^i(B)\}_{i \in B}\right), \tag{1}$$

where $Cone\left(\{r^i(B)\}_{i \in B}\right) := \{x \in \mathbb{R}^n : x = \sum_{i \in B} \lambda_i r^i(B), \lambda_i \geq 0, i \in B\}$ denotes the polyhedral cone generated by the vectors $\{r^i(B)\}_{i \in B}$. Observe that the vectors $\{r^i(B)\}_{i \in B}$ are linearly independent.

We are now in position to derive the intersection cut. Let $D(\pi, \pi_0^1, \pi_0^2)$ be a general split disjunction. Assume all points y in $\bar{x}(B) + L(B)$ violate the disjunction $D(\pi, \pi_0^1, \pi_0^2)$ (Lemma 1 below shows this is the only interesting case). This implies that the linear function $\pi^T x$ is constant on $\bar{x}(B) + L(B)$. Otherwise, the function value can be made as positive (or negative) as we want by choosing particular points in $\bar{x}(B) + L(B)$, which is a contradiction to the assumption that all points in $\bar{x}(B) + L(B)$ violate the disjunction $D(\pi, \pi_0^1, \pi_0^2)$. Define $\epsilon^1(\pi, \pi_0^1, B) := \pi^T \bar{x}(B) - \pi_0^1$ and $\epsilon^2(\pi, \pi_0^2, B) := \pi_0^2 - \pi^T \bar{x}(B)$ to be the amount by which the points in $\bar{x}(B) + L(B)$ violate the first and second terms in the disjunction respectively. Also, for $i \in B$, define

$$\alpha_i(\pi, \pi_0^1, \pi_0^2, B) := \begin{cases} -\epsilon^1(\pi, \pi_0^1, B)/(\pi^T r^i(B)) & \text{if } \pi^T r^i(B) < 0, \\ \epsilon^2(\pi, \pi_0^2, B)/(\pi^T r^i(B)) & \text{if } \pi^T r^i(B) > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

The interpretation of the numbers $\alpha_i(\pi, \pi_0^1, \pi_0^2, B)$, for $i \in B$, is the following. For $\alpha \in \mathbb{R}_+$, let $x^i(\alpha, B) := \bar{x}(B) + \alpha r^i(B)$ denote the half-line starting at $\bar{x}(B)$ in the

direction $r^i(B)$. The value $\alpha_i(\pi, \pi_0^1, \pi_0^2, B)$ is the smallest value of $\alpha \in \mathbb{R}_+$, if any, such that $x^i(\alpha, B)$ satisfies the disjunction $D(\pi, \pi_0^1, \pi_0^2)$ (see Figure 2(c) for an example). If the direction $r^i(B)$ is parallel to the hyperplanes $\pi^T x = \pi_0^1$ and $\pi^T x = \pi_0^2$, $\alpha_i(\pi, \pi_0^1, \pi_0^2, B)$ is defined to be $+\infty$. Given the numbers $\alpha_i(\pi, \pi_0^1, \pi_0^2, B)$ for $i \in B$, the *intersection cut* associated with B and $D(\pi, \pi_0^1, \pi_0^2)$ is given by

$$\sum_{i \in B} (b_i - a_{i.}^T x)/\alpha_i(\pi, \pi_0^1, \pi_0^2, B) \geq 1. \tag{3}$$

The validity of this inequality for $Conv\,(P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$ was proven by Balas [2]. In fact, the intersection cut gives a complete description of the basic disjunctive hull associated with B and $D(\pi, \pi_0^1, \pi_0^2)$.

Lemma 1. *Let $B \in \mathcal{B}_r^*$ and $D(\pi, \pi_0^1, \pi_0^2)$ be a general split disjunction.*

  (i) *If $\pi^T x \notin]\pi_0^1, \pi_0^2[$ for some $x \in \bar{x}(B) + L(B)$, then $Conv\,(P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = P(B)$.*
  (ii) *If $\pi^T x \in]\pi_0^1, \pi_0^2[$ for all $x \in \bar{x}(B) + L(B)$, then $Conv\,(P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = \{x \in P(B) : (3)\}$.*

The proof of Lemma 1 will be given in Sect. 3.

2.2. Split disjunctions and polyhedra

Having completely described the basic disjunctive hulls for general split disjunctions, a natural question is whether considering the intersection of all these sets is enough to describe the disjunctive hull associated with the polyhedron. The main result in this paper gives a positive answer to this question as follows.

Theorem 1. *For every (π, π_0^1, π_0^2) satisfying $\pi_0^1 < \pi_0^2$,*

$$Conv\,(P \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = \bigcap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}). \tag{4}$$

From a convex analysis perspective, this is an unusual result since it is typically not allowed to interchange the intersection operator with the convex hull operator. Note that $P = \cap_{B \in \mathcal{B}_r^*} P(B)$. Theorem 1 states that $Conv\,(\cap_{B \in \mathcal{B}_r^*} P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = \cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$.

An immediate consequence of Theorem 1 is the following characterization of the split closure of a MILP, which implies that the split closure is completely described by intersection cuts.

Corollary 1.

$$SC = \bigcap_{B \in \mathcal{B}_r^*} SC(B). \tag{5}$$

Another consequence is a new proof of the following fact.

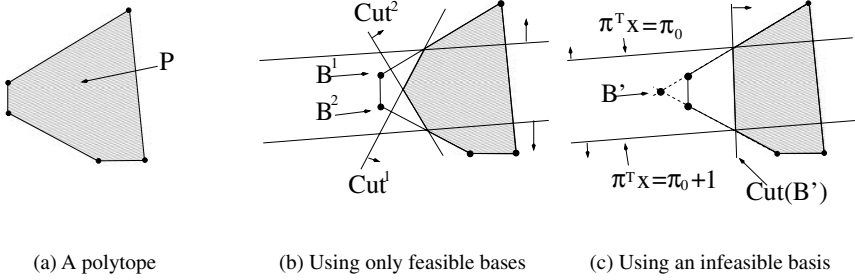**Theorem 2.** *The split closure of MILP is a polyhedron.*

(a) A polytope   (b) Using only feasible bases   (c) Using an infeasible basis

Fig. 3. Infeasible bases are needed in Theorem 1

This was first proved by Cook, Kannan and Schrijver [6]. Our proof is based on showing that $SC(B)$ is a polyhedron. Theorem 2 then follows from Corollary 1.

When $r = n$, $\mathcal{B}_r^*$ is the set of bases of A. Every basis $B \in \mathcal{B}_n^*$ corresponds to a unique basic solution $\bar{x}(B)$, which may or may not be a feasible solution to LP. If $\bar{x}(B)$ is a feasible solution to LP, the basis B is called a feasible basis, otherwise, it is called an infeasible basis. A natural question is whether infeasible bases are needed for Theorem 1 to be true. The example in Figure 3 demonstrates that infeasible bases are necessary. In Figure 3, the two intersection cuts derived from the two feasible bases B^1 and B^2 are both dominated by the cut derived from the infeasible basis B'. In fact, the cut obtained from B' is the only necessary cut to describe $Conv\,(P \cap F_{D(\pi,\pi_0)})$.
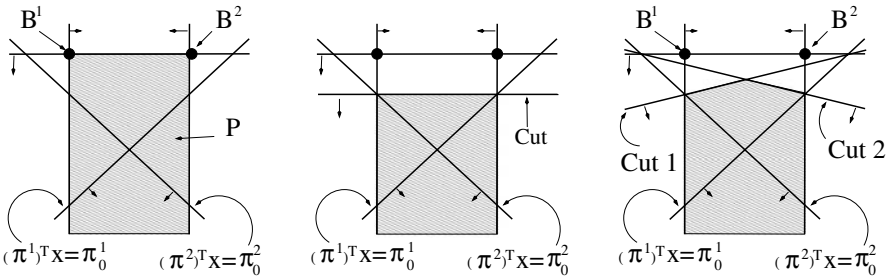
2.3. More general disjunctions

Theorem 1 is proven for split disjunctions but the question of knowing if it generalizes to other disjunctions is still open. The example in Figure 4 demonstrates that Theorem 1 does not generalize from general split disjunctions to other types of disjunctions with two terms. In this example, P has 3 constraints $a_{i.}^T x \leq b_i$, $i = 1, 2, 3$, and two bases B^1 and B^2. D is a two-term disjunction involving the two hyperplanes $(\pi^1)^T x = \pi_0^1$ and $(\pi^2)^T x = \pi_0^2$. In Figure 4(b), the shaded area circumscribed by the cut and the two parallel lines form the constraints of $Conv\,(P \cap F_D)$. In Figure 4(c), the shaded area is $Conv\,(P(B^1) \cap F_D) \cap Conv\,(P(B^2) \cap F_D) = \cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_D)$. In this example, $Conv\,(P \cap F_D) \neq \cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_D)$.

Observe, however, that $Conv\,(P \cap F_D)$ can be generated by considering r-subsets of the constraints ($r = 2$). In fact, in the example, only one 2-subset needs to be considered (the set containing the parallel constraints of P). This does, in fact, hold for general two-term disjunctions D of the form $D^1 x \leq d^1 \vee D^2 x \leq d^2$, where $D^1 \in \mathbb{R}^{m_1 \times n}$, $D^2 \in \mathbb{R}^{m_2 \times n}$, $d^1 \in \mathbb{R}^{m_1}$ and $d^2 \in \mathbb{R}^{m_2}$.

Theorem 3. *Let D be a general two-term disjunction, and suppose A is not of full row-rank, i.e., $|M| = m \geq r + 1$. Define*

(i) $\mathcal{C}_1^* := \{S \subseteq M : |S| = r + 1 \text{ and } r(S) = r\}$.
(ii) $\mathcal{C}_2^* := \{S \subseteq M : |S| = n \text{ and } (r(S) = n \text{ or } r(S) = n - 1)\}$.

(a) A polyhedron and a disjunction

(b) The disjunctive hull

(c) Set obtained from bases

Fig. 4. Example of a general two-term disjunction

We have

$$Conv\,(P \cap F_D) = \bigcap_{T \in \mathcal{C}_1^*} Conv\,(P(T) \cap F_D). \tag{6}$$

Furthermore, if $r = n$,

$$Conv\,(P \cap F_D) = \bigcap_{T \in \mathcal{C}_2^*} Conv\,(P(T) \cap F_D). \tag{7}$$

A different counterexample for the generalization of Theorem 1 is given next. Theorem 1 demonstrates that the split closure of P can be decomposed into split closures of the sets $P(B)$ for $B \in \mathcal{B}_r^*$. Is a similar decomposition result valid for the integer hull of P? In other words, do we have

$$Conv\,(P^I) = \bigcap_{B \in \mathcal{B}_r^*} Conv\,(P^I(B)),$$

where $P^I(B) := \{x \in P(B) : x_j \text{ integer}, j \in N_I\}$? The following example shows that this is not true in 3 dimensions. Consider the polyhedron P defined as the set of $x \in \mathbb{R}^3$ satisfying the following inequalities.

$$2x_1 - 2x_2 + 2x_3 \leq 3, \tag{8}$$

$$-2x_1 + 2x_2 + 2x_3 \leq 3, \tag{9}$$

$$2x_1 + 2x_2 + 2x_3 \leq 5, \tag{10}$$

$$-2x_1 - 2x_2 + 2x_3 \leq 1. \tag{11}$$

It can be verified that $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ is the only point that satisfies all inequalities (8) − (11) with equality. It can also be seen that $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in Conv\,(P^I(B))$ for all four bases $B \in \mathcal{B}_3^*$. However, the inequality $x_3 \leq 0$ is valid for $Conv\,(P^I)$. To see this, observe that the constraints (10) and (11), together with the

integrality of x_3, imply $x_3 \leq 1$. If $x_3 = 1$, by the integrality of x_1 and x_2, the constraints (8) and (9) imply $x_1 - x_2 = 0$, while the constraints (10) and (11) imply $x_1 + x_2 = 1$. Since no integral values of x_1 and x_2 satisfy $x_1 - x_2 = 0$ and $x_1 + x_2 = 1$, P does not have an integral point with $x_3 = 1$.

3. Conic polyhedra, cutting planes and split disjunctions

In this section we focus on conic polyhedra. In Sect. 3.1 we consider conic polyhedra of the form

$$\bar{C} := \{x \in \mathbb{R}^n : x = \bar{x} + \sum\nolimits_{j \in J} \bar{r}^j \mu_j \text{ and } \mu_j \geq 0 \text{ for } j \in J\}, \tag{12}$$

where J is an index set, $\bar{x} \in \mathbb{R}^n$, $\bar{r}^j \in \mathbb{R}^n$ for $j \in J$ and the vectors $\{\bar{r}^j\}_{j \in J}$ are linearly independent. Conic polyhedra of the form $\bar{C}$ above are called *simple* conic polyhedra. We may have $|J| < n$. The focus in Sect. 3.1 is on inequalities that cut off $\bar{x}$, *i.e.*, inequalities $\delta^T x \geq \delta_0$ satisfying $\delta^T \bar{x} < \delta_0$. Inequalities that cut off $\bar{x}$ are called *apex-cuts* for $\bar{C}$.

In Sect. 3.2 we consider simple conic polyhedra and general split disjunctions (π, π_0^1, π_0^2). For split disjunctions that are violated by $\bar{x}$, we derive an intersection cut. Also, the intersection cut is used to describe the set $Conv\,(\bar{C} \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$.

In Sect. 3.3 we prove that, given any subset $\Pi \subseteq \Pi^n(N_I)$ of split disjunctions, the set $SC(\bar{C}, \Pi) := \cap_{(\pi, \pi_0) \in \Pi} Conv\,(\bar{C} \cap F_{D(\pi, \pi_0)})$ is a polyhedron. In words, $SC(\bar{C}, \Pi)$ is defined to be the intersection of the disjunctive hulls defined by $\bar{C}$ and $D(\pi, \pi_0)$ over all the disjunctions (π, π_0) in Π. Since the cardinality of Π can be infinite, it is not trivial to see that $SC(\bar{C}, \Pi)$ is a polyhedron. In particular, when $\Pi = \Pi^n(N_I)$, we know that the split closure of $\bar{C}$ is a polyhedron, as proved in [6].

In Sect. 3.4 we generalize the results of Sect. 3.3 on simple conic polyhedra $\bar{C}$ to conic polyhedra $P(B)$ defined in Sect. 1 and Sect. 2.1, where $B \in \mathcal{B}_r^*$. Given a general split disjunction (π, π_0^1, π_0^2) that is violated by $\bar{x}(B)$, we characterize the disjunctive set obtained from $P(B)$ and (π, π_0^1, π_0^2) as the set of points in $P(B)$ that satisfy the intersection cut. Furthermore, we show that the split closure of $P(B)$ is a polyhedron for all $B \in \mathcal{B}_r^*$.

3.1. Simple conic polyhedra and apex-cuts

We start by considering $\bar{C}$, the simple conic polyhedron defined in (12). The hyperplane associated with an intersection cut has intersection points with the extreme rays of $\bar{C}$. In fact, any apex-cut $\delta^T x \geq \delta_0$ for $\bar{C}$ has this characterization. For $j \in J$, define

$$\alpha'_j(\delta, \delta_0) := \begin{cases} (\delta_0 - \delta^T \bar{x})/(\delta^T \bar{r}^j) & \text{if } \delta^T \bar{r}^j \neq 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{13}$$

Since $\delta^T x \geq \delta_0$ is an apex-cut for $\bar{C}$, $\delta_0 > \delta^T \bar{x}$. As with the intersection cut, $\alpha'_j(\delta, \delta_0)$ denotes the value of α for which the point $\bar{x} + \alpha \bar{r}^j$ is on the hyperplane $\delta^T x = \delta_0$. When

there is no such point, $\alpha'_j(\delta, \delta_0) = +\infty$. In contrast to the intersection cut, however, we might have $\alpha'_j(\delta, \delta_0) < 0$ for some $j \in J$. Let

$$\bar{C}^L := \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R}^{|J|} : x = \bar{x} + \sum_{j \in J} \bar{r}^j \mu_j \text{ and } \mu \geq 0_{|J|}\}$$

be the set obtained from $\bar{C}$ by including the multipliers on the extreme rays of $\bar{C}$ in the description. The "L" in $\bar{C}^L$ is for "Lifted" since $\bar{C}^L$ is an image of $\bar{C}$ in a higher dimensional space (see Balas [3]). We have the following relationship between $\delta^T x \geq \delta_0$ and the scalars $\alpha'_j(\delta, \delta_0)$ for $j \in J$.

Lemma 2. *Let $\delta^T x \geq \delta_0$ be an arbitrary apex-cut for $\bar{C}$.*

$$\{x \in \bar{C} : \delta^T x \geq \delta_0\}$$
$$= \{x \in \bar{C} : \exists \mu \in \mathbb{R}^{|J|} \text{ s.t. } (x, \mu) \in \bar{C}^L \text{ and } \sum_{j \in J} \mu_j / \alpha'_j(\delta, \delta_0) \geq 1\}.$$

Proof. We have $x \in \bar{C}$ and $\delta^T x \geq \delta_0 \iff x = \bar{x} + \sum_{j \in J} \mu_j \bar{r}^j$, where $\mu \geq 0_{|J|}$, and $\delta^T x \geq \delta_0 \iff x = \bar{x} + \sum_{j \in J} \mu_j \bar{r}^j$, where $\mu \geq 0_{|J|}$, and $\sum_{j \in J} \mu_j (\delta^T \bar{r}^j) \geq (\delta_0 - \delta^T \bar{x}) \iff (x, \mu) \in \bar{C}^L$ and $\sum_{j \in J} \mu_j / \alpha'_j(\delta, \delta_0) \geq 1$.   □

Lemma 2 leads to the following notion of dominance between pairs of apex-cuts for $\bar{C}$. Given two apex-cuts $(\delta^1)^T x \geq \delta_0^1$ and $(\delta^2)^T x \geq \delta_0^2$ for $\bar{C}$, we say $(\delta^1)^T x \geq \delta_0^1$ *dominates* $(\delta^2)^T x \geq \delta_0^2$ on $\bar{C}$ if $1/\alpha'_j(\delta^1, \delta_0^1) \leq 1/\alpha'_j(\delta^2, \delta_0^2)$ for all $j \in J$. Furthermore, if $(\delta^1)^T x \geq \delta_0^1$ dominates $(\delta^2)^T x \geq \delta_0^2$ on $\bar{C}$ and $(\delta^2)^T x \geq \delta_0^2$ dominates $(\delta^1)^T x \geq \delta_0^1$ on $\bar{C}$, we say $(\delta^1)^T x \geq \delta_0^1$ and $(\delta^2)^T x \geq \delta_0^2$ are *equivalent* on $\bar{C}$. Two apex-cuts $(\delta^1)^T x \geq \delta_0^1$ and $(\delta^2)^T x \geq \delta_0^2$ are equivalent on $\bar{C}$ if and only if they intersect every ray at the same point, but the hyperplanes $(\delta^1)^T x = \delta_0^1$ and $(\delta^2)^T x = \delta_0^2$ are not necessarily identical as can be illustrated with a cone $\bar{C}$ with two rays in $\mathbb{R}^3$.

Observe that, if $(\delta^1)^T x \geq \delta_0^1$ dominates $(\delta^2)^T x \geq \delta_0^2$ on $\bar{C}$, then Lemma 2 implies $\{x \in \bar{C} : (\delta^1)^T x \geq \delta_0^1\} \subseteq \{x \in \bar{C} : (\delta^2)^T x \geq \delta_0^2\}$. Furthermore, if $(\delta^1)^T x \geq \delta_0^1$ is equivalent to $(\delta^2)^T x \geq \delta_0^2$ on $\bar{C}$, then $\{x \in \bar{C} : (\delta^1)^T x \geq \delta_0^1\} = \{x \in \bar{C} : (\delta^2)^T x \geq \delta_0^2\}$.

We call an apex-cut $\delta^T x \geq \delta_0$ *positive*, if $\alpha'_j(\delta, \delta_0) > 0$ for all $j \in J$. In other words, $\delta^T x \geq \delta_0$ is positive if, for every $j \in J$ such that the hyperplane $\delta^T x = \delta_0$ intersects the line $\{\bar{x} + \alpha \bar{r}^j : \alpha \in \mathbb{R}\}$, the intersection point $\bar{x} + \alpha'_j(\delta, \delta_0)\bar{r}^j$ is on the half-line contained in $\bar{C}$.

Now consider a polyhedron $\bar{P}$ obtained from $\bar{C}$ by adding the positive apex-cuts $(\delta^1)^T x \geq \delta_0^1, (\delta^2)^T x \geq \delta_0^2, \dots, (\delta^p)^T x \geq \delta_0^p$, i.e.,

$$\bar{P} := \{x \in \bar{C} : (\delta^k)^T x \geq \delta_0^k \text{ for } k = 1, 2, \dots, p\}.$$

Observe that $\bar{P} \neq \emptyset$ if and only if, for every $k \in \{1, 2, \dots, p\}$, there exists $j \in J$ such that $\alpha'_j(\delta^k, \delta_0^k) < +\infty$. This is due to Lemma 2, i.e., by viewing the inequality $(\delta^k)^T x \geq \delta_0^k$ in the space of (x, μ) variables. By making the μ variables very large, a point in $\bar{C}$ can be constructed that satisfy the inequality $(\delta^k)^T x \geq \delta_0^k$.

Lemma 3. *Assume $\bar{P} \neq \emptyset$. Let $\delta^T x \geq \delta_0$ be a positive apex-cut that is valid for $\bar{P}$. Then there exists an inequality $(\delta')^T x \geq \delta'_0$, which can be obtained as a convex combination of the apex-cuts $(\delta^k)^T x \geq \delta_0^k$, $k = 1, 2, \dots, p$, that dominates $\delta^T x \geq \delta_0$ on $\bar{C}$.*

Proof. Consider the LP $\min\{\delta^T x : x \in \bar{P}\}$. Since $\bar{P} \neq \emptyset$ and $\delta^T x \geq \delta_0$ is a valid inequality for $\bar{P}$, the LP is feasible and bounded. The dual of the LP gives vectors $\bar{u} \in \mathbb{R}^n$ and $\bar{v} \in \mathbb{R}^p$ such that

(a) $\bar{u}^T \bar{x} + \sum_{k=1}^{p} \delta_0^k \bar{v}_k \geq \delta_0$,
(b) $\bar{u} + \sum_{k=1}^{p} \bar{v}_k \delta^k = \delta$,
(c) $\bar{u}^T \bar{r}^j \geq 0$ for all $j \in J$ and
(d) $\bar{v} \geq 0_q$.

Define $\bar{\delta} := \sum_{k=1}^{p} \bar{v}_k \delta^k$ and $\bar{\delta}_0 := \sum_{k=1}^{p} \delta_0^k \bar{v}_k$. The inequality $\bar{\delta}^T x \geq \bar{\delta}_0$ is a non-negative combination of the inequalities $(\delta^1)^T x \geq \delta_0^1, (\delta^2)^T x \geq \delta_0^2, \ldots, (\delta^p)^T x \geq \delta_0^p$, and is therefore valid for $\bar{P}$. We claim $\bar{\delta}^T x \geq \bar{\delta}_0$ dominates $\delta^T x \geq \delta_0$ on $\bar{C}$.

Let $j \in J$ be arbitrary. Observe that (a) and (b) give $\bar{u}^T \bar{x} + \bar{\delta}_0 \geq \delta_0$ and $\bar{u} + \bar{\delta} = \delta$. This implies $\bar{u}^T \bar{r}^j + \bar{\delta}^T \bar{r}^j = \delta^T \bar{r}^j$. Since from (c) we have $\bar{u}^T \bar{r}^j \geq 0$, we also have $\bar{\delta}^T \bar{r}^j \leq \delta^T \bar{r}^j$. Furthermore, we have $\bar{u}^T \bar{x} + \bar{\delta}^T \bar{x} = \delta^T \bar{x}$ from multiplying $\bar{u} + \bar{\delta} = \delta$ by $\bar{x}$. Substituting the expression $\bar{u}^T \bar{x} = \delta^T \bar{x} - \bar{\delta}^T \bar{x}$ into the inequality $\bar{u}^T \bar{x} + \bar{\delta}_0 \geq \delta_0$, we get $0 < \delta_0 - \delta^T \bar{x} \leq \bar{\delta}_0 - \bar{\delta}^T \bar{x}$.

Observe that the assumption $\alpha'_j(\delta, \delta_0) \geq 0$ implies $\delta^T \bar{r}^j \geq 0$. Now we have $0 \leq (\bar{\delta}^T \bar{r}^j)/(\bar{\delta}_0 - \bar{\delta}^T \bar{x}) \leq (\delta^T \bar{r}^j)/(\bar{\delta}_0 - \bar{\delta}^T \bar{x}) \leq (\delta^T \bar{r}^j)/(\delta_0 - \delta^T \bar{x})$. It follows that $1/\alpha'_j(\bar{\delta}, \bar{\delta}_0) \leq 1/\alpha'_j(\delta, \delta_0)$, so that $\bar{\delta}^T x \geq \bar{\delta}_0$ dominates $\delta^T x \geq \delta_0$ on $\bar{C}$.

Finally, the inequality $\bar{\delta}^T x \geq \bar{\delta}_0$ might not be a convex combination of the inequalities $(\delta^1)^T x \geq \delta_0^1, (\delta^2)^T x \geq \delta_0^2, \ldots, (\delta^p)^T x \geq \delta_0^p$, *i.e.*, we might have $\bar{v}^S := \sum_{k=1}^{p} \bar{v}_k \neq 1$. First we argue that $\bar{v} \neq 0_p$. If $\bar{v} = 0_p$, then (a) and (b) imply $\delta^T \bar{x} \geq \delta_0$. Since $\delta^T x \geq \delta_0$ is an apex-cut for $\bar{C}$, this cannot be the case, so $\bar{v} \neq 0_p$. Now define $\delta' := \bar{\delta}/\bar{v}^S$ and $\delta'_0 := \bar{\delta}_0/\bar{v}^S$. The inequality $(\delta')^T x \geq \delta'_0$ is a convex combination of $(\delta^1)^T x \geq \delta_0^1, (\delta^2)^T x \geq \delta_0^2, \ldots, (\delta^p)^T x \geq \delta_0^p$, and $(\delta')^T x \geq \delta'_0$ is equivalent to $\bar{\delta}^T x \geq \bar{\delta}_0$ on $\bar{C}$, and therefore also dominates $\delta^T x \geq \delta_0$ on $\bar{C}$. $\qquad\square$

3.2. Simple conic polyhedra and split disjunctions

The intersection cut (3) given in Sect. 2 was derived using a description of a conic polyhedron as intersection of half-spaces $a_{i.}^T x \leq b_i$. Since we do not have a description of $\bar{C}$ with half-spaces, we now give an alternative derivation using $\bar{x}$ and the extreme rays $\bar{r}^j$. Recall that the intersection cut goes through the intersection points of the disjunction with the extreme rays of the cone.

Let $(\pi, \pi_0^1, \pi_0^2) \in \mathbb{R}^{n+2}$ satisfy $\pi_0^1 < \pi_0^2$, and suppose $\pi^T \bar{x} \in]\pi_0^1, \pi_0^2[$. Given $j \in J$, define

$$\bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2) := \begin{cases} -\epsilon^1(\pi, \pi_0^1)/(\pi^T \bar{r}^j) & \text{if } \pi^T \bar{r}^j < 0, \\ \epsilon^2(\pi, \pi_0^2)/(\pi^T \bar{r}^j) & \text{if } \pi^T \bar{r}^j > 0, \\ +\infty & \text{otherwise}, \end{cases} \tag{14}$$

where $\epsilon^1(\pi, \pi_0^1) := \pi^T \bar{x} - \pi_0^1$ and $\epsilon^2(\pi, \pi_0^2) := \pi_0^2 - \pi^T \bar{x}$. As mentioned in Sect. 2.1, $\bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2)$, for $j \in J$, denotes the smallest value of $\alpha \geq 0$ such that $\bar{x} + \alpha \bar{r}^j$ satisfies the disjunction $D(\pi, \pi_0^1, \pi_0^2)$ (or $\bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2) = +\infty$ if this does not happen). The intersection cut associated with $\bar{C}$ and $D(\pi, \pi_0^1, \pi_0^2)$ is given by

$$(\delta(\pi, \pi_0^1, \pi_0^2))^T x \geq \delta_0(\pi, \pi_0^1, \pi_0^2), \tag{15}$$

where the vector $\delta(\pi, \pi_0^1, \pi_0^2)$ is one of possibly many solutions to the system

$$\delta^T \bar{r}^j = 1/\bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2), \ \forall j \in J \tag{16}$$

and $\delta_0(\pi, \pi_0^1, \pi_0^2)$ is defined by $\delta_0(\pi, \pi_0^1, \pi_0^2) := (\delta(\pi, \pi_0^1, \pi_0^2))^T \bar{x} + 1$. Note that, since $\bar{r}^j, j \in J$, are linearly independent and the disjunction $D(\pi, \pi_0^1, \pi_0^2)$ gives an intersection cut, the equations (16) have a non-zero solution. The inequality $(\delta(\pi, \pi_0^1, \pi_0^2))^T x \geq \delta_0(\pi, \pi_0^1, \pi_0^2)$ has the property that, for all $j \in J$ satisfying $\pi^T \bar{r}^j \neq 0$, the point $\bar{x} + \bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2)\bar{r}^j$ is on the hyperplane $(\delta(\pi, \pi_0^1, \pi_0^2))^T x = \delta_0(\pi, \pi_0^1, \pi_0^2)$, i.e., the hyperplane of the intersection cut goes through all the intersection points of the disjunction and the cone. Furthermore, we have $(\delta(\pi, \pi_0^1, \pi_0^2))^T \bar{x} < \delta_0(\pi, \pi_0^1, \pi_0^2)$, i.e., the hyperplane cuts off $\bar{x}$. Also observe that, for a general split disjunction (π, π_0^1, π_0^2) and an extreme ray $j \in J$, we have the identity

$$\bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2) = \alpha_j'(\delta(\pi, \pi_0^1, \pi_0^2), \delta_0(\pi, \pi_0^1, \pi_0^2)), \tag{17}$$

where the scalars $\alpha_j'(\delta(\pi, \pi_0^1, \pi_0^2), \delta_0(\pi, \pi_0^1, \pi_0^2))$ are as defined in Sect. 3.1. Given $\bar{C}$ and $D(\pi, \pi_0^1, \pi_0^2)$, when $|J| < n$, the intersection cut is not unique, i.e., the equations (16) have more than just one solution. There can be many intersection cuts that intersect the half-lines of the cone at the same points.

The intersection cut is an apex-cut for $\bar{C}$. Because of Lemma 2, we have

$$\{x \in \bar{C} : (15)\} = \{x \in \bar{C} : \exists \mu \in \mathbb{R}^{|J|} \text{ s.t. } (x, \mu) \in \bar{C}^L, \sum_{j \in J} \mu_j/\bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2) \geq 1\},$$

where $\bar{C}^L$ was defined in Sect. 3.1. The next lemma characterizes $Conv(\bar{C} \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$ for general split disjunctions $D(\pi, \pi_0^1, \pi_0^2)$.

Lemma 4. *Let $(\pi, \pi_0^1, \pi_0^2) \in \mathbb{R}^{n+2}$ be an arbitrary general split disjunction, where $\pi_0^1 < \pi_0^2$.*

   (i) If $\pi^T \bar{x} \notin]\pi_0^1, \pi_0^2[$, then $Conv(\bar{C} \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = \bar{C}$,

   (ii) If $\pi^T \bar{x} \in]\pi_0^1, \pi_0^2[$, then $Conv(\bar{C} \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = \{x \in \bar{C} : (15)\}$.

Proof. (i) Assume $\bar{x} \in F_{D(\pi, \pi_0^1, \pi_0^2)}$. Wlog suppose $\pi^T \bar{x} \leq \pi_0^1$. Clearly $Conv(\bar{C} \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) \subseteq \bar{C}$ since $\bar{C}$ is convex, so we only need to show the other direction. Let $y \in \bar{C}$ be arbitrary. We may write $y = \bar{x} + \sum_{j \in J} s_j \bar{r}^j$, where $s_j \geq 0$ for $j \in J$. If $y \in F_{D(\pi, \pi_0^1, \pi_0^2)}$, then $y \in \bar{C} \cap F_{D(\pi, \pi_0^1, \pi_0^2)}$ and we are done. If not $\pi^T y \in]\pi_0^1, \pi_0^2[$. Define $J^+ := \{j \in J : \pi^T \bar{r}^j > 0 \text{ and } s_j > 0\}$. Since $\pi^T y \in]\pi_0^1, \pi_0^2[$ and $\pi^T \bar{x} \leq \pi_0^1$, we must have $J^+ \neq \emptyset$. Let $z^+ := \sum_{j \in J^+} s_j \bar{r}^j \neq 0_n$, and let $z^- := \sum_{j \in J \setminus J^+} s_j \bar{r}^j$. So $\pi^T z^+ = \sum_{j \in J^+} s_j (\pi^T \bar{r}^j) > 0$. Choose $\lambda \in]0, 1]$ to satisfy $\pi^T (\bar{x} + z^- + \frac{1}{\lambda} z^+) \geq \pi_0^2$. Define $y^1 := \bar{x} + z^- + \frac{1}{\lambda} z^+ \in \bar{C}$ and $y^2 := \bar{x} + z^- \in \bar{C}$. Then $y = \bar{x} + z^- + z^+ = \lambda y^1 + (1 - \lambda) y^2$. By the choice of λ, y^1 satisfies the disjunction $D(\pi, \pi_0^1, \pi_0^2)$. Furthermore, since $\pi^T (\bar{x} + z^-) \leq \pi_0^1$, y^2 satisfies the disjunction $D(\pi, \pi_0^1, \pi_0^2)$. Therefore we have $y^1 \in \bar{C} \cap F_{D(\pi, \pi_0^1, \pi_0^2)}$ and $y^2 \in \bar{C} \cap F_{D(\pi, \pi_0^1, \pi_0^2)}$, so that $y \in Conv(\bar{C} \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$.

(ii) The validity of the intersection cut was proved by Balas [2]. We therefore know that (15) is valid for $Conv\,(\bar{C} \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. Since $\bar{C}$ is convex, we have the inclusion $Conv\,(\bar{C} \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \subseteq \{x \in \bar{C} : (15)\}$. Let $y \in \{x \in \bar{C} : (15)\}$ be arbitrary. If $y \in F_{D(\pi,\pi_0^1,\pi_0^2)}$, we are done, so suppose $\pi^T y \in]\pi_0^1, \pi_0^2[$. We may write $y = \bar{x} + \sum_{j \in J} s_j \bar{r}^j$, where $s_j \geq 0$ for $j \in J$. Observe that Lemma 2 applied on the intersection cut (15) implies that $\sum_{j \in J} s_j/\bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2) \geq 1$.

First assume $\sum_{j \in J} s_j/\bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2) = 1$. Define the set $J^{\neq 0} := \{j \in J : \pi^T \bar{r}^j \neq 0\}$ and define the non-negative scalars $\lambda_j := s_j/\bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2)$ for $j \in J^{\neq 0}$. Clearly $\sum_{j \in J^{\neq 0}} \lambda_j = 1$. Also, let $\bar{z} := \sum_{j \in J \setminus J^{\neq 0}} s_j \bar{r}^j$. We have $y = \bar{x} + \sum_{j \in J} s_j \bar{r}^j = \bar{x} + \bar{z} + \sum_{j \in J^{\neq 0}} s_j \bar{r}^j$. This can be re-written as $y = \sum_{j \in J^{\neq 0}} \lambda_j (\bar{x} + \bar{z} + \bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2) \bar{r}^j)$. Now $y = \sum_{j \in J^{\neq 0}} \lambda_j \bar{x}^j(\pi, \pi_0^1, \pi_0^2)$ is a convex combination of the vectors $\{\bar{x}^j(\pi, \pi_0^1, \pi_0^2)\}_{j \in J}$, which are defined as $\bar{x}^j(\pi, \pi_0^1, \pi_0^2) := \bar{x} + \bar{z} + \bar{\alpha}_j(\pi, \pi_0^1, \pi_0^2) \bar{r}^j$ for $j \in J$. Since all the vectors $\{\bar{x}^j(\pi, \pi_0^1, \pi_0^2)\}_{j \in J}$ satisfy the disjunction $D(\pi, \pi_0^1, \pi_0^2)$, this shows $y \in Conv\,(\bar{C} \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$.

Now suppose $\sum_{j \in J} s_j/\alpha_j(\pi, \pi_0^1, \pi_0^2) > 1$. Let z be the point on the line between y and $\bar{x}$ that satisfies (15) with equality. We have $y = \bar{x} + \mu(z - \bar{x})$ for some $\mu > 1$. Also, we know z can be expressed as $z = \lambda^1 z^1 + \lambda^2 z^2$, where $z^1, z^2 \in \bar{C} \cap F_{D(\pi,\pi_0^1,\pi_0^2)}$, $\lambda^1, \lambda^2 \geq 0$ and $\lambda^1 + \lambda^2 = 1$. Then $y = \lambda^1(\bar{x} + \mu(z^1 - \bar{x})) + \lambda^2(\bar{x} + \mu(z^2 - \bar{x})) \in Conv\,(\bar{C} \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$ since $\bar{x} + \mu(z^i - \bar{x}) \in \bar{C} \cap F_{D(\pi,\pi_0^1,\pi_0^2)}$ for $i = 1, 2$. □

3.3. Polyhedrality of the split closure of a simple conic polyhedron

We now prove that the split closure of a simple conic polyhedron is a polyhedron. Given a set $\Pi \subseteq \Pi^n(N_I)$ of split disjunctions, define $SC(\bar{C}, \Pi) := \cap_{(\pi,\pi_0) \in \Pi} Conv\,(\bar{C} \cap F_{D(\pi,\pi_0)})$. We will prove that $SC(\bar{C}, \Pi)$ is a polyhedron. Wlog assume Π satisfies $\pi^T \bar{x} \in]\pi_0^1, \pi_0^2[$ for all $(\pi, \pi_0) \in \Pi$ (Lemma 4).

Observe that every intersection cut is characterized by its intersection points with the extreme rays of $\bar{C}$. Furthermore, any two intersection cuts that intersect every ray at the same point are equivalent. Therefore, we only need to show that the number of possible intersection points with each extreme ray is finite.

Throughout this section we assume $\bar{x} \in \mathbb{Q}^n$ and $\bar{r}^j \in \mathbb{Z}^n$ for all $j \in J$. This is equivalent to assuming that the simple conic polyhedron $\bar{C}$ is rational. Let $(\pi, \pi_0) \in \Pi$ be an arbitrary split disjunction. For simplicity, let $\delta(\pi, \pi_0)$ and $\delta_0(\pi, \pi_0)$ describe the intersection cut $(\delta(\pi, \pi_0))^T x \geq \delta_0(\pi, \pi_0)$ associated with $\bar{C}$ and $D(\pi, \pi_0)$, and let $\bar{\alpha}_j(\pi, \pi_0)$, $j \in J$, be the coefficients (14) in the intersection cut. Furthermore, let $\epsilon^1(\pi, \pi_0)$ and $\epsilon^2(\pi, \pi_0)$ denote the amounts by which $\bar{x}$ violates the first and second terms of the disjunction $D(\pi, \pi_0)$, respectively.

The following is a key lemma, which gives an expression for $\bar{\alpha}_j(\pi, \pi_0)$ for a disjunction $D(\pi, \pi_0)$ and an extreme ray $j \in J$.

Lemma 5. *Let $(\pi, \pi_0) \in \Pi$ and $j \in J$ satisfy $\bar{\alpha}_j(\pi, \pi_0) < +\infty$. Then there exist integers $p_j(\pi, \pi_0)$, $w_j(\pi, \pi_0)$ and g such that*

(i) $\bar{\alpha}_j(\pi, \pi_0) < 1.$

(ii) $\bar{\alpha}_j(\pi, \pi_0) = \frac{p_j(\pi, \pi_0)}{g w_j(\pi, \pi_0)}$, *where* $0 < p_j(\pi, \pi_0) < g$ *and* $w_j(\pi, \pi_0) > 0.$

Proof. (i) Since $\bar{r}^j$ and π are integers and $\pi^T \bar{r}^j \neq 0$, we must have $|\pi^T \bar{r}^j| \geq 1$. Because every split disjunction is such that $\pi_0^2 - \pi_0^1 = 1$, we have $\epsilon^1(\pi, \pi_0), \epsilon^2(\pi, \pi_0) \in]0, 1[$. Therefore, (14) implies $\bar{\alpha}_j(\pi, \pi_0) < 1$.

(ii) Since $\bar{x}$ is rational, we may write $\bar{x} = (\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n})$, where $p_i \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ for $i = 1, 2, \ldots, n$. Define $d_k := \Pi_{i=1, i \neq k}^n q_i$ for $k = 1, 2, \ldots, n$, and let $g := \Pi_{i=1}^n q_i > 0$. If $\pi^T \bar{r}^j < 0$, let $p_j(\pi, \pi_0) := \sum_{k=1}^n d_k p_k \pi_k - g \pi_0$ and $w_j(\pi, \pi_0) := -\pi^T \bar{r}^j > 0$. Finally, if $\pi^T \bar{r}^j > 0$, let $p_j(\pi, \pi_0) := g + g \pi_0 - \sum_{k=1}^n d_k p_k \pi_k$ and $w_j(\pi, \pi_0) := \pi^T \bar{r}^j > 0$. With these choices, (14) implies $\bar{\alpha}_j(\pi, \pi_0) = \frac{p_j(\pi, \pi_0)}{g w_j(\pi, \pi_0)}$. We also have $0 < p_j(\pi, \pi_0) < g$ because $\epsilon^1(\pi, \pi_0), \epsilon^2(\pi, \pi_0) \in]0, 1[$.   □

In Lemma 5, we observe that the integer g is independent of the disjunction $D(\pi, \pi_0)$ and the extreme ray $j \in J$. The following lemma, which will be used to prove that $\mathrm{SC}(\bar{C}, \Pi)$ is a polyhedron, bounds the number of intersection points the split disjunctions in Π can have with half-lines of the form $\{\bar{x} + \alpha r^j : \alpha \geq \alpha^*\}$, where $j \in J$ and $\alpha^* > 0$ is arbitrary.

Lemma 6. *For any* $j \in J$ *and* $\alpha^* > 0$, *the split disjunctions* $(\pi, \pi_0) \in \Pi$ *only have a finite number of intersection points* $\bar{x} + \bar{\alpha}_j(\pi, \pi_0) r^j$ *with the half-line* $\{\bar{x} + \alpha r^j : \alpha \geq \alpha^*\}.$

Proof. We only have to prove that the number of possible values for $\bar{\alpha}_j(\pi, \pi_0)$ satisfying $\bar{\alpha}_j(\pi, \pi_0) \geq \alpha^*$ is finite. Therefore let $j \in J$ and $(\pi, \pi_0) \in \Pi$ be arbitrary. Wlog assume $\bar{\alpha}_j(\pi, \pi_0) < \infty$, and that $\alpha^* = \frac{p^*}{g w^*}$, where $p^*, w^* \in \mathbb{Z}$, $0 < p^* < g$ and $w^* > 0$.

From Lemma 5 we have that $0 < \alpha^* \leq \bar{\alpha}_j(\pi, \pi_0) = \frac{p_j(\pi, \pi_0)}{g w_j(\pi, \pi_0)} < 1$. Now, there is only a finite number of possible values for $p_j(\pi, \pi_0)$, namely the values $1, 2, \ldots, (g-1)$. Also, for every $p \in \{1, 2, \ldots, (g-1)\}$, a corresponding value w must satisfy $1 > \frac{p}{gw} \geq \frac{p^*}{g w^*}$, which means $\frac{p}{g} < w \leq \frac{p w^*}{p^*}$.   □

We now prove that $\mathrm{SC}(\bar{C}, \Pi)$ is a polyhedron. The proof is by induction on $|J|$. First consider the case of $|J| = 1$ (basic step).

Lemma 7. *Suppose* $|J| = 1$. *Then there exists* $(\pi^*, \pi_0^*) \in \Pi$ *such that* $\mathrm{SC}(\bar{C}, \Pi) = \{x \in \bar{C} : (\delta(\pi^*, \pi_0^*))^T x \geq \delta_0(\pi^*, \pi_0^*)\}.$

Proof. We may assume that $J = \{1\}$. We have $\bar{C} = \{x \in \mathbb{R}^n : x = \bar{x} + \alpha \bar{r}^1, \alpha \geq 0\}$. First suppose there exists $(\pi^*, \pi_0^*) \in \Pi$ such that $(\pi^*)^T \bar{r}^1 = 0$. This implies that the intersection cut associated with (π^*, π_0^*) cuts off the entire ray $\bar{C}$.

Now suppose $\pi^T \bar{r}^1 \neq 0$ for all $(\pi, \pi_0) \in \Pi$. Define $\alpha_1^* := \sup\{\bar{\alpha}_1(\pi, \pi_0) : (\pi, \pi_0) \in \Pi\}$. Then we have $\mathrm{SC}(\bar{C}, \Pi) = \{x \in \mathbb{R}^n : x = \bar{x} + \alpha \bar{r}^1, \alpha \geq \alpha_1^*\}$. Hence the only question is whether the supremum is achieved by some $(\pi, \pi_0) \in \Pi$.

Let $(\pi', \pi_0') \in \Pi$ be arbitrary, and let $p' := p(\pi', \pi_0')$ and $w' = w(\pi', \pi_0')$ be as in Lemma 5. For every integer p satisfying $0 < p < g$ there are only a finite number of integers $w > 0$ for which $\frac{p}{wg} > \frac{p'}{w'g}$, namely integers $w > 0$ satisfying $w < \frac{p w'}{p'}$. Since

there are also only a finite number of integers p satisfying $0 < p < g$, there are only a finite number of possible integers p and w such that $0 < p < g$ and $w > 0$ for which $\frac{p}{wg} > \frac{p'}{w'g}$. Hence the supremum is achieved. □

We now discuss the induction hypothesis. Given $j \in J$, let

$$\bar{C}^j := \bar{x} + Cone\left(\{\bar{r}^k\}_{k \in J \setminus \{j\}}\right),$$

and let $SC(\bar{C}^j, \Pi) := \cap_{(\pi, \pi_0) \in \Pi} Conv\left(\bar{C}^j \cap F_{D(\pi, \pi_0)}\right)$. The induction hypothesis is that there exists a finite set $\Pi^j \subseteq \Pi$ of split disjunctions such that

$$SC(\bar{C}^j, \Pi) = \{x \in \bar{C}^j : (\delta(\pi, \pi_0))^T x \geq \delta_0(\pi, \pi_0) \text{ for all } (\pi, \pi_0) \in \Pi^j\},$$

i.e., $SC(\bar{C}^j, \Pi)$ is the set of points in $\bar{C}^j$ satisfying the intersection cuts associated with $\bar{C}^j$ and split disjunctions (π, π_0) in Π^j. Furthermore, for $(\pi, \pi_0) \in \Pi^j$, $\delta(\pi, \pi_0)$ is a solution to the system

$$\delta^T \bar{r}^k = 1/\bar{\alpha}_k(\pi, \pi_0), \ \forall k \in J \setminus \{j\}, \tag{18}$$

and $\delta_0(\pi, \pi_0) := (\delta(\pi, \pi_0))^T \bar{x} + 1$. Let $(\pi, \pi_0) \in \Pi^j$ be arbitrary. Since the equations in (18) are a subset of those in (16), we can choose $\delta(\pi, \pi_0)$ to be a solution to (16) such that it automatically satisfies (18). In other words, we can choose an intersection cut associated with $\bar{C}^j$ and (π, π_0) which is also an intersection cut associated with $\bar{C}$ and (π, π_0). We now prove that $SC(\bar{C}, \Pi)$ is a polyhedron.

Lemma 8. $SC(\bar{C}, \Pi)$ *is a polyhedron.*

Proof. For each $j \in J$, define $\alpha_j^* := \min\{\bar{\alpha}_j(\pi, \pi_0) : (\pi, \pi_0) \in \Pi^j\} > 0$. The scalar α_j^* is such that $\bar{x} + \alpha_j^* \bar{r}^j$ is the intersection point on the half-line $\bar{x} + \alpha \bar{r}^j$, $\alpha \geq 0$, which is closest to $\bar{x}$ over all disjunctions (π, π_0) in Π^j.

Let $\bar{\Pi} := \cup_{j \in J} \Pi^j$, and let $SC(\bar{C}, \bar{\Pi}) := \cap_{(\pi, \pi_0) \in \bar{\Pi}} Conv\left(\bar{C} \cap F_{D(\pi, \pi_0)}\right)$ denote the approximation to $SC(\bar{C}, \Pi)$ obtained by considering the finite set of split disjunctions in $\bar{\Pi}$.

Consider a split disjunction $(\pi', \pi_0') \in \Pi \setminus \bar{\Pi}$. We will show that, if $\bar{\alpha}_j(\pi', \pi_0') < \alpha_j^*$ for some $j \in J$, then the intersection cut $(\delta(\pi', \pi_0'))^T x \geq \delta_0(\pi', \pi_0')$ is valid for $SC(\bar{C}, \bar{\Pi})$. This will mean that it is only necessary to consider split disjunctions in $\Pi \setminus \bar{\Pi}$ satisfying $\bar{\alpha}_j(\pi, \pi_0) \geq \alpha_j^*$ for all $j \in J$. Since, for every extreme ray $j \in J$ and split disjunction $(\pi, \pi_0) \in \Pi \setminus \bar{\Pi}$, there is only a finite number of possible values for $\bar{\alpha}_j(\pi, \pi_0)$ satisfying $\bar{\alpha}_j(\pi, \pi_0) \geq \alpha_j^*$ (Lemma 6), and since intersection cuts that intersect every extreme ray of $\bar{C}$ at the same points are equivalent, this shows that it is sufficient to consider a finite number of disjunctions in $\Pi \setminus \bar{\Pi}$ to get $SC(\bar{C}, \Pi)$ from $SC(\bar{C}, \bar{\Pi})$. It then follows that $SC(\bar{C}, \Pi)$ is a polyhedron.

Therefore assume $(\pi', \pi_0') \in \Pi \setminus \bar{\Pi}$ and $j \in J$ satisfies $\bar{\alpha}_j(\pi', \pi_0') < \alpha_j^*$. Because $(\delta(\pi', \pi_0'))^T x \geq \delta_0(\pi', \pi_0')$ is an intersection cut associated with $\bar{C}$ and (π', π_0'), it is a positive apex-cut for $\bar{C}$ and $\bar{C}^j$. Since $\bar{C}^j \subseteq \bar{C}$, we have $SC(\bar{C}^j, \Pi) \subseteq SC(\bar{C}, \Pi)$, which implies $(\delta(\pi', \pi_0'))^T x \geq \delta_0(\pi', \pi_0')$ is valid for $SC(\bar{C}^j, \Pi)$. By Lemma 3 there exists an inequality $(\delta')^T x \geq \delta_0'$ that dominates $(\delta(\pi', \pi_0'))^T x \geq \delta_0(\pi', \pi_0')$ on $\bar{C}^j$ and

can be expressed as a convex combination of the inequalities $(\delta(\pi, \pi_0))^T x \geq \delta_0(\pi, \pi_0)$ for $(\pi, \pi_0) \in \Pi^j$. Assume $\delta' = \sum_{(\pi, \pi_0) \in \Pi^j} \delta(\pi, \pi_0) \lambda_{(\pi, \pi_0)}$ and $\delta_0' = \sum_{(\pi, \pi_0) \in \Pi^j} \delta_0(\pi, \pi_0) \lambda_{(\pi, \pi_0)}$, where $\sum_{(\pi, \pi_0) \in \Pi^j} \lambda_{(\pi, \pi_0)} = 1$ and $\lambda_{(\pi, \pi_0)} \geq 0$ for all $(\pi, \pi_0) \in \Pi^j$.

To finish the proof, it suffices to show that $(\delta')^T x \geq \delta_0'$ also dominates $(\delta(\pi', \pi_0'))^T x \geq \delta_0(\pi', \pi_0')$ on $\bar{C}$. Because $(\delta')^T x \geq \delta_0'$ dominates $(\delta(\pi', \pi_0'))^T x \geq \delta_0(\pi', \pi_0')$ on $\bar{C}^j$, by the notation introduced in (13) we know $1/\alpha_i'(\delta', \delta_0') \leq 1/\alpha_i'(\delta(\pi', \pi_0'), \delta_0(\pi', \pi_0'))$ for $i \in J \setminus \{j\}$. We only need to verify $1/\alpha_j'(\delta', \delta_0') \leq 1/\alpha_j'(\delta(\pi', \pi_0'), \delta_0(\pi', \pi_0'))$.

By definition of $\delta_0(\pi, \pi_0)$, $\delta_0(\pi, \pi_0) - (\delta(\pi, \pi_0))^T \bar{x} = 1$ for all $(\pi, \pi_0) \in \Pi^j$. By (16) and the choice of α_j^*, we have $\delta_0(\pi, \pi_0) - (\delta(\pi, \pi_0))^T \bar{x} \geq \alpha_j^*(\delta(\pi, \pi_0))^T \bar{r}^j$. This implies $\sum_{(\pi, \pi_0) \in \Pi^j} \lambda_{(\pi, \pi_0)} (\delta_0(\pi, \pi_0) - (\delta(\pi, \pi_0))^T \bar{x}) \geq \alpha_j^* \sum_{(\pi, \pi_0) \in \Pi^j} \lambda_{(\pi, \pi_0)} (\delta(\pi, \pi_0))^T \bar{r}^j$. Furthermore, we have the identities

(a) $\delta_0' - (\delta')^T \bar{x} = \sum_{(\pi, \pi_0) \in \Pi^j} \lambda_{(\pi, \pi_0)} (\delta_0(\pi, \pi_0) - (\delta(\pi, \pi_0))^T \bar{x})$ and

(b) $(\delta')^T \bar{r}^j = \sum_{(\pi, \pi_0) \in \Pi^j} \lambda_{(\pi, \pi_0)} (\delta(\pi, \pi_0))^T \bar{r}^j$.

From (a), (b) and (13) it follows that $\alpha_j^* \leq \alpha_j'(\delta', \delta_0')$. The choice of α_j^* and (17) imply $\alpha_j'(\delta(\pi', \pi_0'), \delta_0(\pi', \pi_0')) < \alpha_j^*$, so that $\alpha_j'(\delta(\pi', \pi_0'), \delta_0(\pi', \pi_0')) < \alpha_j'(\delta', \delta_0')$. Hence $1/\alpha_j'(\delta', \delta_0') < 1/\alpha_j'(\delta(\pi', \pi_0'), \delta_0(\pi', \pi_0'))$, which shows $(\delta')^T x \geq \delta_0'$ dominates $(\delta(\pi', \pi_0'))^T x \geq \delta_0(\pi', \pi_0')$ on $\bar{C}$. $\qquad\square$

3.4. Conic polyhedra and split disjunctions

In this section we generalize the results of Sect. 3.2 and Sect. 3.3 to conic polyhedra of the form $P(B)$ of Sect. 2.1, where $B \in \mathcal{B}_r^*$. Throughout this section B denotes an element of $\mathcal{B}_r^*$. We mention in Sect. 2.1 that $P(B)$ can be written as $P(B) = \bar{x}(B) + L(B) + Cone(\{r^i(B)\}_{i \in B})$. Observe that the set $P'(B) := \bar{x}(B) + Cone(\{r^i(B)\}_{i \in B})$ is a simple conic polyhedron.

We start by providing a link between the sets $Conv(P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$ and $Conv(P'(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$ for a general split disjunction $D(\pi, \pi_0^1, \pi_0^2)$:

Lemma 9. *Let (π, π_0^1, π_0^2) be a general split disjunction, where $\pi_0^1 < \pi_0^2$.*

(i) *If the linear function $\pi^T x$ is not constant on the affine space $\bar{x}(B) + L(B)$, then $Conv(P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = P(B)$.*

(ii) *If the linear function $\pi^T x$ is constant on the affine space $\bar{x}(B) + L(B)$, then $Conv(P(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = L(B) + Conv(P'(B) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$.*

Proof. (i) Let $x^1, x^2 \in \bar{x}(B) + L(B)$ satisfy $\pi^T x^1 \neq \pi^T x^2$, and let $l := x^1 - x^2 \in L(B)$. We have $\pi^T l \neq 0$. Let x' be a point in $P(B)$ satisfying $\pi^T x' \in]\pi_0^1, \pi_0^2[$. The line $x' + \alpha l$, $\alpha \in \mathbb{R}$, is contained in $P(B)$. Since $\pi^T l \neq 0$, the point x' can be expressed as a convex combination of two points on this line that satisfy the disjunction $D(\pi, \pi_0^1, \pi_0^2)$.

(ii) Now suppose $\pi^T x$ is constant on $\bar{x}(B) + L(B)$. This implies $\pi^T l = 0$ for all $l \in L(B)$. Given a positive integer k, define $\Lambda^k := \{\lambda \in \mathbb{R}^k : \lambda \geq 0_k, \sum_{j=1}^k \lambda_j = 1\}$. We have

$$y \in L(B) + Conv\,(P'(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$$

$$\Updownarrow$$

$$\exists x' \in Conv\,(P'(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}), l \in L(B) : y = x' + l$$

$$\Updownarrow$$

$$\exists k \geq 1, \lambda \in \Lambda^k, \{x^j\}_{j=1}^k \subseteq P'(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}, l \in L(B) : y = \sum_{j=1}^{k} (x^j + l)\lambda_j$$

$$\Updownarrow$$

$$\exists k \geq 1, \lambda \in \Lambda^k, \{y^j\}_{j=1}^k \subseteq P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)} : y = \sum_{j=1}^{k} y^j \lambda_j$$

$$\Updownarrow$$

$$y \in Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}). \qquad\qquad\qquad\qquad\qquad\qquad \square$$

Combining Lemma 9 with Lemma 4 in Sect. 3.2 gives Lemma 1 in Sect. 2.1.

Corollary 2. *Lemma 1 holds.*

Proof. (i) If $\pi^T x$ is *not* constant on $\bar{x}(B) + L(B)$, we are done (Lemma 9(i)). So we assume $\pi^T x$ is constant on $\bar{x}(B) + L(B)$. Let $x' \in \bar{x}(B) + L(B)$ satisfy $x' \in F_{D(\pi,\pi_0^1,\pi_0^2)}$. Since $x' \in F_{D(\pi,\pi_0^1,\pi_0^2)}$, we know $\bar{x}(B) + L(B) \subseteq F_{D(\pi,\pi_0^1,\pi_0^2)}$. In particular $\bar{x}(B) \in F_{D(\pi,\pi_0^1,\pi_0^2)}$. Now, since $P'(B)$ is a simple conic polyhedron, Lemma 4(i) gives $Conv\,(P'(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = P'(B)$. Combining this fact with Lemma 9(ii) gives $Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = P(B)$.

(ii) Suppose $\pi^T x \in]\pi_0^1, \pi_0^2[$ for every $x \in \bar{x}(B) + L(B)$. Then $\pi^T x$ is constant on $\bar{x}(B) + L(B)$, and $\pi^T l = 0$ holds for all $l \in L(B)$. In fact, if $L(B) = \{0\}$, the result is obvious; otherwise, $L(B)$ is a linear space, and we must have $\pi^T l = 0$ to avoid that $x + \alpha l$ satisfies the disjunction for some $\alpha \in \mathbb{R}$.

Now suppose $(\delta(\pi, \pi_0))^T l \neq 0$ for some $l \in L(B)$. Let $x' \in P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}$ be arbitrary, and let $l' \in L(B)$ satisfy $(\delta(\pi, \pi_0))^T l' < 0$. The point $x' + \alpha l'$ is in $P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}$ and violates the inequality $(\delta(\pi, \pi_0))^T x \geq \delta_0(\pi, \pi_0)$ for $\alpha > 0$ sufficiently large. Hence $(\delta(\pi, \pi_0))^T l = 0$ for all $l \in L(B)$.

Since $P'(B)$ is a simple conic polyhedron, Lemma 4(ii) gives $Conv\,(P'(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = \{x \in P'(B) : (15)\}$. Combining this with Lemma 9(ii) gives the identity $Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = L(B) + \{x \in P'(B) : (15)\}$. Finally, since $(\delta(\pi, \pi_0))^T l = 0$ for all $l \in L(B)$, we get $\{x \in P'(B) : (15)\} + L(B) = \{x \in P'(B) + L(B) : (15)\} = \{x \in P(B) : (15)\}$. $\qquad\qquad \square$

By combining Lemma 8 and Lemma 9 we can prove

Corollary 3. *$SC(B)$ is a polyhedron.*

Proof. Let $\Pi_V^n(N_I) := \{(\pi, \pi_0) \in \Pi^n(N_I) : \pi^T x \in]\pi_0, \pi_0 + 1[, \forall x \in \bar{x}(B) + L(B)\}$ denote the set of violated split disjunctions for $P(B)$. The split closure of $P(B)$ is given by $SC(B) = \cap_{(\pi,\pi_0) \in \Pi_V^n(N_I)} Conv\,(P(B) \cap F_{D(\pi,\pi_0)})$. Using Lemma 9 gives $SC(B) =$

$L(B) + \bigcap_{(\pi,\pi_0)\in\Pi_V^n(N_I)} Conv\,(P'(B) \cap F_{D(\pi,\pi_0)})$. Since $P'(B)$ is a simple conic poly-hedron, Lemma 8 implies that $\bigcap_{(\pi,\pi_0)\in\Pi_V^n(N_I)} Conv\,(P'(B) \cap F_{D(\pi,\pi_0)})$ is a polyhedron.
□

Given $B' \in \mathcal{B}_k^*$, where $0 < k < r$, and a general split disjunction $D(\pi, \pi_0^1, \pi_0^2)$ satisfying $\pi^T y \in]\pi_0^1, \pi_0^2[$, $\forall y \in \bar{x}(B') + L(B')$, an intersection cut can be derived from B' and $D(\pi, \pi_0^1, \pi_0^2)$. A question is how the intersection cut derived from B' and $D(\pi, \pi_0^1, \pi_0^2)$ relates to intersection cuts derived from $D(\pi, \pi_0^1, \pi_0^2)$ and sets $B \in \mathcal{B}_r^*$. The next lemma shows that the intersection cut derived from B' and $D(\pi, \pi_0^1, \pi_0^2)$ is equivalent to any intersection cut derived from $D(\pi, \pi_0^1, \pi_0^2)$ and any $B \in \mathcal{B}_r^*$ satisfying $B' \subset B$, i.e., the size of B' is not important.

Lemma 10. *Let $B' \in \mathcal{B}_k^*$, where $0 < k < r$, satisfy $\pi^T y \in]\pi_0^1, \pi_0^2[$ for all $y \in$ $\bar{x}(B') + L(B')$ and $Conv\,(P(B') \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = \{x \in P(B') : \delta^T x \geq \delta_0\}$. Then $Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = \{x \in P(B) : \delta^T x \geq \delta_0\}$ for any $B \in \mathcal{B}_r^*$ satisfying $B \supset B'$.*

Proof. Let B' and B be as stated. Wlog we assume that B and B' only differ in one element, i.e., $B = B' \cup \{q\}$, where $q \in M \setminus B'$. Using the notation in Sect. 2.1, we can choose $r^j(B) = r^j(B')$ for $j \in B'$ and $\bar{x}(B) = \bar{x}(B')$. This implies $\alpha_j(\pi, \pi_0^1, \pi_0^2, B) = \alpha_j(\pi, \pi_0^1, \pi_0^2, B')$ for $j \in B'$, i.e., the intersection cut associated with B only differs from the intersection cut associated with B' in the coefficient corresponding to q. For simplicity, let $\alpha_j := \alpha_j(\pi, \pi_0^1, \pi_0^2, B)$ for $j \in B$ and $\alpha_j' := \alpha_j(\pi, \pi_0^1, \pi_0^2, B')$ for $j \in B'$.

By assumption $\{x \in P(B') : \delta^T x \geq \delta_0\} = Conv\,(P(B') \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. Now $P(B) \subseteq P(B')$, so $Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \subseteq Conv\,(P(B') \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. This implies that the inequality $\delta^T x \geq \delta_0$ is valid for $Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. Therefore, $Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \subseteq \{x \in P(B) : \delta^T x \geq \delta_0\}$.

For the other direction suppose $y \in \{x \in P(B) : \delta^T x \geq \delta_0\}$. Since $P(B) \subseteq P(B')$ and $y \in P(B)$, we have $y \in P(B')$. Also, since $\delta^T y \geq \delta_0$ and $y \in P(B')$, we have $y \in$ $Conv\,(P(B') \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. This implies that y satisfies the intersection cut associated with B' and $D(\pi, \pi_0^1, \pi_0^2)$, i.e., the inequality $\sum_{i\in B'}(b_i - a_{i.}^T y)/\alpha_i' \geq 1$ holds. Finally, using the facts $a_{q.}^T y \leq b_q$, $\alpha_j = \alpha_j'$ for $j \in B'$ and $\alpha_q \geq 0$ gives $\sum_{i\in B}(b_i - a_{i.}^T y)/\alpha_i \geq 1$, i.e., y satisfies the intersection cut associated with B and $D(\pi, \pi_0^1, \pi_0^2)$. From Lemma 1 it follows that $y \in Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$.
□

4. Split closure characterization

In this section we give a geometric argument for the validity of Theorem 1. As mentioned in Sect. 2 this leads to a characterization of the split closure in terms of intersection cuts (Corollary 1).

Throughout this section $D(\pi, \pi_0^1, \pi_0^2)$ denotes a general split disjunction, where $\pi \in \mathbb{R}^n$ and $\pi_0^1 < \pi_0^2$. The following notation will be convenient. Define $P_1 :=$

$P \cap \{x \in \mathbb{R}^n : \pi^T x \le \pi_0^1\}$ and $P_2 := P \cap \{x \in \mathbb{R}^n : \pi^T x \ge \pi_0^2\}$ to be the two subsets of P that satisfy the disjunction $D(\pi, \pi_0^1, \pi_0^2)$. Furthermore, given $B \in \mathcal{B}_r^*$, define $P_1(B) := P(B) \cap \{x \in \mathbb{R}^n : \pi^T x \le \pi_0^1\}$ and $P_2(B) := P(B) \cap \{x \in \mathbb{R}^n : \pi^T x \ge \pi_0^2\}$. The following inclusion is easy to prove.

Lemma 11.

$$Conv\,(P_1 \cup P_2) \subseteq \bigcap_{B \in \mathcal{B}_r^*} Conv\,(P_1(B) \cup P_2(B)). \tag{19}$$

Proof. Let $B \in \mathcal{B}_r^*$. Clearly $P_1 \cup P_2 \subseteq P_1(B) \cup P_2(B)$. Convexifying both sides gives $Conv\,(P_1 \cup P_2) \subseteq Conv\,(P_1(B) \cup P_2(B))$. Since this holds for all $B \in \mathcal{B}_r^*$, the result follows. $\qquad\square$

The remainder of this section is divided into two parts. The focus is on proving the other inclusion $Conv\,(P_1 \cup P_2) \supseteq \bigcap_{B \in \mathcal{B}_r^*} Conv\,(P_1(B) \cup P_2(B))$. In Sect. 4.1 we consider the main case that P, P_1 and P_2 are all full-dimensional. In Sect. 4.2 all remaining cases are considered.

4.1. Main case: P_1, P_2 and P full-dimensional

In this section we assume that the sets P, P_1 and P_2 are full-dimensional. We will show that every facet defining inequality for $Conv\,(P_1 \cup P_2)$ is valid for $\bigcap_{B \in \mathcal{B}_r^*} Conv\,(P_1(B) \cup P_2(B))$. This will show the direction $Conv\,(P_1 \cup P_2) \supseteq \bigcap_{B \in \mathcal{B}_r^*} Conv\,(P_1(B) \cup P_2(B))$. Let $\delta^T x \le \delta_0$ be an arbitrary facet defining inequality for $Conv\,(P_1 \cup P_2)$. Because $\bigcap_{B \in \mathcal{B}_r^*} Conv\,(P_1(B) \cup P_2(B)) \subseteq P$, we can assume that $\delta^T x \le \delta_0$ is not valid for P.

To show $\delta^T x \le \delta_0$ is valid for $\bigcap_{B \in \mathcal{B}_r^*} Conv\,(P_1(B) \cup P_2(B))$, it suffices to prove that there exists $\bar{B} \in \mathcal{B}_r^*$ such that $\delta^T x \le \delta_0$ is valid for $Conv\,(P_1(\bar{B}) \cup P_2(\bar{B}))$. In particular, we prove the stronger fact that there exists $\bar{B} \in \mathcal{B}_r^*$ such that $Conv\,(P_1(\bar{B}) \cup P_2(\bar{B})) = \{x \in P(\bar{B}) : \delta^T x \le \delta_0\}$, *i.e.*, $\delta^T x \le \delta_0$ can be obtained as an intersection cut. Lemma 10 implies that it suffices to find $\bar{B}$ in $\mathcal{B}_k^*$, where $0 < k < r$, such that $Conv\,(P_1(\bar{B}) \cup P_2(\bar{B})) = \{x \in P(\bar{B}) : \delta^T x \le \delta_0\}$.

Let $F := \{x \in Conv\,(P_1 \cup P_2) : \delta^T x = \delta_0\}$ be the facet of $Conv\,(P_1 \cup P_2)$ defined by $\delta^T x \le \delta_0$. Furthermore, let $F_1 := F \cap P_1$ and $F_2 := F \cap P_2$. The assumptions that P_1 and P_2 are non-empty and $\delta^T x \le \delta_0$ is not valid for P imply $F_1 \ne \emptyset$ and $F_2 \ne \emptyset$. We now give some basic properties of the sets F, F_1 and F_2 that we derived from the general split disjunction $D(\pi, \pi_0^1, \pi_0^2)$.

Lemma 12. (i) $F = Conv\,(F_1 \cup F_2)$.
  (ii) For $k \in \{1, 2\}$ we have $F_k \subseteq \{x \in \mathbb{R}^n : \pi^T x = \pi_0^k\}$, i.e., F_k is contained in the hyperplane $\pi^T x = \pi_0^k$.

Proof. (i) is trivial, so we only prove (ii). Wlog let $k = 1$. Assume there exists $x^1 \in F_1$ satisfying $\pi^T x^1 < \pi_0^1$. Note that $\delta^T x^1 = \delta_0$. Since $\delta^T x \le \delta_0$ is not valid for P, there exists $x^2 \in P \setminus (P_1 \cup P_2)$ such that $\delta^T x^2 > \delta_0$. Let $z(\lambda) := x^1 \lambda + (1 - \lambda)x^2$, where $\lambda \in [0, 1]$. Observe that $\delta^T z(\lambda) > \delta_0$ and $z(\lambda) \in P$ for all $\lambda \in]0, 1]$. By choosing λ sufficiently close to zero, we have $\pi^T z(\lambda) < \pi_0^1$ and $\delta^T z(\lambda) > \delta_0$. For such a λ, $z(\lambda) \in P_1$ and $\delta^T z(\lambda) > \delta_0$. This contradicts the validity of $\delta^T x \le \delta_0$ for P_1. $\qquad\square$

Let $k \in \{1, 2\}$ be arbitrary. Observe that, since $\delta^T x \leq \delta_0$ is valid for P_k and $F_k = \{x \in P_k : \delta^T x = \delta_0\}$, F_k is a face of P_k. Let d_k be the dimension of the face F_k of P_k. We have $F_k = P_k \cap Aff(F_k)$, where $Aff(F_k)$ denotes the affine hull of F_k. The set $\bar{B}$ satisfying $Conv(P_1(\bar{B}) \cup P_2(\bar{B})) = \{x \in P(\bar{B}) : \delta^T x \leq \delta_0\}$ that we are about to construct is obtained from constraints of P that can be used to describe $Aff(F_1)$ and $Aff(F_2)$. The following crucial lemma provides the constraints for the set $\bar{B}$.

Lemma 13. *For $k \in \{1, 2\}$, there exists $B_k \in \mathcal{B}^*_{n-d_k-1}$ such that*

(i) *$Aff(F_k) = \{x \in \mathbb{R}^n : a_{i.}^T x = b_i, \forall i \in B_k, \pi^T x = \pi_0^k\}$, i.e., $Aff(F_k)$ is the intersection of the hyperplanes $\pi^T x = \pi_0^k$ and $a_{i.}^T x = b_i$ for $i \in B_k$.*
(ii) *There exist $\bar{u}_0 > 0$ and $\bar{u}_i \geq 0$ for $i \in B_1$ such that $\delta = \bar{u}_0 \pi + \sum_{i \in B_1} \bar{u}_i a_{i.}$ and $\delta_0 = \bar{u}_0 \pi_0^1 + \sum_{i \in B_1} \bar{u}_i b_i$.*
(iii) *There exist $\bar{v}_0 > 0$ and $\bar{v}_i \geq 0$ for $i \in B_2$ such that $\delta = \sum_{i \in B_2} \bar{v}_i a_{i.} - \bar{v}_0 \pi$ and $\delta_0 = \sum_{i \in B_2} \bar{v}_i b_i - \bar{v}_0 \pi_0^2$.*

Proof. Wlog let $k = 1$. Consider the linear program (PLP)

$$\max \quad \delta^T x$$
$$\text{s.t. } a_{i.}^T x \leq b_i, \qquad \forall i \in M, \qquad (u_i) \qquad\qquad (20)$$
$$\pi^T x \leq \pi_0^1, \qquad\qquad\qquad (u_0) \qquad\qquad (21)$$

the problem of maximizing $\delta^T x$ over P_1. Since $\delta^T x \leq \delta_0$ is valid for P_1, and F_1 is the set of points in P_1 satisfying $\delta^T x = \delta_0$, F_1 is the set of optimal solutions to (PLP). The dual of (PLP) is the problem (DLP) given by

$$\min \quad u^T b + \pi_0^1 u_0$$
$$\text{s.t. } \sum_{i \in M} a_{i.} u_i + \pi u_0 = \delta, \qquad (x) \qquad\qquad (22)$$
$$u \geq 0_m, \qquad\qquad\qquad (23)$$
$$u_0 \geq 0. \qquad\qquad\qquad (24)$$

Let $\bar{x}$ be an optimal basic feasible solution to (PLP), and let $(\bar{u}, \bar{u}_0)$ be a corresponding optimal basic feasible solution to (DLP). Since the optimal objective value to (DLP) is δ_0, we have $\delta_0 = b^T \bar{u} + \pi_0^1 \bar{u}_0$.

If $\bar{u}_0 = 0$, then $\sum_{i \in M} a_{i.} \bar{u}_i = \delta$ and $\sum_{i \in M} b_i \bar{u}_i = \delta_0$, which implies that $\delta^T x \leq \delta_0$ is valid for P since $\bar{u} \geq 0_m$ which contradicts our assumption. Hence we must have $\bar{u}_0 > 0$, which means that u_0 is basic.

Let $\tilde{B}_1 := \{i \in M : u_i \text{ basic}\}$. Since the variables $\{u_i\}_{i \in \tilde{B}_1} \cup \{u_0\}$ are all basic, the vectors $\{a_{i.}\}_{i \in \tilde{B}_1} \cup \{\pi\}$ are linearly independent.

Let $M_1^= := \{i \in M : a_{i.}^T x = b_i, \forall x \in F_1\}$ denote the constraints of P satisfied at equality by all points in F_1. We will prove by contradiction that if $i \in M \setminus M_1^=$ then $\bar{u}_i = 0$. Suppose there exists $k \in M \setminus M_1^=$ satisfying $\bar{u}_k > 0$. Consider an inner point x^I of F_1, i.e., x^I satisfies $\pi^T x^I = \pi_0^1$, $a_{i.}^T x^I = b_i$ for $i \in M_1^=$ and $a_{i.}^T x^I < b_i$ for $i \in M \setminus M_1^=$. Since $\delta = \sum_{i \in M} a_{i.} \bar{u}_i + \pi \bar{u}_0$, we have $\delta^T x^I = \sum_{i \in M} a_{i.}^T x^I \bar{u}_i + \pi^T x^I \bar{u}_0 <$

$\sum_{i \in M} b_i \bar{u}_i + \pi_0^1 \bar{u}_0 = \delta_0$, which contradicts $\delta^T x = \delta_0$ for all $x \in F_1$. Therefore we have $\bar{u}_i = 0$ for all $i \in M \setminus M_1^=$.

By choosing a maximal subset B_1 of $M_1^=$ such that $B_1 \supseteq \tilde{B}_1 \cap M_1^=$ and the vectors $\{\pi\} \cup \{a_{i.}\}_{i \in B_1}$ are linearly independent, (i) and (ii) are satisfied. This proves (ii). The proof of (iii) is similar to the proof of (ii). $\square$

Let B_1 and B_2 be as in Lemma 13, and define $\bar{B} := B_1 \cup B_2$. We will show that the vectors $\{a_{i.}\}_{i \in \bar{B}}$ are linearly independent (Lemma 16) and $Conv(P_1(\bar{B}) \cup P_2(\bar{B})) = \{x \in P(\bar{B}) : \delta^T x \le \delta_0\}$ (Lemma 19). This will finish the proof of Theorem 1 for the main case. For $k = 1, 2$, define $A^k := \{x \in \mathbb{R}^n : a_{i.}^T x = b_i, \forall i \in B_k\}$ to be the affine space defined by B_k. First, we prove the following lemma that describes how points in A^1 and A^2 are positioned relative to the hyperplanes $\pi^T x = \pi_0^1$, $\pi^T x = \pi_0^2$ and $\delta^T x = \delta_0$.

Lemma 14. *For $x^1 \in A^1$ and $x^2 \in A^2$,*

(i) $\pi^T x^1 < \pi_0^1 \iff \delta^T x^1 < \delta_0$,
   $\pi^T x^1 = \pi_0^1 \iff \delta^T x^1 = \delta_0$ *and*
   $\pi^T x^1 > \pi_0^1 \iff \delta^T x^1 > \delta_0$.
(ii) $\pi^T x^2 > \pi_0^2 \iff \delta^T x^2 < \delta_0$,
   $\pi^T x^2 = \pi_0^2 \iff \delta^T x^2 = \delta_0$ *and*
   $\pi^T x^2 < \pi_0^2 \iff \delta^T x^2 > \delta_0$.

Proof. By symmetry we only need to prove (i). Let $x^1 \in A^1$, and let $\{\bar{u}_i\}_{i \in B_1}$ and $\bar{u}_0 > 0$ be as in Lemma 13. Since $a_{i.}^T x = b_i$ for all $x \in A^1$, $\sum_{i \in B_1} \bar{u}_i a_{i.}^T x$ is a constant on A^1 that is equal to $c := \sum_{i \in B_1} \bar{u}_i b_i$. We conclude from Lemma 13 that $\delta_0 = c + \bar{u}_0 \pi_0^1$ and $\delta^T x = c + \bar{u}_0 \pi^T x$ for all $x \in A^1$.

Now we have $\delta^T x^1 < \delta_0 \iff \bar{u}_0 \pi^T x^1 + c < \delta_0 \iff \pi^T x^1 < \pi_0^1$. Also, we have $\delta^T x^1 = \delta_0 \iff \bar{u}_0 \pi^T x^1 + c = \delta_0 \iff \pi^T x^1 = \pi_0^1$. $\square$

Now, we show that, if B_1 and B_2 are as in Lemma 13, then the vectors $\{a_{i.}\}_{i \in B_1 \cup B_2}$ are linearly independent. We will show $r(B_1 \cup B_2) = |B_1| + |B_2|$. Observe that this is equivalent to showing that the set $L(B_1) \cap L(B_2)$ is of dimension $n - |B_1| - |B_2|$, where $L(B_k) := \{x \in \mathbb{R}^n : a_{i.}^T x = 0, \forall i \in B_k\}$ for $k = 1, 2$ are as defined in Sect. 2.1.

Lemma 15. *The set $L(B_1) \cap L(B_2) \cap \{x \in \mathbb{R}^n : \pi^T x = 0\}$ is of dimension $n - |B_1| - |B_2|$.*

Proof. Let $Aff(F_1) = L_1 + y^1$ and $Aff(F_2) = L_2 + y^2$, where $y^1 \in Aff(F_1)$, $y^2 \in Aff(F_2)$, and L_1 and L_2 are the linear spaces parallel to $Aff(F_1)$ and $Aff(F_2)$ respectively. Observe that $L(B_1) \cap L(B_2) \cap \{x \in \mathbb{R}^n : \pi^T x = 0\} = L_1 \cap L_2$. Hence we need to show $\dim(L_1 \cap L_2) = n - |B_1| - |B_2|$.

Let $\{x^j\}_{j=1}^n \subseteq F_1 \cup F_2$ be affinely independent and chosen such that the first $d_1 + 1$ points are in F_1, i.e., $Aff(\{x^j\}_{j=1}^{d_1+1}) = Aff(F_1)$. Necessarily, the remaining $(n - d_1 - 1)$ points are in F_2, i.e., $\{x^j\}_{j=d_1+2}^n \subseteq F_2$. Define $S_1 := Aff(\{x^j\}_{j=d_1+2}^n) - y^2$. We have $\dim(S_1) = n - d_1 - 2$. Furthermore, since $S_1 \subseteq L_2$ and $\dim(S_1) = n - d_1 - 2$, we have $d_2 \ge n - d_1 - 2$. Observe that $n - |B_1| - |B_2| = d_2 - (n - d_1 - 2) \ge 0$.

First suppose $d_2 - (n - d_1 - 2) = 0$. Then $L_2 = S_1$. Since $Aff(\{x^j\}_{j=1}^{d_1+1}) - y^1 = L_1$, the vectors in L_1 are linearly independent from those in $L_2 = S_1$, so that $L_1 \cap L_2 = \{0_n\}$. Therefore, $\dim(L_1 \cap L_2) = 0 = d_2 - (n - d_1 - 2) = n - |B_1| - |B_2|$.

Now suppose $d_2 - (n - d_1 - 2) > 0$. Let $d_2^r := d_2 - (n - d_1 - 2)$. Choose d_2^r vectors $\{z^j\}_{j=1}^{d_2^r}$ from $L_2 \setminus S_1$ such that they, together with S_1, span L_2. Define $S_2 := Span(\{z^j\}_{j=1}^{d_2^r})$. By construction, we have $L_2 = S_1 + S_2$ and $S_1 \cap S_2 = \{0_n\}$. Furthermore we have $S_1 \cap L_1 = \{0_n\}$. From this it follows that $L_1 \cap L_2 = S_2$, so that $\dim(L_1 \cap L_2) = d_2^r = n - |B_1| - |B_2|$. □

By combining Lemma 14 and Lemma 15 we get

Lemma 16. *Let B_1 and B_2 be as in Lemma 13. Then*

(i) $B_1 \cap B_2 = \emptyset$,
(ii) *The vectors $\{a_{i\cdot}\}_{i \in B_1 \cup B_2}$ are linearly independent.*

Proof. Lemma 14(i) implies that every solution to the system $a_{i\cdot}^T x = 0, i \in B_1$ satisfying $\pi^T \bar{x} \neq 0$ is either in the set $\{x \in \mathbb{R}^n : \pi^T x < 0, \delta^T x < 0\}$ or in the set $\{x \in \mathbb{R}^n : \pi^T x > 0, \delta^T x > 0\}$. Furthermore, Lemma 14(ii) implies every solution to the system $a_{i\cdot}^T x = 0, i \in B_2$ satisfying $\pi^T \bar{x} \neq 0$ is either in the set $\{x \in \mathbb{R}^n : \pi^T x > 0, \delta^T x < 0\}$ or in the set $\{x \in \mathbb{R}^n : \pi^T x < 0, \delta^T x > 0\}$.

From this we see that every solution to the system $a_{i\cdot}^T x = 0, i \in B_1 \cup B_2$ must satisfy $\pi^T x = 0$. This implies $L(B_1) \cap L(B_2) = L(B_1) \cap L(B_2) \cap \{x \in \mathbb{R}^n : \pi^T x = 0\}$. By applying Lemma 15 we get that $L(B_1) \cap L(B_2)$ is of dimension $n - |B_1| - |B_2|$. Since the description of $L(B_1) \cap L(B_2)$ involves $|B_1| + |B_2|$ constraints, this implies both (i) and (ii). □

In the remainder of this section we show $Conv(P_1(\bar{B}) \cup P_2(\bar{B})) = \{x \in P(\bar{B}) : \delta^T x \leq \delta_0\}$, where $\bar{B} := B_1 \cup B_2$. The following lemma shows the first inclusion.

Lemma 17. *Let B_1 and B_2 be as in Lemma 13, and let $\bar{B} := B_1 \cup B_2$. Then $\delta^T x \leq \delta_0$ is valid for $Conv(P_1(\bar{B}) \cup P_2(\bar{B}))$.*

Proof. We will show that $\delta^T x \leq \delta_0$ is valid for both $P_1(B_1)$ and $P_2(B_2)$. Since $P_1(\bar{B}) \subseteq P_1(B_1)$ and $P_2(\bar{B}) \subseteq P_2(B_2)$, this implies that $\delta^T x \leq \delta_0$ is valid for $Conv(P_1(\bar{B}) \cup P_2(\bar{B}))$. Wlog we only show that $\delta^T x \leq \delta_0$ is valid for $P_1(B_1)$.

Recall that $P_1(B_1) = \{x \in \mathbb{R}^n : a_{i\cdot}^T x \leq b_i, i \in B_1, \pi^T x \leq \pi_0^1\}$. We now derive a representation of $P_1(B_1)$ in terms of extreme points and extreme rays. Let $\bar{x}$ be a solution to the system $\pi^T x = \pi_0^1$ and $a_{i\cdot}^T x = b_i$ for all $i \in B_1$. Also, for $j \in B_1$, let $\bar{r}^j$ be a solution to the system $a_{i\cdot}^T x = 0, \forall i \in B_1 \setminus \{j\}, a_{j\cdot}^T x = -1$ and $\pi^T x = 0$. Finally, let $\bar{r}^\pi$ be a solution to the system $a_{i\cdot}^T x = 0, \forall i \in B_1$, and $\pi^T x = -1$. We have $P_1(B_1) = \bar{x} + L^\pi(B_1) + Cone(\{\bar{r}^j\}_{j \in B_1} \cup \{\bar{r}^\pi\})$, where $L^\pi(B_1) := \{x \in \mathbb{R}^n : a_{i\cdot}^T x = 0, \forall i \in B_1, \pi^T x = 0\}$.

Observe that $\bar{x} + L^\pi(B_1) \subseteq A^1$ (where $A^1 := \{x \in \mathbb{R}^n : a_{i\cdot}^T x = b_i, \forall i \in B_1\}$ as defined earlier). Furthermore $\pi^T x = \pi_0^1$ for every $x \in \bar{x} + L^\pi(B_1)$, which implies that $\delta^T x = \delta_0$ for all $x \in \bar{x} + L^\pi(B_1)$ (Lemma 14(i)). By using the scalars from Lemma 13(ii), we get $\delta^T \bar{r}^j = -\bar{u}_j \leq 0$ for all $j \in B_1$ and $\delta^T \bar{r}^\pi = -\bar{u}_0 < 0$. Since all the extreme rays $\bar{r}^j$ of $P_1(B_1)$ are such that $\delta^T \bar{r}^j \leq 0$ and $\delta^T \bar{x} = \delta_0$, we have that $\delta^T x \leq \delta_0$ is valid for $P_1(B_1)$. □

Lemma 17 proves $Conv\,(P_1(\bar{B}) \cup P_2(\bar{B})) \subseteq \{x \in P(\bar{B}) : \delta^T x \le \delta_0\}$, where $\bar{B} = B_1 \cup B_2$, i.e., that $\delta^T x \le \delta_0$ is valid for $Conv\,(P_1(\bar{B}) \cup P_2(\bar{B}))$. Combining this result with Lemma 10 and Lemma 11 finishes the proof of Theorem 1 for the main case. The remainder of this section is devoted to proving $Conv\,(P_1(\bar{B}) \cup P_2(\bar{B})) \supseteq \{x \in P(\bar{B}) : \delta^T x \le \delta_0\}$. To simplify notation, let $\bar{x} := \bar{x}(\bar{B})$ and $\bar{r}^j := r^j(\bar{B})$ for $j \in \bar{B}$ (see Sect. 2.1). First we prove some basic properties of $P(\bar{B})$.

Lemma 18. *Let* $\bar{B} = B_1 \cup B_2$, *where* B_1 *and* B_2 *are as in Lemma 13.*

(i) $\pi^T x \in]\pi_0^1, \pi_0^2[$ *for all* $x \in A^1 \cap A^2 = \bar{x} + L(\bar{B})$.
(ii) $\pi^T \bar{r}^j \ge 0$ *for all* $j \in B_1$.
(iii) $\pi^T \bar{r}^j \le 0$ *for all* $j \in B_2$.

Proof. (i) The proof is by contradiction. Let $y \in \bar{x} + L(\bar{B})$ and wlog assume $\pi^T y \le \pi_0^1$. Since $y \in A^1$, Lemma 14(i), implies $\delta^T y \le \delta_0$. However, since $y \in A^2$ and $\pi^T y \le \pi_0^1 < \pi_0^2$, Lemma 14(ii) implies $\delta^T y > \delta_0$, which is a contradiction.

(ii)–(iii) We only prove (ii). We know $P(B_1) = A^1 + Cone\,(\{\bar{r}^j\}_{j \in B_1})$. Let $j \in B_1$, and assume $\pi^T \bar{r}^j < 0$ for a contradiction. We have $\bar{x} + \alpha \bar{r}^j \in A^2$ and $\bar{x} + \alpha \bar{r}^j \in P(\bar{B}) \subseteq P(B_1)$ for every $\alpha \ge 0$. Let $\bar{\alpha} > 0$ satisfy $\pi^T(\bar{x} + \bar{\alpha} \bar{r}^j) = \pi_0^1$. Lemma 14(ii) gives $\delta^T(\bar{x} + \bar{\alpha} \bar{r}^j) > \delta_0$. This contradicts the validity of $\delta^T x \le \delta_0$ for $P_1(B_1)$. $\square$

Lemma 18(i) states that an intersection cut can be derived from $D(\pi, \pi_0^1, \pi_0^2)$ and $\bar{B}$ (Lemma 1). Lemma 18(ii) states that rays associated with elements of B_1 point in the direction of satisfaction of the term $\pi^T x \ge \pi_0^2$ of the disjunction $D(\pi, \pi_0^1, \pi_0^2)$, and Lemma 18(iii) states that rays associated with elements of B_2 point in the opposite direction.

To simplify notation, let $\bar{\alpha}_j := \alpha_j(\pi, \pi_0^1, \pi_0^2, \bar{B})$ denote the coefficient in the intersection cut associated with $\bar{B}$ and $D(\pi, \pi_0^1, \pi_0^2)$ for $j \in \bar{B}$. We are now able to prove $Conv\,(P_1(\bar{B}) \cup P_2(\bar{B})) = \{x \in P(\bar{B}) : \delta^T x \le \delta_0\}$.

Lemma 19. *Let* $\bar{B} = B_1 \cup B_2$, *where* B_1 *and* B_2 *are as in Lemma 13. Then*

(i) $\bar{\alpha}_j = (\delta_0 - \delta^T \bar{x})/\delta^T \bar{r}^j$ *for* $j \in \bar{B}$ *satisfying* $\delta^T \bar{r}^j \ne 0$.
(ii) $Conv\,(P_1(\bar{B}) \cup P_2(\bar{B})) = \{x \in P(\bar{B}) : \delta^T x \le \delta_0\}$.

Proof. (i) Let $j \in \bar{B}$ satisfy $\pi^T \bar{r}^j \ne 0$. Wlog assume $\pi^T \bar{r}^j > 0$, so $j \in B_1$ (Lemma 18(ii)). Choose $\bar{\alpha}_j$ such that $\pi^T(\bar{x} + \bar{\alpha}_j \bar{r}^j) = \pi_0^2$. Since $\bar{x} + \bar{\alpha}_j \bar{r}^j \in A^2$, we have $\delta^T(\bar{x} + \bar{\alpha}_j \bar{r}^j) = \delta_0$ (Lemma 14(ii)). Solving for $\bar{\alpha}_j$ gives (i). Notice that, since $\delta^T \bar{x} > \delta_0$ (Lemma 18(i) and Lemma 14), we have $\delta^T \bar{r}^j < 0$.

(ii) We already proved $Conv\,(P_1(\bar{B}) \cup P_2(\bar{B})) \subseteq \{x \in P(\bar{B}) : \delta^T x \le \delta_0\}$ (Lemma 17). Now suppose that $y \in P(\bar{B})$ satisfies $\delta^T y \le \delta_0$. Since $y \in P(\bar{B})$, we may write $y = x' + \sum_{j \in \bar{B}} \bar{r}^j s_j^y$, where $s_j^y \ge 0$ for $j \in \bar{B}$ denotes the slack in the jth constraint, i.e., $s_j^y = b_j - a_{j.}^T y$, and $x' \in \bar{x} + L(\bar{B})$. Multiplying with δ on both sides and deducting δ_0 gives $\delta^T y - \delta_0 = \sum_{j \in \bar{B}} \delta^T \bar{r}^j s_j^y + \delta^T x' - \delta_0$. Since $\delta^T y \le \delta_0$, we have $0 \ge \sum_{j \in \bar{B}} \delta^T \bar{r}^j s_j^y + \delta^T x' - \delta_0$. Dividing with $\delta^T x' - \delta_0 > 0$ on both sides gives $\sum_{j \in \bar{B}} s_j^y (\delta^T \bar{r}^j)/(\delta_0 - \delta^T x') \ge 1$.

Observe that Lemma 18(i) and the fact that $\bar{x} + L(\bar{B}) = A^1 \cap A^2$ is affine implies the linear function $\pi^T x$ is constant on $\bar{x} + L(\bar{B})$. Now, using Lemma 14, this implies the linear function $\delta^T x$ is constant on $\bar{x} + L(\bar{B})$. Therefore $\delta^T x' = \delta^T \bar{x}$.

Now using (i) we get $\sum_{j \in \bar{B}} s_j^y / \bar{\alpha}_j \geq 1$. Hence y satisfies the intersection cut associated with $\bar{B}$ and the disjunction $D(\pi, \pi_0^1, \pi_0^2)$, which implies $y \in Conv\,(P_1(\bar{B}) \cup P_2(\bar{B}))$.

$\square$

4.2. Reduction to the main case

In this section, we consider the remaining cases, *i.e.*, when P or P_1 or P_2 is not full-dimensional. We distinguish between three cases, namely 1) One of the sets is empty 2) P is full-dimensional and 3) P is not full-dimensional. First we prove Theorem 1 when P is empty.

Lemma 20. *Theorem* 1 *holds when P is empty.*

Proof. If P is empty, $Conv\,(P \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = \emptyset$. We know $\cap_{B \in \mathcal{B}_r^*} P(B) = P$. Since $P(B)$ is convex, $\cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \subseteq \cap_{B \in \mathcal{B}_r^*} P(B) = \emptyset$. $\square$

From now on we consider the cases in which P is not empty. Next we consider the case where one of the sets P_1 and P_2 is empty. Wlog assume P_2 is empty.

Lemma 21. *Theorem* 1 *holds when P_2 is empty.*

Proof. Since $P_2 = \emptyset$, $Conv\,(P_1 \cup P_2) = P_1 = P \cap \{x : \pi^T x \leq \pi_0^1\}$. Hence $\pi^T x \leq \pi_0^1$ is valid for $Conv\,(P_1 \cup P_2)$. Furthermore, it is the only inequality needed to describe $Conv\,(P_1 \cup P_2)$ besides the constraints of P. We just need to show that it can be obtained from a set $B \in \mathcal{B}_r^*$. Consider the linear program $z := \max\{\pi^T x : x \in P\}$. We have $z < \pi_0^2$. By the linear programming duality theorem, there exists $B \in \mathcal{B}_r^*$ that satisfies $\pi = \sum_{i \in B} y_i a_i.$, $z = \sum_{i \in B} y_i b_i.$ and $y_i \geq 0$ for $i \in B$. Any $x \in P(B)$ satisfies $\pi^T x = \sum_{i \in B} y_i a_i^T x \leq \sum_{i \in B} y_i b_i. = z$. Therefore, $P(B) \cap \{x : \pi^T x \geq \pi_0^2\} = \emptyset$ and $Conv\,(P_1(B) \cup P_2(B)) = P_1(B)$ for this particular B. $\square$

Next we consider the case where P is full-dimensional.

Lemma 22. *Theorem 1 holds when P is full-dimensional.*

Proof. Because of Lemma 21, we can assume P_1 and P_2 are not empty. Choose $\epsilon > 0$ such that $\pi_0^1 + \epsilon < \pi_0^2 - \epsilon$. Let $P_1^\epsilon := P \cap \{x \in \mathbb{R}^n : \pi^T x \leq \pi_0^1 + \epsilon\}$ and $P_2^\epsilon := P \cap \{x \in \mathbb{R}^n : \pi^T x \geq \pi_0^2 - \epsilon\}$. Similarly, given a set $B \in \mathcal{B}_r^*$, let $P_1^\epsilon(B) := P(B) \cap \{x \in \mathbb{R}^n : \pi^T x \leq \pi_0^1 + \epsilon\}$ and $P_2^\epsilon(B) := P(B) \cap \{x \in \mathbb{R}^n : \pi^T x \geq \pi_0^2 - \epsilon\}$.

Now we prove P_1^ϵ and P_2^ϵ are full-dimensional by showing that P_1^ϵ and P_2^ϵ have interior points. Let $y_1 \in P_1 \neq \emptyset$, let y_2 be an interior point of P and let $y(\lambda) := \lambda y^1 + (1 - \lambda) y^2$, where $\lambda \in [0, 1]$. Because y_2 is an interior point of P, $y(\lambda)$ is in the interior of P for all $\lambda \in [0, 1[$. Choose $\bar{\lambda} \in]0, 1[$ sufficiently close to one such that $\pi^T y(\bar{\lambda}) < \pi_0^1 + \epsilon$. We have $y(\bar{\lambda})$ is an interior point of P_1^ϵ. This shows P_1^ϵ is a full-dimensional polyhedron. A similar proof applies to P_2^ϵ.

Since P_1^ϵ and P_2^ϵ are full-dimensional closed sets in $\mathbb{R}^n$, we have.

$$\bigcap_{B \in \mathcal{B}_r^*} Conv\,(P_1(B) \cup P_2(B))$$

$$= \bigcap_{B \in \mathcal{B}_r^*} lim_{\epsilon \to 0^+} Conv\,(P_1^\epsilon(B) \cup P_2^\epsilon(B))$$

$$= lim_{\epsilon \to 0^+} \bigcap_{B \in \mathcal{B}_r^*} Conv\,(P_1^\epsilon(B) \cup P_2^\epsilon(B))$$

$$= lim_{\epsilon \to 0^+} Conv\,(P_1^\epsilon \cup P_2^\epsilon)$$

$$= Conv\,(P_1 \cup P_2). \qquad\qquad\qquad\qquad \square$$

In the remainder of this section assume $P \neq \emptyset$ and P is not full-dimensional. Let $d_P := \dim(Aff(P)) < n$ denote the dimension of P and let $M^= := \{i \in M : a_{i.}^T x = b_i, \forall x \in P\}$ denote the constraints of P satisfied with equality by all points in P. Since $\dim(P) = d_P$, there exists $B^= \in \mathcal{B}_{n-d_P}^*$ such that $Aff(P) = \{x \in \mathbb{R}^n : a_{i.}^T x = b_i, i \in B^=\}$. Let L denote the linear space parallel to $Aff(P)$, i.e., $L := Aff(P) - x^P$, where $x^P \in P$.

The idea of the proof is to add a linear space $\bar{L}$ to P in such a way that $P' := P + \bar{L}$ is full-dimensional. The space $\bar{L}$ is described in the following lemma.

Lemma 23. *Let* $L^\pi := \{x \in \mathbb{R}^n : \pi^T x = 0\}$. *We may write*

$$L^\pi = (L^\pi \cap L) + \bar{L},$$

where $\dim(L^\pi \cap L) = d_P - 1$, $\dim(\bar{L}) = n - d_P$, $L \cap \bar{L} = \{0_n\}$ *and* $L + \bar{L} = \mathbb{R}^n$.

Proof. Clearly $\dim(L^\pi \cap L) = d_P - 1$ or $\dim(L^\pi \cap L) = d_P$. If $\dim(L^\pi \cap L) = d_P$ we have $\pi \in Span(\{a_{i.}\}_{i \in B^=})$. This implies the function $\pi^T x$ is constant on P, which means that one (or both) of the sets P_1 and P_2 is empty. Since we have assumed this not to be the case, we have $\dim(L^\pi \cap L) = d_P - 1$.

Let $\{b^i\}_{i=1}^{n-1}$ be a basis for L^π chosen such that $\{b^i\}_{i=1}^{d_P-1} \subseteq L^\pi \cap L$. Define $\bar{L} := Span(\{b^i\}_{i=d_P}^{n-1})$. By construction $L^\pi = (L^\pi \cap L) + \bar{L}, \dim(\bar{L}) = n - d_P$ and $L \cap \bar{L} = \{0_n\}$.

To prove $L + \bar{L} = \mathbb{R}^n$, observe that $L^\pi = (L^\pi \cap L) + \bar{L} \subseteq L + \bar{L}$. We cannot have $x \in L^\pi$ for all $x \in L$, since that would imply that $\pi^T x$ is constant on P. $\qquad \square$

Observe that the fact $L + \bar{L} = \mathbb{R}^n$ implies that $P' := P + \bar{L}$ is full-dimensional (we have $Aff(P') = Aff(P) + \bar{L} = x^P + L + \bar{L} = \mathbb{R}^n$). Also observe that the set $\bar{L}$ has been constructed as a subspace of L^π and therefore that $\pi^T \bar{l} = 0$ for all $\bar{l} \in \bar{L}$. We now relate the sets P and P' and the sets $Conv\,(P \cap F_{(\pi,\pi_0^1,\pi_0^2)})$ and $Conv\,(P' \cap F_{(\pi,\pi_0^1,\pi_0^2)})$.

Lemma 24. (i) $P' \cap Aff(P) = P$.
 (ii) $Conv\,(P' \cap F_{(\pi,\pi_0^1,\pi_0^2)}) = Conv\,(P \cap F_{(\pi,\pi_0^1,\pi_0^2)}) + \bar{L}$.

Proof. (ii) follows easily from the facts that $P' = P + \bar{L}$ and $\pi^T \bar{l} = 0$ for all $\bar{l} \in \bar{L}$, so we only prove (i). Since $P \subseteq P'$ and $P \subseteq Aff(P)$, we have $P \subseteq P' \cap Aff(P)$. Let

$\bar{y} \in P' \cap \textit{Aff}(P)$. Since $\bar{y} \in P'$, we have $\bar{y} = z + \bar{l}$, where $z \in P$ and $\bar{l} \in \bar{L}$. Also, since $\bar{y} \in \textit{Aff}(P)$ we have $\bar{y} = x^P + l$, where $l \in L$. This implies $(\bar{y} - x^P) = (z - x^P) + \bar{l}$. Since $(\bar{y} - x^P), (z - x^P) \in L$, we have $(\bar{y} - x^P) - (z - x^P) \in L \cap \bar{L}$. Hence $(\bar{y} - x^P) - (z - x^P) = 0_n$, or, $\bar{y} = z \in P$, which proves that $P' \cap \textit{Aff}(P) \subseteq P$.   □

We now give a description of P' with constraints. The description is based on the constraints of P, and an affine mapping $\tilde{A}_f : \mathbb{R}^n \mapsto \mathbb{R}^n$ with the following properties.

Lemma 25. *There exist $\tilde{D} \in \mathbb{R}^{n \times n}$ and $\tilde{d} \in \mathbb{R}^n$ such that the affine mapping $\tilde{A}_f(x) := \tilde{D}x + \tilde{d}$ and the corresponding linear mapping $\tilde{L}_f(x) := \tilde{D}x$ satisfy*

(i) $\tilde{A}_f(x) = 0_n$ for all $x \in \textit{Aff}(P)$, i.e., $\tilde{A}_f$ maps $\textit{Aff}(P)$ to the set $\{0_n\}$.
(ii) $\tilde{L}_f(x) = 0_n$ for all $x \in L$, i.e., $\tilde{L}_f$ maps L to the set $\{0_n\}$.
(iii) For $x, \bar{l} \in \mathbb{R}^n$, $\bar{l} = \tilde{A}_f(x) \iff \bar{l} \in \bar{L}$ and $(x - \bar{l}) \in \textit{Aff}(P)$.
(iv) For $x, \bar{l} \in \mathbb{R}^n$, $\bar{l} = \tilde{L}_f(x) \iff \bar{l} \in \bar{L}$ and $(x - \bar{l}) \in L$.

Proof. Since $\bar{L}$ is a vector space, there exist linearly independent vectors $\{\bar{b}^k\}_{k \in I}$, where $I := \{1, 2, \ldots, d_P\}$, such that $\bar{L} = \{\bar{l} \in \mathbb{R}^n : (\bar{b}^k)^T \bar{l} = 0, \forall k \in I\}$. Notice that the vectors $\{\bar{b}^k\}_{k \in I}$ define a basis for $\bar{L}^\perp$, and that the vectors $\{a_{i.}\}_{i \in B^=}$ define a basis for $L^\perp$. Since $L + \bar{L} = \mathbb{R}^n$ and $L \cap \bar{L} = \{0_n\}$, it follows that $L^\perp + \bar{L}^\perp = \mathbb{R}^n$, so that the vectors $\{\bar{b}^k\}_{k \in I} \cup \{a_{i.}\}_{i \in B^=}$ are linearly independent. Now we have:

$$\bar{l} \in \bar{L} \text{ and } (x - \bar{l}) \in \textit{Aff}(P) \iff \tag{25}$$

$$\bar{l} \in \bar{L} \text{ and } a_{i.}^T(x - \bar{l}) = b_i, \forall i \in B^= \iff \tag{26}$$

$$a_{i.}^T(x - \bar{l}) = b_i, \forall i \in B^= \text{ and } (\bar{b}^k)^T \bar{l} = 0, \forall k \in I \iff \tag{27}$$

$$\bar{l} = \tilde{D}x + \tilde{d}, \tag{28}$$

where $\tilde{D}$ is the inverse matrix of $[a_{i.} | \bar{b}^k]^T$. Observe that (28) expresses $\bar{l}$ as a linear function of x. Next it is an easy exercise to verify the correctness of (i)-(iv).   □

We now have the following description of P' with constraints

Lemma 26. *Let $\tilde{A}_f$ be as in Lemma 25. Then*

$$P' = \{x \in \mathbb{R}^n : a_{i.}^T x \le b_i + a_{i.}^T \tilde{A}_f(x), \forall i \in M \setminus M^=\}.$$

Proof. We have $x \in P' \iff \exists \bar{l} \in \bar{L}$ such that $a_{i.}^T(x - \bar{l}) \le b_i, \forall i \in M \setminus M^=$, and $a_{i.}^T(x - \bar{l}) = b_i, \forall i \in B^= \iff$ (by Lemma 25(iii)) $\exists \bar{l} \in \bar{L}$ such that $a_{i.}^T(x - \bar{l}) \le b_i, \forall i \in M \setminus M^=$, and $\bar{l} = \tilde{A}_f(x) \iff a_{i.}^T x \le b_i + a_{i.}^T \tilde{A}_f(x), \forall i \in M \setminus M^=$.   □

Lemma 26 shows that P' has one constraint for each element of $M \setminus M^=$. Let r' denote the rank of the corresponding coefficient matrix, *i.e.*, r' is the size of a maximal subset B' of $M \setminus M^=$ such that the vectors $\{a_{i.}^T(I_n - \tilde{D})\}_{i \in B'}$ are linearly independent (I_n denotes the n-by-n identity matrix). Furthermore let $\mathcal{B}'^*_{r'}$ denote the set of all such subsets.

It is easy to see that, if the vectors $\{a_{i.}^T(I_n - \tilde{D})\}_{i \in B'}$ are linearly independent, then so are the vectors $\{a_{i.}\}_{i \in B'}$. Specifically, if the vectors $\{a_{i.}\}_{i \in B'}$ are linearly dependent, then one of them can be expressed as a linear combination of the rest. However, that would

mean that the corresponding vector in $\{a_{i.}^T(I_n - \tilde{D})\}_{i \in B'}$ is a linear combination of the remaining vectors in $\{a_{i.}^T(I_n - \tilde{D})\}_{i \in B'}$, which contradicts their linear independence.

We know that P is not full-dimensional, but that P' is full-dimensional, where $P' := P + \bar{L}$ and $\bar{L}$ is the linear space described by Lemma 23. Similarly, $P(B') \cap Aff(P)$ is not full-dimensional for any $B' \in \mathcal{B}'^*_{r'}$, but we can also add $\bar{L}$ to it to get a full-dimensional conic polyhedron. Given a set $B' \in \mathcal{B}'^*_{r'}$, define $P'(B') := \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i + a_{i.}^T \tilde{A}_f(x), \forall i \in B'\}$. $P'(B')$ denotes the conic polyhedron for P' associated with B'. We now relate the set $P'(B')$ to $P(B')$ for $B' \in \mathcal{B}'^*_{r'}$.

Lemma 27. *Let $B' \in \mathcal{B}'^*_{r'}$. Then*

$$P'(B') = P(B') \cap Aff(P) + \bar{L}$$

Proof. Using Lemma 25(iii) we get $x \in P'(B') \iff a_{i.}^T(x - \bar{l}) \leq b_i, \forall i \in B'$, $a_{i.}^T(x - \bar{l}) = b_i, \forall i \in B^=, \bar{l} \in \bar{L}$ and $\bar{l} = \tilde{A}_f(x) \iff x = y + \bar{l}$, where $y \in P(B') \cap Aff(P), \bar{l} \in \bar{L}$ and $\bar{l} = \tilde{A}_f(x) \iff x \in P(B') \cap Aff(P) + \bar{L}$. □

Next we show that it is possible to find a representation of the conic polyhedron $P'(B')$ such that the extreme point $\bar{x}$ is in $Aff(P)$, and the extreme rays $\{r^k\}_{k \in B'}$ are all in L.

Lemma 28. *Let $B' \in \mathcal{B}'^*_{r'}$. There exist vectors $\bar{x} \in Aff(P)$ and $\{r^k\}_{k \in B'} \subseteq L$ such that*

$$P'(B') = \bar{x} + L'(B') + Cone(\{r^k\}_{k \in B'}),$$

where $L'(B') := \{x \in \mathbb{R}^n : a_{i.}^T x = a_{i.}^T \tilde{L}_f(x), \forall i \in B'\}$, i.e., $L'(B') := \{x \in \mathbb{R}^n : a_{i.}^T(x - \tilde{D}x) = 0, \forall i \in B'\}$. The vector $\bar{x}$ satisfies $a_{i.}^T \bar{x} = b_i$ for all $i \in B' \cup B^=$. Furthermore given $k \in B'$, the vector r^k satisfies

(i) $a_{i.}^T r^k = 0, \forall i \in (B' \cup B^=) \setminus \{k\}$ *and*

(ii) $a_{k.}^T r^k = -1$.

Proof. Since $P'(B')$ is a conic polyhedron we may write (see Sect. 2.1)

$$P'(B') = \bar{y} + L'(B') + Cone(\{s^k\}_{k \in B'})$$

where the vectors $\bar{y}$ and $\{s^k\}_{k \in B'}$ are as follows. The vector $\bar{y}$ satisfies $a_{i.}^T \bar{y} = b_i + a_{i.}^T \tilde{A}_f(\bar{y})$ for all $i \in B'$. Given $k \in B'$, the vector s^k satisfies $a_{i.}^T s^k = a_{i.}^T \tilde{L}_f(s^k), \forall i \in B' \setminus \{k\}$ and $a_{k.}^T s^k = a_{k.}^T \tilde{L}_f(s^k) - 1$.

Define $\bar{l} := \tilde{A}_f(\bar{y})$, and define the vector $\bar{x} := \bar{y} - \bar{l}$. From Lemma 25(iii) we have $\bar{x} \in Aff(P)$ and $\bar{l} \in \bar{L}$. Since $\bar{x} \in Aff(P)$ Lemma 25(i) implies $\tilde{A}_f(\bar{x}) = 0_n$. Furthermore, for $i \in B'$, we have $a_{i.}^T \bar{x} = a_{i.}^T \bar{y} - a_{i.}^T \bar{l} = b_i + a_{i.}^T \tilde{A}_f(\bar{y}) - a_{i.}^T \bar{l} = b_i$. Also, since $\bar{x} \in Aff(P)$, we have $a_{i.}^T \bar{x} = b_i$ for all $i \in B^=$. Finally, since $\tilde{A}_f(\bar{x}) = 0_n$, we have $a_{i.}^T \bar{x} = b_i + a_{i.}^T \tilde{A}_f(\bar{x}), \forall i \in B'$. Hence we can use $\bar{x}$ as an extreme point for $P'(B')$, $\bar{x} \in Aff(P)$ and $a_{i.}^T \bar{x} = b_i$ for all $i \in B' \cup B^=$.

We now define the vectors r^k for $k \in B'$. Let $k \in B'$ be arbitrary, and consider the vector s^k defined above. Let $\bar{l} := \tilde{L}_f(s^k)$ and define r^k as $r^k := s^k - \bar{l}$. From Lemma 25(iv) we have $r^k \in L$. Moreover, since $r^k \in L$, Lemma 25(ii) implies $\tilde{L}_f(r^k) = 0_n$.

Also, for $i \in B' \setminus \{k\}$, we have $a_{i.}^T r^k = a_{i.}^T s^k - a_{i.}^T \bar{l} = a_{i.}^T (\tilde{L}_f(s^k) - \bar{l}) = 0$, and $a_{k.}^T r^k = a_{k.}^T s^k - a_{k.}^T \bar{l} = a_{k.}^T (\tilde{L}_f(s^k) - \bar{l}) - 1 = -1$. Since $r^k \in L$ we have $a_{i.}^T r^k = 0, \forall i \in B^=$. Finally, since $\tilde{L}_f(r^k) = 0_n$, $a_{i.}^T r^k = \tilde{L}_f(r^k), \forall i \in B' \setminus \{k\}$ and $a_{k.}^T r^k = a_{k.}^T \tilde{L}_f(r^k) - 1$. Hence we can use r^k as an extreme ray for $P'(B')$, $r^k \in L$ and r^k satisfies (i) and (ii). $\square$

The choice of extreme point and extreme rays in Lemma 28 implies the following facts about the set $P(B' \cup B)$ and the conic polyhedron $P(B')$.

Lemma 29. *Let $B' \in \mathcal{B}'^*_{r'}$ and let the vectors $\bar{x} \in Aff(P)$ and $\{r^k\}_{k \in B'} \subseteq L$ be as in Lemma 28.*

(i) *The vectors $\{a_{i.}\}_{i \in B' \cup B^=}$ are linearly independent, i.e., $P(B' \cup B)$ is a conic polyhedron.*

(ii) *$P(B') = \bar{x} + L(B') + Cone(\{r^k\}_{k \in B'})$, where $L(B')$ is given by $L(B') = \{x \in \mathbb{R}^n : a_{i.}^T x = 0, \forall i \in B'\}$.*

(iii) *$P(B') \cap Aff(P) = \bar{x} + L(B' \cup B^=) + Cone(\{r^k\}_{k \in B'})$, where $L(B' \cup B^=) = L \cap L(B')$.*

Proof. (ii) and (iii) follow from the properties of the vectors $\bar{x}$ and $\{r^k\}_{k \in B'}$, so we only prove (i). Let $k \in B' \cup B^=$, and assume $a_{k.} = \sum_{i \in (B' \cup B^=) \setminus \{k\}} \mu_i a_{i.}$, where $\mu_i \in \mathbb{R}$ for $i \in (B' \cup B^=) \setminus \{k\}$. From multiplying this equality by $\bar{x}$ and using the fact that $a_{i.}^T \bar{x} = b_i$ for all $i \in B' \cup B^=$, we have $b_k = \sum_{i \in (B' \cup B^=) \setminus \{k\}} \mu_i b_i$. For any $\bar{i} \in B' \setminus \{k\}$ we have $0 = a_{k.}^T r^{\bar{i}} = \sum_{i \in (B' \cup B^=) \setminus \{k\}} \mu_i a_{i.}^T r^{\bar{i}} = -\mu_{\bar{i}}$. Hence $\mu_i = 0$ for all $i \in B' \setminus \{k\}$. Assume by contradiction that $k \in B'$. Then $a_{k.} = \sum_{i \in B^=} \mu_i a_{i.}$ and $b_k = \sum_{i \in B^=} \mu_i b_i$. This implies $k \in M^=$, which contradicts $B' \subseteq M \setminus M^=$. Therefore we can assume $k \in B^=$. The equality $a_{k.} = \sum_{i \in B^= \setminus \{k\}} \mu_i a_{i.}$ contradicts the fact that $B^= \in \mathcal{B}^*_{n-d_P}$ ($B^=$ is defined before Lemma 23). $\square$

Lemma 29(ii)-(iii) gives that both sets $P(B')$ and $P(B') \cap Aff(P)$ can be generated from the vectors $\bar{x}$ and $\{r^k\}_{k \in B'}$ of Lemma 28. The following lemma describes the disjunctive hull associated with a set $B' \in \mathcal{B}'^*_{r'}$ and $D(\pi, \pi_0^1, \pi_0^2)$.

Lemma 30. *Let $B' \in \mathcal{B}'^*_{r'}$.*

(i) *$\pi^T x \in]\pi_0^1, \pi_0^2[, \forall x \in \bar{x} + L'(B') \iff \pi^T x \in]\pi_0^1, \pi_0^2[, \forall x \in \bar{x} + L(B' \cup B^=)$, i.e., an intersection cut can be derived from $P(B' \cup B^=)$ if and only if an intersection cut can be derived from $P'(B')$.*

(ii) *$Conv(P'(B') \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) = Conv(P(B' \cup B^=) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) \cap Aff(P) + \bar{L}$.*

Proof. (i) We first prove that $\bar{x} + L(B' \cup B^=) \subseteq \bar{x} + L'(B')$

$$
\begin{aligned}
y \in \bar{x} + L(B' \cup B^=) &\Rightarrow &\text{(Lemma 28)}\\
a_{i.}^T y = b_i, \forall i \in B^= \cup B' &\Rightarrow &\\
y \in Aff(P) \text{ and } a_{i.}^T y = b_i, \forall i \in B' &\Rightarrow &\text{(Lemma 25(i))}\\
\tilde{A}_f(y) = 0_n \text{ and } a_{i.}^T y = b_i + a_{i.}^T \tilde{A}_f(y), \forall i \in B' &\Rightarrow &\\
y \in \bar{x} + L'(B') &&
\end{aligned}
$$

This shows $\pi^T x \in]\pi_0^1, \pi_0^2[, \forall x \in \bar{x} + L'(B') \Rightarrow \pi^T x \in]\pi_0^1, \pi_0^2[, \forall x \in \bar{x} + L(B' \cup B^=)$.

For the other direction, suppose $y \in \bar{x} + L'(B')$ and $\pi^T y \notin]\pi_0^1, \pi_0^2[$. We will show that this implies there exists $z \in \bar{x} + L(B' \cup B^=)$ satisfying $\pi^T z \notin]\pi_0^1, \pi_0^2[$. Define $\bar{l} := \tilde{A}_f(y)$. From Lemma 25(iii), we have $\bar{l} \in \bar{L}$ and $z := y - \bar{l} \in \mathit{Aff}(P)$. This implies $a_{i.}^T z = b_i, \forall i \in B^= \cup B'$, i.e., $z \in \bar{x} + L(B' \cup B^=)$. Since $\pi^T \bar{l} = 0$ (Lemma 23), we have $\pi^T z = \pi^T(y - \bar{l}) = \pi^T y \notin]\pi_0^1, \pi_0^2[$. This finishes the proof of (i).

(ii) We may assume $\pi^T y \in]\pi_0^1, \pi_0^2[$ for all $y \in \bar{x} + L'(B')$, since otherwise statement (ii) reduces to $P'(B') = P(B') \cap \mathit{Aff}(P) + \bar{L}$ (Lemma 27).

From Lemma 28 and 29 it follows that the intersection cut associated with $P(B' \cup B^=)$ and $D(\pi, \pi_0^1, \pi_0^2)$ is given by

$$\sum_{k \in B'} (b_k - a_{k.}^T x)/\alpha_k \geq 1 \tag{29}$$

for points $x \in \mathit{Aff}(P)$, where

$$\alpha_k := \begin{cases} -\epsilon^1/(\pi^T r^k) & \text{if } \pi^T r^k < 0, \\ \epsilon^2/(\pi^T r^k) & \text{if } \pi^T r^k > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{30}$$

for $k \in B'$, $\epsilon^1 := \pi^T \bar{x} - \pi_0^1$ and $\epsilon^2 := \pi_0^2 - \pi^T \bar{x}$ (see the definition in (2)). Similarly the intersection cut associated with $P'(B')$ and $D(\pi, \pi_0^1, \pi_0^2)$ is given by

$$\sum_{k \in B'} (b_k + a_{k.}^T \tilde{A}_f(x) - a_{k.}^T x)/\alpha_k \geq 1. \tag{31}$$

Now suppose $x \in \mathit{Conv}(P'(B') \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$. Since $x \in P'(B')$, Lemma 27 shows that we may write $x = y + \bar{l}$, where $y \in P(B') \cap \mathit{Aff}(P), \bar{l} := \tilde{A}_f(x)$ and $\bar{l} \in \bar{L}$. From the fact that x satisfies the intersection cut (31) associated with $P'(B')$, we have $\sum_{k \in B'} (b_k + a_{k.}^T \tilde{A}_f(x) - a_{k.}^T x)/\alpha_k \geq 1$. Because $\bar{l} = \tilde{A}_f(x)$, x satisfies $\sum_{k \in B'} (b_k + a_{k.}^T \bar{l} - a_{k.}^T x)/\alpha_k \geq 1$, it follows that $\sum_{k \in B'} (b_k - a_{k.}^T y)/\alpha_k \geq 1$, i.e., y satisfies the intersection cut (29) associated with $P(B' \cap B^=)$.

Finally suppose $x \in \mathit{Conv}(P(B' \cup B^=) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) \cap \mathit{Aff}(P) + \bar{L}$. Write $x = y + \bar{l}$, where $y \in \mathit{Conv}(P(B' \cup B^=) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) \cap \mathit{Aff}(P)$ and $\bar{l} \in \bar{L}$. Since $y = x - \bar{l} \in \mathit{Conv}(P(B' \cup B^=) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) \cap \mathit{Aff}(P)$ and $\bar{l} \in \bar{L}$, we have $\bar{l} = \tilde{A}_f(x)$. From the fact that y satisfies the intersection cut (29) associated with $P(B' \cup B^=)$ and $D(\pi, \pi_0^1, \pi_0^2)$ it follows that x satisfies the intersection cut (31) associated with $P'(B')$ and $D(\pi, \pi_0^1, \pi_0^2)$. $\qquad \square$

We are now able to finish the proof of Theorem 1.

Lemma 31. *Theorem 1 holds when P, P_1 and P_2 are non-empty and not full-dimensional.*

Proof. Let $B' \in \mathcal{B}_{r'}'^*$ be arbitrary. We first prove that $\mathit{Conv}(P(\bar{B}) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}) \subseteq \mathit{Conv}(P(B' \cup B^=) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$. We distinguish two cases:

(a) If $\pi^T y \in]\pi_0^1, \pi_0^2[$ for all $y \in \bar{x} + L(B' \cup B^=)$:

An intersection cut $\bar{\delta}^T x \geq \bar{\delta}_0$ can be derived from $B' \cup B^=$ and $D(\pi, \pi_0^1, \pi_0^2)$. The inequality $\bar{\delta}^T x \geq \bar{\delta}_0$ is also an intersection cut for $\bar{B}$ and $D(\pi, \pi_0^1, \pi_0^2)$ for any $\bar{B} \in \mathcal{B}_r^*$ satisfying $\bar{B} \supseteq B' \cup B^=$ (Lemma 10). Furthermore we have $Conv\,(P(\bar{B}) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \subseteq Conv\,(P(B' \cup B^=) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$.

(b) If $\pi^T y \in]\pi_0^1, \pi_0^2[$ for some $y \in \bar{x} + L(B' \cup B^=)$:

$Conv\,(P(B' \cup B^=) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = P(B' \cup B^=)$. Also $P(B' \cup B^=) \supseteq P(\bar{B}) \supseteq Conv\,(P(\bar{B}) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$ for all $\bar{B} \in \mathcal{B}_r^*$ satisfying $\bar{B} \supseteq B' \cup B^=$.

Therefore for any $B' \in \mathcal{B}_{r'}'^*$ there exists $\bar{B} \in \mathcal{B}_r^*$ s.t. $Conv\,(P(\bar{B}) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \subseteq Conv\,(P(B' \cup B^=) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. Hence we can choose $\bar{\mathcal{B}}_r^* \subseteq \mathcal{B}_r^*$ such that $\cap_{B' \in \mathcal{B}_{r'}'^*} Conv\,(P(B' \cup B^=) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \supseteq \cap_{\bar{B} \in \bar{\mathcal{B}}_r^*} Conv\,(P(\bar{B}) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$.

Since $\bar{\mathcal{B}}_r^*$ is a subset of $\mathcal{B}_r^*$, we have the inclusion

$$\cap_{B' \in \mathcal{B}_{r'}'^*} Conv\,(P(B' \cup B^=) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \supseteq \cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}).$$

Using the fact that P' is full-dimensional, Lemma 22 gives the identity $Conv\,(P' \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = \cap_{B' \in \mathcal{B}_{r'}'^*} Conv\,(P'(B') \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. Also Lemma 24(ii) states $Conv\,(P' \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) = Conv\,(P \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) + \bar{L}$. This implies that $Conv\,(P \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) + \bar{L} = \cap_{B' \in \mathcal{B}_{r'}'^*} Conv\,(P'(B') \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. Using Lemma 30(ii) gives that $\bar{L} + Aff\,(P) \cap (\cap_{B' \in \mathcal{B}_{r'}'^*} Conv\,(P(B' \cup B^=) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})) = \bar{L} + Conv\,(P \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. Since $\cap_{B' \in \mathcal{B}_{r'}'^*} Conv\,(P(B' \cup B^=) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \supseteq \cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$ and $\cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \subseteq P \subseteq Aff\,(P)$ we have $\bar{L} + Conv\,(P \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \supseteq \bar{L} + \cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$. This implies $Conv\,(P \cap F_{D(\pi,\pi_0^1,\pi_0^2)}) \supseteq \cap_{B \in \mathcal{B}_r^*} Conv\,(P(B) \cap F_{D(\pi,\pi_0^1,\pi_0^2)})$ (by intersecting with $Aff\,(P)$ on both sides). Lemma 11 shows the other direction. $\square$

5. Disjunctive hulls derived from polyhedra and two-term disjunctions

We now consider two-term disjunctions D of the form $D^1 x \leq d^1 \vee D^2 x \leq d^2$, where $D^1 \in \mathbb{R}^{m_1 \times n}$, $D^2 \in \mathbb{R}^{m_2 \times n}$, $d^1 \in \mathbb{R}^{m_1}$ and $d^2 \in \mathbb{R}^{m_2}$. The set of points $x \in \mathbb{R}^n$ that satisfy D is denoted F_D. In this section, we will prove a decomposition result for the set $Conv\,(P \cap F_D)$ (Theorem 3). The result states that $Conv\,(P \cap F_D)$ can be written as the intersection of sets $Conv\,(P(T) \cap F_D)$ over sets $T \in C_1^*$, where C_1^* is a family of $(r+1)$-subsets of the constraints and r is the rank of the constraint matrix A. Furthermore, when $r = n$, we show that it suffices to consider r-subsets of the constraints. Finally, we demonstrate how this result can be strengthened to obtain Theorem 1 for the split disjunctions $D(\pi, \pi_0^1, \pi_0^2)$.

We start by proving the following decomposition result on two-term disjunctions.

Theorem 4. *Let $S \subseteq M$. If S satisfies $|S| \geq r(S) + 2$, then*

$$Conv\,(P(S) \cap F_D) = \cap_{i \in S} Conv\,(P(S \setminus \{i\}) \cap F_D). \tag{32}$$

Furthermore, (32) remains true if $r(S) = n$ and $|S| = n + 1$. $\square$

The following inclusion is obvious since $Conv\,(P(S) \cap F_D) \subseteq Conv\,(P(S \setminus \{i\}) \cap F_D)$.

Lemma 32. *Let $S \subseteq M$ be non-empty.*

$$Conv\,(P(S) \cap F_D) \subseteq \cap_{i \in S}\, Conv\,(P(S \setminus \{i\}) \cap F_D). \tag{33}$$

$\square$

However, the proof of the other inclusion involves the idea introduced by Balas [3] of lifting the set $Conv\,(P(S) \cap F_D)$ into a higher dimensional space. Specifically, $Conv\,(P(S) \cap F_D)$ can be described as the projection of the set, described by the constraints $(34) - (40)$, onto the space of x-variables (see [3]).

$$x = x^1 + x^2, \tag{34}$$
$$a_{i.}^T x^1 \le b_i \lambda^1, \qquad \forall i \in S, \tag{35}$$
$$a_{i.}^T x^2 \le b_i \lambda^2, \qquad \forall i \in S, \tag{36}$$
$$\lambda^1 + \lambda^2 = 1, \tag{37}$$
$$D^1 x^1 \le d^1 \lambda^1, \tag{38}$$
$$D^2 x^2 \le d^2 \lambda^2, \tag{39}$$
$$\lambda^1, \lambda^2 \ge 0. \tag{40}$$

The description (34)–(40) can be projected into (x, x^1, λ^1)-space by using constraints (34) and (37). By doing this, we obtain the following characterization of $Conv\,(P(S) \cap F_D)$.

$$-\lambda^1 b_i + a_{i.}^T x^1 \le 0, \quad \forall i \in S,$$
$$\lambda^1 b_i - a_{i.}^T x^1 \le b_i - a_{i.}^T x, \quad \forall i \in S,$$
$$\lambda^1 \le 1,$$
$$-\lambda^1 d^1 + D^1 x^1 \le 0_{m_1},$$
$$\lambda^1 d^2 - D^2 x^1 \le d^2 - D^2 x,$$
$$-\lambda^1 \le 0.$$

The problem of deciding whether a given vector $x \in \mathbb{R}^n$ belongs to the set $Conv\,(P(S) \cap F_D)$ can be decided by solving the following Phase I linear program called $P_{LP}(x, S)$, where the variables are x^1, λ^1 and s.

$$\max\ -s$$
$$-\lambda^1 b_i + a_{i.}^T x^1 \le 0, \qquad\qquad \forall i \in S, \qquad (u_i) \tag{41}$$
$$\lambda^1 b_i - a_{i.}^T x^1 \le b_i - a_{i.}^T x, \qquad \forall i \in S, \qquad (v_i) \tag{42}$$
$$\lambda^1 \le 1, \qquad\qquad\qquad (w_0) \tag{43}$$
$$-\lambda^1 d^1 + D^1 x^1 - s1_{m_1} \le 0_{m_1}, \qquad\qquad (u^0) \tag{44}$$
$$\lambda^1 d^2 - D^2 x^1 - s1_{m_2} \le d^2 - D^2 x, \qquad\qquad (v^0) \tag{45}$$
$$-\lambda^1 \le 0. \qquad\qquad\qquad (t_0) \tag{46}$$
$$-s \le 0. \qquad\qquad\qquad (t_1) \tag{47}$$

Observe that $P_{LP}(x, S)$ is feasible if and only if $x \in P(S)$. Furthermore $P_{LP}(x, S)$ is always bounded above by zero. Finally note that $x \in Conv\,(P(S) \cap F_D)$ if and only if $P_{LP}(x, S)$ is feasible and bounded, and there exists an optimal solution in which the variable s has the value zero.

Note that, in the remainder of this section, we will make extensive use of the dual $D_{LP}(x, S)$ of $P_{LP}(x, S)$. The name of the dual variable associated with each constraint in $P_{LP}(x, S)$ is given next to the constraints in $(41) - (47)$. To prove the other direction of (32), we use the problem $P_{LP}(x, S)$ and its dual $D_{LP}(x, S)$. Specifically we use these problems on sets $S \subseteq M$ satisfying $|S| \geq 2$ and points $x \in P(S)$. Observe that the condition $x \in \cap_{i \in S} Conv\,(P(S \setminus \{i\}) \cap F_D)$, where $|S| \geq 2$, implies $x \in P(S)$.

Throughout this section $\bar{x}$ denotes some element of $P(S)$. Let $(\bar{x}^1, \bar{\lambda}^1, \bar{s})$ denote an optimal basic feasible solution to $P_{LP}(\bar{x}, S)$, and let $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ denote a corresponding optimal basic feasible solution to $D_{LP}(\bar{x}, S)$. Observe that the solutions to $P_{LP}(\bar{x}, S)$ and $D_{LP}(\bar{x}, S)$ depend on the values of $\bar{x}$ and S, i.e., when we write $\bar{u}_i$, say, we mean $\bar{u}_i(\bar{x}, S)$, where the values of $\bar{x}$ and S should be clear from the context.

We now formulate the dual $D_{LP}(\bar{x}, S)$ of $P_{LP}(\bar{x}, S)$. The formulation uses the following quantities. Given $u^0 \geq 0_{m_1}$ and scalars $u_i \geq 0$ for $i \in S$, define $\alpha^1(S, u, u^0) := \sum_{i \in S} u_i a_{i\cdot} + (D^1)^T u^0$ and $\beta^1(S, u, u^0) := \sum_{i \in S} u_i b_i + (d^1)^T u^0$. Observe that the inequality $(\alpha^1(S, u, u^0))^T x \leq \beta^1(S, u, u^0)$ is valid for the set $\{x \in P(S) : D^1 x \leq d^1\}$. Similarly, given $v^0 \geq 0_{m_2}$ and scalars $v_i \geq 0$ for $i \in S$, defining the quantities $\alpha^2(S, v, v^0) := \sum_{i \in S} v_i a_{i\cdot} + (D^2)^T v^0$ and $\beta^2(S, v, v^0) := \sum_{i \in S} v_i b_i + (d^2)^T v^0$, gives the inequality $(\alpha^2(S, v, v^0))^T x \leq \beta^2(S, v, v^0)$, which is valid for $\{x \in P(S) : D^2 x \leq d^2\}$. With these quantities the dual $D_{LP}(\bar{x}, S)$ of $P_{LP}(\bar{x}, S)$ can be formulated as follows.

$$\min \quad \beta^2(S, v, v^0) - (\alpha^2(S, v, v^0))^T \bar{x} + w_0$$

$$\text{s.t.} \quad \alpha^1(S, u, u^0) - \alpha^2(S, v, v^0) = 0, \qquad\qquad (x^1) \quad (48)$$

$$\beta^2(S, v, v^0) - \beta^1(S, u, u^0) + w_0 - t_0 = 0, \qquad (\lambda^1) \quad (49)$$

$$1_{m_1}^T u^0 + 1_{m_2}^T v^0 + t_1 = 1, \qquad\qquad (s) \quad (50)$$

$$u^0 \geq 0_{m_1}, \qquad\qquad (51)$$

$$v^0 \geq 0_{m_2}, \qquad\qquad (52)$$

$$w_0, t_0, t_1 \geq 0, \qquad\qquad (53)$$

$$u_i, v_i \geq 0, \qquad \forall i \in S. \qquad\qquad (54)$$

By our previous observation, $P_{LP}(x, S)$ is feasible if and only if $x \in P(S)$. We now characterize the optimal solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}(\bar{x}, S)$ for sets $S \subseteq M$ and points $\bar{x} \in P(S)$. First we consider the variables u^0 and v^0.

Lemma 33. *Let $S \subseteq M$ be non-empty. Suppose $\bar{x} \in P(S) \setminus Conv\,(P(S) \cap F_D)$. Then $\bar{u}^0 \neq 0_{m_1}$ and $\bar{v}^0 \neq 0_{m_2}$.*

Proof. Let S and $\bar{x}$ be as stated, and suppose firstly that $\bar{v}^0 = 0_{m_2}$. By the definition of $\alpha^2(S, v, v^0)$ and $\beta^2(S, v, v^0)$, the inequality $(\alpha^2(S, \bar{v}, 0_{m_2}))^T x \leq \beta^2(S, \bar{v}, 0_{m_2})$ is valid for $P(S)$. However the optimal objective value to $D_{LP}(\bar{x}, S)$ is negative (since $\bar{x} \notin Conv\,(P(S) \cap F_D)$), and $\bar{w}_0 \geq 0$, so we must have $\beta^2(S, \bar{v}, 0_{m_2}) - (\alpha^2(S, \bar{v}, 0_{m_2}))^T \bar{x} < 0$, a contradiction.

Now suppose $\bar{u}_0 = 0_{m_1}$. The inequality $(\alpha^1(S, \bar{u}, 0_{m_1}))^T x \leq \beta^1(S, \bar{u}, 0_{m_1})$ is valid for $P(S)$, but $\beta^1(S, \bar{u}, 0_{m_1}) - (\alpha^1(S, \bar{u}, 0_{m_1}))^T \bar{x} \leq \beta^2(S, \bar{v}, \bar{v}^0) + w_0 - (\alpha^2(S, \bar{v}, \bar{v}^0))^T \bar{x} < 0$, (from (48) and (49) above). This contradicts $\bar{x} \in P(S)$. □

Observe that Lemma 33 implies that at least two of the variables in (u^0, v^0) are basic when $\bar{x} \in P(S) \setminus Conv\,(P(S) \cap F_D)$.

Define $B_u := \{i \in S : u_i \text{ basic}\}$ and $B_v := \{i \in S : v_i \text{ basic}\}$ to be the set of basic u's and v's respectively. The next lemma describes properties of the variables u_i and v_i for $i \in S$ when $\bar{x}$ belongs to $\cap_{i \in T} Conv\,(P(S \setminus \{i\}) \cap F_D)$, but *not* to $Conv\,(P(S) \cap F_D)$, where T is some subset of S of size at least two.

Lemma 34. *Suppose $\bar{x} \in \cap_{i \in T} Conv\,(P(S \setminus \{i\}) \cap F_D)$ and $\bar{x} \notin Conv\,(P(S) \cap F_D)$, where $T \subseteq S \subseteq M$ and $|T| \geq 2$. Let $B_u(T) := B_u \cap T$ and $B_v(T) := B_v \cap T$, i.e., $B_u(T)$ and $B_v(T)$ denote the basic u's and v's in the set T respectively. Then*

(i) For each $i \in T$, either u_i or v_i has a positive value, i.e., $\bar{u}_i > 0$ or $\bar{v}_i > 0$ for all $i \in T$.

(ii) $B_u(T) \cap B_v(T) = \emptyset$ and $B_u(T) \cup B_v(T) = T$, i.e., exactly one of the variables u_i and v_i is basic for all $i \in T$.

(iii) The vectors $\{[a_{i.}^T, -b_i]\}_{i \in T}$ are linearly independent.

(iv) $|B_u(T) \cup B_v(T)| = |T| \leq r(T) + 1$, i.e., the size of T is bounded by $r(T) + 1$.

Proof. Let S, T and $\bar{x}$ be as stated. We will prove (i) first. Suppose by contradiction $\bar{u}_{i'} = 0$ and $\bar{v}_{i'} = 0$ for some $i' \in T$. Since $\bar{x} \notin Conv\,(P(S) \cap F_D)$, the linear program $P_{LP}(\bar{x}, S)$ is infeasible, and hence its dual $D_{LP}(\bar{x}, S)$ is unbounded. Because $\bar{u}_{i'} = 0$ and $\bar{v}_{i'} = 0$, the linear program $D'_{LP}(\bar{x}, S \setminus \{i'\})$, obtained from $D_{LP}(\bar{x}, S)$ by eliminating $u_{i'}, v_{i'}$ and the normalization constraint (50), is also unbounded. Consequently, the dual of $D'_{LP}(\bar{x}, S \setminus \{i'\})$, the problem $P'_{LP}(\bar{x}, S \setminus \{i'\})$, is infeasible. But $P'_{LP}(\bar{x}, S \setminus \{i'\})$ can be obtained from $P_{LP}(\bar{x}, S)$ by eliminating the variable s and the constraints corresponding to i' in (41) and (42), and we see that $P'_{LP}(\bar{x}, S \setminus \{i'\})$ is feasible if and only if $\bar{x}$ is in $Conv\,(P(S \setminus \{i'\}) \cap F_D)$. The fact that $\bar{x} \in Conv\,(P(S \setminus \{i'\}) \cap F_D)$ implies the feasibility of $P'_{LP}(\bar{x}, S \setminus \{i'\})$, which is a contradiction. Hence $\bar{u}_i > 0$ or $\bar{v}_i > 0$ for all $i \in T$, which gives $B_u(T) \cup B_v(T) = T$. This proves (i). To prove (ii), we show that at most one of u_i and v_i can be basic.

The feasible set for $D_{LP}(\bar{x}, S)$ can be written in the form $\{y \in \mathbb{R}^{n'} : Zy = z^0, y \geq 0_{n'}\}$, where Z and z^0 are of suitable dimensions. Let $i \in T$ be arbitrary. The column of Z corresponding to u_i is given by $[a_{i.}^T, -b_i, 0]^T$. Similarly the column of Z corresponding to v_i is given by $[-a_{i.}^T, b_i, 0]^T$. Hence the columns of Z corresponding to u_i and v_i are linearly dependent. Since the columns of Z corresponding to basic variables must be linearly independent, it follows that at most one of u_i and v_i can be basic. This also shows that the vectors $\{[a_{i.}^T, -b_i]\}_{i \in T}$ are linearly independent. This completes the proof of (i), (ii) and (iii).

Finally, since the vectors $\{[-a_{i.}^T, b_i]\}_{i \in T}$ are linearly independent, there must exist a set $T' \subseteq T$ of cardinality either $|T|$ or $|T| - 1$ such that the vectors $\{a_{i.}\}_{i \in T'}$ are linearly independent. It follows that $|B_u(T) \cup B_v(T)| = |T| \leq r(T) + 1$. Therefore (iv) is also proved. □

With the above lemmas we can prove the missing direction in Theorem 4.

Lemma 35. *Let $S \subseteq M$, and suppose that either $|S| \geq r(S) + 2$ or $r(S) = n$ and $|S| = n + 1$. Then*

$$Conv\,(P(S) \cap F_D) \supseteq \cap_{i \in S}\,Conv\,(P(S \setminus \{i\}) \cap F_D). \tag{55}$$

Proof. Let $\bar{x} \in \cap_{i \in S}Conv\,(P(S \setminus \{i\}) \cap F_D)$ and suppose by contradiction $\bar{x} \notin Conv\,(P(S) \cap F_D)$. If $|S| \geq r(S) + 2$, Lemma 34(iv) with $T = S$ implies that $S \leq r(S) + 1$, a contradiction. If $|S| = n + 1$, since the number of basic variables is bounded by the number of equality constraints in $D_{LP}(\bar{x}, S)$, the number of basic variables in the solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ is at most $n + 2$. The number of basic variables among the variables u_i and v_i for $i \in S$ is $|S| = n + 1$ (Lemma 34 with $T := S$). However, according to Lemma 33, at least two of the variables in $(\bar{u}^0, \bar{v}^0)$ are basic, which gives a total of $n + 3$ basic variables, a contradiction. □

We now strengthen Theorem 4 for the case where $|S| \geq r(S) + 2$. Let $\bar{I}(S)$ be the set of constraints $i \in S$ whose removal from S leaves the rank unchanged, *i.e.*, $\bar{I}(S) := \{i \in S : r(S) = r(S \setminus \{i\})\}$. We have

Theorem 5. *Let $S \subseteq M$ satisfy $|S| \geq r(S) + 2$. Then*

$$Conv\,(P(S) \cap F_D) = \cap_{i \in \bar{I}(S)}\,Conv\,(P(S \setminus \{i\}) \cap F_D). \tag{56}$$

Like in Theorem 4, and with the a similar proof, the inclusion "$\subseteq$" of Theorem 5 is easy to prove. To prove the remaining inclusion, we observe that the condition $|S| \geq r(S) + 1$ and the assumption $a_{i.} \neq 0_n, \forall i \in M$, imply $|\bar{I}(S)| \geq 2$. Therefore, $\cap_{i \in \bar{I}(S)}Conv\,(P(S \setminus \{i\}) \cap F_D) \subseteq P(S)$, so that the problem $P_{LP}(\bar{x}, S)$ is feasible and bounded for a point $\bar{x} \in \cap_{i \in \bar{I}(S)}Conv\,(P(S \setminus \{i\}) \cap F_D)$. The following lemma gives a formal statement and proof of the remaining inclusion.

Lemma 36. *Let $S \subseteq M$ satisfy $|S| \geq r(S) + 2$. Then*

$$Conv\,(P(S) \cap F_D) \supseteq \cap_{i \in \bar{I}(S)}\,Conv\,(P(S \setminus \{i\}) \cap F_D). \tag{57}$$

Proof. Let $\bar{x} \in \cap_{i \in \bar{I}(S)}Conv\,(P(S \setminus \{i\}) \cap F_D)$ and assume by contradiction $\bar{x} \notin Conv\,(P(S) \cap F_D)$. Observe that it suffices to prove that the vectors $\{[a_{i.}^T, -b_i]\}_{i \in S}$ are linearly independent, since that contradicts $|S| \geq r(S) + 2$. Assume for a contradiction that the vectors $\{[a_{i.}^T, -b_i]\}_{i \in S}$ are linearly dependent, and let $k \in S$ satisfy $[a_{k.}^T, -b_k] = \sum_{i \in S \setminus \{k\}} \mu_i[a_{i.}^T, -b_i]$, where $\{\mu_i\}_{i \in S \setminus \{k\}}$ are not all zero. This means $a_{k.} \in Span(\{a_{i.}\}_{i \in S \setminus \{k\}})$, which implies $r(S) = r(S \setminus \{k\})$, *i.e.*, $k \in \bar{I}(S)$.

We have shown that, if $[a_{k.}^T, -b_k] = \sum_{i \in S \setminus \{k\}} \mu_i[a_{i.}^T, -b_i]$, where $k \in S$ and $\{\mu_i\}_{i \in S \setminus \{k\}}$ are not all zero, then $k \in \bar{I}(S)$. Note that this implies $\mu_i = 0$ for all $i \in S \setminus \bar{I}(S)$. To see this, assume for a contradiction that $\mu_l \neq 0$ for some $l \in S \setminus \bar{I}(S)$. Then we may write $[a_{l.}^T, -b_l] = \sum_{i \in S \setminus \{l\}} \mu_i'[a_{i.}^T, -b_i]$, where $\mu_i' = \frac{\mu_i}{\mu_l}$ for $i \in S \setminus \{k, l\}$ and $\mu_k' = \frac{1}{\mu_l}$. As observed above, this implies $l \in \bar{I}(S)$, a contradiction. It follows that $\mu_i = 0$ for all $i \in S \setminus \bar{I}(S)$.

We therefore have $[a_{k.}^T, -b_k] = \sum_{i \in \bar{I}(S) \setminus \{k\}} \mu_i[a_{i.}^T, -b_i]$ and $k \in \bar{I}(S)$. However, according to Lemma 34(iii) with $T := \bar{I}(S)$, the vectors $\{[a_{i.}^T, -b_i]\}_{i \in \bar{I}(S)}$ are linearly independent, a contradiction. □

If we apply Theorem 5 iteratively to sets S satisfying $|S| \geq r(S) + 2$, and Theorem 4 for sets S satisfying $|S| = n + 1$, we obtain Theorem 3 in Sect. 2. The example in Figure 4 in Sect. 2 demonstrates that the assumption $|S| \geq r(S) + 2$ is necessary for (57) to be true.

In the following we prove that (57) remains valid for $|S| = r(S) + 1$ for the special case of a general split disjunction $D(\pi, \pi_0^1, \pi_0^2)$, where $\pi_0^1 < \pi_0^2$. The problem $P_{LP}(\bar{x}, S)$ for the disjunction $D(\pi, \pi_0^1, \pi_0^2)$, which will be called $P_{LP}^S(\bar{x}, S)$ in the following, is obtained from the problem $P_{LP}(\bar{x}, S)$ by replacing (44) and (45) with

$$-\lambda^1 \pi_0^1 + \pi^T x^1 - s \leq 0, \qquad\qquad (u^0) \qquad\qquad (58)$$

$$-\lambda^1 \pi_0^2 + \pi^T x^1 - s \leq -\pi_0^2 + \pi^T \bar{x}. \qquad (v^0) \qquad\qquad (59)$$

The dual of $P_{LP}^S(\bar{x}, S)$ is the problem $D_{LP}^S(\bar{x}, S)$ as follows.

$$\min \quad \sum_{i \in S} v_i (b_i - a_{i.}^T \bar{x}) + w_0 + v^0(\pi^T \bar{x} - \pi_0^2)$$

$$\text{s.t.} \quad \sum_{i \in S} a_{i.}(u_i - v_i) + \pi(u^0 + v^0) = 0_n, \qquad (x^1) \qquad (60)$$

$$\sum_{i \in S} b_i(v_i - u_i) - \pi_0^1 u^0 - \pi_0^2 v^0 + w_0 - t_0 = 0, \qquad (\lambda^1) \qquad (61)$$

$$u^0 + v^0 + t_1 = 1, \qquad (s) \qquad (62)$$

$$u^0, v^0, w_0, t_0, t_1 \geq 0, \qquad (63)$$

$$u_i, v_i \geq 0, \qquad \forall i \in S. \qquad (64)$$

The optimal solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}^S(\bar{x}, S)$, for the case where $|S| = r(S) + 1$, is characterized in the following lemma (also see Lemma 2 in [4]). The statements of this lemma are very similar to the statements made in Lemma 34. However, the assumptions are different than in Lemma 34. Also, we now prove that the vectors $\{a_{i.}\}_{i \in B_u \cup B_v}$ are linearly independent, and this does *not* follow from the statement of Lemma 34 that the vectors $\{[a_{i.}, -b_i]\}_{i \in B_u \cup B_v}$ are linearly independent.

Lemma 37. *Suppose that* $\bar{x} \in \cap_{i \in \bar{I}(S)} Conv\, (P(S \setminus \{i\}) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$ *and* $\bar{x} \notin Conv\, (P(S) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$. *Then* $B_u \cap B_v = \emptyset$, $r(S) = n$, $|B_u \cup B_v| = n$, *and the vectors* $\{a_{i.}\}_{i \in B_u \cup B_v}$ *are linearly independent.*

Proof. The feasible set for $D_{LP}^S(\bar{x}, S)$ can be written in the form $\{y \in \mathbb{R}^{n'} : Zy = z_0, y \geq 0_{n'}\}$, where Z and z_0 are of suitable dimensions. We first argue that the variables w_0, t_0 and t_1 take the value 0. Since $\bar{x} \notin Conv\, (P(S) \cap F_{D(\pi, \pi_0^1, \pi_0^2)})$, the optimal objective value of $P_{LP}^S(\bar{x}, S)$ and $D_{LP}^S(\bar{x}, S)$ is $-\bar{s} < 0$, *i.e.*, s is basic. By complementarity slackness, we deduce that $t_1 = 0$. From Lemma 33 it follows that both u^0 and v^0 are basic. The column in Z corresponding to u^0 is $[\pi^T, -\pi_0^1, 1]^T$ and the column corresponding to v^0 is $[\pi^T, -\pi_0^2, 1]^T$. Subtracting the column corresponding to v^0 from the column corresponding to u^0 gives a constant times e_{n+1}, *i.e.*, a constant times the $(n+1)^{th}$ unit vector in $\mathbb{R}^{n+2}$. Since the column corresponding to both w_0 and $-t_0$ is e_{n+1}, and since basic columns are linearly independent, w_0 and t_0 must be non-basic. Therefore, both w_0 and t_0 take the value 0.

For all $i \in S$, not both v_i and u_i can be in the basis, since their corresponding columns in Z are multiples of each other. Hence $B_u \cap B_v = \emptyset$. Furthermore, since $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ is a basic solution to $D_{LP}^S(\bar{x}, S)$, the following system has a unique solution.

$$\sum_{i \in B_u} a_{i.} u_i - \sum_{i \in B_v} a_{i.} v_i + \pi(u^0 + v^0) = 0_n \tag{65}$$

$$\sum_{i \in B_v} b_i v_i - \sum_{i \in B_u} b_i u_i - \pi_0^1 u^0 - v^0 \pi_0^2 = 0 \tag{66}$$

$$u^0 + v^0 = 1 \tag{67}$$

The system (65)–(67) is of the form $Z^B y = z_0^B$. The number of rows of Z^B (and z_0^B) is $n + 2$ and the number of columns is $|B_u \cup B_v| + 2$. All columns of Z^B are linearly independent. Therefore we must have $|B_u \cup B_v| = n$, since otherwise multiple solutions would exist.

Now we show that the vectors $\{a_{i.}\}_{i \in B_u \cup B_v}$ are linearly independent. Then $r(S) = n$ is also proved. Assume by contradiction that the vectors $\{a_{i.}\}_{i \in B_u \cup B_v}$ are linearly dependent. Then there exists a non-zero solution (u^*, v^*) to the system $\sum_{i \in B_u} a_{i.} u_i^* - \sum_{i \in B_v} a_{i.} v_i^* = 0_n$. Define scalars $u_i(\delta) := \bar{u}_i + \delta u_i^*$ for $i \in B_u$ and $v_i(\delta) := \bar{v}_i + \delta v_i^*$ for $i \in B_v$, where $\delta \in \mathbb{R}$. By solving (65)–(67) for the two remaining variables, we get that $(u(\delta), v(\delta), u^0(\delta), v^0(\delta))$ satisfies (65)–(67) if and only if $u^0(\delta) = 1 - v^0(\delta)$ and $v^0(\delta) = (\pi_0^1 \bar{u}_0 + \pi_0^2 \bar{v}_0 - \pi_0^1 + \delta(\sum_{i \in B_v} b_i v_i^* - \sum_{i \in B_u} b_i u_i^*))/(\pi_0^2 - \pi_0^1)$. Since the numbers $\{u_i^*\}_{i \in B_u} \cup \{v_i^*\}_{i \in B_v}$ are not all zero there must exist $\delta^* \in \mathbb{R}$ such that $(u(\delta^*), v(\delta^*), u^0(\delta^*), v^0(\delta^*))$ is a different solution to (65)-(67) than $(\bar{u}, \bar{v}, \bar{u}_0, \bar{v}_0)$, which is a contradiction. $\qquad \square$

From the above lemma we have the desired extension of Theorem 4 for split disjunctions.

Lemma 38. *Suppose $S \subseteq M$ satisfies $|S| = r(S) + 1$. Then*

$$Conv\left(P(S) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}\right) = \cap_{i \in \bar{I}(S)} Conv\left(P(S \setminus \{i\}) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}\right). \tag{68}$$

Proof. By a similar argument to the proof of Lemma 11, it is easy to show the inclusion $Conv\left(P(S) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}\right) \subseteq \cap_{i \in \bar{I}(S)} Conv\left(P(S \setminus \{i\}) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}\right)$. We prove the remaining inclusion by contradiction. Suppose we have $\bar{x} \in \cap_{i \in \bar{I}(S)} Conv\left(P(S \setminus \{i\}) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}\right)$ and $\bar{x} \notin Conv\left(P(S) \cap F_{D(\pi, \pi_0^1, \pi_0^2)}\right)$. By Lemma 37, $B_u \cap B_v = \emptyset$, $r(S) = n$, $|B_u \cup B_v| = n$, and the vectors $\{a_{i.}\}_{i \in B_u \cup B_v}$ are linearly independent. Let $\{\bar{i}\} = S \setminus (B_u \cup B_v)$. We have that $\bar{i} \notin \bar{I}(S)$, since otherwise Lemma 34(i) with $T := \bar{I}(S)$ would imply $\bar{u}_{\bar{i}} > 0$ or $\bar{v}_{\bar{i}} > 0$, which contradicts $\bar{i} \notin B_u \cup B_v$. But $\bar{i} \in S \setminus \bar{I}(S)$ implies that $\bar{i}$ is in every basis of S, which contradicts $\bar{i} \notin B_u \cup B_v$. $\qquad \square$

From Theorem 5 and Lemma 38, we get Theorem 1 of Sect. 2.

References

1. Andersen, K., Cornuéjols, G., Li, Y.: Reduce-and-split cuts: Improving the performance of mixed integer Gomory cuts. Submitted for publication, 2004
2. Balas, E.: Intersection cuts - a new type of cutting planes for integer programming. Oper. Res. **19**, 19–39 (1971)
3. Balas, E.: Disjunctive programming. Ann. Discrete Math. **5**, 3–51 (1979)
4. Balas, E., Perregaard, M.: A Precise correspondence between, lift-and-project cuts, simple disjunctive cuts and mixed integer Gomory cuts for 0–1 programming. Mathematical Programming B **94**, 221–245 (2003)
5. Bixby, R.E., Gu, Z., Rothberg, E., Wunderling, R.: Mixed integer programming: a progress report. In: M. Grötschel (ed)., The sharpest cut: the impact of Manfred Padberg and his work, MPS/SIAM Series on Optimization 2004, pp. 309–326
6. Cook, W., Kannan, R., Schrijver, A.: Chvátal closures for mixed integer programs. Mathematical Programming **47**, 155–174 (1990)
7. Cornuéjols, G., Li, Y.: Elementary closures for integer programs. Oper. Res. Lett. **28**, 1–8 (2001)
8. Cornuéjols, G., Li, Y.: On the rank of mixed 0, 1 polyhedra. Mathematical Programming **91**, 391–397 (2002)
9. Gomory, R.: An algorithm for the mixed integer problem. Techinical Report RM–2597, The Rand Coporation, 1960
10. Nemhauser, G., Wolsey, L.: A recursive procedure to generate all cuts for 0–1 mixed integer programs. Mathematical Programming **46**, 379–390 (1990)