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A 3-Slope Theorem for the Infinite Relaxation in the Plane

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Abstract In this paper we consider the infinite relaxation of the corner polyhedron with 2 rows. For the 1-row case, Gomory and Johnson proved in their seminal paper a sufficient condition for a minimal function to be extreme, the celebrated 2-Slope Theorem. Despite increased interest in understanding the multiple row setting, no generalization of this theorem was known for this case. We present an extension of the 2-Slope Theorem for the case of 2 rows by showing that minimal 3-slope functions satisfying an additional regularity condition are facets (and hence extreme). Moreover, we show that this regularity condition is necessary, unveiling a structure which is only present in the multi-row setting.

Keywords Integer Programming, Cutting Planes, Corner Polyhedron, Infinite Relaxation, 2-Slope Theorem

Mathematics Subject Classification (2000) 90C10

1 Introduction

For simplicity consider a pure IP set

$$\begin{aligned} Ay &= b \\ y \geq 0, &\quad y \in \mathbb{Z}^d. \end{aligned} \tag{IP}$$

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Rewriting this set in tableaux form with respect to a basis B (i.e., premultiplying the system by B^{-1}) we obtain the equivalent system

$$\begin{aligned} y_B &= \bar{b} - \bar{N} y_N \\ y &\geq 0, \quad y \in \mathbb{Z}^d. \end{aligned} \tag{IP'}$$

In [9] Gomory introduced the *corner polyhedron*, which relaxes the nonnegativity constraints for the basic variables y_B. This relaxation can be conveniently written in the following form:

$$\begin{aligned} f + \sum_{j=1}^{n} r^j s_j &\in \mathbb{Z}^m \\ s \in \mathbb{Z}^n, \quad s &\geq 0. \end{aligned} \tag{CP}$$

The corner polyhedron has been extensively studied in the literature specially in the restricted case $m = 1$ and some interesting results regarding its facial structure are known (see Shim and Johnson [14] for example). Unfortunately, the structure of this IP still heavily relies on the specific choice of r^j's, making it difficult to analyze it.

For interest of tractability, further relaxations of (CP) were proposed, and in particular two have received significant attention of the IP community: the *Andersen et al. relaxation* [1] (which removes the integrality constraint of the s-variables) and the *infinite relaxation* [10]. We focus on the latter relaxation, which is a direct way of reducing the complexity/asymmetry of the system by introducing all possible rays r^j's:

$$\begin{aligned} f + \sum_{r \in \mathbb{R}^m} r s_r &\in \mathbb{Z}^m \\ s_r \in \mathbb{Z}_+ \ \ &\text{for all } r \in \mathbb{R}^m \\ s \text{ has finite support.} \end{aligned} \tag{IR}$$

Note that s is now formally a function $s : \mathbb{R}^m \to \mathbb{Z}_+$, hence imposing that s has finite support has the standard meaning that $s_r \neq 0$ for only finitely many r's. Observe that the set (IR) is completely specified by the choice of f.

Given that (IR) is a relaxation of (CP), some of the valid cuts for the latter may not be valid for the former. Interestingly, many of the important families of cuts for (CP) are actually valid for (IR): Gomory Mixed Integer cuts, and more generally Split cuts, are prominent examples.

We now briefly recall important definitions and results regarding the infinite relaxation; see [3] for a more detailed discussion.

Valid functions. Assume $f \notin \mathbb{Z}^m$. We start by defining the analog of a 'cut' for the infinite relaxation. Let G_f denote the set of feasible solutions to (IR). We say that a function $\pi : \mathbb{R}^m \to \mathbb{R}$ is *valid* for (IR) if $\pi \geq 0$ and the inequality

$$\sum_{r \in \mathbb{R}^m} \pi(r) s_r \geq 1 \tag{1}$$

is satisfied by every $s \in G_f$.

The relevance of the above definition rests on the fact that any valid function yields a valid inequality for the original integer program (IP) by restricting the inequality to the space r^j.

As pointed out in [3], the non-negativity assumption in the definition of valid function might seem artificial at first. If we removed this assumption, there would actually exist 'valid' functions taking negative values. However, any valid function must be nonnegative over rational vectors. Since data in mixed integer linear programs are usually rational and valid functions must be nonnegative over rational vectors, it is natural to focus on nonnegative valid functions. This is the approach that we adopt in this section.

Minimal functions, extreme functions and facets. Given the relationship to valid inequalities for (IP), the general goal is to understand valid functions for (IR). However, at this point they are again too general to have a useful structure. The first way of restricting the set of functions is by studying only those which are minimal. A valid function π is *minimal* if there is no valid function $\pi' \neq \pi$ such that $\pi'(r) \leq \pi(r)$ for all $r \in \mathbb{R}^m$. These are the only valid functions that we need to consider, since for every valid function there is a minimal one which dominates it. Gomory and Johnson give a simple characterization of minimal valid functions using the following concepts.

A function $\pi : \mathbb{R}^m \to \mathbb{R}$ is *periodic* if $\pi(x) = \pi(x+w)$ for all $x \in [0,1]^m$ and $w \in \mathbb{Z}^m$. Also, π is said to satisfy the *symmetry condition* if $\pi(r)+\pi(-f-r) = 1$ for all $r \in \mathbb{R}^m$. Finally, π is *subadditive* if $\pi(a+b) \leq \pi(a) + \pi(b)$.

Theorem 1 (Gomory and Johnson [10]) *Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be a nonnegative function. Then π is a minimal valid function for* (IR) *if and only if $\pi(0) = 0$, π is periodic, subadditive and satisfies the symmetry condition.*

Although minimality helps reducing the number of relevant valid functions that we need to study, it still leaves too many under consideration. Inspired by the importance of facets in the finite dimensional setting, there are two analogous concepts in the infinite dimensional setting. A valid function is *extreme* if it cannot be written as a convex combination of two other valid functions. For the other concept, given a valid function π first define $S(\pi)$ as the set of all s satisfying (IR) which are tight for π, namely $\sum_{r \in \mathbb{R}^m} \pi(r)s_r = 1$. A valid function π is then a *facet* of (IR) if for every other valid function $\theta \neq \pi$, we have $S(\pi) \not\subseteq S(\theta)$. This later concept was introduced by Gomory and Johnson in [12] and it is not difficult to see that every facet is extreme (proof in Appendix B.1).

Lemma 1 *If π is a facet, then π is extreme.*

In general, constructing or even proving that a valid function is extreme or a facet can be a very difficult task. Arguably the deepest result on the infinite relaxation is a sufficient condition for extremality in the restricted setting $m = 1$, the so-called 2-Slope Theorem of Gomory and Johnson [11].

In addition to its theoretical appeal, this result also has practical relevance. Indeed the simplest 2-slope functions give rise to Gomory's mixed integer cuts, which are currently the most effective cuts in integer programming solvers [2].

Theorem 2 (Gomory-Johnson 2-Slope Theorem) *Let* $\pi : \mathbb{R} \to \mathbb{R}$ *be a minimal valid function. If* π *is a continuous piecewise linear function with only two slopes, then* π *is extreme.*

This surprising result was already known in the 70's, and despite the increased efforts in understanding relaxations for (IP) with $m > 1$ no generalization or related result for this case seems to be known; this was posed as an open question by Gomory and Johnson in [12].

Our results. We show that a suitable generalization of the 2-Slope Theorem holds for $m = 2$ (required definitions are presented in the next section).

Theorem 3 (3-Slope Theorem) *Let* $\pi : \mathbb{R}^2 \to \mathbb{R}$ *be a minimal valid function. If* π *is a continuous 3-slope function with 3 directions, then* π *is a facet.*

We remark that the extra condition about the 3 directions of π is required as shown in Section 6.

As an application, our theorem implies that the 3-slope construction of Dey and Richard [7] gives facets, highlighting the properties which are driving this result. We remark that new facets can be derived using the 3-slope functions satisfying the hypotheses of our theorem, for instance via automorphisms [7] or via the sequential-merge procedure introduced by Dey and Richard in [6]. Another observation is that, just as 2-slope functions seem to be the most important extreme functions in the 1-dimensional case, 3-slope functions with 3 directions seem to be important in the 2-dimensional case. In particular Dey and Wolsey [8] characterized the minimal valid functions of the Andersen et al. 2-row relaxation that have a unique lifting in the infinite relaxation (IR): these minimal functions arise from splits, whose lifting are 2-slope functions (they satisfy the Gomory-Johnson theorem), and from so-called triangles of Type 1 and Type 2, whose liftings are 3-slope functions with 3 directions (they satisfy Theorem 3).

The high-level structure of the proof of Theorem 3 is similar to the proof of the 2-Slope Theorem presented in [12]. Consider a valid function $\pi : \mathbb{R}^2 \to \mathbb{R}$ satisfying the assumptions of the theorem. We then consider a valid function θ such that $S(\pi) \subseteq S(\theta)$ and our goal is to show that $\pi = \theta$. In order to achieve this, we write a system of equations which is satisfied by π and θ; the final step is then to argue that this system has a unique solution, which gives the desired equality.

One major departure from the proof of the 2-Slope Theorem is the way the system is constructed: while Gomory and Johnson derived it directly from the valid function at hand, our system can be seen as the 'limit' of systems obtained for suitable approximations of π. In addition, proving that the constructed system has a unique solution is substantially more involved than in the previous proof.

2 Notation and Definitions

Throughout this text, we consider $\mathbb{R}^2$ endowed with the Euclidean metric and standard inner product. For any subset $U \subseteq \mathbb{R}^2$, we use $\overline{U}$ to denote its closure. We say that a collection $\mathcal{F}$ of subsets of $\mathbb{R}^2$ is *locally finite* if every point in $\mathbb{R}^2$ has a neighborhood which intersects only finitely many sets in $\mathcal{F}$.

For a subset S of $\mathbb{R}^2$, we say that a function $\pi : S \to \mathbb{R}$ is *affine* if there is a vector $\nabla \subseteq \mathbb{R}^2$ and a scalar β such that $\pi(x) = \nabla \cdot x + \beta$. Notice that when S has a non-empty interior ∇ is uniquely defined, hence we call it the *gradient* of π.

A subset $P \subseteq \mathbb{R}^2$ is a *polygon* if it satisfies the following properties: (i) P is the union of a locally finite collection of convex polytopes; (ii) P has non-empty and path-connected interior. Note that according to this definition polygons do not need to be convex; in particular, it allows polygons with 'holes'. Also note that unbounded objects, such as splits [4], can also be polygons. However, this definition excludes a single point as a polygon and every polygon is a closed set. As a final remark, we point out that property (ii) in the definition is not actually required in this paper; we add it in order to obtain a more intuitive meaning of polygons.

3-Slope functions. In this part we formally define what we mean for a function $\pi : \mathbb{R}^2 \to \mathbb{R}$ to be 3-slope. The definition is given in general terms since some of the notions introduced here will be useful in the proofs.

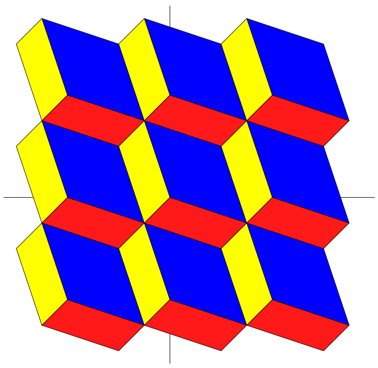

Fig. 1 Example of a 3-partition $\{\mathcal{P}_i\}_{i=1}^3$. Polygons of the same color belong to the same part $\mathcal{P}_i$.

A collection of polygons $\mathcal{P}$ is a *tiling* of a region $R \subseteq \mathbb{R}^2$ if $R = \cup_{P \in \mathcal{P}} P$ and the interiors of the polygons in $\mathcal{P}$ are pairwise disjoint. Given a locally finite tiling $\mathcal{P}$ of $\mathbb{R}^2$, a partition of $\mathcal{P}$ into 3 non-empty subsets $\{\mathcal{P}_i\}_{i=1}^3$ is called a *(polygonal) 3-partition*.

Consider a function $\pi : \mathbb{R}^2 \to \mathbb{R}$ and let D denote the points in $\mathbb{R}^2$ where π is differentiable. We define the equivalence relation $\sim$ over D^2 according to their gradients: $x \sim y$ iff $\nabla\pi(x) = \nabla\pi(y)$. Then let $\{D_i\}_{i \in I}$ be the collection of equivalence classes of D with respect to $\sim$ and let $\mathcal{P}_i^\pi$ contain the closure of the path-connected components of D_i, i.e. $\mathcal{P}_i^\pi = \{\overline{P} : P \text{ is a path-connected component of } D_i\}$. Finally, we reach the definition of a 3-slope function.

Definition 1 We say that $\pi : \mathbb{R}^2 \to \mathbb{R}$ is a *3-slope function* if it is continuous and $\{\mathcal{P}_i^\pi\}_{i \in I}$ is a 3-partition.

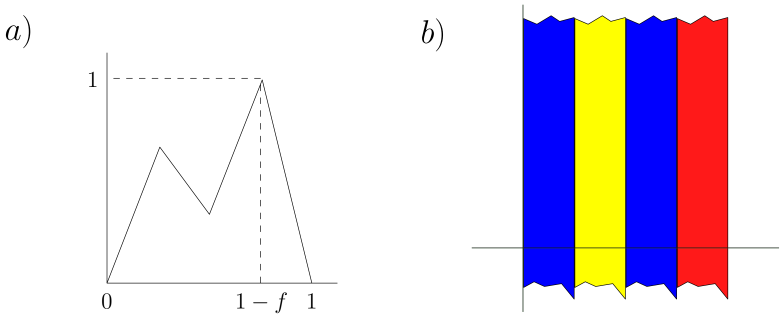

Fig. 2 Example of a 3-slope function and its 3-partition. a) Function π defined in (9), with its projection on x depicted in the interval $[0, 1]$. b) The 3-partition $\{\mathcal{P}_i^\pi\}_{i=1}^3$ formed by 'splits'; polygons of the same color belong to the same part $\mathcal{P}_i^\pi$.

For a 3-slope function π, we use the shorthand $\{\mathcal{P}_i^\pi\}_{i=1}^3$ to denote $\{\mathcal{P}_i^\pi\}_{i \in I}$ and $\mathcal{P}^\pi = \bigcup_{i=1}^3 \mathcal{P}_i^\pi$. We define $\nabla^i \pi$ to be the gradient of the points in $\bigcup_{P \in \mathcal{P}_i^\pi} P$ where π is differentiable. Notice that $\nabla^i \pi$ is the gradient of the restriction $\pi|_P$ for $P \in \mathcal{P}_i^\pi$.

A useful concept when testing if a function is 3-slope is that of compatibility. We say that a function θ is *compatible* with a 3-partition $\{\mathcal{P}_i\}_{i=1}^3$ if : (i) for all $P \in \bigcup_{i=1}^3 \mathcal{P}_i$ the restriction $\theta|_P$ is affine and; (ii) for $i = 1, 2, 3$ and for all $P, P' \in \mathcal{P}_i$ the restrictions $\theta|_P$ and $\theta|_{P'}$ have the same gradient. In this context, we define $\nabla^i \theta$ as the gradient of the points in $\bigcup_{P \in \mathcal{P}_i} P$ where θ is differentiable.

We say that θ is *compatible* with a 3-slope function π if θ is compatible with $\{\mathcal{P}_i^\pi\}_{i=1}^3$. Note that π is itself compatible with $\{\mathcal{P}_i^\pi\}_{i=1}^3$.

Segments and boundary directions. Consider a 3-partition $\{\mathcal{P}_i\}_{i=1}^3$. A *segment* of the 3-partition is a line segment $S \subseteq \mathbb{R}^2$ of non-zero length that belongs to the boundary of a polygon $P \in \bigcup_{i=1}^3 \mathcal{P}_i$. A vector d in the unit circle S^1 is a *direction* of a polygon P if the boundary of P contains a line segment S with endpoints $a \neq b$ such that d is a scaling of $a - b$. We identify antipodal points in S^1 so that d and $-d$ are considered the same direction; for example, a square has only 2 directions. Similarly, d is a direction of $\{\mathcal{P}_i\}_{i=1}^3$ if it is the direction of a polygon in $\bigcup_{i=1}^3 \mathcal{P}_i$. Finally, d is a *direction* of π if it is a direction of $\{\mathcal{P}_i^\pi\}_{i=1}^3$.

Integration along lines. Consider a 3-slope function π, a point $u \in \mathbb{R}^2$ and a vector $v \in \mathbb{R}^2 \setminus \{0\}$. Intuitively, $\pi(u+v)$ is obtained by integrating the 'gradient' of π from u to $u+v$; essentially, $\pi(u+v) = \pi(u) + \sum_i \mu_i(u,v)\nabla^i\pi \cdot v$ where $\mu_i(u,v)$ is the proportion of points with gradient $\nabla^i\pi$ in the segment connecting u and $u+v$. Our main goal in this part is to make the above statement formal, we defer proofs to Section A in the appendix.

Fix a 3-slope function π and $u \in \mathbb{R}^2$. Given $v \in \mathbb{R}^2 \setminus \{0\}$, let $L(u,v)$ denote the line segment connecting u and $u+v$. Consider a polygon $P \in \mathcal{P}^\pi$ and define $\mathcal{I}_P$ to be the set of line segments of non-zero length obtained by intersecting P and $L(u,v)$. Since P is obtained by a union of locally finite polyhedra, only finitely many of them intersect the compact set $L(u,v)$ and hence $\mathcal{I}_P$ is finite. Define $\mathcal{I}' = \bigcup_{P\in\mathcal{P}^\pi} \mathcal{I}_P$. Again notice that $\mathcal{I}'$ is finite since $\mathcal{P}^\pi$ is locally finite and $L(u,v)$ is compact, and therefore only finitely many polygons in $\mathcal{P}^\pi$ intersect $L(u,v)$.

We note that the line segments in $\mathcal{I}'$ may overlap in more than one point. In order to rectify this, we consider the 'shattering' of these segments: we define $\mathcal{I}$ as the collection of all maximal line segments S such that for all $I \in \mathcal{I}'$ either $S \subseteq I$ or S intersects I in at most one point. Now $\mathcal{I}$ is a collection of segments that cover $L(u,v)$ and pairwise intersect in at most one point.

Finally, we partition the segments in $\mathcal{I}$ into 3 families according to the original 3-partition: $\mathcal{I}_1 = \{I \in \mathcal{I} : I \subseteq \bigcup_{P\in\mathcal{P}_1^\pi} P\}$, $\mathcal{I}_2 = \{I \in \mathcal{I} \setminus \mathcal{I}_1 : I \subseteq \bigcup_{P\in\mathcal{P}_2^\pi} P\}$ and $\mathcal{I}_3 = \{I \in \mathcal{I} \setminus (\mathcal{I}_1 \cup \mathcal{I}_2) : I \subseteq \bigcup_{P\in\mathcal{P}_3^\pi} P\}$.

Now we define $\mu_i(u,v)$ as the fraction of $L(u,v)$ which lies in the segments $\mathcal{I}_i$:

$$\mu_i(u,v) = \frac{\sum_{I\in\mathcal{I}_i} \mu(I)}{|v|},$$

where $\mu(I)$ for a line segment $I \subseteq \mathbb{R}^2$ with endpoints a, b is the length $|a-b|$. The following property motivates the careful definition of these structures.

Lemma 2 $\sum_{i=1}^3 \mu_i(u,v) = 1$.

We finally present the most important property of this section.

Lemma 3 $\pi(u+v) = \pi(u) + \sum_{i=1}^3 \mu_i(u,v)\nabla^i\pi \cdot v$.

3 Structural Properties of 3-Partitions and 3-Slope Functions

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be a 3-slope function. Although the 3-partition $\{\mathcal{P}_i^\pi\}_{i=1}^3$ looks arbitrary at first, it actually exhibits a great deal of structure. In this section we present lemmas which capture some of these properties.

Consider a 3-partition $\{\mathcal{P}_i\}_{i=1}^3$. Let S be a segment in this 3-partition and d be the direction of S. Since the 3-partition is a locally finite tiling of $\mathbb{R}^2$, there are sets $P, P' \in \bigcup_{i=1}^3 \mathcal{P}_i$ such that $S \cap (P \cap P')$ is a segment of non-zero length. We then associate the pair $\{i,j\}$ to the direction d. Note that many pairs may be associated to the same direction and that the same pair may be associated to different directions. Note also that pairs $\{i,i\}$ are allowed.

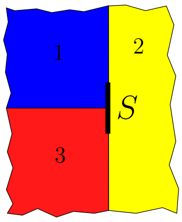

Fig. 3 Association of pairs $\{i,j\}$ and directions d. The 3-partition $\{\mathcal{P}_i\}_{i=1}^3$ has 2 directions: $(0,1)$ and $(1,0)$. A segment S with direction $(0,1)$ is depicted in bold. The pairs $\{1,2\}$ and $\{2,3\}$ are associated to direction $(0,1)$, and the pair $\{1,3\}$ is associated to $(1,0)$.

The importance of this association is given by the following simple lemma which allows us to relate the different gradients of a function.

Lemma 4 *Consider a 3-partition $\{\mathcal{P}_i\}_{i=1}^3$ and a function $\theta : \mathbb{R}^2 \to \mathbb{R}$ compatible with this 3-partition. If d is a direction of $\{\mathcal{P}_i\}_{i=1}^3$ and $\{i,j\}$ is associated to d, then $\nabla^i\theta \cdot d = \nabla^j\theta \cdot d$.*

Proof Consider $P \in \mathcal{P}_i$, $P' \in \mathcal{P}_j$ such that $P \cap P'$ contains a non-degenerate segment S with direction d. To simplify the notation, define $\theta_i = \theta|_P$ and $\theta_j = \theta|_{P'}$. Since θ_i is affine, it is of the form $\theta_i(x) = \nabla\theta_i \cdot x + \beta_i = \nabla^i\theta \cdot x + \beta_i$, and similar for θ_j.

Let a, b be the endpoints of the line segment S. Then $\theta_i(b) - \theta_i(a) = \nabla^i\theta \cdot (b - a) = \nabla^i\theta \cdot \lambda d$, for some $\lambda \neq 0$. Similarly, we obtain that $\theta_j(b) - \theta_j(a) = \nabla^j\theta \cdot \lambda d$. Since $\theta_i(a) = \theta_j(a) = \theta(a)$ and $\theta_i(b) = \theta_j(b) = \theta(b)$, we have $\nabla^i\theta \cdot d = \nabla^j\theta \cdot d$ and the result follows.

We start analyzing the structure of the association of directions and pairs. The following lemma follows directly from the definition of $\{\mathcal{P}_i^\pi\}_{i=1}^3$.

Lemma 5 *Consider a 3-slope function π. If the pair $\{i,j\}$ is associated to a direction of π then $\nabla^i\pi \neq \nabla^j\pi$. In particular, $i \neq j$.*

Consider Figure 3. Notice that there is no 3-slope function π such that the 3-partition depicted equals $\{\mathcal{P}_i^\pi\}_{i=1}^3$: since $\{1,2\}$ and $\{2,3\}$ are associated to the direction $(0,1)$ and $\{1,3\}$ is associated to $(1,0)$, from Lemma 4 we get that $\nabla^1\pi \cdot (0,1) = \nabla^2\pi \cdot (0,1) = \nabla^3\pi \cdot (0,1)$ and $\nabla^1\pi \cdot (1,0) = \nabla^3\pi \cdot (1,0)$; this implies that $\nabla^1\pi = \nabla^3\pi$, contradicting the previous lemma. The next lemma captures the heart of this argument and will be extremely useful in proving properties of 3-partitions.

Lemma 6 *Consider a 3-slope function π and let $\mathcal{D}$ denote the set of directions of π. If $|\mathcal{D}| > 1$ then there is an injection $\zeta : \mathcal{D} \to \{\{i,j\} : i \neq j, 1 \leq i,j \leq 3\}$ such that $\zeta(d)$ is the only pair associated to the direction d.*

Proof First we show that we cannot have a pair $\{i,j\}$ associated to more than one direction. Suppose that $\{i,j\}$ is associated to two distinct directions d_1 and d_2. The vectors d_1 and d_2 are linearly independent and, by Lemma 4,

$\nabla^i\pi \cdot d_1 = \nabla^j\pi \cdot d_1$ and $\nabla^i\pi \cdot d_2 = \nabla^j\pi \cdot d_2$. Lemma 15 then implies that $\nabla^i\pi = \nabla^j\pi$, contradicting Lemma 5.

Now we show that only one pair is associated to a direction. Take a direction d_1 and suppose by contradiction that two distinct pairs $\{i,j\}$ and $\{i',j'\}$ are associated to d_1. Again $i \neq j$ and $i' \neq j'$ by Lemma 5, and by the pigeonhole principle two of these four indices are the same. Without loss of generality assume $i = i'$. Consider a direction d_2 different from d_1. There is a pair associated to d_2, and from the previous paragraph we get that this pair must be $\{j,j'\}$. But again $\nabla^j\pi \cdot d_1 = \nabla^i\pi \cdot d_1 = \nabla^{j'}\pi \cdot d_1$ and $\nabla^j\pi \cdot d_2 = \nabla^{j'}\pi \cdot d_2$, implying that $\nabla^j\pi = \nabla^{j'}\pi$ by Lemma 15. This contradicts Lemma 5.

Obviously, Lemma 6 implies that $|\mathcal{D}| \leq 3$. In fact, the following holds.

Corollary 1 *Consider a 3-slope function π and let $\mathcal{D}$ denote the set of directions of π. Then $|\mathcal{D}| = 1$ or 3.*

Proof Lemma 6 implies that $|\mathcal{D}| \leq 3$. Suppose $|\mathcal{D}| = 2$. Consider a point $x \in \mathbb{R}^2$ where two distinct boundary directions meet (see Figure 4). Since we only have two possible boundary directions, considering a small enough neighborhood around x, we see that the number of possible polygons in $\mathcal{P}^\pi$ containing x is 2, 3 or 4. In the first case, injectivity of Lemma 6 is violated (two different directions are associated with the same pair $\{i,j\}$), a contradiction. In the second case, two different pairs are associated with one of the directions, again contradicting Lemma 6. In the third case, exactly 4 polygons of $\mathcal{P}^\pi$ contain x. Two of these polygons belong to the same family $\mathcal{P}^\pi_i$ for some $i = 1,2,3$. But then two different directions are associated with the same pair $\{i,j\}$, again contradicting Lemma 6.

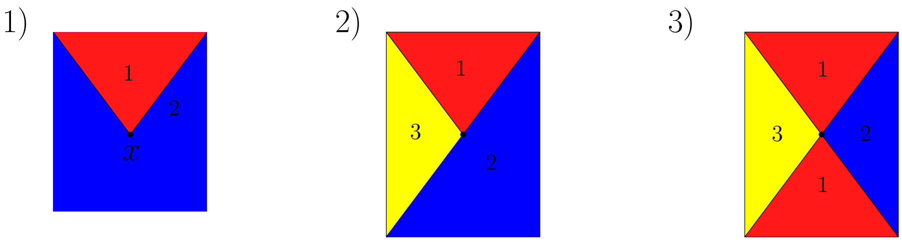

Fig. 4 Cases in the proof of Corollary 1. 1) $\{1,2\}$ is associated to both directions. 2) $\{1,2\}$ and $\{2,3\}$ are associated to the same direction. 3) $\{1,3\}$ is associated to both directions. We note that we cannot exchange the color of the right and bottom polygons due to Lemma 5.

Another corollary of Lemma 6 is that the polygons in $\mathcal{P}^\pi_i$ have at most 2 directions, and that these directions are the same across all of these polygons.

Corollary 2 *Consider a 3-slope function π. Fix $i \in \{1,2,3\}$ and let $\mathcal{D}$ be the set of direction d such that there is a polygon in $\mathcal{P}^\pi_i$ with direction d. Then $|\mathcal{D}| \leq 2$.*

Proof By contradiction suppose not. Then there are three distinct directions associated to pairs of the form $\{i, j\}$, where i is fixed and $j \neq i$ belongs to $\{1, 2, 3\}$. By the pigeon-hole principle we have two directions d, d' associated to the same pair, which contradicts Lemma 6.

Before moving on, we need to make formal an intuitive observation about periodic 3-slope functions.

Lemma 7 *Consider a periodic 3-slope function π. Fix $i \in \{1, 2, 3\}$. Then for every $P \in \mathcal{P}_i^\pi$ and $w \in \mathbb{Z}^2$, the polygon $w + P$ is contained in a polygon in $\mathcal{P}_i^\pi$.*

Proof Fix $i \in \{1, 2, 3\}$, consider $P \in \mathcal{P}_i^\pi$ and $w \in \mathbb{Z}^2$. Let $D_i \subseteq \mathbb{R}^2$ denote the points where π is differentiable and has gradient equal to $\nabla^i \pi$. Using the definition of $\mathcal{P}_i^\pi$, let S be a path-connected component of D_i such that $P = \overline{S}$.

By periodicity of π, notice that $w + S$ belongs to D_i, and hence to a path-connected component S' of D_i. By monotonicity and translation-invariance of closure, we get that $w + P = \overline{(w + S)} \subseteq \overline{S'}$. Again using the definition of $\mathcal{P}_i^\pi$, we have that it contains $\overline{S'}$ and the result follows.

Now we focus on 3-slope functions which are periodic, non-negative and have 3 directions. Notice that the function of interest in the main Theorem 3 satisfies these properties.

Lemma 8 *Consider a periodic 3-slope function $\pi \geq 0$ with 3 directions. Then $\mathbb{R}^2$ is spanned by nonnegative combinations of the gradients $\{\nabla^i \pi\}_{i=1}^3$.*

Proof Since π has 3 directions, the function ζ of Lemma 6 is a bijection. By Lemma 5, at most one of the three gradients equals 0.

First we show that the three gradients cannot be collinear. By contradiction, suppose they are. As noted earlier, at least two of these vectors are non-zero, say $\nabla^2 \pi$, $\nabla^3 \pi$. Since π has 3 directions, at least 2 of these directions are not orthogonal to $\nabla^i \pi$ for $i = 2, 3$. One of these 2 directions, say d, is associated with either $\{1, 2\}$ or $\{1, 3\}$, say $\{1, 2\}$. Then employing Lemma 4 over the pair $\{1, 2\}$ we obtain $\nabla^1 \pi \cdot d = \nabla^2 \pi \cdot d$, and since $\nabla^2 \pi \cdot d \neq 0$ and $\nabla^1 \pi, \nabla^2 \pi$ are collinear, this implies $\nabla^1 \pi = \nabla^2 \pi$, which contradicts Lemma 5.

Now suppose that $\mathbb{R}^2$ is not spanned by non-negative combinations of $\{\nabla^i \pi\}_{i=1}^3$. Since these three vectors are not collinear, this implies that there is a vector d such that $\nabla^i \pi \cdot d \leq 0$ for all i and $\nabla^i \pi \cdot d < 0$ for some i; without loss of generality we assume $\nabla^1 \pi \cdot d < 0$. Intuitively, walking inside a polygon in $\mathcal{P}_1^\pi$ along direction d keeps reducing the value of π. Hence, the high-level idea is to use the periodicity of π to find b and λ such that $\mu_1(b, \lambda d)$ is large; then using Lemma 3 we can contradict $\pi \geq 0$.

So consider a polygon P in $\mathcal{P}_1^\pi$, which has a non-empty interior. Let b be a point in the interior of P, and consider a closed ball B centered at b with radius $r < 1/2$ which is contained in P. Since $r < 1/2$, we have that $(w + B) \cap (w' + B) = \emptyset$ for all $w, w' \in \mathbb{Z}^2$. Moreover, Lemma 7 implies that for every $w \in \mathbb{Z}^2$ the ball $w + B$ centered at $w + b$ belongs to a polygon in $\mathcal{P}_1^\pi$.

Let α be a positive integer to be specified later. Using Lemma 19 repeatedly with increasing values for ℓ, take *distinct* points $\{w_j\}_{j=1}^{\alpha}$ in $\mathbb{Z}^2$ and reals $\{\lambda_j\}_{j=1}^{\alpha}$ such that $|w_j - \lambda_j d|_\infty < r' \doteq r/2$ for $j \in \{1, 2, \ldots, \alpha\}$. Let $\lambda = r + \max_j \lambda_j$ and notice that the distance between w_j and the line $L(0, \lambda d)$ is at most r'. Translating this system by b, we obtain that the distance between $w_j + b$ and the line $L(b, \lambda d)$ is at most r', which implies that the ball $w_j + B$ (centered at $w_j + b$) intersects $L(b, \lambda d)$. Since we chose λ large enough, the intersection $(w_j + B) \cap L(b, \lambda d)$ is actually a line segment, whose endpoints we denote by u_j and v_j. As we chose $r' = r/2$, we get that there exists $\epsilon > 0$ *independent of* α such that $\mu([u_j, v_j]) \geq \epsilon$ for all $j \in \{1, 2 \ldots, \alpha\}$. Since the balls $\{w_j + B\}_{j=1}^{\alpha}$ are disjoint, this implies that $\mu_1(b, \lambda d) \geq \alpha\epsilon/\lambda|d|$.

Using Lemma 3 we obtain that $\pi(b + \lambda d) \leq \pi(b) + \mu_1(b, \lambda d)\nabla^1\pi \cdot \lambda d \leq \pi(b) + (\alpha\epsilon/|d|)\nabla^1\pi \cdot d$, where we used the fact that $\nabla^i\pi \cdot d \leq 0$ for $i = 1, 2, 3$. Since $\nabla^1\pi \cdot d < 0$ and ϵ is independent of α, we can choose α large enough so that $\pi(b + \lambda d) < 0$, contradicting that $\pi \geq 0$. This concludes the proof.

Lemma 9 *Consider a periodic 3-slope function $\pi \geq 0$ with 3 directions and satisfying $\pi(0) = 0$. Then for $i = 1, 2, 3$ there exists a polygon $P_i^0 \in \mathcal{P}_i^\pi$ that contains the origin.*

Proof The proof goes by contradiction. Without loss of generality[1] assume that $0 \notin \bigcup_{P \in \mathcal{P}_3^\pi} P$. Then there is a ball B centered at the origin which does not intersect $\bigcup_{P \in \mathcal{P}_3^\pi} P$: to see this, notice that by locally finiteness there is a neighborhood of 0 which only intersects finitely many polygons in $\mathcal{P}_3^\pi$; all of these have non-zero distance to the origin, so we can define the radius of B to be smaller than the smallest of these distances.

From the previous lemma, $\nabla^1\pi$ and $\nabla^2\pi$ are linearly independent, so there is a vector d such that $\nabla^1\pi \cdot d < 0$ and $\nabla^2\pi \cdot d < 0$. Now scale d such that $d \in B$ and notice that $\mu_3(0, d) = 0$. From Lemma 2 we obtain that $\mu_1(0, d) + \mu_2(0, d) = 1$ and hence from Lemma 3 we get

$$\pi(d) = \pi(0) + \mu_1(0, d)\nabla^1\pi \cdot d + \mu_2(0, d)\nabla^1\pi \cdot d < 0,$$

where the inequality follows since $\pi(0) = 0$. This contradicts the fact that $\pi \geq 0$ and completes the proof.

A *parallelogram* is a convex polygon with non-empty interior and with two distinct boundary directions.

Lemma 10 *Consider a periodic 3-slope function $\pi \geq 0$ with 3 directions and satisfying $\pi(0) = 0$. Fix $i \in \{1, 2, 3\}$ and let P_i^0 be given by the above lemma. Then every $P \in \mathcal{P}_i^\pi$ can be tiled with parallelograms in such a way that each of these parallelograms can be translated to be entirely contained in P_i^0 and to contain the origin as a vertex. Moreover, this tiling is locally finite.*

[1] We remark that this is indeed the case even though the definition of μ_i is not symmetric.

Proof Let $\mathcal{D}$ be the union of the set of direction of P_i^0 and P. From Corollary 2 we have that $|\mathcal{D}| \leq 2$. Consider a parallelogram Q that contains the origin as a vertex, is entirely contained in P_i^0, and whose boundary directions equal $\mathcal{D}$ (in the case $|\mathcal{D}| = 2$) or one of its boundary directions belongs to $\mathcal{D}$ (in the case $|\mathcal{D}| = 1$). Denote by 0, $0 + u$, $0 + v$ and $0 + u + v$ the vertices of Q. The translates $Q + (ku + hv)$ of Q with $k, h \in \mathbb{Z}$ give a locally finite tiling of the plane. By intersecting this tiling of the plane with P, we obtain a tiling $\mathcal{T}$ of P.

Since the set of boundary directions of P is contained in that of Q, the polygons in the tiling $\mathcal{T}$ also have their set of boundary directions contained in that of Q. Moreover, notice that each polygon in $\mathcal{T}$ is bounded and has only finitely many vertices: the latter follows from the fact that, by locally finiteness, only finitely many polyhedra which compose P intersect the compact set $Q + (ku + hv)$. Therefore, whenever a polygon in $\mathcal{T}$ is non-convex, tile it into finitely many parallelograms homothetic to Q using lines with the direction of the boundary directions of Q passing through the various vertices of the polygon.

It then follows that the polygon P can be locally finitely tiled with parallelograms whose boundary directions are the same as in Q. Furthermore, each of these parallelograms is of size no greater than Q. Therefore each can be translated into a parallelogram that contains the origin as a vertex and is entirely contained in P_i^0, which completes the proof.

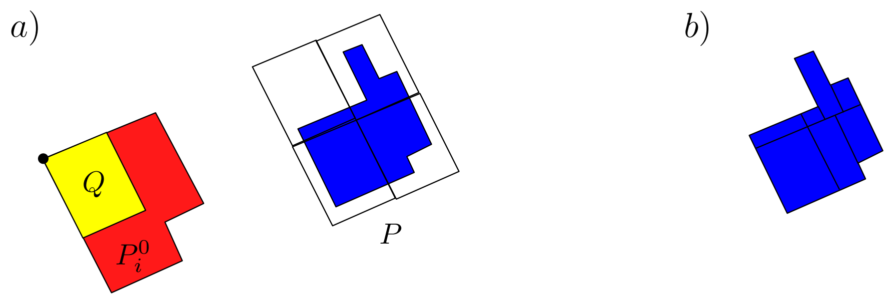

Fig. 5 Illustration of the proof of Lemma 10. a) Polygon P_i^0, parallelogram Q and polygon P. b) Tiling of P with parallelograms.

4 Uniqueness Theorem

The following theorem is the main piece in our argument for the 3-Slope Theorem.

Theorem 4 *(Uniqueness Theorem) Consider a point $f \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ and a periodic 3-slope function $\pi \geq 0$ with 3 directions and satisfying: (i) $\pi(0) = 0$; (ii)*

$\pi(w - f) = 1$ for all $w \in \mathbb{Z}^2$. Suppose that $\theta \geq 0$ is periodic, compatible with π and satisfies conditions (i) and (ii). Then $\pi = \theta$.

4.1 Proof of the Uniqueness Theorem

Let d_1, d_2 and d_3 be the three distinct boundary directions of π. From Lemma 16 we can scale d_1 and d_2 and assume WLOG that $-d_3 = d_1 + d_2$. For every vector $v \in \mathbb{R}^2$ we use $v(1)$ and $v(2)$ to denote the coordinates of v with respect to the canonical basis $\{e_1, e_2\}$.

Let ζ be the injection (which in this case is a bijection) from Lemma 6 with respect to π. Relabel the gradients of π such that $\zeta(d_i) = \{i, i \oplus 1\}$ for all i, where addition is done in a cyclic way: $i \oplus 1 = i + 1$ for $i < 3$ and $3 \oplus 1 = 1$ (to simplify the notation we use the standard plus sign instead of $\oplus$ from now on).

Before proceeding we give an informal idea of the proof strategy. Suppose that $d_i \in \mathbb{Z}^2 - f$ (i.e. $f + d_i \in \mathbb{Z}^2$) for $i = 1, 2, 3$. Condition (ii) then implies that $\pi(d_i) = \theta(d_i) = 1$ for $i = 1, 2, 3$. This leads to the consideration of the the following system on variables $x_1, x_2, x_3 \in \mathbb{R}^2$:

$$x_i \cdot d_i = x_{i+1} \cdot d_i \qquad i = 1, 2, 3 \tag{2}$$

$$\sum_{j=1}^{3} \mu_j(0, d_i) x_j \cdot d_i = 1 \qquad i = 1, 2, 3. \tag{3}$$

Under the above assumption, the gradients of π and θ satisfy this system, namely when setting $x_i = \nabla^i \pi$ for all i or $x_i = \nabla^i \theta$ for all i (via Lemmas 4 and 3). Moreover, using a basis transformation and Lemma 17, it is possible to show that this system has a unique solution, which then implies that $\pi = \theta$. Writing the system for the boundary directions d_i, $i = 1, 2, 3$, is crucial for this step: it imposes the required sign structure on the matrix of the system.

In general the d_i's may not belong to $\mathbb{Z}^2 - f$, hence the gradients of π and θ may not satisfy equations (3). The idea is then to approximate (or perturb) these equations to make the gradients feasible, while maintaining the property that the system has a unique solution (see Lemma 21). More precisely, we use Lemma 20 to find $\lambda > 0$ and for each $i = 1, 2, 3$ a vector $v^i \in \mathbb{Z}^2 - f$ such that $|v^i/\lambda - d_i|_\infty \approx 0$. With these new vectors, the equations

$$\sum_{j=1}^{3} \mu_j(0, v^i) x_j \cdot \frac{v^i}{\lambda} = 1 \qquad i = 1, 2, 3 \tag{4}$$

are satisfied (without any assumptions on the d_i's) by setting $x_i = \lambda \nabla^i \pi$ for all i or setting $x_i = \lambda \nabla^i \theta$ for all i. However, to show that these equations approximate (3) we need to get a better hold on μ_j, which introduces additional technicalities. Now we proceed with the formal proof.

Using Lemma 20, consider a sequence $\{\lambda_n\}$ of positive reals and for each $i = 1, 2, 3$ a sequence $\{v_n^i\}$ in $\mathbb{Z}^2 - f$ such that $\lim_{n \to \infty} \lambda_n = \infty$ and $\lim_{n \to \infty} |v_n^i / \lambda_n -$

$d_i|_\infty = 0$. For a vector $v \in \mathbb{R}^2 \setminus \{0\}$, Lemma 2 guarantees that the vector $(\mu_1(0,v), \mu_2(0,v), \mu_3(0,v))$ has ℓ_1-norm 1. Then $\{(\mu_1(0,v_n^i), \mu_2(0,v_n^i), \mu_3(0,v_n^i))\}_n$ lies in a compact subset of $\mathbb{R}^3$ and thus has a subsequence which converges uniformly to a non-negative vector $(\mu_1^i, \mu_2^i, \mu_3^i)$, also of unit ℓ_1-norm.

Now we do a basis change and write the system that we aim at approximating. Let A be the invertible matrix such that $Ad_1 = e_1$ and $Ad_2 = e_2$. Then consider the system

$$y_i \cdot Ad_i = y_{i+1} \cdot Ad_i \qquad i = 1, 2, 3 \tag{5}$$

$$\sum_{j=1}^{3} \mu_j^i y_j \cdot Ad_i = 1 \qquad i = 1, 2, 3. \tag{6}$$

Notice that setting $y_i = \nabla^i \pi A^{-1}$ for all i or setting $y_i = \nabla^i \theta A^{-1}$ for all i satisfies equations (5). From the definition of A we can write this system in matrix notation as

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \mu_1^1 & 0 & \mu_2^1 & 0 & \mu_3^1 & 0 \\ 0 & \mu_1^2 & 0 & \mu_2^2 & 0 & \mu_3^2 \\ -\mu_1^3 & -\mu_1^3 & -\mu_2^3 & -\mu_2^3 & -\mu_3^3 & -\mu_3^3 \end{bmatrix} \begin{bmatrix} y_1(1) \\ y_1(2) \\ y_2(1) \\ y_2(2) \\ y_3(1) \\ y_3(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let M be the matrix in the left hand side of the above system, let b be the vector in the right hand side and pack $y = (y_1(1), y_1(2), y_2(1), y_2(2), y_3(1), y_3(2))$ to obtain the equivalent system $My = b$. Since for $i = 1, 2, 3$ $(\mu_1^i, \mu_2^i, \mu_3^i)$ is non-negative and has ℓ_1 norm 1, M only has non-zero rows and Lemma 17 implies that the matrix $[M|b]$ has full row rank.

The next lemma gives the desired approximation of equations (6).

Lemma 11 *Fix $\epsilon > 0$. Then there are vectors $v^1, v^2, v^3 \in \mathbb{Z}^2 - f$ and a positive scalar λ such that for each $i, j \in \{1, 2, 3\}$, $|\mu_j(0, v^i)\frac{Av^i}{\lambda} - \mu_j^i Ad_i|_\infty < \epsilon$.*

Proof In hindsight fix ϵ' such that $\epsilon'(\epsilon' + |Ad_i|_\infty + \mu_j^i) < \epsilon$ for all $i, j \in \{1, 2, 3\}$. Recall that $\lim_{n\to\infty} \lambda_n = \infty$, $\lim_{n\to\infty} |v_n^i/\lambda_n - d_i|_\infty = 0$ and $\lim_{n\to\infty, n\in I} \max_{j=1,\ldots,3} |\mu_j(0, v_n^i) - \mu_j^i| = 0$ hold for all $i = 1, 2, 3$ (where $I \subseteq \mathbb{N}$). Moreover, since A is a linear (and hence continuous) transformation we have $\lim_{n\to\infty} |Av_n^i/\lambda_n - Ad_i|_\infty = 0$. Thus, take $n \in I$ large enough and define $v^i = v_n^i$ and $\lambda = \lambda_n$ such that: (i) $|Av^i/\lambda - Ad_i|_\infty < \epsilon'$ for all $i \in \{1, 2, 3\}$ and (ii) $|\mu_j(0, v_n^i) - \mu_j^i| < \epsilon'$ for all $i, j \in \{1, 2, 3\}$.

We show that $|\mu_j(0,v^i)\frac{Av^i}{\lambda} - \mu_j^i Ad_i|_\infty < \epsilon$. Using subadditivity and positive homogeneity of norms we obtain

$$\begin{aligned}\left|\mu_j(0,v^i)\frac{Av^i}{\lambda} - \mu_j^i Ad_i\right|_\infty &\leq \left|\mu_j(0,v^i)\frac{Av^i}{\lambda} - \mu_j^i \frac{Av^i}{\lambda}\right|_\infty + \left|\mu_j^i \frac{Av^i}{\lambda^i} - \mu_j^i Ad_i\right|_\infty \\ &= |\mu_j(0,v^i) - \mu_j^i| \left|\frac{Av^i}{\lambda}\right|_\infty + \mu_j^i \left|\frac{Av^i}{\lambda} - Ad_i\right|_\infty \\ &\leq |\mu_j(0,v^i) - \mu_j^i| \left(\left|\frac{Av^i}{\lambda} - Ad_i\right|_\infty + |Ad_i|_\infty\right) + \mu_j^i \left|\frac{Av^i}{\lambda} - Ad_i\right|_\infty < \epsilon,\end{aligned}$$

where the last inequality follows from the definition of ϵ'. This concludes the proof of the lemma.

According to Lemma 21, there exists $\epsilon > 0$ such that every matrix $[M'|b']$ satisfying $|[M'|b'] - [M|b]|_{\max} < \epsilon$ has full row rank. So consider such ϵ and let v^1, v^2, v^3 and λ be as in the previous claim. Then our perturbed system is given by

$$y_i \cdot Ad_i = y_{i+1} \cdot Ad_i \qquad i = 1,2,3 \tag{7}$$

$$\sum_{j=1}^{3} \mu_j(0,v_i) y_j \cdot \frac{Av^i}{\lambda} = 1 \qquad i = 1,2,3, \tag{8}$$

which now has $y_i = \lambda \nabla^i \pi A^{-1}$ and $y_i = \lambda \nabla^i \theta A^{-1}$ as feasible solutions. If $M'x = b$ is the matrix form of the above system, it follows from the definition of λ and the v^i's that $|[M'|b] - [M|b]|_{\max} < \epsilon$. Thus $[M'|b]$ has full row rank. Since $M'x = b$ has a solution, this implies that it has a unique solution. Since λA^{-1} is invertible we have that $\nabla^i \pi = \nabla^i \theta$ for $i = 1,2,3$ as desired. This implies that $\pi = \theta$ and thus concludes the proof of Theorem 4.

5 The 3-Slope Theorem

In order to prove the 3-Slope Theorem we need two results from the literature.

Lemma 12 (Interval Lemma [7]) *Let U and V be closed sets in $\mathbb{R}^2$. Let g be a real-valued function over U, V and $U+V$. Assume that*

1. *U has a non-empty interior and, for all $u \in U$, the line segment $[0,u] \subseteq U$.*
2. *V is path-connected.*
3. *$g(u) + g(v) = g(u+v)$ for all $u \in U$, $v \in V$.*
4. *For all $S \subseteq U$ with $|S| \leq 3$ and $\sum_{u \in S} u \in U$, g satisfies $\sum_{u \in S} g(u) = g(\sum_{u \in S} u)$.*
5. *$g(u) \geq 0$ for all $u \in U$.*

Then the functions $g|_U$, $g|_V$ and $g|_{U+V}$ are affine with the same gradient.

For a function $\pi : \mathbb{R}^m \to \mathbb{R}$, let $E(\pi) \subseteq \mathbb{R}^m \times \mathbb{R}^m$ denote the set of all possible pairs (r_1, r_2) which satisfy $\pi(r^1) + \pi(r^2) = \pi(r^1 + r^2)$. The next theorem is a strengthening of the Facet Theorem proved in [12], which connects the equality sets $S(.)$ and $E(.)$. Unlike in the original theorem, here we present a version where the required condition only needs to hold for *minimal* valid functions. Its proof is very similar to the original one, but for completeness we present it in Appendix B.2.

Theorem 5 (Facet Theorem) *Let π be a minimal valid function. Suppose that for every other minimal function $\theta \neq \pi$ we have $E(\pi) \not\subseteq E(\theta)$. Then π is a facet.*

5.1 Proof of Theorem 3

Recall that $\pi : \mathbb{R}^2 \to \mathbb{R}$ is a minimal valid function which is also continuous, 3-slope and has 3 boundary directions. In light of Theorem 5, consider a minimal valid function θ such that $E(\pi) \subseteq E(\theta)$; we prove that $\theta = \pi$ using the Uniqueness Theorem.

Since θ is minimal, Theorem 1 implies that $\theta(0) = 0$ and also that $\theta(w - f) = 1$ for all $w \in \mathbb{Z}^2$: from the symmetry condition we have $\theta(0) + \theta(-f) = 1$ and hence $\theta(-f) = 1$, so the observation follows from the periodicity of θ. In order to apply the Uniqueness Theorem, we need to show that θ is compatible with π. Motivated be Lemma 18 we start with the following lemma.

Lemma 13 *Consider $i \in \{1, 2, 3\}$ and $P \in \mathcal{P}_i^\pi$. Also consider the tiling of P given in Lemma 10. Let Q be a parallelogram in this tiling and let y be such that $Q + y$ contains the origin as a vertex and is contained in P_i^0. Then $\theta|_Q$ and $\theta|_{Q+y}$ are affine with the same gradient.*

Proof We employ the Interval Lemma with $U = Q + y$, $V = \{-y\}$ and $g = \theta$. First, recall that by definition of a parallelogram, U has a non-empty interior. Furthermore it contains the origin and is convex, and hence for all $u \in U$, the line segment $[0, u]$ is contained in U. Moreover, V is trivially path-connected and $\theta \geq 0$; thus we satisfy Conditions 1, 2 and 5 in the Interval Lemma.

Now we consider Condition 3. The definitions of U and V imply that $U \subseteq P_i^0$ and $V, U + V \subseteq P \in \mathcal{P}_i$. Then $\pi|_U$ is affine (in fact linear) with gradient $\nabla^i \pi$ and (since $0 \in U$) we have $\pi(x) = \pi(0) + \nabla^i \pi \cdot x$ for all $x \in U$; similarly, $\pi(-y + x) = \pi(-y) + \nabla^i \pi \cdot x$ for all $x \in U$. Recalling that $\pi(0) = 0$ and combining the previous equations gives $\pi(-y + x) = \pi(-y) + \pi(x)$. Since $E(\pi) \subseteq E(\theta)$, we get that $\theta(-y + x) = \theta(-y) + \theta(x)$ for all $x \in U$, hence Condition 3 is satisfied.

For Condition 4, first suppose that $S = \{u_1, u_2\}$ with $u_1, u_2, u_1 + u_2 \in U$. Using the fact that $\pi|_U$ is linear, we get that $\pi(u_1) + \pi(u_2) = \pi(u_1 + u_2)$ and again $\theta(u_1) + \theta(u_2) = \theta(u_1 + u_2)$ as desired. Now suppose that $S = \{u_1, u_2, u_3\}$ with $u_1, u_2, u_3, u_1 + u_2 + u_3 \in U$. We claim that $u_1 + u_2 \in U$. Notice that, since U is a parallelogram containing the origin as a vertex, there is an invertible linear transformation $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\rho(Q) = [0,1]^2$. Then $\rho(u_i) \in [0,1]^2$ for $i = 1, 2, 3$ and $\rho(u_1 + u_2 + u_3) \in [0,1]^2$. Because ρ is linear, $\rho(u_1+u_2) = \rho(u_1)+\rho(u_2) \geq 0$. Also, $\rho(u_1+u_2)+\rho(u_3) = \rho(u_1+u_2+u_3) \leq 1$, so the non-negativity of $\rho(u_3)$ implies $\rho(u_1 + u_2) \leq 1$. Applying the inverse, we get that $u_1 + u_2 \in U$ as claimed. So from the previous part of this item, $\theta(u_1 + u_2) = \theta(u_1) + \theta(u_2)$. Applying it again, we get $\theta((u_1 + u_2) + u_3) = \theta(u_1 + u_2) + \theta(u_3) = \theta(u_1) + \theta(u_2) + \theta(u_3)$, thus Condition 4 holds.

Therefore, we can apply the Interval Lemma with $U = Q + y$, $V = \{-y\}$ and $g = \theta$ to obtain the result.

Now fix $i \in \{1, 2, 3\}$. Consider $P \in \mathcal{P}_i^\pi$; since P is path-connected and its tiling given in Lemma 10 is locally finite, combining the previous lemma with Lemma 18 gives that $\theta|_P$ is affine; in particular $\theta|_{P_i^0}$ is affine. But again employing the previous lemma we get that $\theta|_P$ and $\theta|_{P_i^0}$ have the same gradient: take a parallelogram Q in the tiling of P with $Q + y \subseteq P_i^0$, so $\theta|_P$ and $\theta|_Q$ have the same gradient (since $P \subseteq Q$), $\theta|_{P_i^0}$ and $\theta|_{Q+y}$ have the same gradient (for the same reason) and $\theta|_Q$ and $\theta|_{Q+y}$ have the same gradient. Since P_i^0 is independent of P, this shows that the restriction of θ to every polygon in $\mathcal{P}_i^\pi$ is affine with the same gradient.

This argument shows that θ is compatible with π. Thus, we can now employ the Uniqueness Theorem to obtain that $\pi = \theta$ as desired. This concludes the proof of the theorem.

6 Tightness and Counterexamples

Consider a 3-slope function π and let $\mathcal{D}$ denote the set of directions of π. By Corollary 1, we can only have $|\mathcal{D}| = 1$ or 3. We prove in this section that our assumption in Theorem 3 that $|\mathcal{D}| = 3$ is necessary. In other words, Theorem 3 does not hold when $|\mathcal{D}| = 1$.

Theorem 6 *The condition that π has 3 boundary directions is necessary in the statement of Theorem 3.*

Proof Consider the periodic function $\pi : \mathbb{R}^2 \to \mathbb{R}$ defined as follows on $[0,1]^2$ and elsewhere by periodicity (see Figure 2):

$$\pi(x,y) = \begin{cases} \frac{8}{3}x \text{ for } 0 \leq x \leq \frac{1}{4} \\ 1 - \frac{4}{3}x \text{ for } \frac{1}{4} \leq x \leq \frac{1}{2} \\ -1 + \frac{8}{3}x \text{ for } \frac{1}{2} \leq x \leq \frac{3}{4} \\ 4 - 4x \text{ for } \frac{3}{4} \leq x \leq 1 \end{cases} \tag{9}$$

The function π is a 3-slope function with gradients $(\frac{8}{3}, 0), (-\frac{4}{3}, 0), (-4, 0)$. Note that all the boundary segments are parallel to the line $x = 0$, and therefore $|\mathcal{D}| = 1$.

However, π is a valid function for (IR) (with $f = 1/4$) which is not extreme (and hence by Lemma 1 also not a facet), since it is the convex combination of the two valid functions below. These functions are extensions to 2-d of the classical Gomory function and its 2-cut variant [5].

$$\pi_1(x,y) = \begin{cases} \frac{4}{3}x \text{ for } 0 \le x \le \frac{3}{4} \\ 4-4x \text{ for } \frac{3}{4} \le x \le 1 \end{cases} \quad \text{and} \quad \pi_2(x,y) = \begin{cases} 4x \text{ for } 0 \le x \le \frac{1}{4} \\ 2-4x \text{ for } \frac{1}{4} \le x \le \frac{1}{2} \\ -2+4x \text{ for } \frac{1}{2} \le x \le \frac{3}{4} \\ 4-4x \text{ for } \frac{3}{4} \le x \le 1 \end{cases}$$

Since $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$, this concludes the proof of the theorem.

Regarding 4-slope functions, it is possible to construct continuous 4-slope functions that are not extreme. For example

$$\pi(x,y) = \begin{cases} x+y \text{ for } 0 \le x \le \frac{1}{2},\ 0 \le y \le \frac{1}{2} \\ 1-x+y \text{ for } \frac{1}{2} \le x \le 1,\ 0 \le y \le \frac{1}{2} \\ 1+x-y \text{ for } 0 \le x \le \frac{1}{2},\ \frac{1}{2} \le y \le 1 \\ 2-x-y \text{ for } \frac{1}{2} \le x \le 1,\ \frac{1}{2} \le y \le 1 \end{cases}$$

is a convex combination of the valid Gomory functions (extended to 2-d)

$$\pi_1(x,y) = \begin{cases} 2x \text{ for } 0 \le x \le \frac{1}{2} \\ 2-2x \text{ for } \frac{1}{2} \le x \le 1 \end{cases} \quad \text{and} \quad \pi_2(x,y) = \begin{cases} 2y \text{ for } 0 \le y \le \frac{1}{2} \\ 2-2y \text{ for } \frac{1}{2} \le y \le 1 \end{cases}.$$

This indicates that in order to extend Theorem 3 to 4-slope functions, one might have to impose stricter conditions on the boundary directions. Although we are not aware of any explicitly stated 2-d facet which is continuous, 4-slope and with $|\mathcal{D}| > 1$, it seems that such facets can be constructed, for instance, using the sequential-merge procedure of Dey and Richard [6].

7 Conclusions

In this work we present an extension of the 2-Slope Theorem for the infinite relaxation of the corner polyhedron with 2 rows. Departing from the case $m = 1$, this reveals a geometric condition (the number of boundary directions) different than the number of slopes which influences the faciality of a valid function.

The most direct open problem is to obtain similar sufficient conditions for faciality for the case $m > 2$. The main difficulty in extending our proof lies in finding an analogous of the system (5)-(6). Another important direction is to better understand how to leverage this condition for faciality in order to generate stronger cutting planes.

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A Proofs for Section 2

Our goal is to prove Lemmas 2 and 3. Recall the notation from Section 2.

Notice that the segments in $\bigcup_{i=1}^{3} \mathcal{I}_i$ form a cover of $L(u,v)$ and that their pairwise intersection consists of at most one point. Thus, there are points $\{t_j\}_{j=1}^{k}$ in $L(u,v)$ with $t_1 = u$, $t_k = u+v$, $t_{j+1} = t_j + \lambda_j v$ with $\lambda_j \geq 0$ and such that $\bigcup_{i=1}^{3} \mathcal{I}_i$ equals the collection of line segments $[t_j, t_{j+1}]$.

Proof of Lemma 2: First, $\sum_{i=1}^{3} \mu_i(u,v) = (\sum_{I \in \bigcup_{i=1}^{3} \mathcal{I}_i} \mu(I))/|v|$. Using collinearity, $|v| = |t_k - t_1| = \sum_{j=1}^{k-1} |t_{j+1} - t_j| = \sum_{j=1}^{k-1} \mu([t_j, t_{j+1}]) = \sum_{I \in \bigcup_{i=1}^{3} \mathcal{I}_i} \mu(I)$, and the result follows. ■

Proof of Lemma 3: Consider a segment $[t_j, t_{j+1}]$ which belongs to $\mathcal{I}_i$. By definition this segment belongs to a polygon P in $\mathcal{P}_i^\pi$. The truncation $\pi|_P$ is an affine function with gradient $\nabla^i \pi$; thus, $\pi(t_{j+1}) = \pi(t_j) + \nabla^i \pi \cdot (t_{j+1} - t_j) = \pi(t_j) + \nabla^i \pi \cdot v \mu([t_j, t_{j+1}])/|v|$.

Chaining this observation over all j gives

$$\pi(u+v) = \pi(u) + \sum_{i=1}^{3} \nabla^i \pi \cdot v \frac{\sum_{[t_j, t_{j+1}] \in \mathcal{I}_i} \mu([t_j, t_{j+1}])}{|v|} = \pi(u) + \sum_{i=1}^{3} \mu_i(u,v) \nabla^i \pi \cdot v.$$

■

B Faciality

B.1 Proof of Lemma 1

Suppose π is a facet and let $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. We observe that $S(\pi) \subseteq S(\pi_1)$ and $S(\pi) \subseteq S(\pi_2)$. Let $s \in S(\pi)$. Then

$$1 = \sum_{r \in \mathbb{R}^k} \pi(r) s_r = \frac{1}{2} \sum_{r \in \mathbb{R}^k} \pi_1(r) s_r + \frac{1}{2} \sum_{r \in \mathbb{R}^k} \pi_2(r) s_r \geq \frac{1}{2} + \frac{1}{2} = 1,$$

so equality must hold throughout and in particular, $\sum_{r \in \mathbb{R}^k} \pi_i(r) s_r = 1$ for both $i = 1, 2$. Therefore $s \in S(\pi_i)$ for both $i = 1, 2$. Since π is a facet, by definition this implies $\pi = \pi_1 = \pi_2$.

B.2 Facet Theorem

The next lemma shows that a weaker condition than that in the definition of a facet is enough to guarantee faciality.

Lemma 14 *Let π be minimal valid function. Suppose that for every minimal valid function $\theta \neq \pi$ we have that $S(\pi) \not\subseteq S(\theta)$. Then π is a facet.*

Proof Consider any valid function θ (not necessarily minimal) such that $S(\pi) \subseteq S(\theta)$; we show that $\theta = \pi$.

Suppose to the contrary that there exists $r_1 \in \mathbb{R}^k$ such that $\pi(r_1) \neq \theta(r_1)$. We claim that actually there is r_2 such $\pi(r_2) > \theta(r_2)$. To see this, first notice that the symmetry condition of π (via Theorem 1) guarantees that $\pi(r_1) + \pi(-f - r_1) = 1$. Moreover, it is clear that the solution $\bar{s}$ given by $\bar{s}_{r_1} = \bar{s}_{-f-r_1} = 1$ and $\bar{s}_r = 0$ otherwise is feasible; together, these observations imply that $\bar{s} \in S(\pi)$. Since $S(\pi) \subseteq S(\theta)$, we have that $\bar{s} \in S(\theta)$ and hence

$$\theta(r_1) + \theta(-f - r_1) = \sum_{r \in \mathbb{R}^k} \theta(r) \bar{s}_r = 1 = \pi(r_1) + \pi(-f - r_1).$$

Since $\pi(r_1) \neq \theta(r_1)$, it follows that either $\pi(r_1) > \theta(r_1)$ or $\pi(-f - r_1) > \theta(-f - r_1)$, and the claim holds.

Now consider a minimal valid function $\theta^* \leq \theta$. Notice that $S(\theta) \subseteq S(\theta^*)$: for $\bar{s} \in S(\theta)$, using its validity we get $1 \leq \sum_{r \in \mathbb{R}^k} \theta^*(r)\bar{s}_r \leq \sum_{r \in \mathbb{R}^k} \theta(r)\bar{s}_r = 1$, hence equality hold throughout and $\bar{s} \in S(\theta^*)$. Since $S(\pi) \subseteq S(\theta)$, we get that $S(\pi) \subseteq S(\theta^*)$. However, $\pi \neq \theta^*$, since there is r_2 such that $\pi(r_2) > \theta(r_2) \geq \theta^*(r_2)$. This contradicts the assumptions on π, which concludes the proof.

Proof of Theorem 5 By Lemma 14, all we need to show is that for every minimal valid function θ, $S(\pi) \subseteq S(\theta)$ implies $\theta = \pi$. We simply show that for every minimal valid function θ, $S(\pi) \subseteq S(\theta)$ implies $E(\pi) \subseteq E(\theta)$, and the result then follows from the assumption on π.

So let θ be a minimal valid function with $S(\pi) \subseteq S(\theta)$. Consider any $(r_1, r_2) \in E(\pi)$, namely such that $\pi(r_1) + \pi(r_2) = \pi(r_1 + r_2)$. Notice that the solution $\bar{s}$ given by $\bar{s}_{r_1} = \bar{s}_{r_2} = \bar{s}_{-f-r_1-r_2} = 1$ and $\bar{s}_r = 0$ is feasible. Moreover, using symmetry condition of π we get that $\bar{s} \in S(\pi)$. Indeed,

$$\sum_{r \in \mathbb{R}^k} \pi(r)\bar{s}_r = \pi(r_1) + \pi(r_2) + \pi(-f - (r_1 + r_2)) = \pi(r_1 + r_2) + \pi(-f - (r_1 + r_2)) = 1.$$

Since $S(\pi) \subseteq S(\theta)$, the solution $\bar{s}$ also belongs to $S(\theta)$, and now the symmetry condition of θ gives

$$1 = \sum_{r \in \mathbb{R}^k} \theta(r)\bar{s}_r = \theta(r_1) + \theta(r_2) + \theta(-f - r_1 - r_2) = \theta(r_1) + \theta(r_2) + (1 - \theta(r_1 + r_2)).$$

Thus, $\theta(r_1) + \theta(r_2) = \theta(r_1 + r_2)$ and $(r_1, r_2) \in E(\theta)$. This concludes the proof.

C Technical Lemmas

C.1 Linear algebra

Lemma 15 *Consider two linear independent vectors $d_1, d_2 \in \mathbb{R}^2$ and two vectors $u, v \in \mathbb{R}^2$. If $d_1 \cdot u = d_1 \cdot v$ and $d_2 \cdot u = d_2 \cdot v$, then $u = v$*

Proof It suffices to prove that $t \doteq u - v$ equals 0. The vectors d_1, d_2 form a basis and hence $t = \alpha_1 d_1 + \alpha_2 d_2$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Expanding the first appearance of t and using linearity we get $t \cdot t = \alpha_1 d_1 \cdot t + \alpha_2 d_2 \cdot t = 0$, which implies that $t = 0$.

Lemma 16 *Consider vectors $d_1, d_2, d_3 \in \mathbb{R}^2$ such that no two are linearly dependent. Then there are $\alpha, \beta \neq 0$ such that $-d_3 = \alpha d_1 + \beta d_2$.*

Proof Since d_1 and d_2 are linearly independent there are α, β such that $-d_3 = \alpha d_1 + \beta d_2$. Since d_3 is linearly independent of d_1 and d_2 we get that $\alpha, \beta \neq 0$.

Lemma 17 *Consider a matrix $[M|b]$ with the sign pattern*

$$M = \left[\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ \oplus & 0 & \oplus & 0 & \oplus & 0 & 1 \\ 0 & \oplus & 0 & \oplus & 0 & \oplus & 1 \\ \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & 1 \end{array}\right]$$

where $\oplus/\ominus$ denotes that the entry is non-negative/non-positive. If M has no zero rows then $[M|b]$ has full row rank.

Proof Assume that M has no zero rows. Pivoting $[M|b]$ on entries $(1,3)$ and $(2,4)$ we obtain that $[M|b]$ is of full row rank iff the matrix $[M'|b']$ is of full row rank, and the latter has the following sign pattern:

$$[M'|b'] = \left[\begin{array}{cccc|c} -1 & -1 & 1 & 1 & 0 \\ \oplus & 0 & \oplus & 0 & 1 \\ 0 & \oplus & 0 & \oplus & 1 \\ \ominus & \ominus & \ominus & \ominus & 1 \end{array}\right].$$

Also notice that M' has no zero rows.

Let m'_i denote the ith row of M'. By means of contradiction, suppose that there are λ_i's not all zero such that $\sum_i \lambda_i [m'_i|b'_i] = 0$. We consider a few different cases.

First, assume that $\lambda_1 \neq 0$; then we can assume without loss of generality that $\lambda_1 > 0$. If $\lambda_4 \geq 0$, the first and the second column give that $\lambda_2 > 0$ and $\lambda_3 > 0$; however, from the last column we get $\sum_i \lambda_i b'_i > 0$, raising a contradiction. If $\lambda_4 < 0$ then from the third and forth column we get $\lambda_2 < 0$ and $\lambda_3 < 0$, which gives $\sum_i \lambda_i b'_i < 0$ and again raises a contradiction.

Finally, suppose that $\lambda_1 = 0$. We may assume $\lambda_4 \geq 0$ without loss of generality. Since M' has no zero rows, we can infer that $\lambda_2 \geq 0$ and $\lambda_3 \geq 0$. But then from the last column we obtain that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, which contradicts our hypothesis. This concludes the proof.

Lemma 18 *Consider a locally finite tiling $\mathcal{P}$ of a path-connected region $R \subseteq \mathbb{R}^2$. Assume that $\theta : R \to \mathbb{R}$ is such that $\theta|_P(x) = \nabla \cdot x + \beta_P$ for each $P \in \mathcal{P}$, where ∇ is independent of P. Then θ is a function in R of the form $\theta(x) = \nabla \cdot x + \beta$.*

Proof Consider $P_0 \in \mathcal{P}$ and let $R_0 = \bigcup_{P \in \mathcal{P} : \beta_P = \beta_{P_0}} P$. We claim that $R = R_0$, which gives the desired result. By means of contradiction suppose not and consider $x \in R_0$ and $y \in R \setminus R_0$. Since R is path-connected, consider a path Π from x to y in R. Since Π is compact and $\mathcal{P}$ locally finite, there is a finite number of polygons in $\mathcal{P}$ intersecting Π. Let $S_0 = \bigcup_{P \in \mathcal{P} : \beta_P = \beta_{P_0}, P \cap \Pi \neq \emptyset} P$ and $S_1 = \bigcup_{P \in \mathcal{P} : \beta_P \neq \beta_{P_0}, P \cap \Pi \neq \emptyset} P$. Notice that both S_0 and S_1 are closed, since they are finite unions of closed sets. Since S_0 and S_1 cover Π, $x \in S_0$ and $y \in S_1$, it is easy to see that there is a point z in Π which belongs to both S_0 and S_1. However, this means that z belongs to a polygon P with $\beta_P = \beta_{P_0}$, and hence $\theta(z) = \theta|_P(z) = \nabla \cdot z + \beta_{P_0}$. Similarly, z belongs to a polygon P' with $\beta_{P'} \neq \beta_{P_0}$ and hence $\theta(z) = \nabla \cdot \beta_{P'} \neq \theta(z)$, which is a contradiction.

C.2 Approximation Theory

Lemma 19 *Consider real numbers $d_1, d_2, \ldots, d_n$. Then for all $k, \ell > 0$ there is $\lambda > \ell$ and integers $w_1, w_2, \ldots, w_n$ such that $|w_i - \lambda d_i| < 1/k$ for $i = 1, 2, \ldots, n$.*

Proof If all the d_i's are rational then this is clearly true: take λ as a large enough multiple of Δ, where Δ is the product of the denominators of the d_i's (say, in its lowest terms) and let $w_i = \lambda d_i$. So suppose without loss of generality that d_1 is not rational. From Dirichlet's theorem for simultaneous approximation [13], we have a sequence $\{\lambda_m\} > 0$ and sequences $\{w^i_m\} \in \mathbb{Z}$ for $i = 1, \ldots, n$ such that $\lim_{m \to \infty} |w^i_m - \lambda_m d_i| = 0$ for all i. Since d_1 is not rational, $\{w^1_m\}$ takes infinitely many distinct values, which implies that there is m such that $|w^i_m - \lambda_m d_i| < 1/k$ for all i and $|w^1_m| > \ell|d| + 1/k$. To see that $\lambda_m > \ell$ we employ the inequalities

$$\ell|d_1| + \frac{1}{k} < |w^1_m| \leq |w^1_m - \lambda_m d_1| + |\lambda_m d_1| < \frac{1}{k} + \lambda_m |d_1|,$$

and the result follows by rearranging the first and last terms.

Lemma 20 *Consider real numbers $f_1, f_2, \ldots, f_n$ and $d_1, d_2, \ldots, d_n$. Then for all $k > 0$, there is $\lambda > k$ and integers $w_1, w_2, \ldots, w_n$ such that $|\frac{w_i - f_i}{\lambda} - d_i| < 1/k$ for $i = 1, 2, \ldots, n$.*

Proof Take $\ell > \max\{k, 2, 2k \max_i |f_i|\}$ and let λ and $\{w_i\}$ be as in the previous lemma. We have that for all i

$$\left|\frac{w_i - f_i}{\lambda} - d_i\right| = \frac{|w_i - f_i - \lambda d_i|}{\lambda} \leq \frac{|w_i - \lambda d_i|}{\lambda} + \frac{|f_i|}{\lambda} < \frac{1}{k\lambda} + \frac{\ell}{2k\lambda}.$$

Since $\lambda \geq \ell > 2$, we have $\frac{1}{k\lambda} + \frac{\ell}{2k\lambda} < \frac{1}{k}$ and the result follows.

Lemma 21 *Let A be an $n \times m$ matrix with full row rank. Then there exists $\epsilon > 0$ such that every $n \times m$ matrix A' satisfying $|A - A'|_{\max} < \epsilon$ is of full row rank.*

Proof Consider an $n \times n$ nonsingular submatrix B of A, and let B' be the corresponding submatrix of A'. The determinant $\det(B)$ is a continuous function of the entries of A. Since $\det(B) \neq 0$, we also have $\det(B') \neq 0$ for a small enough perturbation A' of A.