

Cutting Planes from Two-Term Disjunctions

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Abstract

We consider a polyhedron intersected by a two-term disjunction, and we characterize the polyhedron resulting from taking its closed convex hull. This generalizes an earlier result of Conforti, Wolsey and Zambelli on split disjunctions. We also recover as a special case the valid inequalities derived by Judice, Sherali, Ribeiro and Faustino for linear complementarity problems.

Keywords: polyhedron, two-term disjunction, convex hull, cutting plane

1 Introduction

We consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and a two-term disjunction $c^1x \leq d_1 \vee c^2x \leq d_2$. Let $P_1 = P \cap \{c^1x \leq d_1\}$ and $P_2 = P \cap \{c^2x \leq d_2\}$. The main result of this paper is a description of $\overline{\text{conv}}(P_1 \cup P_2)$, the closed convex hull of $P_1 \cup P_2$. The standard disjunctive programming approach to this question is to express the closed convex hull of $P_1 \cup P_2$ in a higher dimensional space and to project the resulting polyhedron onto the original x -space. In this paper, we follow a different approach introduced by Conforti, Wolsey and Zambelli [1].

To avoid trivial cases, we assume that $c^1, c^2 \neq 0$, (c^1, d_1) is not a multiple of (c^2, d_2) , and P_1 and P_2 are nonempty. Before stating our main result, we give an easy sufficient condition.

For $w \in \mathbb{R}^m$, let w^+ be the vector of \mathbb{R}^m defined by $w_i^+ = \max(0, w_i)$, and let $w^- \in \mathbb{R}^m$ be defined by $w_i^- = \max(0, -w_i)$.

Lemma 1. For any $(w, u_0, v_0) \in \mathbb{R}^{m+2}$ such that $wA + u_0c^1 - v_0c^2 = 0$ and $u_0, v_0 \geq 0$, the inequality $(w^+A + u_0c^1)x \leq \max\{w^+b + u_0d_1, w^-b + v_0d_2\}$ is valid for $\overline{\text{conv}}(P_1 \cup P_2)$.

Proof. Since P_1 and P_2 are nonempty, the inequality $\alpha x \leq \beta$ is valid for $\overline{\text{conv}}(P_1 \cup P_2)$ if and only if there exists multipliers $u, v \in \mathbb{R}^m$, $u_0, v_0 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha &= uA + u_0c^1 \\ \alpha &= vA + v_0c^2 \\ \beta &\geq ub + u_0d_1 \\ \beta &\geq vb + v_0d_2 \\ u, v, u_0, v_0 &\geq 0.\end{aligned}$$

Given (w, u_0, v_0) such that $wA + u_0c^1 - v_0c^2 = 0$ and $u_0, v_0 \geq 0$, choose $u = w^+$, $v = w^-$, $\alpha = w^+A + u_0c^1$ and $\beta = \max\{w^+b + u_0d_1, w^-b + v_0d_2\}$ and notice that this satisfies the system above. \square

Given a set of vectors $X \subseteq \mathbb{R}^m$, we say that X is *minimally dependent* if the vectors in X are linearly dependent but the vectors in every proper subset of X are linearly independent.

Let $\mathcal{W}(A, c^1, c^2)$ be the set of all $(w, u_0, v_0) \in \mathbb{R}^{m+2}$ such that: (i) $u_0, v_0 > 0$; (ii) $wA + u_0c^1 - v_0c^2 = 0$; (iii) the set of vectors comprising c^1, c^2 and the rows of A relative to nonzero entries of w is minimally dependent. Note that $\mathcal{W}(A, c^1, c^2)$ is a finite set of vectors up to positive scaling. Indeed, finitely many row sets are considered by Condition (iii), and each of these sets gives rise to one ray.

Let $\mathcal{W}^a = \mathcal{W}(A, c^1, c^2)$ and $\mathcal{W}^b = \mathcal{W}((A, b), (c^1, d_1), (c^2, d_2))$ where (A, b) denotes the augmented matrix.

Theorem 2. The polyhedron $\overline{\text{conv}}(P_1 \cup P_2)$ is the set of all points in P satisfying

$$(w^+A + u_0c^1)x \leq \max\{w^+b + u_0d_1, w^-b + v_0d_2\} \quad \text{for every } (w, u_0, v_0) \in \mathcal{W}^a \cup \mathcal{W}^b. \quad (1)$$

Theorem 2 implies the following result of Conforti, Wolsey and Zambelli [1].

Corollary 3. Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and a split disjunction $\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$ where $\pi \in \mathbb{Z}^n, \pi_0 \in \mathbb{Z}$. Let $P_1 = P \cap \{\pi x \leq \pi_0\}$ and $P_2 = P \cap \{\pi x \geq \pi_0 + 1\}$. The polyhedron $\overline{\text{conv}}(P_1 \cup P_2)$ is the set of all points in P satisfying the inequalities

$$\pi x - s^- \frac{b - Ax}{\lceil sb \rceil - sb} \leq \pi_0,$$

for all $s \in \mathbb{R}^m$ such that $sA = \pi$, $\pi_0 < sb < \pi_0 + 1$, and the rows of A relative to nonzero entries of s are linearly independent.

This is the formula of the MIR inequalities of Nemhauser and Wolsey [3].

Proof. The case $(w, u_0, v_0) \in \mathcal{W}^a$ cannot occur in Theorem 2 since no valid inequality for $\overline{\text{conv}}(P_1 \cup P_2)$ has $w = 0$ and $c^1 = \pi$ and $c^2 = -\pi$ are already dependent. Thus Theorem 2 reduces to: $\overline{\text{conv}}(P_1 \cup P_2)$ is the set of all points in P satisfying the inequalities $(w^+ A + u_0 \pi)x \leq w^+ b + u_0 \pi_0$ for every $(w, u_0, v_0) \in \mathbb{R}^{m+2}$ such that: (i) $u_0, v_0 > 0$; (ii) $wA + u_0 \pi + v_0 \pi = 0$, $wb + u_0 \pi_0 + v_0(\pi_0 + 1) = 0$; (iii) the set of vectors comprising (π, π_0) , $(\pi, \pi_0 + 1)$ and the rows of (A, b) relative to nonzero entries of w is minimally dependent.

Since the inequality $(w^+ A + u_0 \pi)x \leq w^+ b + u_0 \pi_0$ is defined up to a positive scaling factor, we may assume $u_0 + v_0 = 1$. Set $s = -w$. Thus (ii) implies $sA = \pi$ and $sb = \pi_0 + v_0$. Since $u_0, v_0 > 0$ and $\pi_0 \in \mathbb{Z}$, it follows that $\pi_0 < sb < \pi_0 + 1$. Thus $(w^+ A + u_0 \pi)x \leq w^+ b + u_0 \pi_0$ can be rewritten as $\pi x - s^- \frac{b - Ax}{|sb| - sb} \leq \pi_0$. Note that (iii) and the equation $sA = \pi$ imply that the rows of A corresponding to nonzero entries of s are linearly independent. \square

We first prove Theorem 2 for the homogeneous case where all the right-hand sides are zero.

2 The Homogeneous Case

We consider a polyhedral cone $P = \{x \in \mathbb{R}^n : Ax \leq 0\}$ and a two-term disjunction $c^1 x \leq 0 \vee c^2 x \leq 0$. Let $P_1 = P \cap \{c^1 x \leq 0\}$ and $P_2 = P \cap \{c^2 x \leq 0\}$. Since both P_1 and P_2 are nonempty, an inequality $\alpha x \leq 0$ is valid for $\overline{\text{conv}}(P_1 \cup P_2)$ if and only if there exist multipliers $u, v \in \mathbb{R}^m$, $u_0, v_0 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha &= uA + u_0 c^1 \\ \alpha &= vA + v_0 c^2 \\ u, v, u_0, v_0 &\geq 0.\end{aligned}$$

We denote by C the pointed cone defined by the vectors (α, u, u_0, v, v_0) satisfying the above linear system. We will use the following facts.

Fact 4. Consider a point $(\alpha, u, u_0, v, v_0) \in C$. If either $u_0 = 0$ or $v_0 = 0$ then the inequality $\alpha x \leq 0$ is implied by $Ax \leq 0$.

Fact 5. Let M be a matrix. The rows of M are minimally dependent if and only if there exists a vector \bar{x} all of whose components are nonzero such that every solution of the system $xM = 0$ is a multiple of \bar{x} .

Lemma 6. A vector (α, u, u_0, v, v_0) in C such that $u_0, v_0 > 0$ is an extreme ray of C if and only if $u_i v_i = 0$, $i = 1, \dots, m$, and the following holds: the set of vectors comprising c^1, c^2 and the rows of A corresponding to positive entries of $u + v$ is minimally dependent.

Proof. Let (α, u, v, u_0, v_0) be a vector in C such that $u_0, v_0 > 0$. Then (α, u, v, u_0, v_0) is an extreme ray of C if and only if (u, v, u_0, v_0) is an extreme ray of the cone defined by

$$(u - v)A + u_0 c^1 - v_0 c^2 = 0 \tag{2}$$

$$u, v \geq 0 \tag{3}$$

By Fact 5, this is the case if and only if the set of rows of the matrix $\begin{pmatrix} A \\ -A \\ c^1 \\ -c^2 \end{pmatrix}$ corresponding to positive entries of (u, v, u_0, v_0) is minimally dependent. Since $u_0, v_0 > 0$, this happens if and only if $u_i v_i = 0$ for all $i = 1, \dots, m$ and the set of vectors comprising c^1, c^2 and the rows of A corresponding to positive entries of $u + v$ is minimally dependent. \square

Theorem 7. *The cone $\overline{\text{conv}}(P_1 \cup P_2)$ is the set of all points in P satisfying the inequalities*

$$(w^+ A + u_0 c^1)x \leq 0 \quad \text{for every } (w, u_0, v_0) \in \mathcal{W}(A, c^1, c^2).$$

Proof. To obtain a description of $\overline{\text{conv}}(P_1 \cup P_2)$ it suffices to consider inequalities $\alpha x \leq 0$ for which there exist multipliers u, v, u_0, v_0 such that (α, u, v, u_0, v_0) is an extreme ray of C . Furthermore, we can restrict our attention to inequalities $\alpha x \leq 0$ that are not valid for P . By Fact 4, this implies that $u_0, v_0 > 0$.

Lemma 6 implies that a vector $(\alpha, u, v, u_0, v_0) \in C$ such that $u_0, v_0 > 0$ is an extreme ray of C if and only if $u = w^+$ and $v = w^-$ for some $(w, u_0, v_0) \in \mathcal{W}(A, c^1, c^2)$. Given that $\alpha = uA + u_0 c^1$, the theorem follows. \square

3 The Polyhedral Case

Now we consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and a two-term disjunction $c^1 x \leq d_1 \vee c^2 x \leq d_2$. Let $P_1 = P \cap \{c^1 x \leq d_1\}$ and $P_2 = P \cap \{c^2 x \leq d_2\}$. Our assumptions are that $c^1, c^2 \neq 0$, (c^1, d_1) is not a multiple of (c^2, d_2) , and P_1 and P_2 are nonempty.

Let \tilde{P} be the following homogenized version of P : $\tilde{P} = \{(x, x_0) \in \mathbb{R}^{n+1} : Ax - bx_0 \leq 0, x_0 \geq 0\}$. Similarly, let $\tilde{P}_1 = \tilde{P} \cap \{c^1 x - d_1 x_0 \leq 0\}$ and $\tilde{P}_2 = \tilde{P} \cap \{c^2 x - d_2 x_0 \leq 0\}$.

Lemma 8. *The inequality $\alpha x \leq \beta$ is valid for $\overline{\text{conv}}(P_1 \cup P_2)$ if and only if the inequality $\alpha x - \beta x_0 \leq 0$ is valid for $\overline{\text{conv}}(\tilde{P}_1 \cup \tilde{P}_2)$.*

Proof. Assume $\alpha x \leq \beta$ is valid for $\overline{\text{conv}}(P_1 \cup P_2)$. Take $i \in \{1, 2\}$; we know that $\alpha x \leq \beta$ is valid for P_i , we show that $\alpha x - \beta x_0 \leq 0$ is valid for \tilde{P}_i . To see this, take $(x, x_0) \in \tilde{P}_i$. If $x_0 > 0$, then $(\frac{x}{x_0}, 1) \in P_i$ and thus $\alpha \frac{x}{x_0} \leq \beta$, or equivalently $\alpha x - \beta x_0 \leq 0$. If $x_0 = 0$, then $\begin{pmatrix} A \\ c^i \end{pmatrix} x \leq 0$ and so x is in the recession cone of P_i . Validity then implies that $\alpha x \leq 0$, or equivalently $\alpha x - \beta x_0 \leq 0$.

The converse is direct. \square

Proof of Theorem 2.

Let $\tilde{A} = \begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix}$, $\tilde{c}^1 = (c^1, -d_1)$ and $\tilde{c}^2 = (c^2, -d_2)$. From these definitions and employing Theorem 7 we have that

$$\overline{\text{conv}}(\tilde{P}_1 \cup \tilde{P}_2) = \tilde{P} \cap \left\{ (x, x_0) : (\tilde{w}^+ \tilde{A} + u_0 \tilde{c}^1) \begin{pmatrix} x \\ x_0 \end{pmatrix} \leq 0 \text{ for all } (\tilde{w}, u_0, v_0) \in \mathcal{W}(\tilde{A}, \tilde{c}^1, \tilde{c}^2) \right\}.$$

Now using Lemma 8 we get that

$$\begin{aligned} \overline{\text{conv}}(P_1 \cup P_2) &= P \cap \left\{ x : (w, w_0)^+ \begin{pmatrix} A \\ 0 \end{pmatrix} x + (w, w_0)^+ \begin{pmatrix} -b \\ -1 \end{pmatrix} + u_0 c^1 x - u_0 d_1 \leq 0 \right. \\ &\quad \left. \text{for all } ((w, w_0), u_0, v_0) \in \mathcal{W}(\tilde{A}, \tilde{c}^1, \tilde{c}^2) \right\} \\ &= P \cap \left\{ x : (w^+ A + u_0 c^1) x \leq w^+ b + u_0 d_1 + w_0^+ \right. \\ &\quad \left. \text{for all } ((w, w_0), u_0, v_0) \in \mathcal{W}(\tilde{A}, \tilde{c}^1, \tilde{c}^2) \right\}. \end{aligned}$$

We claim that for $((w, w_0), u_0, v_0) \in \mathcal{W}(\tilde{A}, \tilde{c}^1, \tilde{c}^2)$ we have $w^+ b + u_0 d_1 + w_0^+ = \max\{w^+ b + u_0 d_1, w^- b + v_0 d_2\}$. This is because

$$(w, w_0) \begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} + u_0(c^1, -d_1) - v_0(c^2, -d_2) = 0,$$

which implies that $w_0 = -wb - u_0 d_1 + v_0 d_2$; noticing that $w = w^+ - w^-$, we get $w_0 = (w^- b + v_0 d_2) - (w^+ b + u_0 d_1)$, which gives the claim.

To complete the proof of the theorem, it remains to show that if $((w, w_0), u_0, v_0) \in \mathcal{W}(\tilde{A}, \tilde{c}^1, \tilde{c}^2)$ then $(w, u_0, v_0) \in \mathcal{W}^a \cup \mathcal{W}^b$. If $w_0 = 0$, then it follows from definitions that $(w, u_0, v_0) \in \mathcal{W}^b$. Now consider the case $w_0 > 0$. Let (A', b') be the submatrix of (A, b)

corresponding to the nonzero entries of w . We claim that if $N = \begin{pmatrix} 0 & -1 \\ A' & -b' \\ c^1 & -d_1 \\ c^2 & -d_2 \end{pmatrix}$ is minimally

dependent, then $M = \begin{pmatrix} A' \\ c^1 \\ c^2 \end{pmatrix}$ is also minimally dependent. Suppose that the rows of M

are not minimally dependent and let λ be a nonzero vector with $\lambda_j = 0$ such that $\lambda M = 0$. Using the same λ , we can find a multiplier λ_0 for the row $(0, -1)$ such that $(\lambda_0, \lambda)N = 0$, contradicting the fact that N is minimally dependent. \square

4 Complementarity Problems and Separation

In this section, we relate Theorem 2 to previous work of Judice, Serali, Ribeiro and Faustino [2], who proposed a valid inequality for the complementarity problem.

We first observe that our two-term disjunction problem is equivalent to the following complementarity problem: $Ax = b$, $x \geq 0$ and $x_1 \cdot x_2 = 0$.

We show how to transform the model $Ax \leq b$ with the disjunction $c^1 x \leq d_1 \vee c^2 x \leq d_2$ into the above form. First, unrestricted variables x_i can be replaced by the difference of two nonnegative variables. By adding nonnegative slacks, the inequality system $Ax \leq b$ can be transformed into equality form. Next consider the disjunction $c^1 x \leq d_1 \vee c^2 x \leq d_2$. We add the two equations $c^1 x + s_1^+ - s_1^- = d_1$ and $c^2 x + s_2^+ - s_2^- = d_2$ to the current system, with $s_1^+, s_1^-, s_2^+, s_2^- \geq 0$, and we impose $s_1^- \cdot s_2^- = 0$.

Conversely, in a complementarity problem $Ax = b$, $x \geq 0$, $x_1 \cdot x_2 = 0$, the complementarity condition $x_1 \cdot x_2 = 0$ can be formulated as the disjunction $x_1 \leq 0 \vee x_2 \leq 0$.

Thus Theorem 2 can be used to describe the convex hull of feasible solutions to $Ax = b$, $x \geq 0$, $x_1 \cdot x_2 = 0$. Multipliers satisfying (ii) in $\mathcal{W}\left(\left(\begin{smallmatrix} A \\ -I \end{smallmatrix}\right), \begin{pmatrix} b \\ 0 \end{pmatrix}, (e^1, 0), (e^2, 0)\right)$ give valid inequalities for $Ax \leq b$, $x \geq 0$, $x_1 \cdot x_2 = 0$ and therefore for our complementarity problem (here the unit vectors e^1 and e^2 are considered as row vectors). Condition (ii) is

$$\mu A = \nu - u_0 e^1 + v_0 e^2 \quad (4)$$

$$\mu b = 0. \quad (5)$$

where μ are the multipliers associated with the constraint $Ax \leq b$ and ν the multipliers associated with $-x \leq 0$.

Using $Ax = b$, cut (1) becomes

$$\sum_j \nu_j^+ x_j - u_0 x_1 \geq 0 \quad (6)$$

Assume that we have a tableau for problem $Ax = b$, $x \geq 0$ where x_1 and x_2 are both basic and strictly positive:

$$x_1 + \sum_{j \in N} \bar{a}_{1j} x_j = \bar{b}_1 \quad (7)$$

$$x_2 + \sum_{j \in N} \bar{a}_{2j} x_j = \bar{b}_2. \quad (8)$$

We can use formula (6) to cut off the current solution \bar{x} which is infeasible since $\bar{x}_1 \cdot \bar{x}_2 = \bar{b}_1 \cdot \bar{b}_2 > 0$. Multiplying (7) by $1/\bar{b}_1$ and (8) by $1/\bar{b}_2$ and subtracting we get

$$-\frac{x_1}{\bar{b}_1} + \frac{x_2}{\bar{b}_2} + \sum_{j \in N} \left(\frac{\bar{a}_{2j}}{\bar{b}_2} - \frac{\bar{a}_{1j}}{\bar{b}_1} \right) x_j = 0. \quad (9)$$

This equality is obtained from the system $Ax = b$ using multipliers μ such that $\mu b = 0$. Let ν, u_0, v_0 satisfy (4). Since μA are the coefficients in the left-hand-side of (9), this implies $u_0 = 1/\bar{b}_1$, $v_0 = 1/\bar{b}_2$, and $\nu_j = \left(\frac{\bar{a}_{2j}}{\bar{b}_2} - \frac{\bar{a}_{1j}}{\bar{b}_1} \right)$ for all $j \in N$. As observed above, these multipliers will produce a valid inequality using formula (6). The corresponding cut is

$$-\frac{x_1}{\bar{b}_1} + \sum_{j \in N} \left(\frac{\bar{a}_{2j}}{\bar{b}_2} - \frac{\bar{a}_{1j}}{\bar{b}_1} \right)^+ x_j \geq 0.$$

This is the cut of Judice, Sherali, Ribeiro and Faustino [2].

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