Slow and Stale Gradients Can Win the Race: Error-Runtime Trade-offs in Distributed SGD

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Abstract

Distributed Stochastic Gradient Descent (SGD) when run in a synchronous manner, suffers from delays in waiting for the slowest learners (stragglers). Asynchronous methods can alleviate stragglers, but cause gradient staleness that can adversely affect convergence. In this work we present the first theoretical characterization of the speed-up offered by asynchronous methods by analyzing the trade-off between the error in the trained model and the actual training runtime (wall-clock time). The novelty in our work is that our runtime analysis considers random straggler delays, which helps us design and compare distributed SGD algorithms that strike a balance between stragglers and staleness. We also present a new convergence analysis of asynchronous SGD variants without bounded or exponential delay assumptions, and a novel learning rate schedule to compensate for gradient staleness.

1 INTRODUCTION

Stochastic gradient descent (SGD) is the backbone of most state-of-the-art machine learning algorithms. Thus, improving the stability and convergence rate of SGD algorithms is critical for making machine learning algorithms fast and efficient.

Traditionally SGD is run serially at a single node. However, for massive datasets, running SGD serially at a single server can be *prohibitively* slow. A solution that has proved successful in recent years is to parallelize the training across many learners (processing units).

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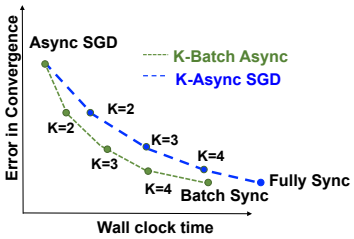

Figure 1: SGD variants span the error-runtime trade-off between fully Sync-SGD and fully Async-SGD. K is the number of learners or mini-batches the PS waits for before updating the model parameters, as we elaborate in Section 2.

This method was first used at a large-scale in Google's DistBelief [Dean et al., 2012] which used a central parameter server (PS) to aggregate gradients computed by learner nodes. While parallelism dramatically speeds up training, distributed machine learning frameworks face several challenges such as:

Straggling Learners. In synchronous SGD, the PS waits for all learners to push gradients before it updates the model parameters. Random delays in computation (referred to as straggling) are common in today's distributed systems [Dean and Barroso, 2013]. Waiting for slow and straggling learners can diminish the speed-up offered by parallelizing the training.

Gradient Staleness. To alleviate the problem of stragglers, SGD can be run in an asynchronous manner, where the central parameters are updated without waiting for all learners. However, learners may return *stale* gradients that were evaluated at an older version of the model, and this can make the algorithm unstable.

The key contributions of this work are listed below.

1. Most SGD algorithms optimize the trade-off between training error, and the number of iterations or epochs. However, the wallclock time per iteration is a random variable that depends on the gradient aggregation algorithm. We present the first rigorous analysis of the trade-off between error and the

actual runtime (instead of iterations). This analysis is then used to compare different SGD variants such as K-sync SGD, K-async SGD and K-batch-async SGD, as illustrated in Figure 1.

2. We present a new convergence analysis of asynchronous SGD and its variants, where we relax several commonly made assumptions such as bounded delays and gradients, exponential service times, and independence of the staleness process.

3. We propose a novel learning rate schedule to compensate for gradient staleness, and improve the stability and convergence of asynchronous SGD, while preserving its fast runtime.

1.1 RELATED WORKS

Single Node SGD: Analysis of gradient descent dates back to classical works [Boyd and Vandenberghe, 2004] in the optimization community. The problem of interest is the minimization of empirical risk of the form:

$$\min_{\mathbf{w}} \left\{ F(\mathbf{w}) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{w}, \xi_n) \right\}. \tag{1}$$

Here, ξ_n denotes the $n-$th data point and its label where $n = 1, 2, \ldots, N$, and $f(\mathbf{w}, \xi_n)$ denotes the composite loss function. Gradient descent is a way to iteratively minimize this objective function by updating the parameter $\mathbf{w}$ in the opposite direction of the gradient of $F(\mathbf{w})$ at every iteration, as given by:

$$\mathbf{w}_{j+1} = \mathbf{w}_j - \eta \nabla F(\mathbf{w}_j) = \mathbf{w}_j - \frac{\eta}{N} \sum_{n=1}^{N} \nabla f(\mathbf{w}_j, \xi_n).$$

The computation of $\sum_{n=1}^{N} \nabla f(\mathbf{w}_j, \xi_n)$ over the entire dataset is expensive. Thus, stochastic gradient descent [Robbins and Monro, 1951] with mini-batching is generally used in practice, where the gradient is evaluated over small, randomly chosen subsets of the data. Smaller mini-batches result in higher variance of the gradients, which affects convergence and error floor [Bottou et al., 2016, Dekel et al., 2012, Li et al., 2014]. Algorithms such as AdaGrad [Duchi et al., 2011] and Adam [Kingma and Ba, 2015] gradually reduce learning rate to achieve a lower error floor. Another class of algorithms includes stochastic variation reduction techniques that include SVRG [Johnson and Zhang, 2013], SAGA [Roux et al., 2012] and their variants listed out in [Nguyen et al., 2017]. For a detailed survey of different SGD variants, refer to [Ruder, 2016].

Synchronous SGD and Stragglers: To process large datasets, SGD is parallelized across multiple learners with a central PS. Each learner processes one mini-batch, and the PS aggregates all the gradients. The convergence of synchronous SGD is same as mini-batch SGD, with a P-fold larger mini-batch, where P is the number of learners. However, the time per iteration grows with the number of learners, because some straggling learners that slow down randomly [Dean and Barroso, 2013]. Thus, it is important to juxtapose the error reduction per iteration with the runtime per iteration to understand the true convergence speed of distributed SGD.

To deal with stragglers and speed up machine learning, system designers have proposed several straggler mitigation techniques such as [Harlap et al., 2016] that try to detect and avoid stragglers. An alternate direction of work is to use redundancy techniques as proposed in [Dutta et al., 2016, Lee et al., 2017, Tandon et al., 2017, Wang et al., 2015] to ignore the stragglers altogether.

Asynchronous SGD and Staleness: A complementary approach to deal with the issue of straggling is to use asynchronous SGD. In asynchronous SGD, any learner can evaluate the gradient and update the central PS without waiting for the other learners. Asynchronous variants of existing SGD algorithms have also been proposed and implemented in systems [Cipar et al., 2013, Dean et al., 2012, Gupta et al., 2016].

In general, analyzing the convergence of asynchronous SGD with the number of iterations is difficult in itself because of the randomness of gradient staleness. There are only a few pioneering works such as [Chaturapruek et al., 2015, Lian et al., 2015, Mania et al., 2017, Mitliagkas et al., 2016, Recht et al., 2011, Tsitsiklis et al., 1986] in this direction. In [Tsitsiklis et al., 1986], a fully decentralized analysis was proposed that considers no central PS. In [Recht et al., 2011], a new asynchronous algorithm called Hogwild was proposed and analyzed under bounded gradient assumption that has been followed upon by several works such as [Lian et al., 2015, Mania et al., 2017]. In Hogwild, every learner only updates a part of the central parameter vector $\mathbf{w}$ and is thus essentially different in spirit from conventional asynchronous SGD [Lian et al., 2015] where every learner operates on the entire $\mathbf{w}$.

1.2 OUR CONTRIBUTIONS

Existing machine learning algorithms mostly try to optimize the trade-off of error with the number of iterations, epochs or "work complexity" [Bottou et al., 2016]. Time to complete a task has traditionally been calculated in terms of work complexity measures [Sedgewick and Wayne, 2011], where the time taken to complete a task is a deterministic function of the size of the task (number of operations). However, due to straggling and synchronization bottle-necks in the system, the same task can often take different time to compute

across different learners or iterations. To the best of our knowledge, the theoretical trade-off of error with runtime modelling runtimes as random variables has not been studied. We bring statistical perspective to the traditional work complexity analysis that incorporates the randomness introduced due to straggling. In this paper, we provide a systematic approach to analyze the error with runtime for both synchronous and asynchronous SGD, and some variants like K-sync, K-batch-sync, K-async and K-batch-async SGD.

We also propose a new error convergence analysis for Async and K-async SGD that holds for strongly convex objectives and can also be extended to non-convex formulations. In this analysis we relax the bounded delay assumption in [Chaturapruek et al., 2015, Lian et al., 2015] and the bounded gradient assumption in [Recht et al., 2011]. We also remove the assumption of exponential computation time and the staleness process being independent of the parameter values [Mitliagkas et al., 2016] as we will elaborate in Section 3.2. Interestingly, our analysis also brings out the regimes where asynchrony can be better or worse than synchrony in terms of speed of convergence. Further, we propose a new learning rate schedule to compensate for staleness, and stabilize asynchronous SGD.

The rest of the paper is organized as follows. Section 2 describes our problem formulation introducing the system model and assumptions. Section 3 provides the main results of the paper – analytical characterization of runtime, new convergence analysis for Async and K-async SGD and the proposed learning rate schedule to compensate for staleness. The analysis of runtime is elaborated further in Section 4. Proofs and detailed discussions are presented in the Supplement.

2 PROBLEM FORMULATION

Our objective is to minimize the risk function of the parameter vector $\mathbf{w}$ as mentioned in (1) given N training samples. Let S denote the total set of N training samples, *i.e.*, a collection of some data points with their corresponding labels or values. We use the notation ξ to denote a random seed $\in S$ which consists of either a single data and its label or a single mini-batch (m samples) of data and their labels.

2.1 SYSTEM MODEL

We assume that there is a central parameter server (PS) with P parallel learners as shown in Figure 2. The learners fetch the current parameter vector $\mathbf{w}_j$ from the PS as and when instructed in the algorithm. Then they compute gradients using one mini-batch and push their gradients back to the PS as and when

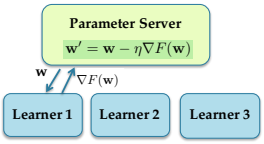

Figure 2: Parameter Server Model

instructed in the algorithm. At each iteration, the PS aggregates the gradients computed by the learners and updates the parameter $\mathbf{w}$. Based on how these gradients are fetched and aggregated, we have different variants of synchronous or asynchronous SGD.

The time taken by a learner to compute gradient of one mini-batch is denoted by random variable X_i for $i = 1, 2, \ldots, P$. We assume that the X_is are i.i.d. across mini-batches and learners.

2.2 PERFORMANCE METRICS

There are two metrics of interest: Runtime and Error

Definition 1 (Runtime)**.** *The runtime of J iterations is the expected time to perform a total of J iterations.*

Definition 2 (Error)**.** *The Error after j iterations is defined as $\mathbb{E}\left[F(\mathbf{w}_j) - F^*\right]$, the expected gap of the risk function from its optimal value.*

Our aim is to determine the trade-off between the error (measures the accuracy of the algorithm) and the runtime for the different SGD variants.

2.3 VARIANTS OF SGD

We now describe the SGD variants considered in this paper. Please refer to Figure 3 and Figure 4 for a pictorial illustration.

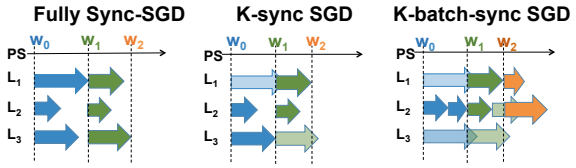

Figure 3: For $K = 2$ and $P = 3$, we illustrate the K-sync and K-batch-sync SGD in comparison with fully synchronous SGD. Lightly shaded arrows indicate straggling gradient computations that are cancelled.

K-sync SGD: This is a generalized form of synchronous SGD, also suggested in [Chen et al., 2016, Gupta et al., 2016] to offer some resilience to straggling as the PS does not wait for all the learners to finish. The PS only waits for the first K out of P learners to push their gradients. Once it receives K gradients, it updates $\mathbf{w}_j$ and cancels the remaining learners. The updated parameter vector $\mathbf{w}_{j+1}$ is sent to all P learners

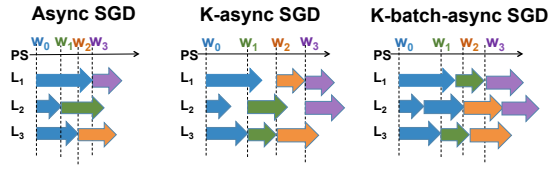

Figure 4: For $K = 2$ and $P = 3$, we illustrate the K-async and K-batch-async algorithms in comparison with fully asynchronous SGD.

for the next iteration. The update rule is given by:

$$\mathbf{w}_{j+1} = \mathbf{w}_j - \frac{\eta}{K} \sum_{l=1}^{K} g(\mathbf{w}_j, \xi_{l,j}). \tag{2}$$

Here $\xi_{l,j}$ denotes the mini-batch of m samples used by the l-th learner at the j-th iteration and $g(\mathbf{w}_j, \xi_{l,j}) = \frac{1}{m} \sum_{\xi \in \xi_{l,j}} \nabla f(\mathbf{w}_j, \xi)$ denotes the average gradient of the loss function evaluated over the mini-batch $\xi_{l,j}$ of size m. For $K = P$, the algorithm is exactly equivalent to a fully synchronous SGD with P learners.

K-batch-sync: In K-batch-sync, all the P learners start computing gradients with the same $\mathbf{w}_j$. Whenever any learner finishes, it pushes its update to the PS and evaluates the gradient on the next mini-batch at the same $\mathbf{w}_j$. The PS updates using the first K mini-batches that finish and cancels the remaining learners. Theoretically, the update rule is still the same as (2) but here l now denotes the index of the mini-batch instead of the learner. However K-batch-sync will offer advantages over K-sync in runtime as no learner is idle.

K-async SGD: This is a generalized version of asynchronous SGD, also suggested in [Gupta et al., 2016]. In K-async SGD, all the P learners compute their respective gradients on a single mini-batch. The PS waits for the first K out of P that finish first, but it does not cancel the remaining learners. As a result, for every update the gradients returned by each learner might be computed at a stale or older value of the parameter $\mathbf{w}$. The update rule is thus given by:

$$\mathbf{w}_{j+1} = \mathbf{w}_j - \frac{\eta}{K} \sum_{l=1}^{K} g(\mathbf{w}_{\tau(l,j)}, \xi_{l,j}). \tag{3}$$

Here $\xi_{l,j}$ is one mini-batch of m samples used by the l-th learner at the j-th iteration and $\tau(l,j)$ denotes the iteration index when the l-th learner last read from the central PS where $\tau(l,j) \leq j$. Also, $g(\mathbf{w}_{\tau(l,j)}, \xi_{l,j}) = \frac{1}{m} \sum_{\xi \in \xi_{l,j}} \nabla f(\mathbf{w}_{\tau(l,j)}, \xi_{l,j})$ is the average gradient of the loss function evaluated over the mini-batch $\xi_{l,j}$ based on the stale value of the parameter $\mathbf{w}_{\tau(l,j)}$. For $K = 1$, the algorithm is exactly equivalent to fully asynchronous SGD, and the update rule can be simplified as:

$$\mathbf{w}_{j+1} = \mathbf{w}_j - \eta g(\mathbf{w}_{\tau(j)}, \xi_j). \tag{4}$$

Here ξ_j denotes the set of samples used by the learner that updates at the j-th iteration such that $|\xi_j| = m$ and $\tau(l,j)$ denotes the iteration index when that particular learner last read from the central PS. Note that $\tau(j) \leq j$.

K-batch-async: Observe in Figure 4 that K-async also suffers from some learners being idle while others are still working on their gradients until any K finish. In K-batch-async (proposed in [Lian et al., 2015]), the PS waits for K mini-batches before updating itself but irrespective of which learner they come from. So wherever any learner finishes, it pushes its gradient to the PS, fetches current parameter at PS and starts computing gradient on the next mini-batch based on the current value of the PS. Surprisingly, the update rule is again similar to (3) theoretically except that now l denotes the indices of the K mini-batches that finish first instead of the learners and $\mathbf{w}_{\tau(l,j)}$ denotes the version of the parameter when the learner computing the $l-$th mini-batch last read from the PS. While the error convergence of K-batch-async is similar to K-async, it reduces runtime as no learner is idle.

2.4 ASSUMPTIONS

Closely following [Bottou et al., 2016], we also make the following assumptions:

1. $F(\mathbf{w})$ is an $L-$ smooth function. Thus,

$$||\nabla F(\mathbf{w}_1) - \nabla F(\mathbf{w}_2)||_2 \leq L||\mathbf{w}_1 - \mathbf{w}_2||_2. \tag{5}$$

2. $F(\mathbf{w})$ is strongly convex with parameter c. Thus,

$$2c(F(\mathbf{w}) - F^*) \leq ||\nabla F(\mathbf{w})||_2^2 \quad \forall\ \mathbf{w}. \tag{6}$$

   Refer to supplement for discussion on strong convexity and extension to non-convex objectives.

3. The stochastic gradient is an unbiased estimate of the true gradient:

$$\mathbb{E}_{\xi_j|\mathbf{w}_k}\left[g(\mathbf{w}_k, \xi_j)\right] = \nabla F(\mathbf{w}_k) \quad \forall\ k \leq j. \tag{7}$$

   Observe that this is slightly different from the assumption stated in [Bottou et al., 2016] which says $\mathbb{E}_{\xi_j}\left[g(\mathbf{w}, \xi_j)\right] = \nabla F(\mathbf{w})$ for all $\mathbf{w}$. Observe that all $\mathbf{w}_j$ for $j > k$ is actually not independent of the data ξ_j. We thus make the assumption more rigorous by conditioning on $\mathbf{w}_k$ for $k \leq j$. Our requirement $k \leq j$ means that $\mathbf{w}_k$ is the value of the parameter at the PS before the data ξ_j was accessed and can thus be assumed to be independent of the data ξ_j.

4. Similar to the previous assumption, we also assume that the variance of the stochastic update given $\mathbf{w}_k$

Table 1: List of Notations

Mini-batch Size	m
Total Iterations	J
Number of learners (Processors)	P
Number of learners to wait for	K
Learning rate	η
Lipschitz Constant	L
Strong-convexity parameter	c
Runtime of a learner for one mini-batch	X_i
Total runtime	T

at iteration k before the data point was accessed is also bounded as follows:

$$\mathbb{E}_{\xi_j|\mathbf{w}_k}\left[||g(\mathbf{w}_k,\xi_j)-\nabla F(\mathbf{w}_k)||_2^2\right] \leq \frac{\sigma^2}{m}+\frac{M_G}{m}||\nabla F(\mathbf{w}_k)||_2^2 \;\forall\; k\leq j. \quad (8)$$

3 MAIN RESULTS

3.1 RUNTIME ANALYSIS

We compare the theoretical wall clock runtime of the different SGD variants to illustrate the speed-up offered by different asynchronous and batch variants. A detailed discussion is provided in Section 4.

Theorem 1. *Let the wall clock time of each learner to process a single mini-batch be i.i.d. random variables $X_1, X_2, \ldots, X_P$. Then the ratio of the expected time of synchronous to asynchronous SGD is*

$$\frac{\mathbb{E}\left[T_{Sync}\right]}{\mathbb{E}\left[T_{Async}\right]} = P\frac{\mathbb{E}\left[X_{P:P}\right]}{\mathbb{E}\left[X\right]}$$

where $X_{(P:P)}$ is the P^{th} order statistic of P i.i.d. random variables $X_1, X_2, \ldots, X_P$.

This is the first result that analytically characterizes the speed-up offered by asynchronous SGD. To prove this result, we use ideas from renewal theory as we discuss in Section 4. In the following corollary, we highlight this speed-up for the special case of exponential computation time.

Corollary 1. *Let the wall clock time of each learner to process a single mini-batch be i.i.d. exponential random variables $X_1, X_2, \ldots, X_P \sim \exp(\mu)$. Then the ratio of the expected time of synchronous to asynchronous is approximately given by $P\log P$.*

Thus, the speed-up scales with P and can diverge to infinity for large P. We illustrate the speed-up for different distributions in Figure 5.

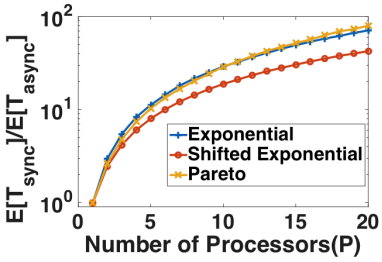

Figure 5: Plot of the speed-up using asynchronous over synchronous: $\log \frac{\mathbb{E}[T_{Sync}]}{\mathbb{E}[T_{Async}]}$ with P for different distributions - $\exp(1)$, $1+\exp(1)$ and $Pareto(2,1)$.

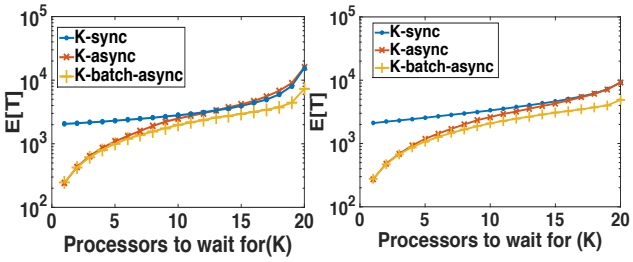

Figure 6: Plot of runtime $\mathbb{E}[T]$ for 2000 iterations: (Left) Pareto distribution $Pareto(2,1)$ and (Right) Shifted exponential distribution $1+\exp(1)$.

The next result illustrates the advantages offered by K-batch-sync and async over their corresponding counterparts K-sync and K-async respectively.

Theorem 2. *Let the wall clock time of each learner to process a single mini-batch be i.i.d. exponential random variables $X_1, X_2, \ldots, X_P \sim \exp(\mu)$. Then the ratio of the expected time of K-async (or sync) SGD to K-batch-async (or sync) SGD is given by*

$$\frac{\mathbb{E}\left[T_{K-async}\right]}{\mathbb{E}\left[T_{K-batch-async}\right]} = \frac{P\mathbb{E}\left[X_{K:P}\right]}{K\mathbb{E}\left[X\right]} \approx \frac{P\log\frac{P}{P-K}}{K}$$

where $X_{K:P}$ is the K^{th} order statistic of i.i.d. random variables $X_1, X_2, \ldots, X_P$.

To prove this, we derive an exact expression for the runtime of K-batch-async SGD, for *any* given distribution X, not necessarily exponential. The runtime is given by $\frac{JK\mathbb{E}[X]}{P}$ as we derive in Section 4 (Lemma 4) using ideas from renewal theory.

Theorem 2 shows that as $\frac{K}{P}$ increases, the speed-up using K-batch-async increases and can be upto $\log P$ times higher. For non-exponential distributions, we simulate the behaviour of $\mathbb{E}[T]$ in Figure 6 for K-sync, K-async and K-batch-async respectively for Pareto and Shifted Exponential.

3.2 ERROR ANALYSIS UNDER FIXED LEARNING RATE

Theorem 3 below gives a convergence analysis of K-async SGD for fixed η, relaxing the following assumptions in existing literature.

- [Mitliagkas et al., 2016] assumes that X_i's are exponentially distributed. Our analysis holds for any general service time X_i.
- [Mitliagkas et al., 2016] also assumes that the staleness process is independent of $\mathbf{w}$. While this assumption simplifies the analysis greatly, it is not true in practice. For instance, for a two learner case, the parameter $\mathbf{w}_2$ after 2 iterations depends on whether the update from $\mathbf{w}_1$ to $\mathbf{w}_2$ was based on a stale gradient at $\mathbf{w}_0$ or the current gradient at $\mathbf{w}_1$, depending on which learner finished first. In this work, we remove this independence assumption.
- Instead of the bounded delay assumption in [Lian et al., 2015], we use a general staleness bound $\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \leq \gamma \mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right]$, which allows for large, but rare delays.
- In [Recht et al., 2011], the norm of the gradient is assumed to be bounded. However, if we assume that $||\nabla F(\mathbf{w})||_2^2 \leq M$ for some constant M, then using (6) we obtain $||\mathbf{w} - \mathbf{w}^*||_2^2 \leq \frac{2}{c}(F(\mathbf{w}) - F^*) \leq \frac{M}{c^2}$ implying that $\mathbf{w}$ itself is bounded which is a very strong and restrictive assumption, that we relax in this result.

Theorem 3. *Suppose the objective $F(\mathbf{w})$ is c-strongly convex and the learning rate $\eta \leq \frac{1}{2L(\frac{M_G}{Km}+\frac{1}{K})}$. Also assume that $\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \leq \gamma \mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right]$, for some $\gamma \leq 1$. Then, the error of K-async SGD after J iterations is,*

$$\mathbb{E}\left[F(\mathbf{w}_J)\right] - F^* \leq \frac{\eta L \sigma^2}{2c\gamma' Km} + (1 - \eta c\gamma')^J \left(\mathbb{E}\left[F(\mathbf{w}_0)\right] - F^* - \frac{\eta L \sigma^2}{2c\gamma' Km}\right) \quad (9)$$

where $\gamma' = 1 - \gamma + \frac{p_0}{2}$ and p_0 is a lower bound on the conditional probability that $\tau(l,j) = j$, given all the past delays and parameters.

Here, γ is a measure of staleness of the gradients returned by learners; smaller γ indicates a less staleness.

We use the following lemma to prove Theorem 3. The proof is given in the Supplement.

Lemma 1. *Suppose that $p_0^{(l,j)}$ is the conditional probability that $\tau(l,j) = j$ given all the past delays and all the previous $\mathbf{w}$, and $p_0 \leq p_0^{(j)}$ for all j. Then,*

$$\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \geq p_0 \mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right]. \quad (10)$$

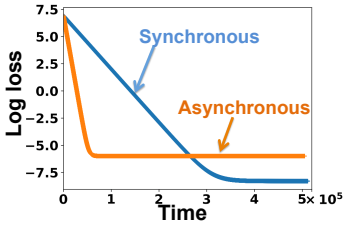

Figure 7: Theoretical error-runtime trade-off for Sync and Async-SGD with same η. Async-SGD has faster decay with time but a higher error floor.

Proof. By the law of total expectation,

$$\begin{aligned}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] &= p_0^{(l,j)} \mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2 | \tau(j) = j\right] \\ &\quad + (1 - p_0^{(l,j)}) \mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2 | \tau(j) \neq j\right] \\ &\geq p_0 \mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right].\end{aligned}$$

□

For the exponential distribution, p_0 is invariant of j and is equal to $\frac{1}{P}$ as we discuss in Supplement. Thus, p_0 can be taken as $\frac{1}{P}$. For non-exponential distributions, it is a constant in $[0, 1]$. For some special classes of distributions like new-longer-than-used Definition 3 (new-shorter-than-used) we can show that p_0 lies in $[0, \frac{1}{P}]$ ($[\frac{1}{P}, 1]$) respectively as discussed in Supplement.

For K-batch-async, the update rule is same as K-async except that the index l denotes the index of the mini-batch. Thus, the error analysis will be exactly similar. Our analysis can also be extended to non-convex $F(\mathbf{w})$ as we show in the Supplement.

Now let us compare with K-sync SGD. We observe that the analysis of K-sync SGD is same as serial SGD with mini-batch size Km. Thus,

Lemma 2 (Error of K-sync)**.** *[Bottou et al., 2016] Suppose that the objective $F(\mathbf{w})$ is c-strongly convex and learning rate $\eta \leq \frac{1}{2L(\frac{M_G}{Km}+1)}$. Then, the error after J iterations of K-sync SGD is*

$$\mathbb{E}\left[F(\mathbf{w}_J) - F^*\right] \leq \frac{\eta L \sigma^2}{2c(Km)} + (1 - \eta c)^J \left(F(\mathbf{w}_0) - F^* - \frac{\eta L \sigma^2}{2c(Km)}\right).$$

Can stale gradients win the race? For the same η, observe that the error given by Theorem 3 decays at the rate $(1 - \eta c(1 - \gamma + \frac{p_0}{2}))$ for K-async or K-batch-async SGD while for K-sync, the decay rate with number of iterations is $(1 - \eta c)$. Thus, depending on the values of γ and p_0, the decay rate of K-async or K-batch-async SGD can be faster or slower than K-sync SGD. The decay rate of K-async or K-batch-async SGD is faster if $\frac{p_0}{2} > \gamma$. As an example, one might consider an exponential or new-shorter-than-used service time

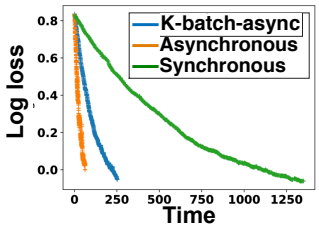

Figure 8: Error-runtime trade-off comparison of different SGD variants for logistic regression on MNIST, with $X_i \sim \exp(1)$, $P = 8$, $K = 4$, $\eta = 0.01$ and $m = 1$. K-batch-async gives intermediate performance, between Async and sync-SGD.

where $p_0 \geq \frac{1}{P}$ and γ can be made smaller by increasing K. It might be noted that asynchronous SGD can still be faster than synchronous SGD with respect to wall clock time even if its decay rate with respect to number of iterations is lower as every iteration is much faster in asynchronous SGD (Roughly $P \log P$ times faster for exponential service times).

The maximum allowable learning rate for synchronous SGD is $\max\{\frac{1}{c}, \frac{1}{2L(\frac{M_G}{Pm}+1)}\}$ which can be much higher than that for asynchronous SGD,*i.e.*, $\max\{\frac{1}{c(1-\gamma+\frac{p_0}{2})}, \frac{1}{2L(\frac{M_G}{m}+1)}\}$. Similarly the error-floor for synchronous is $\frac{\eta L \sigma^2}{2cPm}$ as compared to asynchronous whose error floor is $\frac{\eta L \sigma^2}{2c(1-\gamma+\frac{p_0}{2})m}$.

In Figure 7, we compare the theoretical trade-offs between synchronous ($K = P$ in Lemma 2) and asynchronous SGD ($K = 1$ in Theorem 3). Async-SGD converges very quickly, but to a higher floor. Figure 8 shows the same comparison on the MNIST dataset, along with K-batch-async SGD.

3.3 VARIABLE LEARNING RATE FOR STALENESS COMPENSATION

The staleness of the gradient is random, and can vary across iterations. Intuitively, if the gradient is less stale, we want to weigh it more while updating the parameter $\mathbf{w}$, and if it is more stale we want to scale down its contribution to the update. With this motivation, we propose the following condition on the learning rate at different iterations.

$$\eta_j \mathbb{E}\left[||\mathbf{w}_j - \mathbf{w}_{\tau(j)}||_2^2\right] \leq C \tag{11}$$

for a constant C. This condition is also inspired from our error analysis in Theorem 3, because it helps remove the assumption $\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \leq \gamma \mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right]$. Using (11), we obtain the following convergence result.

Theorem 4. *Suppose the learning rate in the j-th iteration $\eta_j \leq 1/2L(\frac{M_G}{m}+1)$, and $\eta_j \mathbb{E}\left[||\mathbf{w}_j - \mathbf{w}_{\tau(j)}||_2^2\right] \leq C$ for some constant C. Then, we have*

$$\mathbb{E}\left[F(\mathbf{w}_J)\right] - F^* \leq \Delta + \left(\mathbb{E}\left[F(\mathbf{w}_0)\right] - F^*\right) \prod_{j=1}^{J} (1 - \rho_j)$$

where $\rho_j = \eta_j(1 + \frac{p_0}{2})c$, and the error floor $\Delta = \Delta_J + (1-\rho_J)\Delta_{J-1} + \cdots + \prod_{j=1}^{J}(1-\rho_j)\Delta_0$, where $\Delta_j = \frac{\eta_j^2 L \sigma^2}{2m} + \frac{CL^2}{2}$.

The proof is provided in the Supplement. In our analysis of Asynchronous SGD, we observe that the term $\frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right]$ is the most difficult to bound. For fixed learning rate, we had assumed that $\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right]$ is bounded by $\gamma||\nabla F(\mathbf{w}_j)||_2^2$. However, if we impose the condition (11) on η, we do not require this assumption. Our proposed condition actually provides a bound for the staleness term as follows:

$$\frac{\eta_j}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \leq \frac{\eta_j L^2}{2}\mathbb{E}\left[||\mathbf{w}_j - \mathbf{w}_{\tau(j)}||_2^2\right] \leq \frac{CL^2}{2}. \tag{12}$$

Proposed Algorithmic Modification Inspired by this analysis, we propose the learning rate schedule,

$$\eta_j = \min\left\{\frac{C}{||\mathbf{w}_j - \mathbf{w}_{\tau(j)}||_2^2}, \eta_{max}\right\}, \tag{13}$$

where η_{max} is a suitably large ceiling on learning rate. It ensures stability when the first term in (13) becomes large due to the staleness $||\mathbf{w}_j - \mathbf{w}_{\tau(j)}||_2$ being small. The C is chosen of the same order as the desired error floor. To implement this schedule, the PS needs to store the last read model parameters for every learner. In Figure 9 we illustrate how this schedule can stabilize asynchronous SGD.

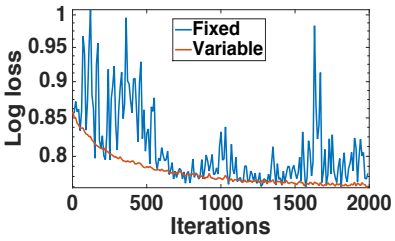

Figure 9: Async-SGD on CIFAR10 dataset, with $X \sim \exp 20$ and $P = 40$ learners. We compare fixed $\eta = 0.01$, and the variable schedule given in (13) for $\eta_{max} = 0.01$ and $C = 0.005\eta_{max}$. Observe that the proposed schedule can give fast convergence, and also maintain stability, while the fixed η algorithm becomes unstable.

4 RUNTIME ANALYSIS

In this section, we provide our analysis of the runtime of different variants of SGD. These lemmas are used in

the proofs of Theorem 1 and Theorem 2.

4.1 RUNTIME OF K-SYNC SGD

Lemma 3 (Runtime of K-sync SGD)**.** *The expected time taken by K-sync SGD to complete J iterations is,*

$$\mathbb{E}[T] = J\mathbb{E}[X_{K:P}] \tag{14}$$

where $X_{K:P}$ is the K^{th} order statistic of P i.i.d. random variables $X_1, X_2, \ldots, X_P$.

Comment: For $X_i \sim \exp(\mu)$, the runtime for J iterations is $\mathbb{E}[T] = \frac{J}{\mu}\sum_{i=P-K+1}^{P}\frac{1}{i} \approx \frac{J}{\mu}\left(\log\frac{P}{P-K}\right)$ [Sheldon, 2002]. Refer to supplement for justifications.

The runtime of K-batch-sync SGD is not tractable in general, but for $X_i \sim \exp(\mu)$, the time per iteration is distributed as $Erlang(K, P\mu)$. Thus, $\mathbb{E}[T] = J\frac{K}{P\mu}$.

4.2 RUNTIME OF K-BATCH-ASYNC SGD

Lemma 4 (Runtime of K-batch-async SGD)**.** *The expected time taken by K-batch-async SGD for J iterations is given by:*

$$\mathbb{E}[T] = J\frac{K\mathbb{E}[X]}{P}. \tag{15}$$

To prove the result we use ideas from renewal theory. Refer to Supplement for discussion.

Proof of Lemma 4. For the i-th learner, let $\{N_i(t), t > 0\}$ be the number of times the i-th learner pushes its gradient to the PS over in time t. The time between two pushes is an independent realization of X_i. Thus, the inter-arrival times $X_i^{(1)}, X_i^{(2)}, \ldots$ are i.i.d. with mean inter-arrival time $\mathbb{E}[X_i]$. Using the elementary renewal theorem [Gallager, 2013, Chapter 5] we have,

$$\lim_{t\to\infty}\frac{\mathbb{E}[N_i(t)]}{t} = \frac{1}{\mathbb{E}[X_i]}. \tag{16}$$

Thus, the rate of gradient pushes by the i-th learner is $1/\mathbb{E}[X_i]$. As there are P learners, the rate of gradient pushes to the PS is

$$\lim_{t\to\infty}\sum_{i=1}^{P}\frac{\mathbb{E}[N_i(t)]}{t} = \sum_{i=1}^{P}\frac{1}{\mathbb{E}[X_i]} = \frac{P}{\mathbb{E}[X]}. \tag{17}$$

Every K pushes are one iteration. Thus, the time to complete J iterations or effectively JK pushes is given by $\mathbb{E}[T] = \frac{JK\mathbb{E}[X]}{P}$. □

For $K = 1$, K-batch-async reduces to asynchronous SGD, and its runtime $\mathbb{E}[T] = J\frac{\mathbb{E}[X]}{P}$.

Proof of Theorem 1. By taking the ratio of the runtimes in Lemma 3 with $K = P$ and Lemma 4 with $K = 1$, we get the result in Theorem 1. □

Corollary 1 also follows by substituting in Theorem 1 that for $X_i \sim \exp(\mu)$, $\mathbb{E}[X_{P:P}] = \sum_{i=1}^{P}\frac{1}{i\mu} \approx \frac{\log P}{\mu}$.

4.3 RUNTIME OF K-ASYNC SGD

The runtime of K-async SGD is not tractable for non-exponential X_i, but we obtain an upper bound on it for "new-longer-than-used" distributions, defined below.

Definition 3 (New-longer-than-used)**.** *A random variable is said to have a new-longer-than-used distribution if the following holds for all $t, u \geq 0$:*

$$\Pr(U > u + t | U > t) \leq \Pr(U > u)$$

Most of the continuous distributions we encounter like normal, exponential, gamma, beta are new-longer-than-used. Alternately, the hyper exponential distribution is new-shorter-than-used and it satisfies $\Pr(U > u+t|U > t) \geq \Pr(U > u)$ for all $t, u \geq 0$.

Lemma 5 (Runtime of K-async SGD)**.** *Suppose that each X_i has a new longer than used distribution. Then, the expected time taken to complete J iterations by K-async is upper-bounded as*

$$\mathbb{E}[T] \leq J\mathbb{E}[X_{K:P}] \tag{18}$$

where $X_{K:P}$ is the K^{th} order statistic of P i.i.d. random variables $X_1, X_2, \ldots, X_P$.

Proof of Theorem 2. For the exponential X_i, equality holds in (18) in Lemma 5, as we justify in the Supplement. The expectation can be derived as $\mathbb{E}[X_{K:P}] = \sum_{i=P-K+1}^{P}\frac{1}{i\mu} \approx \frac{\log(P/P-K)}{\mu}$. And for exponential computation times, the runtime of K-batch-async is given by $\mathbb{E}[T] = J\frac{K\mathbb{E}[X]}{P} = J\frac{K}{\mu P}$ from Lemma 4. □

5 CONCLUSIONS

The speed of distributed SGD depends on the error reduction per iteration, as well as the runtime per iteration. To the best of our knowledge, this paper presents the first runtime analysis of synchronous and asynchronous SGD, and their variants. When juxtaposed with the error analysis, we get error-runtime trade-offs that can be used to compare different SGD algorithms. We also give a new analysis of asynchronous SGD by relaxing some commonly made assumptions, and a novel learning rate schedule to compensate for gradient staleness. In the future we plan to explore methods to gradually increase synchrony, so that we can achieve fast convergence as well as low error floor.

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Slow and Stale Gradients Can Win the Race: Error-Runtime Trade-offs in Distributed SGD Supplement

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6 STRONG CONVEXITY DISCUSSION

Definition 4 (Strong-Convexity)**.** *A function $h(\mathbf{u})$ is defined to be c-strongly convex, if the following holds for all $\mathbf{u}_1$ and $\mathbf{u}_2$ in the domain:*

$$h(\mathbf{u}_2) \geq h(\mathbf{u}_1) + [\nabla h(\mathbf{u}_1)]^T(\mathbf{u}_2 - \mathbf{u}_1) + \frac{c}{2}||\mathbf{u}_2 - \mathbf{u}_1||_2^2$$

For strongly convex functions, the following result holds for all $\mathbf{u}$ in the domain of $h(.)$.

$$2c(h(\mathbf{u}) - h^*) \leq ||\nabla h(\mathbf{u})||_2^2 \tag{19}$$

The proof is derived in [Bottou et al., 2016]. For completeness, we give the sketch here.

Proof. Given a particular $\mathbf{u}$, let us define the quadratic function as follows:

$$q(\mathbf{u}') = h(\mathbf{u}) + \nabla h(\mathbf{u})^T(\mathbf{u}' - \mathbf{u}) + \frac{c}{2}||\mathbf{u}' - \mathbf{u}||_2^2$$

Now, $q(\mathbf{u}')$ is minimized at $\mathbf{u}' = \mathbf{u} - \frac{1}{c}\nabla h(\mathbf{u})$ and the value is $h(\mathbf{u}) - \frac{1}{2c}||\nabla h(\mathbf{u})||_2^2$. Thus, from the definition of strong convexity we now have,

$$\begin{aligned} h^* &\geq h(\mathbf{u}) + \nabla h(\mathbf{u})^T(\mathbf{u}' - \mathbf{u}) + \frac{c}{2}||\mathbf{u}' - \mathbf{u}||_2^2 \\ &\geq h(\mathbf{u}) - \frac{1}{2c}||\nabla h(\mathbf{u})||_2^2 \quad \text{[minimum value of } q(\mathbf{u}')\text{]} \end{aligned}$$

□

7 RUNTIME ANALYSIS PROOFS

Here we provide proofs for all results in Section 4.

7.1 Runtime of K-sync SGD

Proof of Lemma 3. We assume that the P learners have an i.i.d. computation times. When all the learners start together, and we wait for the first K out of P i.i.d. random variables to finish, the expected computation time for that iteration is $\mathbb{E}[X_{K:P}]$, where $X_{K:P}$ denotes the K-th statistic of P i.i.d. random variables $X_1, X_2, \ldots, X_P$. Thus, for J iterations, the runtime is given by $J\mathbb{E}[X_{K:P}]$. □

K-th statistic of exponential distributions Here we give a sketch of why the K-th order statistic of P exponentials scales as $\log(P/P-K)$. A detailed derivation can be obtained in [Sheldon, 2002]. Consider P i.i.d. exponential distributions with parameter μ. The minimum $X_{1:P}$ of P independent exponential random variables with parameter μ is exponential with parameter $P\mu$. Conditional on $X_{1:P}$, the second smallest value $X_{2:P}$ is distributed like the sum of $X_{1:P}$ and an independent exponential random variable with parameter $(P-1)\mu$. And so on, until the K-th smallest value $X_{K:P}$ which is distributed like the sum of $X_{(K-1):P}$ and an independent exponential random variable with parameter $(P-K+1)\mu$. Thus,

$$X_{K:P} = Y_P + Y_{P-1} + \cdots + Y_{P-K+1}$$

where the random variables Y_is are independent and exponential with parameter $i\mu$. Thus,

$$\mathbb{E}[X_{K:P}] = \sum_{i=P-K+1}^{P} \frac{1}{i\mu} = \frac{H_P - H_{P-K}}{\mu} \approx \frac{\log \frac{P}{P-K}}{\mu}.$$

Here H_P and H_{P-K} denote the P-th and $(P-K)$-th harmonic numbers respectively.

For the case where $K = P$, the expectation is given by,

$$\mathbb{E}[X_{P:P}] = \frac{1}{\mu}\sum_{i=1}^{P}\frac{1}{i} = \frac{1}{\mu}H_P \approx \frac{1}{\mu}\log P.$$

7.2 Runtime of K-batch-async SGD

The proof of Lemma 4, which gives the runtime of K-batch-async SGD is already provided in the main

paper in Section 4 using ideas from renewal theory. Here we include a discussion on renewal processes for completeness.

Definition 5 (Renewal Process). *A renewal process is an arrival process where the inter-arrival intervals are positive, independent and identically distributed random variables.*

Lemma 6 (Elementary Renewal Theorem). *[Gallager, 2013, Chapter 5] Let $\{N(t), t > 0\}$ be a renewal counting process denoting the number of renewals in time t. Let $\mathbb{E}[Z]$ be the mean inter-arrival time. Then,*

$$\lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} = \frac{1}{\mathbb{E}[Z]} \tag{20}$$

Observe that for asynchronous SGD or K-batch-async SGD, every gradient push by a learner to the PS can be thought of as an arrival process. The time between two consecutive pushes by a learner follows the distribution of X_i and is independent as computation time has been assumed to be independent across learners and mini-batches. Thus the inter-arrival intervals are positive, independent and identically distributed and hence, the gradient pushes are a renewal process.

7.3 Runtime of K-async SGD

Proof of Lemma 5. For new-longer-than-used distributions observe that the following holds:

$$\Pr(X_i > u + t | X_i > t) \leq \Pr(X_i > u) \tag{21}$$

Thus the random variable $X_i - t | X_i > t$ is thus stochastically dominated by X_i. Now let us assume we want to compute the expected computation time of one iteration of K-async starting at time instant t_0. Let us also assume that the learners last read their parameter values at time instants $t_1, t_2, \dots t_P$ respectively where any K of these $t_1, t_2, \dots t_P$ are equal to t_0 as K out of P learners were updated at time t_0 and the remaining $(P-K)$ of these $t_1, t_2, \dots t_P$ are $< t_0$. Let $Y_1, Y_2, \dots Y_P$ be the random variables denoting the computation time of the P learners starting from time t_0. Thus,

$$Y_i = X_i - (t_0 - t_i) | X_i > (t_0 - t_i) \ \forall\, i = 1, 2, \dots, P \tag{22}$$

Now each of the Y_i s are independent and are stochastically dominated by X_i s.

$$\Pr(Y_i > u) \leq \Pr(X_i > u) \ \forall\ i, j = 1, 2, \dots, P \tag{23}$$

The expectation of the K-th statistic of $\{Y_1, Y_2, \dots, Y_P\}$ is the runtime of the iteration. Let us denote $h_K(x_1, x_2, \dots, x_P)$ as the K-th statistic of P numbers $(x_1, x_2, \dots, x_P)$. And let us us denote $g_{K,\boldsymbol{s}}(x)$ as the K-th statistic of P numbers where $P-1$ of them are given as $\boldsymbol{s}_{1\times(P-1)}$ and x is the $P-$th number. Thus

$$g_{K,\boldsymbol{s}}(x) = h_K(x, s(1), s(2), \dots, s(P-1))$$

First observe that $g_{K,\boldsymbol{s}}(x)$ is an increasing function of x since given the other $P-1$ values, the K-th order statistic will either stay the same or increase with x. Now we use the property that if Y_i is stochastically dominated by X_i, then for any increasing function $g(.)$, we have

$$\mathbb{E}_{Y_1}[g(Y_1)] \leq \mathbb{E}_{X_1}[g(X_1)].$$

This result is derived in [Kreps, 1990] .

This implies that for a given $\boldsymbol{s}$,

$$\mathbb{E}_{Y_1}[g_{K,\boldsymbol{s}}(Y_1)] \leq \mathbb{E}_{X_1}[g_{K,\boldsymbol{s}}(X_1)]$$

This leads to,

$$\begin{aligned}&\mathbb{E}_{Y_1|Y_2=s(1),Y_3=s(2)\dots Y_P=s(P-1)}[h_K(Y_1, Y_2, \dots Y_P)]\\ &\leq \mathbb{E}_{X_1|Y_2=s(1),Y_3=s(2)\dots Y_P=s(P-1)}[h_K(X_1, Y_2, \dots Y_P)]\end{aligned} \tag{24}$$

From this,

$$\begin{aligned}&\mathbb{E}[h_K(Y_1, Y_2, \dots Y_P)]\\ &= \mathbb{E}_{Y_2,\dots,Y_P}\left[\mathbb{E}_{Y_1|Y_2,Y_3\dots Y_P}[h_K(Y_1, Y_2, \dots Y_P)]\right]\\ &\leq \mathbb{E}_{Y_2,\dots,Y_P}\left[\mathbb{E}_{X_1|Y_2,Y_3\dots Y_P}[h_K(X_1, Y_2, \dots Y_P)]\right]\\ &= \mathbb{E}[h_K(X_1, Y_2, \dots Y_P)]\end{aligned} \tag{25}$$

This step proceeds inductively. Thus, similarly

$$\begin{aligned}&\mathbb{E}[h_K(X_1, Y_2, \dots Y_P)]\\ &= \mathbb{E}_{X_1,Y_3,\dots,Y_P}\left[\mathbb{E}_{Y_2|X_1,Y_3\dots Y_P}[h_K(X_1, Y_2, \dots Y_P)]\right]\\ &\leq \mathbb{E}_{X_1,Y_3,\dots,Y_P}\left[\mathbb{E}_{X_2|X_1,Y_3\dots Y_P}[h_K(X_1, X_2, Y_3, \dots Y_P)]\right]\\ &= \mathbb{E}[h_K(X_1, X_2, Y_3 \dots Y_P)]\end{aligned} \tag{26}$$

Thus, finally combining, we have,

$$\begin{aligned}&\mathbb{E}[h_K(Y_1, Y_2, \dots Y_P)]\\ &\leq \mathbb{E}[h_K(X_1, Y_2, \dots Y_P)]\\ &\leq \mathbb{E}[h_K(X_1, X_2, Y_3 \dots Y_P)] \leq \dots\\ &\leq \mathbb{E}[h_K(X_1, X_2, X_3 \dots X_P)]\end{aligned} \tag{27}$$

□

Exponential Computation time: For exponential distributions, the inequality in Lemma 5 holds with equality. This follows from the memoryless property of exponentials. Let us consider the scenario of the proof

of Lemma 5 where we similarly define $Y_i = X_i - (t_0 - t_i)|X_i > (t_0 - t_i)$. From the memoryless property of exponentials [Sheldon, 2002], if $X_i \sim \exp(\mu)$, then $Y_i \sim \exp(\mu)$. Thus, the expectation of the K-th statistic of Y_is can be easily derived as all the Y_is are now i.i.d. with distribution $\exp(\mu)$. Thus, the runtime for J iterations is given by,

$$\mathbb{E}[T] = J\mathbb{E}[Y_{K:P}] = \frac{J}{\mu}\sum_{i=P-K+1}^{P}\frac{1}{i} \approx \frac{J}{\mu}\log\frac{P}{P-K}.$$

Comparison of K-async and K-batch-async SGD: We compare the error runtime trade-off of K-async with K-batch-async SGD in Figure 10 as follows.

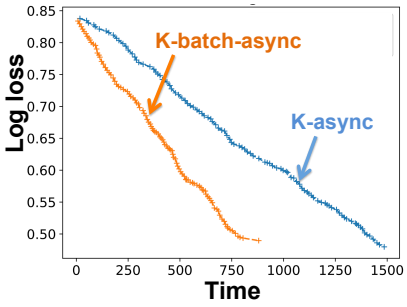

Figure 10: Accuracy Runtime Trade-off on MNIST Dataset: Comparison of K-async with K-batch-async under exponential computation time with $X_i \sim \exp(1)$. As derived theoretically, the K-batch-async has a sharper fall with time as compared to K-async even though the error attained is similar.

8 ASYNC-SGD ANALYSIS PROOFS

8.1 Async-SGD with Fixed learning rate

In this section, we provide a proof of the error convergence of asynchronous SGD. While this is actually a corollary of the more general Theorem 3, we prove this first for the ease of understanding and simplicity as compared to Theorem 3.

Corollary 2. *Suppose that the objective function $F(\mathbf{w})$ is strongly convex with parameter c and the learning rate $\eta \leq \frac{1}{2L(\frac{M_G}{m}+1)}$. Also assume that $\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \leq \gamma\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right]$ for some constant $\gamma \leq 1$. Then, the error after J iterations of Async SGD is given by,*

$$\mathbb{E}[F(\mathbf{w}_J)] - F^* \leq \frac{\eta L\sigma^2}{2c\gamma' m} + (1-\eta c\gamma')^J(\mathbb{E}[F(\mathbf{w}_0)] - F^* - \frac{\eta L\sigma^2}{2c\gamma' m})$$

where $\gamma' = 1 - \gamma + \frac{p_0}{2}$ and p_0 is a non-negative lower bound on the conditional probability that $\tau(j) = j$ given all the past delays and parameters.

To prove the result, we will use the following lemma.

Lemma 7. *Let us denote $\mathbf{v}_j = g(\mathbf{w}_{\tau(j)}, \xi_j)$, and assume that $\mathbb{E}_{\xi_j|\mathbf{w}}[g(\mathbf{w}, \xi_j)] = \nabla F(\mathbf{w})$. Then,*

$$\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \mathbf{v}_j||_2^2\right] \leq \mathbb{E}\left[||\mathbf{v}_j||_2^2\right] - \mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] + \mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right]$$

Proof of Lemma 7. Observe that,

$$\begin{aligned}
&\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \mathbf{v}_j||_2^2\right] \\
&= \mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)}) + \nabla F(\mathbf{w}_{\tau(j)}) - \mathbf{v}_j||_2^2\right] \\
&= \mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\
&\qquad + \mathbb{E}\left[||\mathbf{v}_j - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \qquad (28)
\end{aligned}$$

The last line holds since the cross term is 0 as derived below.

$$\begin{aligned}
&\mathbb{E}\left[(\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})^T(\mathbf{v}_j - \nabla F(\mathbf{w}_{\tau(j)}))\right] \\
&= \mathbb{E}_{\mathbf{w}_{\tau(j)},\mathbf{w}_j}[(\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})^T \\
&\qquad \mathbb{E}_{\xi_j|\mathbf{w}_{\tau(j)},\mathbf{w}_j}\left[(\mathbf{v}_j - \nabla F(\mathbf{w}_{\tau(j)}))\right]] \\
&= \mathbb{E}_{\mathbf{w}_{\tau(j)},\mathbf{w}_j}[(\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})^T \\
&\qquad (\mathbb{E}_{\xi_j|\mathbf{w}_{\tau(j)}}[\mathbf{v}_j] - \nabla F(\mathbf{w}_{\tau(j)}))] = 0
\end{aligned}$$

Here again the last line follows from Assumption 2 in Section 2 which states that

$$\mathbb{E}_{\xi_j|\mathbf{w}_{\tau(j)}}[\mathbf{v}_j] = \nabla F(\mathbf{w}_{\tau(j)})).$$

Returning to (28), observe that the second term can be further decomposed as,

$$\begin{aligned}
&\mathbb{E}\left[||\mathbf{v}_j - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\
&= \mathbb{E}_{\mathbf{w}_{\tau(j)}}\left[\mathbb{E}_{\xi_j|\mathbf{w}_{\tau(j)}}\left[||\mathbf{v}_j - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right]\right] \\
&= \mathbb{E}_{\mathbf{w}_{\tau(j)}}\left[\mathbb{E}_{\xi_j|\mathbf{w}_{\tau(j)}}\left[||\mathbf{v}_j||_2^2\right]\right] \\
&\qquad - 2\mathbb{E}_{\mathbf{w}_{\tau(j)}}\left[\mathbb{E}_{\xi_j|\mathbf{w}_{\tau(j)}}\left[\mathbf{v}_j^T\nabla F(\mathbf{w}_{\tau(j)})\right]\right] \\
&\qquad + \mathbb{E}_{\mathbf{w}_{\tau(j)}}\left[\mathbb{E}_{\xi_j|\mathbf{w}_{\tau(j)}}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right]\right] \\
&= \mathbb{E}\left[||\mathbf{v}_j||_2^2\right] - 2\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] + \mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\
&= \mathbb{E}\left[||\mathbf{v}_j||_2^2\right] - \mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right]
\end{aligned}$$

□

We also prove a K-learner version of this lemma in the Appendix to prove Theorem 3. Now we proceed to provide the proof of Corollary 2.

Proof of Corollary 2.

$$\begin{aligned}F(\mathbf{w}_{j+1}) &\leq F(\mathbf{w}_j) + (\mathbf{w}_{j+1} - \mathbf{w}_j)^T \nabla F(\mathbf{w}_j) \\ &\quad + \frac{L}{2}||\mathbf{w}_{j+1} - \mathbf{w}_j||_2^2 \\ &= F(\mathbf{w}_j) + (-\eta \mathbf{v}_j)^T \nabla F(\mathbf{w}_j) + \frac{L\eta^2}{2}||\mathbf{v}_j||_2^2 \\ &= F(\mathbf{w}_j) - \frac{\eta}{2}||\nabla F(\mathbf{w}_j)||_2^2 - \frac{\eta}{2}||\mathbf{v}_j||_2^2 \\ &\quad + \frac{\eta}{2}||\nabla F(\mathbf{w}_j) - \mathbf{v}_j||_2^2 + \frac{L\eta^2}{2}||\mathbf{v}_j||_2^2 \qquad (29)\end{aligned}$$

Here the last line follows from $2\boldsymbol{a}^T\boldsymbol{b} = ||\boldsymbol{a}||_2^2 + ||\boldsymbol{b}||_2^2 - ||\boldsymbol{a} - \boldsymbol{b}||_2^2$. Taking expectation,

$$\begin{aligned}\mathbb{E}\left[F(\mathbf{w}_{j+1})\right] &\leq \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \\ &\quad - \frac{\eta}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] + \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \mathbf{v}_j||_2^2\right] \\ &\quad + \frac{L\eta^2}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \\ &\overset{(a)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] - \frac{\eta}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \\ &\quad + \frac{\eta}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] - \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\ &\quad + \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\ &\quad + \frac{L\eta^2}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \qquad (30)\end{aligned}$$

Here, (a) follows from Lemma 7 that we just derived. Now, again bounding from (30), we have

$$\begin{aligned}&\mathbb{E}\left[F(\mathbf{w}_{j+1})\right] \qquad (31)\\ &\overset{(b)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] - \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\ &\quad + \frac{\eta}{2}\gamma\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \frac{L\eta^2}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \\ &\overset{(c)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}(1-\gamma)\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \frac{L\eta^2\sigma^2}{2m} \\ &\quad - \frac{\eta}{2}\left(1 - L\eta(\frac{M_G}{m} + 1)\right)\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\ &\overset{(d)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}(1-\gamma)\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \frac{L\eta^2\sigma^2}{2m} \\ &\quad - \frac{\eta}{4}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\ &\overset{(e)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}(1-\gamma)\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \frac{L\eta^2\sigma^2}{2m} \\ &\quad - \frac{\eta}{4}p_0\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \qquad (32)\end{aligned}$$

Here (b) follows from the statement of the theorem that

$$\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \leq \gamma\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right]$$

for some constant $\gamma \leq 1$. The next step (c) follows from Assumption 4 in Section 2 which lead to

$$\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \leq \frac{\sigma^2}{m} + \left(\frac{M_G}{m} + 1\right)\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right].$$

Step (d) follows from choosing $\eta < \frac{1}{2L(\frac{M_G}{m}+1)}$ and finally (e) follows from Lemma 1.

Now one might recall that the function $F(w)$ was defined to be strongly convex with parameter c. Using the standard result of strong-convexity (6) in (32), we obtain the following result.

$$\begin{aligned}\mathbb{E}\left[F(\mathbf{w}_{j+1})\right] - F^* &\leq \frac{\eta^2 L\sigma^2}{2m} \\ &\quad + (1 - \eta c(1 - \gamma + \frac{p_0}{2}))(\mathbb{E}\left[F(\mathbf{w}_j)\right] - F^*)\end{aligned}$$

Let us denote $\gamma' = (1 - \gamma + \frac{p_0}{2})$. Then, using the above recursion, we thus have,

$$\begin{aligned}\mathbb{E}\left[F(\mathbf{w}_J)\right] - F^* &\leq \frac{\eta L\sigma^2}{2c\gamma' m} + \\ &\quad (1 - \eta\gamma' c)^J(\mathbb{E}\left[F(\mathbf{w}_0)\right] - F^* - \frac{\eta L\sigma^2}{2c\gamma' m})\end{aligned}$$

□

Discussion on range of p_0: Let us denote the conditional probability of $\tau(j) = j$ given all the past delays and parameters as $p_0^{(j)}$. Now $p_0 \leq p_0^{(j)}\ \forall j$. Clearly the value of $p_0^{(j)}$ will differ for different distributions and accordingly the value of p_0 will differ. Here we include a brief discussion on the possible values of p_0 for different distributions. These also hold for K-async and K-batch-async SGD.

Lemma 8 (Bounds of p_0). *Define $p_0 = \inf_j p_0^{(j)}$, i.e. the largest constant such that $p_0 \leq p_0^{(j)}\ \forall\ j$.*

- *For exponential computation times, $p_0^{(j)} = \frac{1}{P}$ for all j and is thus invariant of j and $p_0 = \frac{1}{P}$.*
- *For new-longer-than-used (See Definition 3) computation times, $p_0^{(j)} \leq \frac{1}{P}$ and thus $p_0 \leq \frac{1}{P}$.*
- *For new-shorter-than-used computation times, $p_0^{(j)} \geq \frac{1}{P}$ and thus $p_0 \geq \frac{1}{P}$.*

Proof of Lemma 8. Let t_0 be the time when the j-th iteration occurs, and suppose that learner i' pushed its gradient in the j-th iteration. Now similar to the proof of Lemma 5, let us also assume that the learners last read their parameter values at time instants $t_1, t_2, \ldots t_P$ respectively where $t'_i = t_0$ and the remaining $(P-1)$ of these t_is are $< t_0$. Let $Y_1, Y_2, \ldots Y_P$ be the random variables denoting the computation

time of the P learners starting from time t_0. Thus, $Y_i = X_i - (t_0 - t_i)|X_i > (t_0 - t_i)$. For exponentials, from the memoryless property, all these Y_i s become i.i.d. and thus from symmetry the probability of i' finishing before all the others is equal, *i.e.* $\frac{1}{P}$. Thus, $p_0^{(j)} = p_0 = \frac{1}{P}$. For new-longer-than-used distributions, as we have discussed before all the Y_is with $i \neq i'$ will be stochastically dominated by $Y_{i'} = X_{i'}$. Thus, probability of is with $i \neq i'$ finishing first is higher than i'. Thus, $p_0^{(j)} \leq \frac{1}{P}$ and so is p_0. Similarly, for new-shorter-than-used distributions, $Y_{i'}$ is stochastically dominated by all the Y_is and thus probability of i' finishing first is more. So, $p_0^{(j)} \geq \frac{1}{P}$ and so is p_0. □

8.2 K-async SGD under fixed learning rate

In this subsection, we provide a proof of Theorem 3.

Before we proceed to the proof of this theorem, we first extend our Assumption 4 from the variance of a single stochastic gradient to sum of stochastic gradients in the following Lemma.

Lemma 9. *If the variance of the stochastic updates is bounded as* $\mathbb{E}_{\xi_j|\mathbf{w}_{\tau l,j}}\left[||g(\mathbf{w}_{\tau(l,j)},\xi_{l,j}) - \nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \leq \frac{\sigma^2}{m} + \frac{M_G}{m}||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2 \ \forall\ \tau(l,j) \leq j$ *, then for K-async, the variance of the sum of stochastic updates given all the parameter values* $\mathbf{w}_{\tau(l,j)}$ *is also bounded as follows:*

$$\mathbb{E}_{\xi_{1,j},\ldots,\xi_{K,j}|\mathbf{w}_{\tau(1,j)}\ldots\mathbf{w}_{\tau(K,j)}}\left[||\sum_{l=1}^{K} g(\mathbf{w}_{l,j},\xi_{l,j})||_2^2\right] \leq \frac{K\sigma^2}{m} + (\frac{M_G}{m} + K)||\sum_{l=1}^{K}\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2 \quad (33)$$

Proof. First let us consider the expectation of any cross term such that $l \neq l'$. For the ease of writing, let $\Omega = \{\mathbf{w}_{\tau(1,j)}\ldots\mathbf{w}_{\tau(K,j)}\}$. Now observe the conditional expectation of the cross term as follows.

$$\begin{aligned}
&\mathbb{E}_{\xi_{1,j},\ldots,\xi_{K,j}|\Omega}[(g(\mathbf{w}_{l,j},\xi_{l,j}) - \nabla F(\mathbf{w}_{\tau(l,j)}))^T \\
&\qquad ((g(\mathbf{w}_{l',j},\xi_{l',j}) - \nabla F(\mathbf{w}_{\tau(l',j)}))] \\
&= \mathbb{E}_{\xi_{l,j},\xi_{l',j}|\Omega}[(g(\mathbf{w}_{l,j},\xi_{l,j}) - \nabla F(\mathbf{w}_{\tau(l,j)}))^T \\
&\qquad ((g(\mathbf{w}_{l',j},\xi_{l',j}) - \nabla F(\mathbf{w}_{\tau(l',j)}))] \\
&= \mathbb{E}_{\xi_{l',j}|\Omega}[\mathbb{E}_{\xi_{l,j}|\xi_{l',j},\Omega}[(g(\mathbf{w}_{l,j},\xi_{l,j}) - \nabla F(\mathbf{w}_{\tau(l,j)}))^T] \\
&\qquad (g(\mathbf{w}_{l',j},\xi_{l',j}) - \nabla F(\mathbf{w}_{\tau(l',j)})] \\
&= \mathbb{E}_{\xi_{l',j}|\Omega}[0^T(g(\mathbf{w}_{l',j},\xi_{l',j}) - \nabla F(\mathbf{w}_{\tau(l',j)})] = 0 \quad (34)
\end{aligned}$$

Thus the cross terms are all 0. So the expression simplifies as,

$$\begin{aligned}
&\mathbb{E}_{\xi_{1,j},\ldots,\xi_{K,j}|\Omega}\left[||\sum_{l=1}^{K} g(\mathbf{w}_{l,j},\xi_{l,j}) - F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \\
&\overset{(a)}{=} \sum_{l=1}^{K}\mathbb{E}_{\xi_{1,j},\ldots,\xi_{K,j}|\Omega}\left[||g(\mathbf{w}_{l,j},\xi_{l,j}) - F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \\
&\leq \sum_{l=1}^{K}\frac{\sigma^2}{m} + \frac{M_G}{m}||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2 \quad (35)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}_{\xi_{1,j},\ldots,\xi_{K,j}|\Omega}\left[||\sum_{l=1}^{K} g(\mathbf{w}_{l,j},\xi_{l,j})||_2^2\right] \\
&= \mathbb{E}_{\xi_{1,j},\ldots,\xi_{K,j}|\Omega}\left[||\sum_{l=1}^{K} g(\mathbf{w}_{l,j},\xi_{l,j}) - F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \\
&+ \mathbb{E}_{\xi_{1,j},\ldots,\xi_{K,j}|\Omega}\left[||\sum_{l=1}^{K} F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \\
&\leq \frac{K\sigma^2}{m} + \sum_{l=1}^{K}\frac{M_G}{m}||F(\mathbf{w}_{\tau(l,j)})||_2^2 + ||\sum_{l=1}^{K} F(\mathbf{w}_{\tau(l,j)})||_2^2 \\
&\leq \frac{K\sigma^2}{m} + \sum_{l=1}^{K}\frac{M_G}{m}||F(\mathbf{w}_{\tau(l,j)})||_2^2 + \sum_{l=1}^{K} K||F(\mathbf{w}_{\tau(l,j)})||_2^2 \\
&\qquad (36)
\end{aligned}$$

Now we return to the proof of the theorem. □

Proof of Theorem 3. Let $\mathbf{v}_j = \frac{1}{K}\sum_{l=1}^{K} g(\mathbf{w}_{l,j},\xi_{l,j})$. Following steps similar to the Async-SGD proof, from Lipschitz continuity we have the following.

$$\begin{aligned}
F(\mathbf{w}_{j+1}) &\leq F(\mathbf{w}_j) + (\mathbf{w}_{j+1} - \mathbf{w}_j)^T\nabla F(\mathbf{w}_j) \\
&\quad + \frac{L}{2}||\mathbf{w}_{j+1} - \mathbf{w}_j||_2^2 \\
&= F(\mathbf{w}_j) - \frac{\eta}{K}\sum_{l=1}^{K} g(\mathbf{w}_{l,j},\xi_{l,j})^T\nabla F(\mathbf{w}_j) + \frac{L}{2}||\eta\mathbf{v}_j||_2^2 \\
&\overset{(a)}{=} F(\mathbf{w}_j) - \frac{\eta}{2K}\sum_{l=1}^{K}||\nabla F(\mathbf{w}_j)||_2^2 - \frac{\eta}{2K}\sum_{l=1}^{K}||g(\mathbf{w}_{l,j},\xi_{l,j})||_2^2 \\
&\quad + \frac{\eta}{2K}\sum_{l=1}^{K}||g(\mathbf{w}_{l,j},\xi_{l,j})||_2^2 - \frac{\eta}{2K}\sum_{l=1}^{K}||\nabla F(\mathbf{w}_j)||_2^2 \\
&\quad + \frac{L\eta^2}{2}||\mathbf{v}_j||_2^2 \\
&= F(\mathbf{w}_j) - \frac{\eta}{2}||\nabla F(\mathbf{w}_j)||_2^2 - \frac{\eta}{2K}\sum_{l=1}^{K}||g(\mathbf{w}_{l,j},\xi_{l,j})||_2^2 \\
&\quad + \frac{\eta}{2K}\sum_{l=1}^{K}||g(\mathbf{w}_{l,j},\xi_{l,j}) - \nabla F(\mathbf{w}_j)||_2^2 \\
&\quad + \frac{L\eta^2}{2}||\mathbf{v}_j||_2^2 \quad (37)
\end{aligned}$$

Here (a) follows from $2\boldsymbol{a}^T\boldsymbol{b} = ||\boldsymbol{a}||_2^2 + ||\boldsymbol{b}||_2^2 - ||\boldsymbol{a}-\boldsymbol{b}||_2^2$. Taking expectation,

$$\begin{aligned}
\mathbb{E}\left[F(\mathbf{w}_{j+1})\right] &\leq \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \\
&\quad - \frac{\eta}{2K}\sum_{l=1}^{K}\mathbb{E}\left[||g(\mathbf{w}_{l,j},\xi_{l,j})||_2^2\right] \\
&\quad + \frac{\eta}{2K}\sum_{l=1}^{K}\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - g(\mathbf{w}_{l,j},\xi_{l,j})||_2^2\right] \\
&\quad + \frac{L\eta^2}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \\
&\overset{(a)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \\
&\quad - \frac{\eta}{2K}\sum_{l=1}^{K}\mathbb{E}\left[||g(\mathbf{w}_{l,j},\xi_{l,j})||_2^2\right] + \\
&\quad \frac{\eta}{2K}\sum_{l=1}^{K}\mathbb{E}\left[||g(\mathbf{w}_{l,j},\xi_{l,j})||_2^2\right] \\
&\quad - \frac{\eta}{2K}\sum_{l=1}^{K}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \\
&\quad + \frac{\eta}{2K}\sum_{l=1}^{K}\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \\
&\quad + \frac{L\eta^2}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \qquad (38) \\
&\overset{(b)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \\
&\quad - \frac{\eta}{2K}\sum_{l=1}^{K}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \\
&\quad + \frac{\eta}{2}\gamma\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \frac{L\eta^2}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \\
&\overset{(c)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}(1-\gamma)\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \frac{L\eta^2\sigma^2}{2Km} \\
&\quad - \frac{\eta}{2K}\sum_{l=1}^{K}\left(1 - L\eta(\frac{M_G}{Km} + \frac{1}{K})\right)\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \\
&\overset{(d)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}(1-\gamma)\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \frac{L\eta^2\sigma^2}{2Km} \\
&\quad - \frac{\eta}{4K}\sum_{l=1}^{K}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \\
&\overset{(e)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta}{2}(1-\gamma)\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \frac{L\eta^2\sigma^2}{2Km} \\
&\quad - \frac{\eta}{4}p_0\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \qquad (39)
\end{aligned}$$

Here step (a) follows from Lemma 7 and step (b) follows from the assumption that $\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \leq \gamma\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right]$ for some constant $\gamma \leq 1$. The next step (c) follows from the Lemma 9 that bounds the variance of the sum of stochastic gradients. Step (d) follows from choosing $\eta < \frac{1}{2L(\frac{M_G}{Km}+\frac{1}{K})}$ and finally (e) follows from Lemma 1 in Section 3 that says $\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(l,j)})||_2^2\right] \geq p_0\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right]$ for some non-negative constant p_0 which is a lower bound on the conditional probability that $\tau(l,j) = j$ given all past delays and parameter values.

Finally, since $F(\mathbf{w})$ is strongly convex, using the inequality $2c(F(\mathbf{w}) - F^*) \leq ||\nabla F(\mathbf{w})||_2^2$ in (39), we finally obtain the desired result. □

Extension to Non-Convex case The analysis can be extended to provide weaker guarantees for non-convex objectives. Let $\gamma' = 1 - \gamma + \frac{p_0}{2}$

For non-convex objectives, we have the following result.

Theorem 5. *For non-convex objective function, we have the following ergodic convergence result given by:*

$$\frac{1}{J+1}\sum_{j=0}^{J}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \leq \frac{2(F(\mathbf{w}_0) - F^*)}{(J+1)\eta\gamma'} + \frac{L\eta\sigma^2}{Km\gamma'}$$

where $F^* = \min_{\mathbf{w}} F(\mathbf{w})$.

Proof. Recall the recursion derived in the last proof in (39). After re-arrangement, we obtain the following:

$$\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \leq \frac{2(\mathbb{E}\left[F(\mathbf{w}_j)\right] - \mathbb{E}\left[F(\mathbf{w}_{j+1})\right]))}{\eta\gamma'} + \frac{L\eta\sigma^2}{Km\gamma'} \qquad (40)$$

Taking summation from $j = 0$ to $j = J$, we get,

$$\begin{aligned}
&\frac{1}{J+1}\sum_{j=0}^{J}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \\
&\quad \leq \frac{2(\mathbb{E}\left[F(\mathbf{w}_0)\right] - \mathbb{E}\left[F(\mathbf{w}_J)\right])}{(J+1)\eta\gamma'} + \frac{L\eta\sigma^2}{Km\gamma'} \\
&\quad \overset{(a)}{\leq} \frac{2(F(\mathbf{w}_0) - F^*)}{(J+1)\eta\gamma'} + \frac{L\eta\sigma^2}{Km\gamma'} \qquad (41)
\end{aligned}$$

Here (a) follows since we assume $\mathbf{w}_0$ to be known and also from $\mathbb{E}\left[F(\mathbf{w}_J)\right] \geq F^*$. □

8.3 Variable Learning Rate Schedule

We propose a new heuristic for learning rate schedule that is more stable than fixed learning rate for asynchronous SGD. Our learning rate schedule is $\eta_j = \min\left\{\frac{C}{||\mathbf{w}_j - \mathbf{w}_{\tau(j)}||_2^2}, \eta_{max}\right\}$, where η_{max} is a suitably large value of learning rate beyond which the convergence diverges. This heuristic is inspired from the assumption in Theorem 4 given by $\eta_j\mathbb{E}\left[||\mathbf{w}_j - \mathbf{w}_{\tau(j)}||_2^2\right] \leq C$. In this section, we derive the accuracy trade-off mentioned in Theorem 4 based on this assumption.

Proof of Theorem 4. Following steps similar to (29), we first obtain the following:

$$F(\mathbf{w}_{j+1}) \leq F(\mathbf{w}_j) - \frac{\eta_j}{2}||\nabla F(\mathbf{w}_j)||_2^2 - \frac{\eta_j}{2}||\mathbf{v}_j||_2^2 + \frac{\eta_j}{2}||\nabla F(\mathbf{w}_j) - \mathbf{v}_t||_2^2 + \frac{L\eta_j^2}{2}||\mathbf{v}_j||_2^2 \quad (42)$$

Now taking expectation, we obtain the following result.

$$\begin{aligned}
&\mathbb{E}\left[F(\mathbf{w}_{j+1})\right] \\
&\overset{(a)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta_j}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] - \frac{\eta_j}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \\
&\quad + \frac{\eta_j}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] - \frac{\eta_j}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\
&\quad + \frac{\eta_j}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j) - \nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\
&\quad + \frac{L\eta_j^2}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \\
&\overset{(b)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta_j}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \\
&\quad - \frac{\eta_j}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] + \frac{CL^2}{2} + \frac{L\eta_j^2}{2}\mathbb{E}\left[||\mathbf{v}_j||_2^2\right] \\
&\overset{(c)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta_j}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \frac{CL^2}{2} + \frac{L\eta_j^2\sigma^2}{2m} \\
&\quad - \frac{\eta_j}{2}\left(1 - L\eta_j(\frac{M_G}{m} + 1)\right)\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \\
&\overset{(e)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta_j}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \\
&\quad + \frac{CL^2}{2} + \frac{\eta_j^2 L\sigma^2}{2m} - \frac{\eta_j}{4}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] \quad (43)
\end{aligned}$$

Here (a) follows from (30), (b) follows from (12), (c) follows from Assumption 4 and (d) follows as $\eta_j \leq \frac{1}{2L(\frac{M_G}{m}+1)}$. Let us define $\Delta_j = \frac{CL^2}{2} + \frac{\eta_j^2 L\sigma^2}{2m}$. Thus, the recursion can be written as,

$$\begin{aligned}
\mathbb{E}\left[F(\mathbf{w}_{j+1})\right] &\leq \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta_j}{2}\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] \\
&\quad - \frac{\eta_j}{4}\mathbb{E}\left[||\nabla F(\mathbf{w}_{\tau(j)})||_2^2\right] + \Delta_j \\
&\overset{(e)}{\leq} \mathbb{E}\left[F(\mathbf{w}_j)\right] - \frac{\eta_j}{2}(1 + \frac{p_0}{2})\mathbb{E}\left[||\nabla F(\mathbf{w}_j)||_2^2\right] + \Delta_j \quad (44)
\end{aligned}$$

Here (e) follows from Lemma 1. If the loss function $F(\mathbf{w})$ is strongly convex with parameter c, then for all $\mathbf{w}$, we have $2c(F(\mathbf{w}) - F^*) \leq ||\nabla F(\mathbf{w})||_2^2$. Using this result, we obtain

$$\begin{aligned}
\mathbb{E}\left[F(\mathbf{w}_{j+1})\right] - F^* &\leq (1 - \eta_j(1 + \frac{p_0}{2})c)(\mathbb{E}\left[F(\mathbf{w}_j)\right] - F^*) + \Delta_j \\
&\leq (1 - \eta_j(1 + \frac{p_0}{2})c)(1 - \eta_{j-1}(1 + \frac{p_0}{2})c)(\mathbb{E}\left[F(\mathbf{w}_{j-1})\right] - F^*) \\
&\quad + (1 - \eta_j(1 + \frac{p_0}{2})c)\Delta_{j-1} + \Delta_j \\
&\leq (1 - \rho_j)(1 - \rho_{j-1})\dots(1 - \rho_0)(\mathbb{E}\left[F(\mathbf{w}_0)\right] - F^*) + \Delta \quad (45)
\end{aligned}$$

where $\rho_j = \eta_j(1 + \frac{p_0}{2})c$ and $\Delta = \Delta_j + (1 - \rho_j)\Delta_{j-1} + \dots + (1 - \rho_j)(1 - \rho_{j-1})\dots(1 - \rho_1)\Delta_0$. □

9 SIMULATION SETUP DETAILS

MNIST [LeCun, 1998]: For the simulations on MNIST dataset, we first convert the 28×28 images into single vectors of length 784. We use a single layer of neurons followed by soft-max cross entropy with logits loss function. Thus effectively the parameters consist of a weight matrix $\boldsymbol{W}$ of size 784×10 and a bias vector $\boldsymbol{b}$ of size 1×10. We use a regularizer of value 0.01, mini-batch size $m = 1$, and learning rate $\eta = 0.01$. For implementation we used Tensorflow with Python3. Thus, the model is as follows:

```
X=tf.placeholder(tf.float32,[None,784])
Y=tf.placeholder(tf.float32,[None,10])
W=tf.Variable(tf.random_normal(shape=[784,10],
              stddev=0.01), name="weights")
b=tf.Variable(tf.random_normal(shape=[1,10],
              stddev=0.01),  name="bias")

logits=tf.matmul(X,W) + b
entropy=tf.nn.softmax_cross_entropy_with
        _logits(logits=logits,labels=Y) +
                lamda*tf.square(tf.norm(W))

loss=tf.reduce_mean( entropy)
```

For the run-time simulations, we generate random variables from the respective distributions in python to represent the computation times.

CIFAR10 [Krizhevsky and Hinton, 2009]: For the CIFAR10 simulations, similar to MNIST, we convert the images into vectors of length 1024. We combine the three colour variants in the ratio $[0.2989, 0.5870, 0.114]$ to generate a single vector of length 1024 for every image. We use a single layer of neurons again followed by soft-max cross entropy with logits in tensorflow. Thus, the parameters consist of a weight matrix $\boldsymbol{W}$ of size 1024×10 and a bias vector $\boldsymbol{b}$ of size 1×10. We use a mini-batch size of 250, regularizer of 0.05.

We use a similar model as follows:

```
X=tf.placeholder(tf.float32,[None,1024])
Y=tf.placeholder(tf.float32,[None,10])
W=tf.Variable(tf.random_normal(shape=[1024,10],
            stddev= 0.01),name="weights")
b=tf.Variable(tf.random_normal(shape=[1,10],
            stddev = 0.01),name="bias")

logits=tf.matmul(X,W) +  b
entropy=tf.nn.softmax_cross_entropy_with
          _logits(logits=logits,labels=Y) +
```

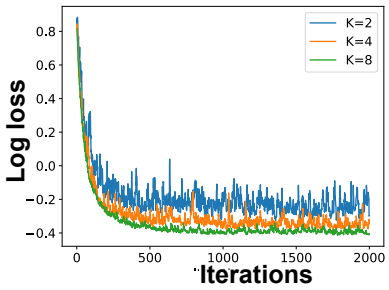

Figure 11: Error-Iterations tradeoff on MNIST dataset: Simulation of K-sync SGD for different values of K. Observe that accuracy improves with increasing K which means increasing effective batch size ($\eta = 0.05$).

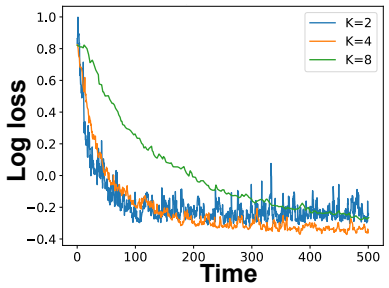

Figure 12: Error Runtime tradeoff on MNIST dataset: Simulation of K-sync SGD for different values of K ($\eta = 0.05$).

```
            lamda*tf.square(tf.norm(W))
loss=tf.reduce_mean(entropy)
```

The computation time as each learner is generated from exponential distribution.

10 CHOICE OF HYPERPARAMETERS

Our analysis techniques can also inform the choice of hyperparameters synchronous and K-sync SGD.

10.1 Varying K in K-sync

We first perform some simulations of K-sync SGD applied on the MNIST dataset. For the simulation set-up, we consider 8 parallel learners with fixed mini-batch size $m = 1$ and fixed learning rate 0.05. The number of learners to wait for in K-sync, *i.e.* K is varied and the error runtime trade-off is observed. The runtimes are generated from a shifted exponential distribution given by $X_i \sim m + \exp \mu$.

Observe that in the plot of error with the number of iterations in Figure 11, the error improves with increasing K, which means increasing the effective mini-batch and reducing the variability in the gradient. However, if we look at the same error plotted against runtime (See Figure 12) instead of the number of iterations, observe that increasing K naively does not always lead to a better trade-off. As K increases, the central PS has to wait for more learners to finish at every iteration, thus suffering from increased straggler effect. The best error runtime trade-off is obtained at an intermediate $K = 4$. Thus, the current analysis informs the optimal choice of K to achieve a good error runtime trade-off.

10.2 Varying mini-batch m

We consider the training of Alexnet on ImageNet dataset [Krizhevsky et al., 2012] using $P = 4$ learners. For this simulation, we perform fully synchronous SGD, *i.e.* K-sync with $K = P = 4$. We fix the learning rate and vary the mini-batch used for training. The runtimes are generated from a shifted exponential distribution given by $X_i \sim m + \exp \mu$, that depends on the mini-batch size. Intuitively, this distribution makes sense since to compute one mini-batch, a processor would atleast need a time m (Work Complexity). However, due to delays, it has the additional exponential tail. The error runtime trade-offs are observed in Figure 13 and Figure 14.

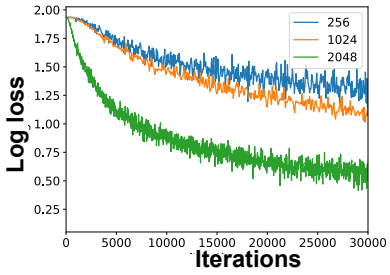

Figure 13: Error-Iterations tradeoff on IMAGENET dataset: Simulation of fully synchronous SGD ($K = P = 4$) for different values of mini-batch m. Observe that accuracy improves with increasing m which means increasing effective batch size.

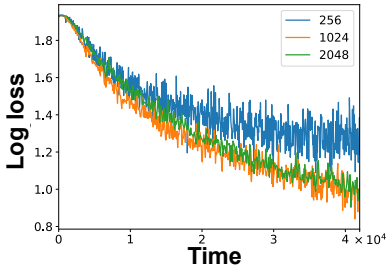

Figure 14: Error Runtime tradeoff on IMAGENET dataset: Same simulation of fully synchronous SGD ($K = P = 4$) for different values of mini-batch m plotted against time. Observe that higher m does not necessarily mean the best trade-off with runtime as higher mini-batch also has longer time.

Again, observe that the plot of error with the number of iterations improves with the mini-batch size, as also expected from theory. However, increasing the mini-batch also changes the runtime distribution. Thus, when we plot the same error against runtime, we again observe that increasing the mini-batch size naively does not necessarily lead to the best trade-off. Instead, the best error runtime trade-off is observed with an intermediate mini-batch value of 1024. Thus, our analysis informs the choice of the optimal mini-batch.