

Downlink Erlang Capacity of Cellular OFDMA

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Abstract—In this paper, we present a novel approach to evaluate the downlink Erlang capacity of a cellular Orthogonal Frequency Division Multiple Access (OFDMA) system with 1:1 frequency reuse. Erlang capacity analysis of traditional cellular systems like Global System for Mobile communications (GSM) cannot be applied to cellular OFDMA because in the latter, each incoming call requires a random number of subcarriers. To address this problem, we divide incoming calls into classes according to their subcarrier requirement. Then, we model the system as a multi-dimensional Markov chain and evaluate the Erlang capacity. We draw an interesting analogy between the problem considered, and the concept of stochastic knapsack, a generalization of the classical knapsack problem. Techniques used to solve the stochastic knapsack problem simplify the analysis of the multi-dimensional Markov chain.

Index Terms—Cellular OFDMA, Blocking Probability, Erlang Capacity.

I. INTRODUCTION

Recent years have witnessed the emergence of Orthogonal Frequency Division Multiple Access (OFDMA) as one of the dominant Medium Access Control (MAC) techniques for next-generation wireless networks [1]. OFDMA employs multicarrier modulation to combat frequency selective fading. Each base station (BS) has a set of orthogonal subcarriers, subsets of which are allocated to users in the cell. Due to limited availability of spectrum, a 1:1 frequency reuse factor is most common in multi-cell OFDMA architecture. In 1:1 reuse, by allocating a random permutation of subcarriers to users in each cell, the inter-cell interference may be averaged out and hence may not affect the system performance severely.

Erlang capacity corresponds to the traffic load that a cell can support while providing acceptable service to the users. It is an important parameter from the capacity planning perspective and is used as a performance metric for admission control algorithms. Erlang capacity is a well studied topic for the traditional Global System for Mobile communications (GSM) cellular systems [2]. The capacity of these systems for a given blocking probability is determined by the Erlang-B formula. It has also been studied extensively in the context of Code Division Multiple Access (CDMA) systems [3]–[5]. Unlike GSM in which a user is blocked if all the time or frequency channels at the BS are occupied, in CDMA, an incoming user

is blocked if it increases the interference and causes outage conditions for the existing users.

In this paper, we determine the downlink Erlang capacity of cellular OFDMA. The main idea is to take into account the fact that each incoming user, depending upon its position in the cell, channel fading and inter-cell interference, experiences a random signal to interference plus noise ratio (SINR) on each subcarrier. Thus, each user requires a random number of subcarriers to satisfy its required data rate. The idea of incoming users requiring random number of resources has been previously addressed in operations research [6]. The authors consider a queueing system with Poisson arrivals of customers and exponentially distributed service times and determine the probability distribution of the waiting times.

Only a few studies focus on determining the Erlang capacity of cellular OFDMA [7], [8]. In [7], the uplink capacity of relay-assisted cellular networks is analyzed. The authors present a joint algorithm to determine the bandwidth distribution between BS and relays and the Erlang capacity for given values of blocking and outage probabilities. In [8], Erlang capacity is used as a performance metric for comparison of various adaptive resource allocation algorithms. However, in both these papers, the random subcarrier requirement of a user is not considered in the derivation of blocking probability.

In this paper, we present a novel approach to evaluate the downlink Erlang capacity of cellular OFDMA. Incoming users are divided into service classes based on their subcarrier requirement. To determine the arrival rate for each class we compute the probability distribution of the number of subcarriers required by an incoming user. We then model the system as a multi-dimensional Markov chain and compute blocking probability and Erlang capacity of the system.

The rest of the paper is organized as follows. In Section II, we present the system model. In Section III, we formulate the capacity evaluation problem for cellular OFDMA. The solution strategy is discussed in Section IV where we obtain the probability distribution of the user's subcarrier requirement and use it to determine the Erlang capacity. We present simulation setup and results in Section V. We conclude the paper in Section VI and provide directions for further investigations.

II. SYSTEM MODEL

We consider a cellular OFDMA system with cells of radius R as shown in Fig. 1. Cell 0 is the reference cell and cells 1 to 6 are its neighbors. Call arrivals in each cell are Poisson distributed with rate λ , and the call holding times are exponentially distributed with mean $\frac{1}{\mu}$. We assume that all incoming calls have a rate requirement R_{req} bits/sec. Though we do not consider multi-service traffic where each class of users has a different rate requirement, our analysis can be extended to such a scenario.

The BS in each cell has N subcarriers available for allocation to mobile stations (MSs) in the cell for downlink data transmission. The BS transmits at constant power P_{tx} on each subcarrier. The downlink signal to noise plus interference ratio (SINR) on each subcarrier depends upon location of the MS in the reference cell, channel fading and interference from neighboring cells. We consider log-normal shadowing on each subcarrier, with mean 0 and standard deviation σ . Fast fading is not considered since our aim is to evaluate the Erlang capacity from a long term capacity planning perspective. For modeling inter-cell interference, we consider a 1:1 frequency reuse pattern, and assume that interference occurs from only the first-tier of neighboring cells, i.e cells 1 to 6 in Fig. 1.

III. PROBLEM FORMULATION

Our aim is to determine the Erlang capacity of cellular OFDMA. Erlang capacity is defined as the traffic load in Erlangs supported by the cell while ensuring that blocking probability is below a certain value. By determining the blocking probability as a function of the offered load in Erlangs, we can evaluate the Erlang capacity of the cell.

In any standard circuit switched system which can support N calls simultaneously, every call occupies only one circuit and releases it on departure. The system is said to be in state i when it has i active calls. This system state can be modeled as a one-dimensional continuous time Markov chain. The blocking probability P_B is the steady state probability of being in state N and is given by the standard Erlang-B formula $P_B = \frac{\rho^N / N!}{\sum_{i=0}^N \rho^i / i!}$ where $\rho = \frac{\lambda}{\mu}$ is the offered Erlang load. However, this analysis cannot be applied to cellular OFDMA

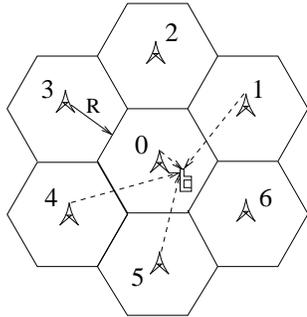


Fig. 1. Illustration of the inter-cell interference on a subcarrier allocated to the MS in cell 0. The interference comes from the neighboring BSs where that subcarrier is in use (BSs 1, 4 and 5 in this illustration)

where the number of subcarriers required, n_{req} is a function of the downlink SINR on each subcarrier. Let SINR_i be the downlink SINR on i^{th} subcarrier in the reference cell. Then,

$$\text{SINR}_i = \frac{P_{\text{tx}} D^{-\eta} 10^{\frac{\xi}{10}}}{I_i + N_0}, \quad 1 \leq i \leq N, \quad (1)$$

where P_{tx} is the power transmitted by the BS on i^{th} subcarrier, D is the radial distance of the MS from the BS and η is the path loss exponent. ξ models the shadowing on the BS-MS link and it is a Gaussian with mean 0 and standard deviation σ . N_0 is the thermal noise level and I_i is the total interference received on i^{th} subcarrier. Let $I_{i,j}$ be the interference from j^{th} neighboring cell where $1 \leq j \leq 6$. Then,

$$I_i = \sum_{j=1}^6 I_{i,j} \mathbf{1}_A(i, j) \quad (2)$$

$$= \sum_{j=1}^6 P_{\text{tx}} D_j^{-\eta} 10^{\frac{\xi_j}{10}} \mathbf{1}_A(i, j) \quad (3)$$

where $\mathbf{1}_A(i, j)$ is an indicator random variable which takes the value 1 if i^{th} subcarrier has been allocated to an MS in j^{th} neighboring cell, and 0 otherwise. D_j is the distance between the BS of j^{th} neighboring cell and the MS in the reference cell and ξ_j is the shadowing factor on this link.

Let us assume that each subcarrier is of unit bandwidth, 1Hz. As a result, the data rate which is achieved by allocation of the i^{th} subcarrier to a user is given by $\log_2(1 + \text{SINR}_i)$. When there is a call arrival in the reference cell, the BS allocates a set of subcarriers to the MS in order to just satisfy the user's rate requirement R_{req} . Thus the number of subcarriers required, n_{req} is random and is such that,

$$\sum_{i=1}^{n_{\text{req}}} \log_2(1 + \text{SINR}_i) \geq R_{\text{req}}. \quad (4)$$

An incoming call is blocked if the number of subcarriers available at BS is less than n_{req} . Further, we assume that once a set of subcarriers is allocated to the MS, it uses them for the entire call duration, and the subcarriers are released when the call departs.

IV. SOLUTION STRATEGY

In this section, we describe our approach to determine the blocking probability for the cellular OFDMA system. The key idea is to divide incoming calls into classes according to their subcarrier requirement n_{req} . We first determine the probability distribution function of n_{req} , and then use it to evaluate the blocking probability of an incoming call.

A. Distribution of number of subcarriers required

We assume the random subcarrier allocation scheme given in Algorithm 1 where the BS arbitrarily chooses n_{req} subcarriers from the available set and allots them to an incoming user¹.

¹We assume that subcarrier allocation is not in chunks, and the BS can allocate any integer number of subcarriers to the users. However, the analysis can be extended to chunk allocation.

The BS chooses one subcarrier randomly from the available set and checks whether the user's rate requirement is satisfied by it, that is whether $\log_2(1 + SINR)$ for that subcarrier is greater than or equal to R_{req} . If not, it continues to add randomly chosen subcarriers until the total rate over the set of subcarriers becomes greater than or equal to R_{req} . This set of n_{req} subcarriers is then allocated to the user. If the available set of subcarriers cannot meet the rate requirement, the call is blocked. We assume that the MS uses the set of allocated subcarriers for the entire duration of the call. When the call ends, the subcarriers are released into the available set.

We determine $p[n_{req}]$, the discrete probability distribution of the number of subcarriers required by an incoming call for $1 \leq n_{req} \leq N$, by performing system level simulations, which will be elaborated in Section V. For an n_{req} , $p[n_{req}]$ is the fraction of incoming calls that require n_{req} subcarriers to satisfy their rate requirement.

Algorithm 1 Subcarrier allocation scheme

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for each call arrival do
   $R_{temp} \leftarrow 0$ 
   $n_{allot} \leftarrow 0$ , number of subcarriers to be allocated
   $\mathbf{S}_{avail}$  : set of available subcarriers in the cell
   $\mathbf{S}_{allot} \leftarrow \{\}$ , set of subcarriers to be allocated
   $\mathbf{S}_{temp} \leftarrow \mathbf{S}_{avail}$ 
  while  $\mathbf{S}_{temp} \neq \{\}$  do
    Choose randomly subcarrier  $s \in \mathbf{S}_{temp}$ 
     $\mathbf{S}_{temp} \leftarrow \mathbf{S}_{temp} - s$ 
     $n_{allot} \leftarrow n_{allot} + 1$ 
     $\mathbf{S}_{allot} \leftarrow \mathbf{S}_{allot} + s$ 
     $R_{temp} \leftarrow R_{temp} + \log_2(1 + SINR_s)$ 
    if  $R_{temp} \geq R_{req}$  then
      Allocate  $\mathbf{S}_{allot}$  to arriving call
       $\mathbf{S}_{avail} \leftarrow \mathbf{S}_{avail} - \mathbf{S}_{allot}$ 
       $n_{req} \leftarrow n_{allot}$ 
      break
    else if  $\mathbf{S}_{temp} = \{\}$  then
      Block the arriving call
    end if
  end while
end for

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B. Computation of blocking probability

In this section, we use the probability distribution function determined previously, to evaluate the Erlang capacity of the cell. Let $p[n_{req}]$ be non-zero for K values of n_{req} , $\{n_1, n_2, \dots, n_K\}$. Hence we divide incoming calls into K classes such that a call belongs to class- i if it requires $n_{req} = n_i$ subcarriers to satisfy its rate requirement. The rate of Poisson call arrivals in the cell is λ . Thus, by the probability distribution $p[n_{req}]$, we can infer that the arrival rate of calls of class- i is $\lambda_i = \lambda p[n_i]$. The holding time for all classes of calls is exponentially distributed with mean $\frac{1}{\mu}$. Thus, the offered load for class- i is $\rho_i = \rho p[n_i]$, where $\rho = \lambda/\mu$.

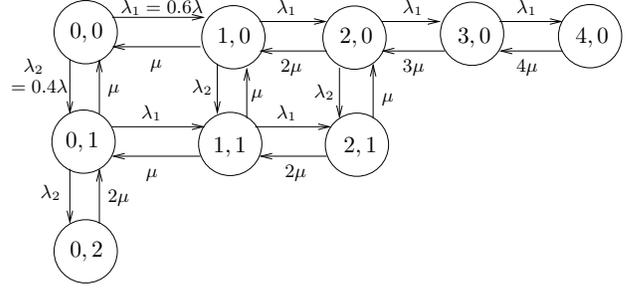


Fig. 2. Markov chain model for a system with $K = 2$ and $N = 4$. Incoming calls are divided into 2 classes since n_{req} takes 2 possible values.

We model the system by a K -dimensional Markov chain where the system state is defined as $\mathbf{X} = (X_1, X_2, \dots, X_K)$ where X_i is the number of active calls belonging to class i (i.e. using n_i subcarriers each). N is the total number of subcarriers available in the cell. As an illustrative example, consider the Markov chain shown in Fig. 2 with $N = 4$ subcarriers and $K = 2$ classes of calls. Calls require $n_1 = 1$ subcarrier with probability $p[1] = 0.6$ and $n_2 = 2$ subcarriers with probability $p[2] = 0.4$. Thus, the arrival rate of class-1 calls is 0.6λ and that of class-2 is 0.4λ .

The steady state probability of state $\mathbf{x} = (x_1, x_2, \dots, x_K)$ is defined as $\pi(\mathbf{x}) = P(\mathbf{X} = \mathbf{x})$. When $\mathbf{X} = \mathbf{x}$, the number of subcarriers in use by the active calls in the system is,

$$N_{used} = \sum_{i=1}^K n_i x_i = \mathbf{n} \cdot \mathbf{x}, \quad (5)$$

where $\mathbf{n} = (n_1, n_2, \dots, n_K)$. Thus, the set of possible states is, $\mathcal{S} := \{\mathbf{X} = \mathbf{x} : \mathbf{n} \cdot \mathbf{x} \leq N\}$. Let \mathcal{S}_i be the subset of states in which an incoming call of class i will be blocked. Thus, $\mathcal{S}_i := \{\mathbf{x} \in \mathcal{S} : \mathbf{n} \cdot \mathbf{x} > N - n_i\}$.

For example, in Fig. 2 the subset of states in which an incoming call of class-1 will be blocked is $(4, 0)$, $(2, 1)$ and $(0, 2)$. The sum of the steady state probabilities of this subset is equal to the blocking probability for class-1. For the cellular OFDMA system with K classes, the blocking probability for class- i arrivals can be expressed as,

$$P_{B_i} = \sum_{\mathbf{x} \in \mathcal{S}_i} \pi(\mathbf{x}) = \frac{\sum_{\mathbf{x} \in \mathcal{S}_i} \prod_{j=1}^K \frac{\rho_j^{x_j}}{x_j!}}{\sum_{\mathbf{x} \in \mathcal{S}} \prod_{j=1}^K \frac{\rho_j^{x_j}}{x_j!}} \quad (6)$$

where, (6) is a standard result whose proof is given in [9]. Due to large size of the state spaces, the evaluation of the blocking probability by brute force computation of all the steady state probabilities in (6) is practically infeasible. We solve this problem by drawing an analogy between the cellular OFDMA system and the stochastic knapsack in [9].

The classical knapsack problem involves a knapsack of capacity N resource units, and objects belonging to K different classes. An object of class i occupies n_i resource units, and a reward r_i is gained on placing it in the knapsack. The objective is to fit objects in the knapsack so as to maximize the total reward. The stochastic knapsack is a generalization of

the classical knapsack problem. Like the classical knapsack, objects are of K heterogeneous classes, but they have random arrivals and departures. Objects of class i have Poisson arrivals with rate λ_i and remain in the knapsack for exponentially distributed times, with mean $\frac{1}{\mu_i}$. There are no class dependent rewards. Arriving objects are accommodated as long as there is room in the knapsack, and blocked otherwise. The blocking probability of objects of class i is same as that of calls in class i in cellular OFDMA as derived in (6).

We model the cellular OFDMA system as a stochastic knapsack by choosing $\lambda_i = \lambda p[n_i]$ and $\mu_i = \mu$. Thus, the Erlang load of class i calls is $\rho_i = \rho p[n_i]$. An elegant recursive algorithm to evaluate the blocking probability of class i objects without brute force computation is presented in [9]. The authors define a term $q(c)$ for a $c \in [1, N]$, as the probability of exactly c resource units being used. We can derive the following recursive relation for $q(c)$,

$$cq(c) = \sum_{i=1}^K n_i \rho_i q(c - n_i), \quad c \in [1, N]. \quad (7)$$

We recursively compute $q(c)$ for all $c \in [1, N]$, and evaluate the blocking probability P_{B_i} of class i calls, such that $P_{B_i} = \sum_{N-n_i+1}^N q(c)$. The average blocking probability for the system is,

$$P_B = \sum_{i=1}^N P_{B_i} p[n_i], \quad 1 \leq i \leq K,$$

where, $p[n_i]$ is the probability that an incoming call belongs to class i . Thus, we have obtained the blocking probability for a given value of offered load ρ . It is used to evaluate the Erlang capacity of the cellular OFDMA system.

V. EXPERIMENTAL EVALUATION

In this section, we describe the system-level simulations performed to determine $p[n_{\text{req}}]$ and present the numerical results of blocking probability and Erlang capacity. Specifically, we demonstrate the effect of rate requirement and power transmitted by BSs on the Erlang capacity of the system.

A. Simulation Setup

We consider downlink transmissions in a cellular OFDMA network with cells of radius R . The values of system parameters are chosen as given in Table I. We generate random arrival and departure instants of a large number of calls in the reference cell and each of the 6 neighboring cells. Call arrivals are Poisson with rate λ and holding times are

TABLE I

SYSTEM PARAMETERS	
Cell Radius	$R = 1000$ m
Subcarriers available in each cell	$N = 512$
Power transmitted by BS per subcarrier	$P_{\text{tx}} = 10$ dBm
Path loss exponent	$\eta = 3.5$
Shadowing standard deviation	$\sigma = 8$ dB
Thermal noise level	$N_0 = -80$ dBm

exponentially distributed with mean $\frac{1}{\mu}$. Users are assumed to be uniformly distributed over the area of each cell. To simplify the generation of uniform spatial distribution of users in each cell, we approximate the hexagonal cell by a circular cell of radius R . We generate the radial position (D, θ) of the MS by drawing θ uniformly from the interval $[-\pi, \pi]$, and D according to the distribution $f_D(d) = \frac{2d}{R^2}$ for $0 \leq d \leq R$.

For every call arrival, we use the subcarrier allocation scheme in Algorithm 1 to allocate n_{req} subcarriers to the incoming call. For this, we need to determine the $SINR$, and hence the interference on each of the available subcarriers. To evaluate the interference I_i on subcarrier i , we check whether it has been allocated to an MS in any of the neighboring cells. For each call admission, we compute the increase in interference caused to neighboring cells on the allotted set of subcarriers. For every call departure, the set of subcarriers allocated to it are released into the pool of available subcarriers, and the inter-cell interference caused to the neighboring cells on these subcarriers is set to zero.

We obtain a large number of values of n_{req} for the given system parameters. For a given n_{req} , the value of the probability $p[n_{\text{req}}]$ is the fraction of call arrivals for which the number of subcarriers required is n_{req} . We obtain the call arrival rates of each class of users from this distribution as explained in Section IV. The arrival rate for class i is $\lambda p[n_i]$.

B. Results

The probability distribution $p[n_{\text{req}}]$ is plotted in Fig. 3 for different values of rate requirement R_{req} . For higher values of R_{req} , the average number of subcarriers required by users is higher. The calls requiring larger number of subcarriers correspond to the users either located at the cell edge, or experiencing a deep fade. Using $p[n_{\text{req}}]$ we determine the blocking probability as a function of the offered load ρ as shown in Fig. 4. For a higher value of R_{req} , users need more subcarriers on an average and hence experience a higher blocking probability. In Fig. 5, we plot the Erlang capacity as a function of R_{req} for 2% and 5% blocking probabilities. The

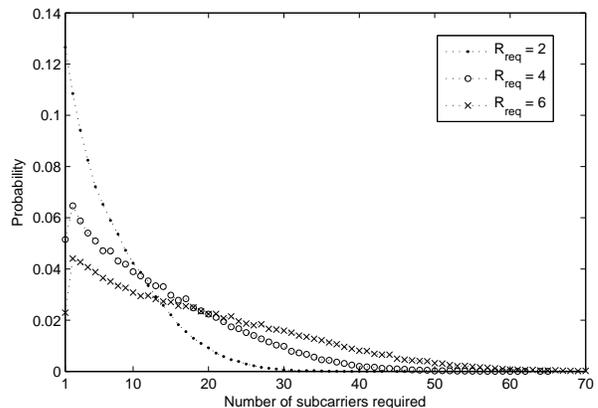


Fig. 3. Probability distribution $p[n_{\text{req}}]$ of the number of subcarriers required by an incoming call for the rate requirement $R_{\text{req}} = 2, 4$ and 6 bits/sec

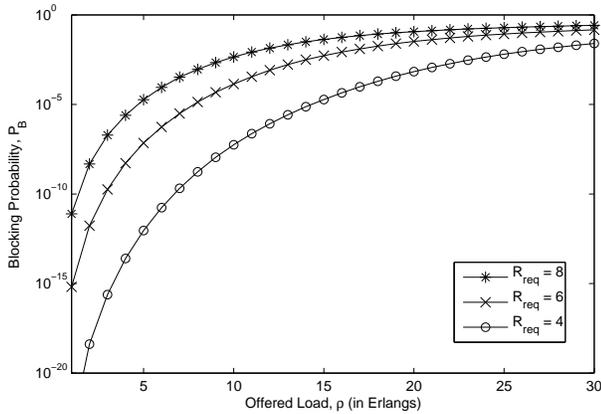


Fig. 4. Blocking probability versus ρ for $R_{\text{req}} = 4, 6$ and 8 bits/sec

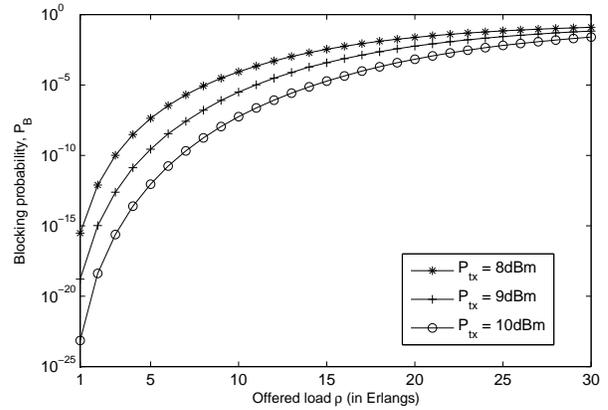


Fig. 6. Blocking probability versus ρ for $P_{\text{tx}} = 8, 9$ and 10 dBm

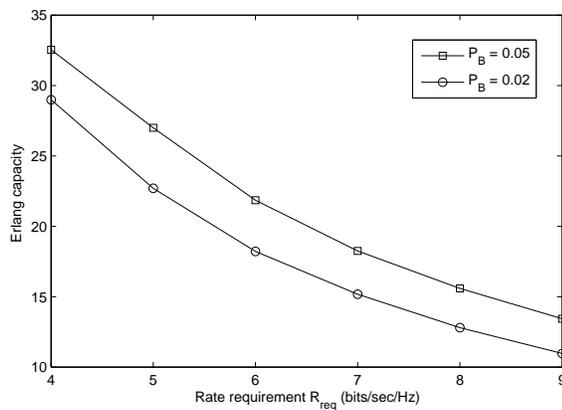


Fig. 5. Erlang capacity versus R_{req} for $P_B = 2\%$ and 5%

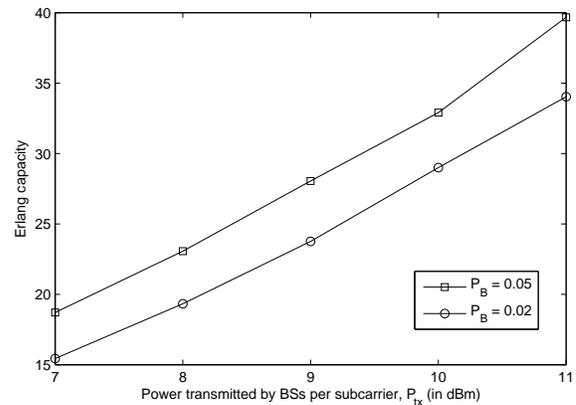


Fig. 7. Erlang capacity versus P_{tx} for $P_B = 2\%$ and 5%

Erlang capacity is the offered load corresponding to $P_B = 2\%$ and 5% in Fig. 4. In Fig. 6 and Fig. 7, we present similar results for fixed $R_{\text{req}} = 4$ bits/sec and different values of transmit power P_{tx} . Fig. 6 shows a decrease in blocking probability with P_{tx} . This is because, for a higher value of P_{tx} , users need less subcarriers and hence experience a lower blocking probability. In Fig. 7, we plot the Erlang capacity as a function of P_{tx} for 2% and 5% blocking probabilities.

VI. CONCLUSIONS

In this paper, we have determined the downlink Erlang capacity of a cellular OFDMA system with 1:1 frequency reuse. We have divided incoming calls into classes according to their subcarrier requirement. Then, we have modeled the system as a multi-dimensional Markov chain and applied the techniques used to solve the analogous stochastic knapsack problem to simplify the computation of blocking probability.

We have evaluated the worst case Erlang capacity under the assumption that the allotted subcarriers are used by the MS for the entire call duration. However if voice activity factor is taken into account, the inter-cell interference will be reduced, thus causing an increase in the Erlang capacity. This could be

a possible avenue for future investigations. Another research direction could be to determine the capacity of relay-assisted cellular systems by the proposed approach.

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