

Cartan Geometry in Modal Homotopy Type Theory

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Abstract

In this article, some Differential Geometry is developed synthetically in a Modal Homotopy Type Theory. While Homotopy Type Theory is used to reason about general ∞ -toposes, the “Modal” extension we are using here, is concerned with special ∞ -toposes with the extra structure of an idempotent monad with some additional properties. On the type theory side, the extension is realized by adding well known axioms of a *monadic modality*. In the applications we have in mind, this monadic modality corresponds to monads exhibiting infinitesimal information of the objects of special ∞ -toposes of spaces. There are two main lines of examples of these toposes, one containing smooth manifolds, the other algebraic varieties. Since we use Homotopy Type Theory, stacks from both of these worlds are naturally included in our discussion. We will make use of this in developing some new higher differential geometry. Much of the higher differential geometry in this article is aimed at making our construction of moduli spaces of G -structures and torsionfree G -structures possible in a useful way. As a basic example, this abstract construction of moduli spaces may be instantiated for a topos containing manifolds and the orthogonal group to give a construction of the moduli stack of Riemannian Metrics on a smooth manifold. The G -structures are developed along the lines of Urs Schreiber’s Higher Cartan Geometry.

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1 Introduction

This article uses Modal Homotopy Type Theory to reason about objects with differential geometric structure. Homotopy Type Theory on its own, is known to be a suitable language for Homotopy Types, or ∞ -groupoids. It is also a suitable language for objects, which carry more structure than just the ∞ -groupoidal one. This article is concerned with objects, which carry a differential geometric structure and an ∞ -groupoidal structure at the same time. Those objects are also called ∞ -stacks in the literature.

If one is not particularly interested in ∞ -stacks or their applications, the theory in this article can also be read as an *elementary* theory of objects carrying differential geometric structure. One interesting aspect of this perspective is, that we do not make any reference to numbers at all. This indicates, that there is some fundamental difference to other elementary approaches to differential geometry, like, for example Synthetic Differential Geometry. The latter is still very close to our approach, which we will use below to explain how our setup works.

It is also possible to read everything in this article as statements about an abstract *monadic modality* in Homotopy Type Theory. A monadic modality is a concept close to idempotent monads in category theory. The modality we postulate throughout this article, will be denoted \mathfrak{S} and comes with a map $\iota_X : X \rightarrow \mathfrak{S}X$ for all X . It will satisfy a universal property, which is a dependent version of the following: For all Y such that ι_Y is an equivalence and all maps $f : X \rightarrow Y$, there is a unique $\psi : \mathfrak{S}X \rightarrow Y$, such that the diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & \mathfrak{S}X \\ & \searrow f & \downarrow \exists! \psi \\ & & Y \end{array}$$

The dependent version of this universal property will be axiom 2.5.

We will now start to explain how the differential geometric structure is presented in our setting, by working from the basics of Synthetic Differential Geometry to one particular view of our setup. After that, we will turn to Homotopy Type Theory.

In Synthetic Differential Geometry, a ring object R in, say, a topos posing as “the affine line” is postulated and the *infinitesimal line segment* D given as ¹

$$D := \{ d \in R \mid d^2 = 0 \}$$

plays an essential role in the development of the basic geometric notions, assuming that the *Kock-Lawvere Axiom* ² holds. For example, a tangent vector at $x : 1 \rightarrow X$, for some object X , is just a map

$$t : D \rightarrow X$$

¹We suggestively use Set Theory notation, that the reader is left to translate to Category Theory. In this instance, D is the pullback of $x \mapsto x^2 : R \rightarrow R$ and $0 : 1 \rightarrow R$.

²See [Koc06] for an exposition of Synthetic Differential Geometry.

such that $t(0) = x$. Since all maps are differentiable in this setting, for any map $f : X \rightarrow Y$ there has to be a differential df_x of f at x , mapping tangent vectors at x to tangent vectors at $f(x)$. A first glimpse of the nice features of Synthetic Differential Geometry is that this map df_x is easily defined by $df_x(t) := f \circ t$.

The setup of this article is not the same as any flavor of Synthetic Differential Geometry. Yet there are similarities and we will eventually, for example, also define the differential of a function. The axioms we will use, are of a more abstract and categorial nature and centered around a relation of infinitesimal closeness. We will now show, how this relation arises in Synthetic Differential Geometry.

For $x, y \in R$, we define the following equivalence relation “ \sim ”:

$$x \sim y, \text{ if and only if } (x - y)^n = 0 \text{ for some } n \in \mathbb{N}.$$

If $x \sim y$ holds, we say that x and y are *infinitesimally close*. If our category is a *first order* model, which is a condition we will explain below, the previously defined D will be equal to the collection of all elements of R , which are infinitesimally close to 0:

$$D = \{d \in R \mid d \sim 0\}$$

If the first-order condition is dropped, maps from the new D will correspond to arbitrary *jets* at a point and not just tangent vectors. The important point is that D can be defined from the relation \sim without using any special properties of R . A curious question is, what the quotient R/\sim looks like. More precisely, we construct the relation as a subobject

$$\iota : \{(x, y) \in R \times R \mid x \sim y\} \rightarrow R \times R$$

to get the quotient as the coequalizer of $\pi_1 \circ \iota$ and $\pi_2 \circ \iota$. Let us answer this question in a simple model, supporting just the very basic properties we need for our discussion. Let k be a field and $k\text{-Alg}$ the category of finitely generated commutative unital k -algebras, such that any nilpotent element in any $A \in k\text{-Alg}$ squares to zero. A model of set-valued functors on algebras with the latter property is called a *first-order* model. Let the line R be the representable functor $k\text{-Alg}(k[X], _)$ on $k\text{-Alg}$. A map of k -algebras $\varphi : k[X] \rightarrow A$ is uniquely determined by the value $\varphi(X) \in A$ and each $x \in A$ defines a map $\varphi_x : k[X] \rightarrow A$ with $\varphi_x(X) = x$, so we may identify $R(A)$ with A . The quotient R/\sim can be computed pointwise:

$$R/\sim(A) = R(A)/\sim \quad \text{for all } A \in k\text{-Alg}$$

Where $R(A)/\sim$ is the coequalizer in Set of the diagram

$$\{(x, y) \in R \times R \mid x \sim y\}(A) \rightrightarrows R(A)$$

Or equivalently:

$$\{(x, y) \in A \times A \mid (x - y)^n = 0, \text{ for some } n \in \mathbb{N}\} \rightrightarrows A$$

The coequalizer $R(A)/\sim$ is nothing else than the *reduction* of A , i.e. the quotient $A/\sqrt{0}$ by the ideal $\sqrt{0}$ of nilpotent elements of A . Reducing algebras is a reflection into the subcategory of algebras without nilpotent elements. We have

$$R(A)/\sim \cong A/\sqrt{0} \cong R(A/\sqrt{0})$$

– so taking the quotient of R by \sim may be described as precomposition with the reduction functor. This is something we can do with any functor $X : k\text{-Alg} \rightarrow \text{Set}$. Let us denote the resulting endofunctor on $(k\text{-Alg} \rightarrow \text{Set})$ with \mathfrak{S} :

$$(\mathfrak{S}X)(A) := X(A/\sqrt{0})$$

The reduction admits an extension from representables to a coreflection on all functors $k\text{-Alg} \rightarrow \text{Set}$ and \mathfrak{S} is its right adjoint. It turns out, that \mathfrak{S} itself also has a right adjoint and is a reflection or put differently, an idempotent monad. This adjoint triple of endofunctors is the differential part of a *Differential Cohesive Topos* a notion due to Urs Schreiber [Scha], extending Lawvere’s Axiomatic Cohesion [Law07]. More precisely, Schreiber requires less properties from the Cohesive structure but uses it for $(\infty, 1)$ -toposes only, while the differential structure is also used on toposes of Set-valued sheaves [KS17].

Let us denote the unit of the monad \mathfrak{S} by ι . The map $\iota_R : R \rightarrow \mathfrak{S}R$ is the quotient map and we can recover the infinitesimal line segment D from ι_R by taking the fiber at $\iota_R(0)$:

$$\begin{array}{ccc} D & \longrightarrow & 1 \\ \downarrow & \text{(pb)} & \downarrow \iota_R(0) \\ R & \xrightarrow{\iota_R} & \mathfrak{S}R \end{array}$$

We will later use that as the definition of a *formal disk* at a point. In a model, say functors $(k\text{-Alg} \rightarrow \text{Set})$ without restrictions on the order of nilpotence, formal disks may turn out to be formal schemes. This can be seen, by calculating the pullback we used above to define the formal disk D . In the pullback diagram below, the formal disk D is already replaced by its representation given by homomorphisms of the topological ring $k[[X]]$ with topology generated by the powers of the ideal (X) to k -algebras in $k\text{-Alg}$ carrying the discrete topology.

$$\begin{array}{ccc} k\text{-Alg}_{\text{top}}(k[[X]], _) & \longrightarrow & 1 \\ \downarrow & \text{(pb)} & \downarrow \iota_{k\text{-Alg}(k[[X]], _)}(0) \\ k\text{-Alg}(k[X], _) & \xrightarrow{\iota_{k\text{-Alg}(k[X], _)}} & \mathfrak{S}(k\text{-Alg}(k[X], _)) \end{array}$$

When we say something is a *model* of the type theory we use in the following, this is not meant to be read in any technical sense. So far, none of the models we are interested in is known to be a model for all of the rules and axioms we will use.

Another strain of models which contain smooth n -manifolds are *formal smooth ∞ -groupoids*. Here, one starts with the category of \mathbb{R} -algebras of the form $\mathcal{C}^\infty(\mathbb{R}^n)$ for some $n \in \mathbb{N}$ and enlarges it in the following way³:

$$\{\mathcal{C}^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} (\mathbb{R} \oplus V) \mid V^n = 0, n \in \mathbb{N}, \dim_{\mathbb{R}}(V) < \infty, V \text{ ideal of some } A \in \mathbb{R}\text{-Alg}\}$$

³This is discussed in more detail in [KS17].

Let C be the opposite category of these \mathbb{R} -algebras with all \mathbb{R} -algebra morphisms between them. Then

$$\mathbb{R}^n \times \mathbb{D}_V := (\mathcal{C}^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} (\mathbb{R} \oplus V))^{\text{op}}$$

is a meaningful notation, since “ $\otimes_{\mathbb{R}}$ ” corresponds to the product and the space $(\mathbb{R} \oplus V)^{\text{op}}$ behaves like an infinitesimal, or formal, disk. A topology is given by jointly surjective immersions of open subsets with contractible intersections only on the non-infinitesimal parts and identities on the infinitesimal disks. The Set-sheaves with respect to this topology are called *formal smooth sets* and the higher sheaves with values in ∞ -groupoids are called *formal smooth ∞ -groupoids*. The infinitesimal extensions allow us to define a coreduction for sheaves $X \in \text{Sh}(C)$:

$$(\mathfrak{S}X)(\mathbb{R}^n \times \mathbb{D}_V) := X(\mathbb{R}^n)$$

We can do the same construction with a different choice of infinitesimals, one important example are *first order formal smooth ∞ -groupoids*, where the condition $V^n = 0$ is replaced by $V^2 = 0$. This is an important model to keep in mind, since here, smooth n -manifolds are included in the 0-types, the notion formal disk is just an infinitesimal version of the tangent space and notions we will define in section 3.1, like formal disk bundle, or the differential of a function are respectively an infinitesimal version of the tangent bundle and the usual differential of a function. As we already stated at the beginning of this introduction, we work with an arbitrary *monadic modality*, a dependent type theoretic analogue of an idempotent monad. Assuming that \mathfrak{S} satisfies the axioms of a monadic modality [Uni13, Section 7.7] will be enough to admit all the differential geometric constructions we need in this article. This entails, that all the theorems in this article also apply to any other monadic modality, which turned out to be interesting at least for the most basic notions and especially for modalities from Real-Cohesive Homotopy Type Theory [Shu15a].

Similar to [Shu15a], familiar objects from Geometry, like manifolds, are supposed to be included as *0-types* in our type theory. The most basic topos this article applies to in the intended way is formed by the first-order formal smooth ∞ -groupoids, as introduced above. Smooth n -manifolds are fully faithfully embedded in the 0-truncated objects of this topos. This means, there is a smooth manifold \mathbb{S}^1 different from the homotopy type S^1 , given as a higher inductive type in [Uni13]. The two circles are related by the shape modality \int from [Shu15a] by “ $\int \mathbb{S}^1 \simeq S^1$ ”. Mostly⁴, Shulman’s type theory is compatible with assuming \mathfrak{S} as a modality whose modal types contain the \int -modal types. However, we will not use any other modalities in this article.

But since we use no property of \mathfrak{S} beyond being an abstract modality, it is possible to ask, what the theory developed here means for other modalities. The table below compares some motivating models with models with other modalities. We will

⁴The most common models admitting the coreduction \mathfrak{S} and a shape \int forbid the construction of \int as nullification at the Dedekind-Reals, but it might still be defined as the nullification at an abstract ring-object.

illustrate the meaning of switching the modality by listing what the *formal disks* will be in the models. The categories appearing are in order of appearance, sheaves on the Zariski-site, formal smooth ∞ -groupoids with first order infinitesimals, general formal smooth ∞ -groupoids, ∞ -cohesive toposes like sheaves on the formal cartesian spaces and ∞ -groupoids.

Model	Formal disks
\mathfrak{S} on $\text{Sh}(\mathfrak{Zar})$	Formal neighborhoods
\mathfrak{S}_1 on FSGrp_1	Almost tangent spaces
\mathfrak{S} on FSGrp	Jets through a point
$\ _{-}\ _n$ on Homotopy Types	n -connective covers

The coreduction \mathfrak{S} has also been used to characterize formally étale maps [ST], [KR] — what we will do as well in section 3.2. This notion of formally étale maps coincides with at least for sheaves on a site given by finitely generated algebras. In the case of general rings, the formally étale maps in the sense of [EGAIV4, Definition 17.1.1.1] contain the class of maps we will defined in 3.17. It is not known to the author, if this inclusion is strict.

In one model containing smooth n -manifolds, the formally étale maps corresponding to this definition will be local diffeomorphisms. A quite different example is the inclusion of a formal disk.

We give again a table of the different meaning of formally étale maps for some modalities:

Model	Formally étale maps
\mathfrak{S} on $\text{Sh}(\mathfrak{Zar})$	A special case of formally étale maps
\mathfrak{S}_1 on FSGrp_1	Local diffeomorphisms (if domain and codomain are manifolds)
\mathfrak{S} on FSGrp	Local diffeomorphisms (if domain and codomain are manifolds)
$\ _{-}\ _n$ on Homotopy Types	Maps inducing equivalences on all n -connective covers of the codomain
\sharp on Spaces ⁵	Maps where the domain carries the induced topology

The restriction to first order infinitesimals made above serves only the purpose of presenting \mathfrak{S} for commonly known differential geometric concepts. We can make the same definition for a category of arbitrary k -algebras. It works, whenever we have a reduction which is a reflection. Furthermore, if we want sheaves instead of presheave on $k\text{-Alg}^{\text{op}}$, the construction of \mathfrak{S} still works, if covers are sent to covers by the reduction. The latter is the case for the Zariski, étale and Nisnevich topologies.

Since this functor \mathfrak{S} , that we will call *coreduction* from now on, allows us to build at least some abstract differential geometry relative to it, one might ask what role it plays in conventional geometry. The answer is, that concepts very close to it appear very early in the Grothendieck school of Algebraic Geometry, which is no

surprise at all, since algebras with nilpotent elements were specifically used to admit reasoning with this kind of infinitesimals. However, the functor itself leaves the impression of a rather exotic concept under the names of deRham prestack [GR14], deRham stack, deRham space or infinitesimal shape and is usually used to represent D-modules over a smooth scheme or algebraic stack X as quasicoherent sheaves over $\mathfrak{S}X$. A functor \mathfrak{S} also exists in meaningful ways in non-commutative geometry [KR]. In the face of these rather advanced use cases of the coreduction \mathfrak{S} , it might be irritating that we use it as a basis for differential geometry. But we will actually never really study $\mathfrak{S}X$ as a space — what we are interested in is the unit or quotient map $\iota_X : X \rightarrow \mathfrak{S}X$, that provides us with the notion of “*infinitesimally close*” we discussed above.

We use Homotopy Type Theory like presented in [Uni13] throughout the article assuming a monadic modality [Uni13, Section 7.7]. For an introduction to Homotopy Type Theory that goes very well with the goals of this article, see [Shu17]. Homotopy Type Theory, was shown to be a suitable language for *Synthetic Homotopy Theory*. Another way to put the latter in more precise terms, is that Homotopy Type Theory has an interpretation in the ∞ -category of ∞ -groupoids ([LKV14]).

It is conjectured that a similar thing holds for arbitrary ∞ -toposes in the sense of [Lur09]. So far, this has been settled for some classes of ∞ -toposes ([Shu15b], [Shu15c]) to which the ∞ -toposes of interest in this article do not belong. However, all the rules we use except those for the strict univalent universe we assume starting in section 4.1 are known to be translatable in a consistent way to locally presentable locally cartesian closed ∞ -categories [GK12]. Furthermore, the conjecture is already known to hold for a weaker version of univalence universes. It might be useful to know, that everything before section 4.1 doesn’t rely on univalence and could be interpreted in an ordinary 1-topos.

When we speak of sheaves on a site C with values in ∞ -groupoids, what we mean is the ∞ -category represented by a right proper local model structure on the category ($C^{\text{op}} \rightarrow \text{sSet}$) of functors on C^{op} with values in simplicial sets. We can make the following translations between Category Theory and Homotopy Type Theory (HoTT):

HoTT	Notation	Category Theory
Identity Type	$x =_A y$	Path object
Universe	\mathcal{U}	Object classifier
Dependent type	$B : A \rightarrow \mathcal{U}$	Fibration
Substitution	$B(f(x))$	$f^* \equiv$ Pullback along f
Dependent Sum	$\sum_{x:A} B(x)$	Left adjoint of f^*
Dependent Product	$\prod_{x:A} B(x)$	Right adjoint of f^*
n -truncation	$\ A\ _n$	Pointwise n th coskeleton

The statements and proofs concerning the modality we will postulate may be translated to ∞ -toposes with a modality, for example in the sense of [Ane+17] or using the machinery discussed in the last section of [RSS17].

In Homotopy Type Theory 0-truncated types are sometimes called *Sets*. In our setup, it is better to think of them as Set-valued sheaves forming a subcategory of a topos of general ∞ -groupoid-valued sheaves. So we prefer to speak of 0-types or 0-truncated types to avoid wrong impressions.

Smooth n -manifolds are included in the formal smooth ∞ -groupoids as their representable functors, which take values in Set. So the ordinary spaces we are interested in when doing Differential Geometry are included as 0-types in our modal Homotopy Type Theory. Examples of spaces appearing in Geometry that can be 1-types, are quotient stacks. If we take such a quotient of the manifold \mathbb{R} by the action of $\mathbb{Z}/2\mathbb{Z}$ by changing the sign, the quotient is almost a ray $\mathbb{R}_{>0}$ with the only difference being at 0. Since 0 gets mapped to itself, it will be identified with itself in a non-trivial way, which means that the equality type $0 = 0$ is equivalent to $\mathbb{Z}/2\mathbb{Z}$. Spaces like this quotient are naturally included in our framework.

The univalent universe is used to construct and reason about classifying stacks for fiber bundles. In section 4.3, we start with an informal discussion how higher structures in the form of the equality types in Homotopy Type Theory admit reasoning about fiber bundles and access to additional structures on manifolds. Later on, we use more specifically the fact ⁶ that actions of a group G on spaces can be encoded as dependent types over a classifying stack BG and that the homotopy quotient of an action $\rho : BG \rightarrow \mathcal{U}$ is just $\sum_{x:BG} \rho(x)$. This is crucial for the goal of the article to construct various moduli stacks as homotopy quotients of actions given in this way.

Important advantages of Homotopy Type Theory for this work include the unusual clarity for a higher categorical framework. Furthermore, a proof-assistant software, in this case *Agda* can be used to check definitions and proofs written out in Homotopy Type Theory. This was of great help to the author during the development of the theory in this article and while learning the subject. The partial formalization can be viewed at <https://github.com/felixwellen/DCHoTT-Agda>.

Contribution. This is an improved and extended version of the type theoretic content of the authors dissertation [Wel17]. The dissertation solved a problem presented by Urs Schreiber in 2015 to the Homotopy Type Theory community at the meeting of the german mathematical society (DMV) in Hamburg ([Schb, Theorem 3.6]). The first part of the problem was to prove the triviality of the formal disk bundle over a group in a modal Homotopy Type Theory, a theorem generalizing the triviality of the tangent bundle over a Lie-group. This is theorem 3.11 in this article, where the more general class of *homogeneous types* was used. This notion is defined in 3.8 and is related to but not the same as any common notion of “homogeneous space” in Geometry.

The second part of the problem was the associated bundle construction for the formal disk bundle of a manifold. This is implicit in lemma 4.16. Along the way some geometric notions were developed in a way Schreiber suggested and later replaced with more type theoretic counterparts. The same happened with proof

⁶See [NSS15, Definition 3.1] for a version using $(\infty, 1)$ -categories and [BvR18, Section 4.2] for the Homotopy Type Theory version.

ideas the author learned from Schreiber. All were replaced eventually by more direct versions using the dependency built into Homotopy Type Theory which is hard to use in Category Theory.

Theorem 3.12 is not necessary for the main goals of the article, but gives some additional examples of manifolds (lemma 4.15). It states that any homogeneous type A sits in a sequence

$$\mathbb{D}_e \xrightarrow{\iota_e} A \xrightarrow{\iota_A} \mathfrak{S}A$$

of homogeneous types, where \mathbb{D}_e is the formal disk (definition 3.3) at the unit of A . Keeping in mind that in most⁷ cases, the unit ι_A is an epimorphism, this is close to a sequence used in the theory of algebraic groups ([Dem72, p. 34]). If we take the left-exactness of \mathfrak{S} into account and assume that ι_A is epi, then the sequence is exact and 0-truncated, whenever A is 0-truncated. If A is not 0-truncated, we get a natural generalization of the classical result to stacks by continuing the homotopy fiber sequence of ι_A on the left.

Urs Schreiber provided the author with more ideas building on top of the first two results, how what he calls *Higher Cartan Geometry* can be developed in this abstract setup. This was done to the most part already in [Wel17]. Along these lines G -structures are defined, a concept explained informally at the beginning of section 4.3. See 4.3 for a short overview, which structures on a manifold can be encoded as G -structures. Infinitesimally trivial G -structures, or *torsionfree* G -structures, are also treated. While the collection of G -structures on a manifold is easily given as a sum-type, the actual moduli spaces of G -structures, i.e. the homotopy quotients by the diffeomorphisms of the underlying manifold, proved to be hard to construct in a nice way with the setup in [Wel17].

To remedy the situation, the theory developed in [Wel17] was changed. Some changes to the theory of fiber bundles in section 4.1 yielded new characterizations and constructions. Most notably, it is shown, that from a classifying morphism

$$\chi : M \rightarrow \mathbf{BAut}(F)$$

of a fiber bundle $E \rightarrow M$, a trivializing cover of M can be constructed (4.10). Especially important to the goals of this article is a characterization of fiber bundles which is clearly a proposition (definition 4.9). This admits a definition of G -structures as a dependent type over the delooped diffeomorphism group of a manifold, which turns the homotopy quotient into a simple sum-type. The article concludes with the construction of moduli types of torsionfree G -structures precisely as such a homotopy quotient.

All of the main theorems in the original dissertation [Wel17] were formalized in Agda. For the present version, everything that builds on the chain rule 3.5 has not yet been included in the formalization. This includes the revised definition of torsionfree G -structures.

⁷The author is not aware of a counterexample. In differential cohesion, this is always true. In the topos of Zariski-sheaves on finitely generated algebras over an algebraically closed field, this is also true for any sheaf.

Acknowledgments

The idea of using modalities in Homotopy Type Theory in the way present in this work is due to Urs Schreiber and Mike Shulman [Schb] [SS14], Schreiber was one of the supervisors of the author's thesis. He provided the author with all the categorical versions of the important geometric definitions, as well as the main theorems and category theoretic proofs leading to the type theoretic version of his Higher Cartan Geometry presented in this article. Adaptions to Homotopy Type Theory of Schreiber's original proofs are included in the author's thesis [Wel17]. The proofs in this article make more use of type theoretic dependency which shortens the arguments a lot in most cases. Some concepts needed reformulation and additional theorems were needed to make the main result, the construction of moduli spaces of torsionfree G -structures possible.

Schreiber explained a lot of mathematics important to this work to the author on his many visits in Bonn and answered countless questions via email.

During the time of writing his thesis and on later occasional visits, the author profited a lot from his working groups in Karlsruhe. This work wouldn't be the same without the discussions with and the Algebra knowledge of Tobias Columbus and Fabian Januszewski and the support of Frank Herrlich, Stefan Kühnlein and other members of the Algebra group and the Didactics group. On a couple of visits in Darmstadt, Ulrik Buchholtz, Thomas Streicher and Jonathan Weinberger listened carefully to various versions of the theory in this article and made lots of helpful comments. Two questions of Ulrik Buchholtz led directly to propositions in this article (part of 4.10 and 4.15).

A short visit in Nottingham and discussion with Paolo Capriotti, Nicolai Kraus and Thorsten Altenkirch also helped in the early stages of the theory and had an impact on the authors agda knowledge.

The discussion with the Mathematics Research Community group, helped the author a lot to understand Differential Cohesion better. The research events in this line were sponsored by the National Science Foundation under Grant Number DMS 1641020. The group work for the Differential Cohesion group at this event was organized by Dan Licata and Mike Shulman. The group member Max S. New later read part of the thesis and made an important suggestion for an improvement of the definition of fiber bundle.

The improvements on this work were developed on a Postdoc position in Steve Awodey's group at Carnegie Mellon University, sponsored by The United States Air Force Research Laboratory under agreement number FA9550-15-1-0053. The good atmosphere with lots of opportunities of discussion with local homotopy type theorists as well as the many visitors helped a lot. Steve Awodey gave the author lot's of opportunities to present his work and new ideas to the locals and the guests and made lots of helpful discussions possible. One consequence important to this work was a joint, successful effort with Egbert Rijke, to understand formally étale maps better — another countless discussions with Jonas Frey about abstract Geometry, the role of higher categorical structures therein and Type and Category Theory in general.

Comments of and discussions with the visitors Mike Shulman, Mathieu Anel, An-

dré Joyal, Eric Finster, Dan Christensen and Marcelo Fiore led to improvements and helped the author to understand many things important to this article better.

2 Modal Homotopy Type Theory

2.1 Terminology and notation

Mostly, we use the same terminology and notation as the HoTT-Book [Uni13]. However, there are a few exceptions. To denote terms of type $\prod_{x:A} B(x)$ we use the notation for λ -expressions from pure mathematics, i.e. $x \mapsto f(x)$. There are no implicit propositional truncations. If the propositional truncation of a statement is used, it is indicated by the word “merely”. Phrases like “for all” and “there is” are to be interpreted as \prod - and \sum -types. For example, the sentence

For all $x:A$ we have $t:B(x)$.

is to be read as the statement describing the term $(x:A) \mapsto t$ of type $\prod_{x:A} B(x)$. We sometimes write f_a for the application of a dependent function $f: \prod_{x:A} B(x)$ to $a:A$, instead of $f(a)$.

Furthermore, similar to [Shu15a], when dealing with identity types, we avoid topology and geometry related words. For example, we write “equality” instead of “path” and “2-cell” instead of “homotopy”, to avoid confusion with the notions of paths and homotopies for the classical geometric objects we like to study by including them in our theory as 0-types. Similarly we say that x is unique with some properties, if the type of all x with these properties is contractible.

2.2 Preliminaries from Homotopy Type Theory

We use a fragment of the Type Theory from [Uni13]. Function extensionality is always assumed to hold. We need either -1-truncations or images to exist. And, we need either univalence or higher inductive types to construct Eilenberg-MacLane spaces, to make everything in the last couple of sections work. We chose to use -1-truncations and univalence in the presentation. Furthermore, later on, we use univalence a lot more than is necessary. We believe this is just convenience and everything essential could also be done with higher inductive types and we care about this, because it might simplify the applications to some ∞ -toposes.

In the next section we will give axioms for a modality, which will be assumed throughout the article. Some knowledge of the basic concepts in [Uni13] is assumed. In addition, we will use more facts about pullbacks than presented in [Uni13], which we will list in this section.

It is very useful to switch between pullback squares and equivalences *over* a morphism. We start with the latter concept.

Definition 2.1

Let $f: A \rightarrow B$ be a map and $P: A \rightarrow \mathcal{U}$, $Q: B \rightarrow \mathcal{U}$ be dependent types.

(a) A *morphism over f* or *fibred morphism* is a

$$\varphi : \prod_{x:A} P(x) \rightarrow Q(f(x)).$$

(b) An *equivalence over f* or *fibred equivalence* is a

$$\varphi : \prod_{x:A} P(x) \simeq Q(f(x)).$$

For every morphism over $f : A \rightarrow B$ as above, we can construct a square ⁸

$$\begin{array}{ccc} \sum_{x:A} P(x) & \longrightarrow & \sum_{x:B} Q(x) \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ A & \xrightarrow{f} & B \end{array}$$

where the top map is given as $(a, p_a) \mapsto (f(a), \varphi_a(p_a))$. This square will turn out to be a pullback in the sense we are going to describe now, if and only if φ is an equivalence over f .

For a cospan given by the maps $f : A \rightarrow C$ and $g : B \rightarrow C$, we can construct a pullback square:

$$\begin{array}{ccc} \sum_{x:A, y:B} f(x) = g(y) & \xrightarrow{\pi_2} & B \\ \downarrow \pi_1 & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

Then, for any other completion of the cospan to a square

$$\begin{array}{ccc} X & \longrightarrow & B \\ \downarrow & \nearrow \eta & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

where $\eta : \prod_{x:X} g(x) = f(x)$ is a 2-cell letting it commute, an induced map to the pullback is given by $x \mapsto (\varphi_A(x), \varphi_B(x), \eta_x)$.

Definition 2.2

A square is a *pullback square* if the induced map described above is an equivalence.

To reverse the construction of a square for a morphism over “ f ” above, we can start with a general square:

⁸By stating that it is a “square” we implicitly assume that there is a 2-cell letting it commute, which is considered to be part of the square. In this particular case, the 2-cell is trivial.

$$\begin{array}{ccc}
X & \longrightarrow & Y \\
p_A \downarrow & \swarrow_{\eta} & \downarrow p_B \\
A & \xrightarrow{f} & B
\end{array}$$

Let $P : A \rightarrow \mathcal{U}$ and $Q : B \rightarrow \mathcal{U}$ be the fiber types of the vertical maps, i.e. $P(x : A) := \sum_{y:Y} p_A(y) = x$ and Q respectively. Then, for all $x : A$, a morphism $\varphi_x : P(x) \rightarrow Q(x)$ is given as

$$\varphi_x((y, p_y)) := (f(y), f(p_y)).$$

So φ is a morphism from P to Q over f . The following statement is quite useful and will be used frequently in this article:

Lemma 2.3

- (a) A square is a pullback if and only if the induced fibered morphism is an equivalence.
- (b) A fibered morphism is an equivalence, if and only if the corresponding square is a pullback.

Now, the following corollary can be derived by using the fact that equivalences are stable under pullback:

Corollary 2.4

Let $f : A \rightarrow B$ be an equivalence, $P : A \rightarrow \mathcal{U}$, $Q : B \rightarrow \mathcal{U}$ dependent types and $\varphi : \prod_{x:A} P(x) \rightarrow Q(f(x))$ an equivalence over f . Then the induced map

$$\left(\sum_{x:A} P(x) \right) \rightarrow \left(\sum_{x:B} Q(x) \right)$$

is an equivalence.

2.3 The Coreduction \mathfrak{S}

From this section on, we will postulate the existence of a *modality* \mathfrak{S} . We use the definition of a uniquely eliminating modality from [RSS17], which is equivalent to the definition given in [Uni13, Section 7.7]. More on modalities and their relation to concepts in category theory can be found in [RSS17].

Axiom 2.5

From this point on, we assume existence of a map $\mathfrak{S} : \mathcal{U} \rightarrow \mathcal{U}$ and maps $\iota_A : A \rightarrow \mathfrak{S}A$ for all types A , subject to this condition: For any $B : \mathfrak{S}A \rightarrow \mathcal{U}$, the map

$$- \circ \iota_A : \left(\prod_{a:\mathfrak{S}A} \mathfrak{S}B(a) \right) \rightarrow \left(\prod_{a:A} \mathfrak{S}B(\iota_A(a)) \right)$$

is an equivalence.

We call the inverse of the equivalence \mathfrak{S} -*elimination*. Note that the equivalence specializes to the universal property of a reflection if the family B is constant:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathfrak{S}A \\ & \searrow f & \downarrow \exists! \psi \\ & & \mathfrak{S}B \end{array}$$

i.e. for all types B and all $f : A \rightarrow \mathfrak{S}B$, we get a unique ψ letting the triangle commute up to a 2-cell. Unique means here, there is a contractible type of maps with 2-cells letting the triangle commute. That type is also a fiber of the equivalence “ $_ \circ \iota_A$ ”, so we do know it is contractible.

We will make use of this in showing, that \mathfrak{S} is idempotent in the following sense:

Proposition 2.6

For all types A , the map $\iota_{\mathfrak{S}A} : \mathfrak{S}A \rightarrow \mathfrak{S}(\mathfrak{S}A)$ is an equivalence.

Proof By the universal property we just discussed, we get a candidate for an inverse to $\iota_{\mathfrak{S}A}$, which we call φ :

$$\begin{array}{ccc} \mathfrak{S}A & \xrightarrow{\iota_{\mathfrak{S}A}} & \mathfrak{S}(\mathfrak{S}A) \\ & \searrow \text{id} & \downarrow \exists! \varphi \\ & & \mathfrak{S}A \end{array}$$

By construction, φ is already a left inverse of $\iota_{\mathfrak{S}A}$. To see that we consider the diagram

$$\begin{array}{ccccc} & & \mathfrak{S}(\mathfrak{S}A) & & \\ & \nearrow \iota_{\mathfrak{S}A} & \downarrow \varphi & & \\ \mathfrak{S}A & \xrightarrow{\text{id}} & \mathfrak{S}A & & \\ & \searrow \iota_{\mathfrak{S}A} & \downarrow \iota_{\mathfrak{S}A} & \curvearrowright \text{id} & \\ & & \mathfrak{S}(\mathfrak{S}A) & & \end{array}$$

and conclude that φ is also a right inverse by uniqueness.

Like reflections determine a subcategory, \mathfrak{S} determines a subuniverse of the universe \mathcal{U} of all types ⁹.

Definition 2.7

- (a) A type A is *coreduced*, if ι_A is an equivalence.
- (b) The *universe of coreduced types* is

$$\mathcal{U}_{\mathfrak{S}} := \sum_{A:\mathcal{U}} (A \text{ is coreduced})$$

⁹We implicitly assume a hierarchy of universes \mathcal{U}_i , but only mention indices if there is something interesting to say about them.

From what we proved above, all types $\mathfrak{S}A$ will be coreduced.

We call \mathfrak{S} *Coreduction*. A common name in geometry for $\mathfrak{S}(X)$ is the *deRham-stack* of X . However, as we explained in the introduction, we will not be very interested in spaces of the form “ $\mathfrak{S}X$ ”, but more in the “quotient map” $\iota_X : X \rightarrow \mathfrak{S}X$, which we will view as identifying infinitesimally close points.

It might seem unreasonable to have a special name for a general modality and its modal types, but the names of definitions given later just make sense in intended models, so it might be good to remind ourselves of this and tie some particular pictures to this modality.

Like a functor, \mathfrak{S} extends to maps and we get a naturality squares for ι :

Definition 2.8

- (i) For any function $f : A \rightarrow B$ between arbitrary types A and B , we have a function:

$$\mathfrak{S}f : \mathfrak{S}A \rightarrow \mathfrak{S}B$$

given by \mathfrak{S} -elimination.

- (ii) For any function $f : A \rightarrow B$ between arbitrary types A and B , there is a 2-cell η witnessing that the following commutes:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathfrak{S}A \\ f \downarrow & \swarrow \eta_f & \downarrow \mathfrak{S}f \\ B & \xrightarrow{\iota_B} & \mathfrak{S}B \end{array}$$

It is also straight forward to prove that the application of \mathfrak{S} to maps commutes with composition of maps up to equality and preserves identities up to equality. Like it is often the case in situations like this, it seems to be the case that coherence of these equalities can always be proved for what is needed in practice.

Remark 2.9

For any 2-cell $\eta : f \Rightarrow g$, we have a 2-cell between the images:

$$\mathfrak{S}\eta : \mathfrak{S}f \Rightarrow \mathfrak{S}g.$$

Coreduced types have various closedness properties, which we review in the following lemma.

Proposition 2.10

Let A be any type and $B : A \rightarrow \mathcal{U}$ such that for all $a : A$ the type $B(a)$ is coreduced.

- (a) Retracts of coreduced types are coreduced.
- (b) The dependent product

$$\prod_{a:A} B(a)$$

is coreduced. Note that A is not required to be coreduced here and this implies all function spaces with coreduced codomain are coreduced.

(c) If A is coreduced, the sum

$$\sum_{a:A} B(a)$$

is coreduced.

(d) Coreduced types have coreduced identity types.

Proof (a) A type R is a retract of B , if there are maps $r:B \rightarrow R$ and $\iota:R \rightarrow B$, such that $r \circ \iota$ is equal to the identity. For all coreduced B and retracts R of B we have the following diagram:

$$\begin{array}{ccccc} & & \text{id} & & \\ & \curvearrowright & & \curvearrowleft & \\ R & \xrightarrow{\iota} & B & \xrightarrow{r} & R \\ \downarrow \iota_R & & \downarrow \iota_B & & \downarrow \iota_R \\ \mathfrak{S}R & \xrightarrow{\mathfrak{S}\iota} & \mathfrak{S}B & \xrightarrow{\mathfrak{S}r} & \mathfrak{S}R \\ & \curvearrowleft & & \curvearrowright & \\ & & \text{id} & & \end{array}$$

Since ι_B is an equivalence, it has an inverse and by the diagram, $r \circ \iota_B^{-1} \circ \mathfrak{S}\iota$ is a biinverse to ι_R .

(b) This is proved, up to equivalence, in [Uni13, Theorem 7.7.7].

(c) This is [Uni13, Theorem 7.7.4].

(d) This is [RSS17, Lemma 1.25].

One immediate consequence is $\mathfrak{S}1 \simeq 1$ – this is the only provably coreduced type, since the operation mapping every type to 1 is a modality. Furthermore, 2.10 entails the possibility to prove for some propositions about coreduced types, that the proposition is coreduced and may hence be proved by using \mathfrak{S} -elimination. The following is a slight variation of [RSS17][Lemma 1.24]:

Proposition 2.11

Let A be a type and $B:\mathfrak{S}A \rightarrow \mathcal{U}$ a dependent type. Then the induced map is an equivalence:

$$\mathfrak{S} \left(\sum_{x:A} B(\iota_A(x)) \right) \simeq \left(\sum_{x:\mathfrak{S}A} \mathfrak{S}(B(x)) \right).$$

3 A basis for Differential Geometry

3.1 Formal disks

We will start to build geometric notions on top of the coreduction \mathfrak{S} and its unit ι . This modality provides us with the notion of infinitesimal proximity. To see if two points x, y in some type A are infinitesimally close to each other, we map them to $\mathfrak{S}A$ and ask if the images are equal.

Definition 3.1

Let $x, y: A$. Then we have a type which could be read “ x is *infinitesimally close* to y ” and is given as:

$$x \sim y := (\iota_A(x) = \iota_A(y)).$$

Of course, this is in general not a proposition, but it is a useful way to think about $\iota_A(x) = \iota_A(y)$ in this way.

It turns out, all morphisms of types already respect that notion of closedness, i.e. if two points are infinitesimally close to each other, their images are close as well.

Remark 3.2

If $x, y: A$ are infinitesimally close, then for any map $f: A \rightarrow B$, the images $f(x)$ and $f(y)$ are infinitesimally close. More precisely, we have an induced function

$$\tilde{f}: (x \sim y) \rightarrow (f(x) \sim f(y))$$

Proof We construct a map between the two types $\iota_A(x) = \iota_A(y)$ and $\iota_B(f(x)) = \iota_B(f(y))$. By 2.8 we can apply \mathfrak{S} to maps and get a map $\mathfrak{S}f: \mathfrak{S}A \rightarrow \mathfrak{S}B$. So we can apply $\mathfrak{S}f$ to an equality $\gamma: \iota_A(x) = \iota_A(y)$ to get an equality

$$\mathfrak{S}f(\gamma): \mathfrak{S}f(\iota_A(x)) = \mathfrak{S}f(\iota_A(y))$$

By 2.8 again, we know that we have a naturality square:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathfrak{S}A \\ f \downarrow & \swarrow \eta_f & \downarrow \mathfrak{S}f \\ B & \xrightarrow{\iota_B} & \mathfrak{S}B \end{array}$$

and hence equalities $\eta_f(x): \mathfrak{S}f(\iota_A(x)) = \iota_B(f(x))$ and $\eta_f(y): \mathfrak{S}f(\iota_A(y)) = \iota_B(f(y))$. This yields an equality of the desired type:

$$\eta_f(x)^{-1} \bullet \mathfrak{S}f(\gamma) \bullet \eta_f(y)$$

A formal disk at a point is the “collection” of all other points infinitesimally close to it:

Definition 3.3

Let A be a type and $a:A$. The type \mathbb{D}_a defined below in three equivalent ways is called the *formal disk at a* .

- (i) \mathbb{D}_a is the sum of all points infinitesimally close to a , i.e.:

$$\mathbb{D}_a := \sum_{x:A} \iota_A(x) = \iota_A(a)$$

- (ii) \mathbb{D}_a is the fiber of ι_A at $\iota_A a$.

- (iii) \mathbb{D}_a is defined by the following pullback square:

$$\begin{array}{ccc}
\mathbb{D}_a & \longrightarrow & 1 \\
\downarrow & \text{(pb)} & \downarrow * \mapsto \iota_A(a) \\
A & \xrightarrow{\iota_A} & \mathfrak{S}A
\end{array}$$

The characterization (iii) is a verbatim translation of its topos theoretic analog [Scha][Definition 5.3.50] to Homotopy Type Theory. Therefore, among a lot of more general concepts, it also subsumes an analogue of tangent spaces.

As morphisms of manifolds induce maps on tangent spaces, maps of types induce morphisms on formal disks:

Remark 3.4

If $f : A \rightarrow B$ is a type, there is a dependent function:

$$df : \prod_{x:A} \mathbb{D}_x \rightarrow \mathbb{D}_{f(x)}$$

We denote the evaluation at $a:A$ with

$$df_a : \mathbb{D}_a \rightarrow \mathbb{D}_{f(a)}$$

and call it *the differential of f at a* .

Proof To define df we take the sum over the map from 3.2:

$$df_a \equiv (x, \epsilon) \mapsto (f(x), \eta_f^{-1}(x) \bullet \mathfrak{S}f(\epsilon) \bullet \eta_f(x))$$

– where $\eta_f(x)$ is the equality from the naturality of ι .

It would also make sense to denote the differential by \hat{f} and call it completion, but we will stick to terminology fitting the first order models. Also, the names we are choosing are necessarily somewhat arbitrary since they are relative to an abstract modality.

Some of the familiar rules for differentiation can be derived in this generality. We will need only the chain rule:

Lemma 3.5

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be a maps. Then the following holds for all $x : A$

$$d(g \circ f)_x = (dg)_{f(x)} \circ df_x.$$

Proof Note that, in general, the differential df_x is equal to the map induced by the universal property of $\mathbb{D}_{f(x)}$ as a pullback. We can use this to get the desired “functoriality”:

$$\begin{array}{ccccc}
\mathbb{D}_x & \longrightarrow & A & \xrightarrow{\iota_A} & \mathfrak{S}A \\
\downarrow df_x & & \downarrow f & & \downarrow \mathfrak{S}f \\
\mathbb{D}_{f(x)} & \longrightarrow & B & \xrightarrow{\iota_B} & \mathfrak{S}B \\
\downarrow (dg)_{f(x)} & & \downarrow g & & \downarrow \mathfrak{S}g \\
\mathbb{D}_{g(f(x))} & \longrightarrow & C & \xrightarrow{\iota_C} & \mathfrak{S}C
\end{array}$$

$d(g \circ f)_x$ (indicated by a dashed arrow from \mathbb{D}_x to $\mathbb{D}_{g(f(x))}$)

– the induced map $d(g \circ f)_x$ and the composition $(dg)_{f(x)} \circ df_x$ solve the same factorization problem, so they are equal.

In Differential Geometry, the tangent bundle is an important basic construction consisting of all the tangent spaces in a manifold. We can mimic the construction in this abstract setting, by combining all the formal disks of a space to a bundle.

Definition 3.6

Let A be a type. The type $T_\infty A$ defined in one of the equivalent ways below is called the *formal disk bundle* of A .

(i) $T_\infty A$ is the sum over all the formal disks in A :

$$T_\infty A \equiv \sum_{x:A} \mathbb{D}_x$$

(ii) $T_\infty A$ is defined by the following pullback square:

$$\begin{array}{ccc} T_\infty A & \longrightarrow & A \\ \downarrow & \text{(pb)} & \downarrow \iota_A \\ A & \xrightarrow{\iota_A} & \mathfrak{S}A \end{array}$$

Note that despite the seemingly symmetric second definition, we want $T_\infty A$ to be a bundle having formal disks as its fibers, so it is important to distinguish between the two projections and their meaning. If we look at $T_\infty A$ as a bundle, meaning a morphism $p: T_\infty A \rightarrow A$, we always take p to be the first projection in both cases. This convention agrees with the first definition – taking the sum yields a bundle with fibers of the first projection equivalent to the \mathbb{D}_a we put in.

For any $f: A \rightarrow B$ we defined the induced map df on formal disks. This extends to formal disk bundles.

Definition 3.7

For a map $f: A \rightarrow B$ there is an induced map on the formal disk bundles, given as

$$T_\infty f \equiv (a, \epsilon) \mapsto (f(a), df_a(\epsilon))$$

In Differential Geometry, the tangent bundle may or may not be trivial. This is some interesting information about a space. If we have a smooth group structure on a manifold G , i.e. a Lie-group, we may consistently translate the tangent space at the unit to any other point. This may be used to construct an isomorphism of the tangent bundle with the projection from the product of G with the tangent space at the unit.

It turns out, that this generalizes to formal disk bundles and the group structure may be replaced by the weaker notion of a homogeneous type.

The notion of homogeneous type was developed by the author to satisfy two needs. The first is to match the intuition of a pointed space, that is equipped with a continuous family of translations that map the base point to any given point.

The second need is to have just the right amount of data in all the proofs and constructions concerning homogeneous types. It has not been investigated in what circumstances this definition of homogeneous spaces coincides with the various notions of homogeneous spaces in Geometry – apart from the obvious examples given below.

Definition 3.8

A type A is *homogeneous*, if there are terms of the following types:

- (i) $e:A$
- (ii) $t: \prod_{x:A} A \simeq A$
- (iii) $p: \prod_{x:A} t_x(e) = x$

Where t is called the *family of translations*.

Examples 3.9

- (a) Let G be a group in the sense of [Uni13][6.11], then G is a homogeneous type with $x \bullet _$ or $_ \bullet x$ as its family of translations.
- (b) Let G be an h-group, i.e. a type with a unit, operation and inversion that satisfy the group axioms up to a 2-cell. Then G is a homogeneous type in the same two ways as above.
- (c) As a notable special case, for any type A and $\ast:A$, the loop space $\ast =_A \ast$ is homogeneous.
- (d) Let X be a connected H-space, then X is homogeneous, again in two ways. See [Uni13][8.5.2] and [LF14][Section 4].
- (e) Let Q be a type with a quasigroup-structure, i.e. a binary operation $_ \bullet _$ such that all equations $a \bullet x = b$ and $x \bullet a = b$ have a contractible space of solutions, then Q is homogeneous if it has a left or right unit.

In the following we will build a family of equivalences from one formal disk of a homogeneous type to any other formal disk of the space. We start by observing how equivalences and equalities act on formal disks.

Lemma 3.10

- (a) If $f: A \rightarrow B$ is an equivalence, then

$$df_x: \mathbb{D}_x \rightarrow \mathbb{D}_{f(x)}$$

is an equivalence for all $x: A$.

- (b) Let A be a type and $x, y: A$ two points. For any equality $\gamma: x = y$, we get an equivalence $\mathbb{D}_x \simeq \mathbb{D}_y$.

Proof (a) Let us first observe, that for any $x, y: A$ the map $\iota_A(x) = \iota_A(y) \rightarrow \iota_B(f(x)) = \iota_B(f(y))$ is an equivalence. This follows since it is equal to the composition of two equivalences. One is the conjugation with the equalities

from naturality of ι , the other is the equivalence of path spaces induced by the equivalence $\mathfrak{S}f$.

Now, for a fixed $a: A$ we have two dependent types, $\iota_A(a) = \iota_A(x)$ and $\iota_B(f(a)) = \iota_B(f(x))$ and an equivalence over f between them. The sum of this equivalence over f is by definition df and by 2.4 a sum of a fibered equivalence is an equivalence.

- (b) The equivalence is just the transport in the dependent type $x \mapsto \mathbb{D}_x$.

We are now ready to state and prove the triviality theorem.

Theorem 3.11

Let V be a homogeneous type and \mathbb{D}_e the formal disk at its unit. Then the following is true:

- (a) For all $x:V$, there is an equivalence

$$\psi_x: \mathbb{D}_x \rightarrow \mathbb{D}_e$$

- (b) $T_\infty V$ is a trivial bundle with fiber \mathbb{D}_e , i.e. we have an equivalence $T_\infty V \rightarrow V \times \mathbb{D}_e$ and a homotopy commutative triangle

$$\begin{array}{ccc} T_\infty V & \xrightarrow{\simeq} & V \times \mathbb{D}_e \\ & \searrow \pi_1 & \swarrow \pi_1 \\ & V & \end{array}$$

Proof (a) Let $x \in V$ be any point in V . The translation t_x given by the homogeneous structure on V is an equivalence. Therefore, we have an equivalence $\psi'_x: \mathbb{D}_e \rightarrow \mathbb{D}_{t_x(e)}$ by 3.10. Also directly from the homogeneous structure, we get an equality $t_x(e) = x$ and transporting along it yields an equivalence $\mathbb{D}_{t_x(e)} \rightarrow \mathbb{D}_x$. So we can compose and invert to get the desired ψ_x .

- (b) By the first definition 3.6 of the formal disk bundle, we have

$$T_\infty V \equiv \sum_{x:V} \mathbb{D}_x$$

We define a morphism $\varphi: T_\infty V \rightarrow V \times \mathbb{D}_e$ by

$$\varphi((x, \epsilon_x)) \equiv (x, \psi_x(\epsilon_x))$$

and its inverse by

$$\varphi^{-1}((x, \epsilon_x)) \equiv (x, \psi_x^{-1}(\epsilon_x)).$$

Now, to see φ is an equivalence with inverse φ^{-1} , one has to provide equalities of types

$$\begin{aligned} (x, \epsilon_x) &= \varphi^{-1}(\varphi(x, \epsilon_x)) = (x, \psi^{-1}(\psi(\epsilon_x))) \\ \text{and } (x, \epsilon_e) &= \varphi(\varphi^{-1}(x, \epsilon_e)) = (x, \psi(\psi^{-1}(\epsilon_e))) \end{aligned}$$

– which exist since the ψ_x are equivalences by (a).

In geometry, it is usually possible to add tangent vectors. Our formal disks can at least inherit the group like properties of a homogeneous type:

Theorem 3.12

Let A be homogeneous with unit $e : A$. Then \mathbb{D}_e is homogeneous.

Proof We look at the sequence

$$\mathbb{D}_e \xrightarrow{\iota_e} A \xrightarrow{\iota_A} \mathfrak{S}A$$

where ι_e is the inclusion of the formal disk, given as the first projection. We will proceed by constructing a homogeneous structure on $\mathfrak{S}A$, note some properties of ι_A which could be part of a definition of *morphism of homogeneous types* and finally give some “kernel”-like construction of the structure on \mathbb{D}_e .

For $x : A$, there is a translation $t_x : A \simeq A$, since \mathfrak{S} preserves equivalences, this yields a $\mathfrak{S}t_x : \mathfrak{S}A \simeq \mathfrak{S}A$. By \mathfrak{S} -induction, this extends to a family of translations

$$t' : \prod_{y:\mathfrak{S}A} \mathfrak{S}A \simeq \mathfrak{S}A, \text{ with } t'_{\iota_A(x)} = \mathfrak{S}t_x.$$

Application of \mathfrak{S} to maps is defined by induction, so we can compute the application of $\mathfrak{S}t$ as $\mathfrak{S}t_{\iota_A(x)}(\iota_A(y)) = \iota_A(t_x(y))$. Let $e' \equiv \iota_A(e)$, then $\mathfrak{S}A$ is homogeneous if we can produce a

$$p' : \prod_{y:\mathfrak{S}A} t'_y(e') = y.$$

Inducting on y admits application of our computation, which reduces the problem to construct a $q : \prod_{x:A} \iota_A(t_x(e)) = \iota_A(x)$. But this is given by applying ι_A to the corresponding witness $p : \prod_{x:A} t_x(e) = x$ of the homogeneous structure on A .

We start to construct the homogeneous structure on \mathbb{D}_e by letting $e'' \equiv (e, \text{refl})$ be the unit. For the translations, we look at the dependent type $(x : A) \mapsto \iota_A(e) = \iota_A(x)$ and establish the following chain of equivalences for $y : A$ with $\iota_A(e) = \iota_A(y)$:

$$\begin{aligned} \iota_A(e) &= \iota_A(x) \\ &\simeq t'_{\iota_A(y)} \iota_A(e) = t'_{\iota_A(y)} \iota_A(x) \\ &\simeq t'_{\iota_A(y)} \iota_A(e) = \iota_A(t_y(x)) \\ &\simeq \iota_A(y) = \iota_A(t_y(x)) \\ &\simeq \iota_A(e) = \iota_A(t_y(x)) \end{aligned}$$

The resulting equivalence, is an equivalence over t_y . So by 2.4 this induces an equivalence on the sum, which is \mathbb{D}_e .

To see that the resulting translations t'' satisfy $\prod_{x:\mathbb{D}_e} t''_x(e'') = x$, we can calculate the value of the equivalence above step by step for $x \equiv e$ on $\text{refl} : \iota_A(e) = \iota_A(e)$. After the first step above, which is application of $t'_{\iota_A(y)}$ we have refl . The second step concatenates an equality that computes the application, which we denote by

$$c_y : t'_{\iota_A(y)}(\iota_A(e)) = \iota_A(t_y(e)).$$

The third step concatenates $p'_{\iota_A(y)}$ from the left and the last step a given $\gamma : \iota_A(e) = \iota_A(y)$. But what we really need to look at, is transport of what we have so far along $p_y : t_y(e) = y$, which is $\gamma \cdot p'^{-1}_{\iota_A(y)} \cdot c_y \cdot \iota_A(p_y)$. If the latter turns out to be γ , we are done, so it is enough to show

$$c_y \cdot \iota_A(p_y) = p'_{\iota_A(y)}.$$

But p' was constructed by induction using the left hand side, so this equality holds.

Now, to conclude the section, let us look at some analoga of classical notions. To define vector fields and 1-forms, the author saw no other way than to base them on an “affine line”. So far, no reason came up to consider anything more special than a homogeneous type as the affine line:

Axiom 3.13

There is a homogeneous type \mathbb{A}^1 , which we call the *affine line*.

We will not use this axiom for anything essential in this article, it merely serves us to draw some connections to classical theory and indicate how more classical material could be imported to this setting. The following definitions merely serve this purpose as well.

Definition 3.14

- (a) The *unit disk* is the formal disk \mathbb{D}_e at the unit $e : \mathbb{A}$.
- (b) Let $\tau_{\mathbb{A}} : \prod_{x:\mathbb{A}} \mathbb{D}_x \simeq \mathbb{D}_e$ be the term we get by applying 3.11 to \mathbb{A} .

For the names given to the concepts in the remainder of this section, we assume a smooth first order model. One way to define the tangent vectors at a point in differential geometry, is to quotient the set of all curves moving through the point by equality of first derivatives. So let M be a type and $x : M$. If we accept curves to be maps $\gamma : \mathbb{A} \rightarrow M$ and say that “moving through x ” means $\gamma(e) = x$, the classes of curves moving through x up to equality of differentials at x may be identified with differentials of the form $d\gamma : \mathbb{D}_e \rightarrow \mathbb{D}_x$ realized by some curve. Let us assume for the sake of simplicity, that the spaces we are interested in are nice enough, such that all differentials are realized. Then the following definition is sensible:

Definition 3.15

A *vector field* on a type M is a term $\chi : \prod_{x:M} \mathbb{D}_e \rightarrow \mathbb{D}_x$.

Differential forms and the differential of a function may be defined similarly:

Definition 3.16

- (a) A *differential 1-form* is a term $\omega : \prod_{x:M} \mathbb{D}_x \rightarrow \mathbb{D}_e$. We denote their type by $\Omega(M)$.
- (b) There is a $d : (M \rightarrow \mathbb{A}) \rightarrow \Omega(M)$ given by $f \mapsto (x \mapsto \tau_{\mathbb{A}}(x) \circ df(x))$.

There is a pairing of vector fields χ and 1-forms ω given by the dependent composition $(x : M) \mapsto \omega(x) \circ \chi(x)$ of type $M \rightarrow (\mathbb{D}_e \rightarrow \mathbb{D}_e)$. Since linear endomorphisms of a 1-dimensional vector space are isomorphic to the base field, it makes sense that the pairing produces values in $\mathbb{D}_e \rightarrow \mathbb{D}_e$.

3.2 Formally étale maps

In Algebraic Geometry, formally étale maps are supposed to be analogous to local diffeomorphisms in Differential Geometry. Below, we will give a definition which corresponds to a stronger notion in algebraic settings ¹⁰ and coincides with the local diffeomorphisms between manifolds in the case of Differential Geometry ¹¹.

Definition 3.17

A map $f: A \rightarrow B$ is *formally étale*, if its naturality square is a pullback:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathfrak{S}A \\ f \downarrow & & \downarrow \mathfrak{S}f \\ B & \xrightarrow{\iota_B} & \mathfrak{S}B \end{array}$$

Lemma 3.18

- (a) If $f: A \rightarrow B$ and $g: B \rightarrow C$ are formally étale, their composition $g \circ f$ is formally étale. If the composition $g \circ f$ and g are formally étale, then f is formally étale.
- (b) Equivalences are formally étale.
- (c) Maps between coreduced types are formally étale.
- (d) All fibers of a formally étale map are coreduced.

Proof (a) By pullback pasting.

- (b) The naturality square for an equivalence is a commutative square with equivalences on opposite sides. Those squares are always pullback squares.
- (c) This is, again, a square with equivalences on opposite sides.
- (d) The pullback square witnessing $f: A \rightarrow B$ being formally étale yields an equivalence over ι_B . So, each fiber of f is equivalent to some fiber of $\mathfrak{S}f$. But fibers of maps between coreduced types are always coreduced by 2.10 (c), hence each fiber of f is equivalent to a coreduced type, thus itself coreduced.

Together with the following, we have all the properties of formally étale maps needed in this article:

Lemma 3.19

Let $f: A \rightarrow B$ be formally étale, then the following is true:

- (a) For all $x:A$, the differential df_x is an equivalence.
- (b) There is a pullback square of the following form:

¹⁰This is in [Wel17] for Zariski-sheaves, in the discussion following Remark 4.4.2.

¹¹See [KS17, Proposition 3.2] for a precise statement in an intended model.

$$\begin{array}{ccc}
T_\infty A & \longrightarrow & T_\infty B \\
\downarrow & \text{(pb)} & \downarrow \\
A & \xrightarrow{f} & B
\end{array}$$

Proof (a) The pullback square witnessing that f is formally étale can be reformulated as:

For all $x : \mathfrak{S}A$, the induced map between the fibers of ι_A and ι_B is an equivalence. But these fibers are just the formal disks, so this can be applied to any $\iota_A(y)$ to see that df_y is an equivalence.

(b) This is just a reformulation.

The following is part of ongoing work with Egbert Rijke, which also contains more on formally étale maps.

Theorem 3.20

Let $f : A \rightarrow B$ be formally étale and

$$\begin{array}{ccc}
A' & \longrightarrow & A \\
f' \downarrow & \text{(pb)} & \downarrow f \\
B' & \longrightarrow & B
\end{array}$$

a pullback square. Then f' is formally étale.

Proof Let us denote the bottom map with $\psi : B' \rightarrow B$. We start by describing A' as a pullback:

$$\begin{aligned}
A' &\simeq \left(\sum f'^{-1} \right) \simeq \left(\sum f^{-1} \circ \psi \right) \simeq \left(\sum (\mathfrak{S}f)^{-1} \circ \iota_B \circ \psi \right) \\
&\simeq \left(\sum (\mathfrak{S}f)^{-1} \circ \mathfrak{S}\psi \circ \iota_{B'} \right)
\end{aligned}$$

Now we can apply 2.11 to compute $\mathfrak{S}A'$:

$$\mathfrak{S}A' \simeq \mathfrak{S} \left(\sum (\mathfrak{S}f)^{-1} \circ \mathfrak{S}\psi \circ \iota_{B'} \right) \simeq \left(\sum (\mathfrak{S}f)^{-1} \circ \mathfrak{S}\psi \right)$$

Note that the right hand side is the pullback of $\mathfrak{S}A$ along $\mathfrak{S}\psi$. This means that applying \mathfrak{S} to the pullback square given in the statement of the theorem, is again a pullback and by pullback pasting the naturality square of f' is a pullback.

Corollary 3.21

(a) Let X be a type and $x : X$. The inclusion $\iota_x : \mathbb{D}_x \rightarrow X$ of the formal disk at x is a formally étale map.

(b) Any pullback of a map between modal types is formally étale.

Proof All maps between modal types are formally étale. Hence the second statement follows from the theorem and the first follows as the special case for the map $\iota_X(x) : 1 \rightarrow \mathfrak{S}X$.

We will put formally étale maps to use in section 4.2. The next section makes no reference to \mathfrak{S} .

4 Structures on manifolds

4.1 Fiber bundles

As mentioned in the introduction, the spaces we have in mind carry both differential geometric and homotopical information. This section is about maps, that correspond to fiber bundles concerning the homotopical structure. We will occasionally hint at how the notion might be extended to fiber bundles in the spatial sense. In this section, we will give four definitions of these fiber bundles and prove they are equivalent. It will be useful in section 4.3 to switch between the different definitions.

A classical ∞ -topos-theoretic motivation for the first version of this account of fiber bundles in [Wel17] may be found in [NSS15]. Some of the following definitions of fiber bundles were also used early in the short history of Homotopy Type Theory at least by Mike Shulman, Ulrik Buchholtz and Egbert Rijke.

For the following statements about fiber bundles, we will make a lot of unavoidable use of a univalent universe \mathcal{U} and propositional truncation. We will frequently use that all maps of types $p : E \rightarrow B$ appear in a pullback square

$$\begin{array}{ccc} E & \longrightarrow & \tilde{\mathcal{U}} \\ p \downarrow & \text{(pb)} & \downarrow \\ B & \xrightarrow{p^{-1}} & \mathcal{U}, \end{array}$$

where $\tilde{\mathcal{U}}$ is called the *universal family* and obtained by summing over the dependent type $(A:\mathcal{U}) \mapsto A$. The bottom map p^{-1} determines p up to canonical equivalence over B and is called the *classifying map* of p . If E is a sum over a dependent type $q:B \rightarrow \mathcal{U}$, and p the projection to B , then q is the classifying map.

This way of using a univalent universe corresponds to looking at it as a *moduli space* or *classifying space* of types. We could replace the \mathcal{U} with some other moduli space to get specialized notions of fiber-bundles with additional structure on the fibers.

Before we start, we will look at some preliminaries about surjective and injective maps. A surjective map is a map with merely inhabited fibers, or in other words a $\|_-\|_{-1}$ -connected map. An injective map has $\|_-\|_{-1}$ -truncated fibers.¹²

Definition 4.1

Let $f : A \rightarrow B$ be a map of types.

- (a) The map f is *surjective* if

$$\prod_{b:B} (\|f^{-1}(b)\|_{-1} \simeq 1).$$

¹²Note that in a sheaf-topos, this notion corresponds to epimorphisms and not to a pointwise surjective map. In [Uni13, chapter 7], surjective maps are called (-1) -connected or also surjective, if their domain and codomain are 0-types. Topos theoretic analogs are defined in [Lur09, pp. 6.5.1.10, 5.5.6.8] and are called 0-connective and (-1) -truncated. In the terminology of Urs Schreiber, e.g. at [Scha] or [nLab] and in [Wel17] surjective maps would be 1-epimorphisms and injective maps 1-monomorphisms.

We write $f : A \twoheadrightarrow B$ in this case.

(b) The map f is *injective* if

$$\prod_{b:B} (f^{-1}(b) \text{ is } -1\text{-truncated}).$$

We write $f : A \hookrightarrow B$ in this case.

Lemma 4.2

Surjective and injective maps are preserved by pullbacks.

Proof This is immediate by passing from pullback squares to fibered equivalences.

Examples 4.3

(a) Let $f : A \rightarrow B$ be an equivalence of types. Then f is surjective and injective since all fibers of f are contractible.

(b) Let $P : A \rightarrow \mathcal{U}$ be a proposition. Then the projection

$$\pi_1 : \sum_{a:A} P(a) \rightarrow A$$

is injective.

(c) For the higher inductive type S^1 , the inclusion of the base point is a surjection.

Lemma 4.4

For any map $f : A \rightarrow B$ there is a unique triangle:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow e & \nearrow m \\ & \text{image}(f) & \end{array}$$

where e is surjective, m injective and $\text{image}(f)$ is given by

$$\text{image}(f) := \sum_{b:B} \left\| \sum_{a:A} f(a) = b \right\|_{-1}.$$

A proof of the general case of $\|_-\|_n$ may be found in [Uni13, chapter 7.6].

In Topology, an F -fiber bundle is a map $p : E \rightarrow B$, that is locally trivial and all its fibers are isomorphic to F . Local triviality means, that B may be covered by open sets U_i , such that on each U_i the restricted map $p|_{p^{-1}(U_i)}$ is isomorphic to the projection $F \times U_i \rightarrow U_i$. We may rephrase this in a more economical way: From our cover, we construct a surjective map $w : \coprod_{i \in I} U_i \rightarrow B$. Then the local triviality translate to the pullback of p along w being isomorphic to the product projection $F \times \coprod_{i \in I} U_i \rightarrow \coprod_{i \in I} U_i$.

For fiber bundles in geometry, we would require more from a general surjective map, or *cover*, $w : W \rightarrow B$ than that pulling back along it turns p into a product projection. However, for the notion we discuss in this section, this turns out to be already enough.

Definition 4.5

Let $p: E \rightarrow B$ be a map of types. For another map $w: W \rightarrow B$ we say w trivializes p , if w is a surjective map and there is a pullback square:

$$\begin{array}{ccc} W \times F & \longrightarrow & E \\ \pi_1 \downarrow & \text{(pb)} & \downarrow p \\ W & \xrightarrow{w} & B \end{array}$$

The map p is called an F -fiber bundle in this case.

Following a suggestion from Max New¹³, we give an equivalent dependent version of this definition, which will be a lot easier to work with:

Definition 4.6

Let $E: B \rightarrow \mathcal{U}$ be a dependent type. We say a surjection $w: W \rightarrow B$ trivializes E , if

$$\prod_{x:W} E(w(x)) \simeq F.$$

The dependent type E is called an F -fiber bundle in this case.

We can switch between the two definitions in the usual way: Given an F -fiber bundle $p: E \rightarrow B$ in the first sense, the dependent type of its fibers $p^{-1}: B \rightarrow \mathcal{U}$ will be an F -fiber bundle in the second sense, by direct application of 2.3. To go back, we take the projection from the sum of an F -fiber bundle $E: B \rightarrow \mathcal{U}$.

Note that both definitions do not immediately provide us with a nice way to define the *type* of F -fiber bundles, since we ask for at least one non-unique datum, the trivializing map. Of course, we could truncate appropriately to remove the choice of map. We will later see that we could have defined F -fiber bundles more easily with their classifying maps to a type called $\text{BAut}(F)$, providing us directly with the type of F -fiber bundles. However, in those definitions, it is unclear how we may require that the surjective map has additional properties. One example, where we are interested in special surjections, is the definition of a V -manifold, where we will use generalizations of local diffeomorphisms.

We review the type $\text{BAut}(F)$ now, which will be used to give the alternative definition of fiber bundles mentioned above:

Definition 4.7

Let F be a type and $t_F: 1 \rightarrow \mathcal{U}$ the map given by $* \mapsto F$.

- (a) Let $\text{BAut}(F) \equiv \text{image}(t_F)$.
- (b) We also have the injection $\iota_{\text{BAut}(F)}: \text{BAut}(F) \rightarrow \mathcal{U}$.
- (c) The map $\pi: F//\text{Aut}(F) \rightarrow \text{BAut}(F)$ is given as the first projection of the dependent sum over

$$((F', |\varphi|): \text{BAut}(F)) \mapsto F'$$

¹³<http://maxsnew.github.io/>

The map $\pi: F//\text{Aut}(F) \rightarrow \text{BAut}(F)$ is the universal F -fiber bundle, meaning all F -fiber bundles with any base will turn out to be pullbacks of this map. We are now ready to give yet another definition of fiber bundles:

Definition 4.8

A map $p: E \rightarrow B$ is an F -fiber bundle, if and only if there is a map $\chi: B \rightarrow \text{BAut}(F)$, such that there is a pullback square

$$\begin{array}{ccc} E & \longrightarrow & F//\text{Aut}(F) \\ p \downarrow & \text{(pb)} & \downarrow \pi \\ B & \xrightarrow{\chi} & \text{BAut}(F). \end{array}$$

In this case, χ is called the *classifying map* of p .

This definition also has a surprisingly easy dependent variant, which is obviously a mere proposition:

Definition 4.9

Let $E: B \rightarrow \mathcal{U}$ be a dependent type. We say E is an F -fiber bundle, if

$$\prod_{b:B} \|E(b) \simeq F\|_{-1}.$$

Again, we will switch between the dependent and non-dependent version by taking fibers of p and the sum respectively. To arrive at the dependent version, we can directly use the classifying morphism χ of an F -fiber bundle $p: E \rightarrow B$ to construct a term of

$$\prod_{b:B} \|p^{-1}(b) \simeq F\|_{-1},$$

since all points $\chi(b): \text{BAut}(F)$ are of the form (F', γ) , with $F' \simeq p^{-1}(b)$ by the pullback square and γ a proof that F' is merely equivalent to F .

Now, for the converse, let

$$E: B \rightarrow \mathcal{U}$$

be an F -fiber bundle, by $t: \prod_{b:B} \|E(b) \simeq F\|_{-1}$. Then the classifying map is given by $(x: B) \mapsto t_x$ and the pullback square is given by pasting: ¹⁴

$$\begin{array}{ccccc} \sum E & \longrightarrow & F//\text{Aut}(F) & \longrightarrow & \tilde{\mathcal{U}} \\ \pi_1 \downarrow & & \pi \downarrow & \text{(pb)} & \downarrow \\ B & \xrightarrow{\chi} & \text{BAut}(F) & \longrightarrow & \mathcal{U}. \end{array}$$

We will conclude this section by showing that all our definitions of fiber bundles are logically equivalent and discussion of some examples. This is most efficiently done by establishing the equivalence of the two dependent definitions:

¹⁴Note that the outer rectangle is a pullback for all dependent types.

Theorem 4.10

Let F be a type and $E:B \rightarrow \mathcal{U}$ be a dependent type, then

$$\prod_{b:B} \|E(b) \simeq F\|_{-1}$$

if and only if there is a type W and a surjective $w:W \rightarrow B$ such that

$$\prod_{x:W} E(w(x)) \simeq F.$$

For the proof, we need to construct a trivializing cover at some point. The author has to thank Ulrik Buchholtz for asking if such a cover always exists. The construction we use is similar to the universal cover and interesting on its own:

Definition 4.11

Let $E:B \rightarrow \mathcal{U}$ be an F -fiber bundle by $t: \prod_{b:B} \|E(b) \simeq F\|_{-1}$, then

$$W: \equiv \sum_{b:B} E(b) \simeq F$$

together with its projection to B is the *canonical trivializing cover* of p .

The given t directly proves that this projection is surjective. Let us denote this projection by $w:W \rightarrow B$, then for all $x:W$, with $x = (b, e)$ we have

$$E(w(x)) \simeq E(\pi_1(b, e)) \simeq F$$

by transport and $e:E(b) \simeq F$ itself.

Proof (of 4.10) With the definition and remark above, it remains to show the converse. Let $E:B \rightarrow \mathcal{U}$ and $w:W \rightarrow B$ such that $t: \prod_{x:W} E(w(x)) \simeq F$. Now, for any $b:B$ and $x_b:w^{-1}(b)$, we get an equivalence $t_{\pi_1(x_b)}:E(w(\pi_1(x_b))) \simeq F$. By general properties of fibers, we have $w(\pi_1(x_b)) = b$ yielding $E(b) \simeq F$. By surjectivity of w , we merely have a $x_b:w^{-1}(b)$ for any $b:B$, therefore we merely have an equivalence $E(b) \simeq F$.

Examples 4.12

- (a) Let A be a pointed connected type, then any $E:A \rightarrow \mathcal{U}$ is an $E(*)$ -fiber bundle.¹⁵
- (b) The map $1 \rightarrow S^1$ is a \mathbb{Z} -fiber bundle.
- (c) More general, for a pointed connected type A , the homotopical universal cover $\sum_{x:A} x = *$ is an ΩA -fiber bundle and $\sum_{x:A} \|x = *\|_1$ a $\pi_1(A, *)$ -fiber bundle.

¹⁵Thanks to Egbert for pointing this out.

- (d) The canonical trivializing map $w:W \rightarrow B$ of an F -fiber bundle is an $\text{Aut}(F)$ -fiber bundle – to see this, let us first note, that W could also be written as $\sum_{b:B} E(b) =_{\text{BAut}(F)} F$, since for any $X, Y : \text{BAut}(F)$,

$$(X =_{\text{BAut}(F)} Y) \simeq (\iota(X) =_{\mathcal{U}} \iota(Y)) \simeq (\iota(X) \simeq \iota(Y)).$$

This means W is the homotopy fiber of E , so we can apply the equivalence below known to hold for iterated homotopy fibers¹⁶ for all $b:B$ and merely pointed maps $B \rightarrow \text{BAut}(F)$ induced by them:

$$\text{Aut}(F) \simeq \Omega \text{BAut}(F) \rightarrow W \rightarrow B \rightarrow \text{BAut}(F)$$

So the fibers of w are merely equivalent to $\text{Aut}(F)$.

4.2 V -manifolds

A smooth manifold is a space that is locally diffeomorphic to \mathbb{R}^n , hausdorff and second countable. The definition used in this article just mimics the first property. A covering $(U_i)_{i \in I}$ with $U_i \simeq \mathbb{R}^n$ of a manifold M yields a surjective local diffeomorphism

$$\coprod_{i \in I} U_i \rightarrow M.$$

This is generalized by the following internal definition:

Definition 4.13

Let V be a homogeneous type. A type M is a V -manifold, if there is a span

$$\begin{array}{ccc} & U & \\ \text{ét} \swarrow & & \searrow \text{ét} \\ V & & M \end{array}$$

where the left map is formally étale and the right map formally étale and surjective.

There is one trivial example:

Example 4.14

Let V be a homogeneous type, then V is a V -manifold witnessed by the span:

$$\begin{array}{ccc} & V & \\ \text{id} \swarrow & & \searrow \text{id} \\ V & & V \end{array}$$

Less obvious are the following two ways of producing new V -manifolds. However, without adding anything to our type theory making the modality \mathfrak{S} more specific, we cannot hope for examples that are not given as homogeneous types. What could be added will be discussed at the beginning of the next section.

The statement in (a) is a variant of the classical fact that the tangent bundle of a manifold is a manifold, but in our case, the infinitesimal or tangent information, is kept separate. Statement (c) was a question by Ulrik Buchholtz.

¹⁶See [Uni13, Section 8.4].

Lemma 4.15

Let V be homogeneous and M be a V -manifold.

- (a) The formal disk bundle $T_\infty M$ of M is a $(V \times \mathbb{D}_e)$ -manifold.
- (b) For any formally étale map $\varphi : N \rightarrow M$, N is a V -manifold.
- (c) If V' is a homogeneous V -manifold and N a V' -manifold, then N is also a V -manifold.

Proof (a) We can pull back the span witnessing that M is a V -manifold along the projection $T_\infty M \rightarrow M$:

$$\begin{array}{ccccc}
 V \times \mathbb{D}_e & \xleftarrow{\text{ét}} & T_\infty U & \xrightarrow{\text{ét}} & T_\infty M \\
 \downarrow & & \downarrow & & \downarrow \\
 & & \text{(pb)} & & \text{(pb)} \\
 V & \xleftarrow{\text{ét}} & U & \xrightarrow{\text{ét}} & M
 \end{array}$$

Formally étale maps are preserved by pullbacks by 3.20 and surjective maps by 4.2. In 3.12 we showed that \mathbb{D}_e is homogeneous, so $V \times \mathbb{D}_e$ is homogeneous by giving it a componentwise structure.

- (b) Pullback along φ and composition give us the following:

$$\begin{array}{ccccc}
 & & \varphi^* U & \xrightarrow{\text{ét}} & N \\
 & \swarrow \text{ét} & \downarrow \text{ét} & & \downarrow \varphi \\
 & & \text{(pb)} & & \text{(pb)} \\
 V & \xleftarrow{\text{ét}} & U & \xrightarrow{\text{ét}} & M
 \end{array}$$

- (c) That N is a V -manifold is witnessed by the following diagram using preservation of surjections and formally étale maps under pullbacks:

$$\begin{array}{ccccc}
 & & U_V \times_{V'} U_N & & \\
 & \swarrow \text{ét} & & \searrow \text{ét} & \\
 & U_V & & U_N & \\
 & \swarrow \text{ét} & \text{(pb)} & \searrow \text{ét} & \\
 V & & V' & & N
 \end{array}$$

One important special case of part (b) of the lemma is that any formal disk \mathbb{D}_x of M is a V -manifold.

In the following, let V be homogeneous and M a fixed V -manifold. The definition of V -manifolds entails a stronger local triviality condition on the formal disk bundle of M than was discussed in the last section about F -fiber bundles, since there has to be a formally étale trivializing covering. This property of the trivializing covering will *not* be used in the following lemma.

Lemma 4.16

- (a) The formal disk bundle of the covering U is trivial and there is a pullback square:

$$\begin{array}{ccc} U \times \mathbb{D}_e & \longrightarrow & T_\infty M \\ \downarrow & \text{(pb)} & \downarrow \\ U & \longrightarrow & M \end{array}$$

- (b) The formal disk bundle of M has a *classifying morphism* $\tau: M \rightarrow \text{BAut}(\mathbb{D}_e)$, i.e. there is a pullback square:

$$\begin{array}{ccc} T_\infty M & \longrightarrow & \mathbb{D}_e // \text{Aut}(\mathbb{D}_e) \\ \downarrow & \text{(pb)} & \downarrow \pi \\ M & \xrightarrow{\tau_M} & \text{BAut}(\mathbb{D}_e) \end{array}$$

Proof (a) By 3.19, there is a pullback square for the formally étale map to V :

$$\begin{array}{ccc} T_\infty V & \longrightarrow & T_\infty U \\ \downarrow & \text{(pb)} & \downarrow \\ V & \longrightarrow & U \end{array}$$

Since V is homogeneous, by 3.11 its formal disk bundle is trivial. This is preserved by pullback, so $T_\infty U$ is trivial. The pullback square in the proposition is again given by 3.19.

- (b) The statement (a) tells us, that $T_\infty M$ is a \mathbb{D}_e -fiber bundle by definition 4.5. And (b) is just another way to state that fact, namely definition 4.8.

The classifying morphism τ_M is compatible with formally étale maps in the sense of the following remark.

Remark 4.17

Let $\varphi: N \rightarrow M$ be formally étale, then N is also a V -manifold by 4.15. There is a 2-cell given by the differential of φ :

$$\begin{array}{ccc} M & \xrightarrow{\tau_M} & \text{BAut}(\mathbb{D}_e) \\ \varphi \uparrow & \swarrow d\varphi & \nearrow \\ N & \xrightarrow{\tau_N} & \end{array}$$

Proof Since φ is formally étale, the differential is a fibered equivalence

$$d\varphi: \prod_{x:N} \mathbb{D}_x \simeq \mathbb{D}_{\varphi(x)}$$

and therefore a 2-cell of the given type.

This will be useful when we work with G -structures in the next section.

4.3 G -structures

The classifying morphism $\tau_M : M \rightarrow \mathbf{BAut}(\mathbb{D}_e)$ of a V -manifold M describes how the formal disk bundle is glued together using automorphisms of \mathbb{D}_e .

We will start this last section by informally ¹⁷ discussing for a simple example, what morphisms of type $M \rightarrow \mathbf{BAut}(\mathbb{D}_e)$ could look like. Let us assume for this example, that we have a ring object \mathbb{R} , that behaves like the real line as a smooth manifold. This enables us to construct \mathbb{R}^n and \mathbb{S}^n as smooth manifolds. The intended model of formal smooth ∞ -groupoids from [Scha, Section 6.5] ¹⁸, admits such an \mathbb{R} and construction of these manifolds.

Its ring structure, or more precisely its abelian group structure, turns \mathbb{R} into a homogeneous type. The universal covering $\mathbb{R} \rightarrow \mathbb{S}^1$ is a local diffeomorphism, and therefore a formally étale map ¹⁹. So the span

$$\begin{array}{ccc} & \mathbb{R} & \\ \text{id} \swarrow & & \searrow \text{ét} \\ \mathbb{R} & & \mathbb{S}^1 \end{array}$$

proves that \mathbb{S}^1 is a \mathbb{R} -manifold.

It might be somewhat surprising, that a morphism from a 0-type like \mathbb{S}^1 to the 0-connected $\mathbf{BAut}(\mathbb{D}_0)$ can be non-trivial. One way to get a handle on morphisms of type $\mathbb{S}^1 \rightarrow \mathbf{BAut}(\mathbb{D}_0)$ is to write \mathbb{S}^1 as a pushout and use the universal property or its recursion rule to describe the maps.

So let $U, V \subseteq \mathbb{S}^1$ be subspaces such that we have the following pushout square:

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & \text{(po)} & \downarrow \\ V & \longrightarrow & \mathbb{S}^1 \end{array}$$

Then we have an equivalence by the recursion rule for the pushout:

$$\begin{aligned} & \sum_{f:U \rightarrow \mathbf{BAut}(\mathbb{D}_0)} \sum_{g:V \rightarrow \mathbf{BAut}(\mathbb{D}_0)} \prod_{x:U \cap V} f(x) = g(x) \\ & \simeq \mathbb{S}^1 \rightarrow \mathbf{BAut}(\mathbb{D}_0) \end{aligned}$$

This description would give us a handle on those maps, if we understand $\mathbf{Aut}(\mathbb{D}_0)$ well. In the motivating model of formal smooth ∞ -groupoids with first order infinitesimals, $\mathbf{Aut}(\mathbb{D}_0)$ will be an infinitesimal extension of $\mathbf{GL}_1(\mathbb{R})$ or \mathbb{R}^\times as an \mathbb{R} -manifold. If we map into $\mathbf{Aut}(\mathbb{D}_0)$ or $\mathbf{BAut}(\mathbb{D}_0)$ from a reduced space like \mathbb{S}^1 ,

¹⁷The following arguments leave some gaps and the theory used here is not developed far enough to fill those gaps. The example was included to make the actual content of this section more understandable.

¹⁸This model is based on ideas from [Dub79] and in [KS17] a 1-categorical version is presented in a way that might be most helpful for a reader interested in learning more about these models.

¹⁹[KS17, Proposition 3.2] states that formally étale maps between manifolds correspond to local diffeomorphisms.

the infinitesimal part will not make a difference, so let us ignore it for now. So let us assume this to continue our calculation. Let us also assume for simplicity, that the function spaces $U \rightarrow \text{BAut}(\mathbb{D}_0)$ and $V \rightarrow \text{BAut}(\mathbb{D}_0)$ are contractible - which would be true in the motivating model if U and V are contractible. Then, if we take into account, that we still can transport along equalities of maps $U \rightarrow \text{BAut}(\mathbb{D}_0)$, we get:

$$\begin{aligned} & \mathbb{S}^1 \rightarrow \text{BAut}(\mathbb{D}_0) \\ \simeq & \sum_{_:\text{BAut}(\mathbb{D}_0)} \sum_{_:\text{BAut}(\mathbb{D}_0)} \prod_{x:U \cap V} \mathbb{D}_0 = \mathbb{D}_0 \\ \simeq & \sum_{_:\text{BAut}(\mathbb{D}_0)} \sum_{_:\text{BAut}(\mathbb{D}_0)} (U \cap V \rightarrow \mathbb{R}^\times) \end{aligned}$$

The equalities we get by transporting along paths in $\text{BAut}(\mathbb{D}_0)$ amount to multiplying a function $U \cap V \rightarrow \mathbb{R}^\times$ with a restriction of a function of type $U \rightarrow \mathbb{R}^\times$. So:

$$(\mathbb{S}^1 \rightarrow \text{BAut}(\mathbb{D}_0)) \simeq 2$$

- which corresponds to the classical fact that up to isomorphism, there are just two \mathbb{R} -vector bundles on \mathbb{S}^1 .

Now, to steer more to the topic of this section, we can ask for lifts of one specific morphism of type $\mathbb{S}^1 \rightarrow \text{BAut}(\mathbb{D}_0)$, the classifying morphism $\tau_{\mathbb{S}^1}$ of $T_\infty \mathbb{S}^1$ along a delooped group homomorphism, for example the delooped inclusion $\iota : \text{BO}(1) \rightarrow \text{BAut}(\mathbb{D}_e)$ or $\iota : \text{B}\{\pm 1\} \rightarrow \text{B}\mathbb{R}^\times$. Here a lift means, that we have a map $\varphi : \mathbb{S}^1 \rightarrow \text{BO}(1)$ and a 2-cell η :

$$\begin{array}{ccc} & & \text{B}\{\pm 1\} \\ & \nearrow \varphi & \downarrow \iota \\ \mathbb{S}^1 & \xrightarrow{\tau_{\mathbb{S}^1}} & \text{B}\mathbb{R}^\times \\ & \searrow \eta & \end{array}$$

There are just two maps $\varphi : \mathbb{S}^1 \rightarrow \text{B}\{\pm 1\}$ ²⁰, but fixing a φ , there are lots of degrees of freedom for η : a priori it can be any smooth family of non-zero real numbers indexed by \mathbb{S}^1 . Taking the equality in $\sum_{\varphi:\mathbb{S}^1 \rightarrow \text{B}\{\pm 1\}} \iota \circ \varphi \Rightarrow \tau_{\mathbb{S}^1}$ into account, these are families of positive reals modulo scalar factors and therefore precisely the Riemannian metrics on \mathbb{S}^1 .

In general, for \mathbb{R}^n -manifolds as sketched in the beginning of this section, $\text{BAut}(\mathbb{D}_0)$ corresponds to an infinitesimal extension of $\text{GL}_n(\mathbb{R})$ and the classifying map τ_M can be thought of as the datum telling us, with which elements from $\text{GL}_n(\mathbb{R})$ the formal disk bundle may be glued from trivial pieces. The infinitesimal part of $\text{BAut}(\mathbb{D}_0)$ corresponds to the translation part of the automorphisms of \mathbb{R}^n and mapping into it from reduced spaces like ordinary manifolds yields the same result as mapping into a delooping of the pointed automorphisms, which would literally be $\text{BGL}_n(\mathbb{R})$.

²⁰Since $\text{BO}(1)$ is a classifying space for \mathbb{R} -vector bundles, or since $\|\mathbb{S}^1 \rightarrow \text{B}\{\pm 1\}\|_0 \simeq \text{H}^1(\mathbb{S}^1, \mathbb{Z}/2\mathbb{Z})$.

In Urs Schreiber’s Higher Cartan Geometry lifts like those we discussed above encode geometric structure on manifolds. For example we could in general lift from $\mathrm{GL}_n(\mathbb{R})$ to $\mathrm{O}(n)$ to obtain $\mathrm{O}(n)$ -structures on a \mathbb{R}^n -manifold M which turn out to be Riemannian Metrics on M .

There are lots of structures on manifolds that can be encoded as G -structures. We give a list of examples, what group morphisms – which are almost always inclusions of subgroups – encode structures on a smooth n -manifold as G -structures. Some of the examples assume $n = 2d$.

$G \rightarrow \mathrm{GL}(n)$	G -structure
$\mathrm{O}(n) \rightarrow \mathrm{GL}(n)$	Riemannian metric
$\mathrm{GL}^+(n) \rightarrow \mathrm{GL}(n)$	orientation
$\mathrm{O}(n-1, 1) \rightarrow \mathrm{GL}(n)$	pseudo-Riemannian metric
$\mathrm{SO}(n, 2) \rightarrow \mathrm{GL}(n)$	conformal structure
$\mathrm{GL}(d, \mathbb{C}) \rightarrow \mathrm{GL}(2d, \mathbb{R})$	almost complex structure
$\mathrm{U}(d) \rightarrow \mathrm{GL}(2d, \mathbb{R})$	almost Hermitian structure
$\mathrm{Sp}(d) \rightarrow \mathrm{GL}(2d, \mathbb{R})$	almost symplectic structure
$\mathrm{Spin}(n) \rightarrow \mathrm{GL}(n)$	spin structure

For a classical definition of $\mathrm{O}(n)$ - and $\mathrm{GL}(d, \mathbb{C})$ -structures, see [Che66]. Note that in all of the above examples, G is a 1-group²¹, yet our theory also supports higher groups. The string 2-group and the fivebrane 6-group are examples of higher G -structures of interest in physics. See [SSS09] for details and references. The notion of torsionfree G -structures will describe, for example, the almost symplectic structures that are actually symplectic and analogous for complex and hermitian structures.

We will now turn to the formal treatment of G -structures on V -manifolds and the construction of the moduli spaces of these structures. Let V be a homogeneous type from now on. As we learned in the last section in 4.16, the formal disk bundle of a V -manifold M is always classified by a morphism $\tau_M : M \rightarrow \mathrm{BAut}(\mathbb{D}_e)$, where \mathbb{D}_e is the formal disk at the unit $e : V$. Since this is the only feature of a V -manifold that we need for the constructions in this section, we will work with the following more general class of spaces.

Definition 4.18

A type M is called *formal D -space* if its formal disk bundle is a D -fiber bundle.

The name was invented by Urs Schreiber and the author for the present purpose.

Remark 4.19

- (a) Any V -manifold M is a formal \mathbb{D}_e -space.
- (b) Being a formal D -space is a proposition.

Proof (a) This is 4.16.

²¹We call a 0-type with a group structure a 1-group.

(b) One of the equivalent definitions of D -fiber bundle, 4.9, was directly a proposition:

$$(P : A \rightarrow \mathcal{U} \text{ is a } D\text{-fiber bundle}) := \prod_{x:A} \|P(x) \simeq D\|_{-1}$$

We are interested in the case $D \equiv \mathbb{D}_e$ for $e : V$ meaning that M is a formal \mathbb{D}_e -space if $\prod_{x:M} \|\mathbb{D}_x \simeq \mathbb{D}_e\|_{-1}$. In 4.15 we saw, that we can “pullback” the structure of a V -manifold along a formally étale map. Formal \mathbb{D}_e -spaces behave the same way by virtue of the 2-cell we already saw in 4.17.

Lemma 4.20

Let M be a formal \mathbb{D}_e -space. For any formally étale $\varphi : N \rightarrow M$, N is also a formal \mathbb{D}_e -space and there is the triangle:

$$\begin{array}{ccc} M & \xrightarrow{\tau_M} & \mathbf{BAut}(\mathbb{D}_e) \\ \varphi \uparrow & \swarrow d\varphi & \nearrow \\ N & \xrightarrow{\tau_N} & \end{array}$$

Proof First, the triangle in the statement exists for a formally étale map between any types, if $\mathbf{BAut}(\mathbb{D}_e)$ is replaced with the universe:

$$\begin{array}{ccc} M & \xrightarrow{x \mapsto \mathbb{D}_x} & \mathcal{U} \\ \varphi \uparrow & \swarrow d\varphi & \nearrow \\ N & \xrightarrow{x \mapsto \mathbb{D}_x} & \end{array}$$

By assumption we know, that $(x : M) \mapsto \mathbb{D}_x$ lands in $\mathbf{BAut}(\mathbb{D}_e)$. But φ is formally étale, so we have $d\varphi : \prod_{x:N} \mathbb{D}_x \simeq \mathbb{D}_{f(x)}$. The latter may be truncated and composed with $\tau_M \prod_{x:M} \|\mathbb{D}_x \simeq \mathbb{D}_e\|_{-1}$ to get $\tau_N : \prod_{x:N} \|\mathbb{D}_x \simeq \mathbb{D}_e\|_{-1}$. So both maps to \mathcal{U} factor over $\mathbf{BAut}(\mathbb{D}_e)$.

Now we start to define G -structures or *reductions of the structure group* a synonym hinting that in a lot of cases G is a subgroup of $\mathbf{Aut}(\mathbb{D}_e)$. We will not restrict ourselves to reductions to subgroups and look at general pointed maps $\mathbf{BG} \rightarrow \mathbf{BAut}(\mathbb{D}_e)$. These maps correspond to group homomorphisms $G \rightarrow \mathbf{Aut}(\mathbb{D}_e)$, if \mathbf{BG} is a pointed connected type with $(* =_{\mathbf{BG}} *) \simeq G$. We will not impose any conditions on connected or truncatedness of the type \mathbf{BG} below, so “ \mathbf{BG} ” is just a name indicating the intended use case.

Definition 4.21

Let $\iota : \mathbf{BG} \rightarrow \mathbf{BAut}(\mathbb{D}_e)$ be a pointed map and M a formal \mathbb{D}_e -space. A G -structure on M is a map $\varphi : M \rightarrow \mathbf{BG}$ together with a 2-cell $\eta : \iota \circ \varphi \Rightarrow \tau_M$:

$$\begin{array}{ccc} & \varphi \curvearrowright & \mathbf{BG} \\ & \eta \swarrow & \downarrow \iota \\ M & \xrightarrow{\tau_M} & \mathbf{BAut}(\mathbb{D}_e) \end{array}$$

We write

$$G\text{-str}(M) := \sum_{\varphi: M \rightarrow \text{BG}} (\iota \circ \varphi \Rightarrow \tau_M)$$

for the type of G -structures on M .

The special case B1 turns out to be interesting – a 1-structure on a formal \mathbb{D}_e -space is nothing else than a trivialization of the formal disk bundle, like we produced in 3.11 for any homogeneous type. This provides us with an example of a 1-structure, whose construction is in spite of the name we will give below, not entirely trivial.

Definition 4.22

The *trivial 1-structure* on V is the trivialization $\psi : \prod_{x:V} \mathbb{D}_e \simeq \mathbb{D}_x$ constructed in 3.11:

$$\begin{array}{ccc} & \xrightarrow{- \mapsto *} & \text{B1} := 1 \\ & \searrow \psi & \downarrow * \mapsto \mathbb{D}_e \\ V & \xrightarrow{\tau_V} & \text{BAut}(\mathbb{D}_e) \end{array}$$

Since we have pointed maps, there is a triangle for any $\iota : \text{BG} \rightarrow \text{BAut}(\mathbb{D}_e)$:

$$\begin{array}{ccc} \text{B1} & \xrightarrow{* \mapsto *} & \text{BG} \\ * \mapsto \mathbb{D}_e \downarrow & \rightrightarrows & \uparrow \iota \\ \text{BAut}(\mathbb{D}_e) & & \end{array}$$

So we can define a trivial structure in the same way as above for arbitrary G . Let us fix a pointed map $\iota : \text{BG} \rightarrow \text{BAut}(\mathbb{D}_e)$ from now on.

Definition 4.23

Let $\mathcal{T} : \mathbb{D}_e \simeq \iota(*)$ be the transport along the equality witnessing that ι is pointed. The *trivial G -structure* on V is given by $\psi'_x := \psi_x \circ \mathcal{T}$:

$$\begin{array}{ccc} & \xrightarrow{- \mapsto *} & \text{BG} \\ & \searrow \psi' & \downarrow \iota \\ V & \xrightarrow{\tau_V} & \text{BAut}(\mathbb{D}_e) \end{array}$$

An important notion we will introduce in the end of this section, is a *torsionfree* G -structure. In some sense to be made precise, these G -structures will be trivial on all formal disks. Before we can do this, we need to be able to restrict G -structures to formal disks, or more generally, to pull them back along formally étale maps.

Definition 4.24

- (a) For M a formal \mathbb{D}_e -space and $f : N \rightarrow M$ a formally étale map from some type N , there is a map $f^* : G\text{-str}(M) \rightarrow G\text{-str}(N)$.
- (b) For the special case of formal disk inclusions $\iota_x : \mathbb{D}_x \rightarrow M$ and $\Theta : G\text{-str}(M)$, we call $\iota_x^* \Theta$ the *restriction* of Θ to the formal disk at x .

Construction (of f^*) Let $\Theta \equiv (\varphi, \eta) : G\text{-str}(M)$. Then we can paste the triangle constructed in 4.17 to the triangle given by (φ, η) :

$$\begin{array}{ccc}
 & & \text{BG} \\
 & \nearrow \varphi & \downarrow \iota \\
 M & \xrightarrow{\tau_M} & \text{BAut}(\mathbb{D}_e) \\
 \uparrow f & \swarrow df & \nearrow \\
 N & \xrightarrow{\tau_N} &
 \end{array}$$

We define the result of the pasting to be $f^*(\varphi, \eta) : G\text{-str}(N)$. Or, put differently:

$$f^*(\varphi, \eta) := (\varphi \circ f, (y : N) \mapsto \eta_{f(y)} \bullet df_y^{-1}).$$

Pulling back G -structures is 1-functorial in the following sense.

Remark 4.25

Let $f : N \rightarrow M$, $g : L \rightarrow N$ be formally étale and M a formal \mathbb{D}_e -space then there is a triangle

$$\begin{array}{ccc}
 G\text{-str}(M) & \xrightarrow{(f \circ g)^*} & G\text{-str}(L) \\
 \searrow f^* & \Downarrow & \nearrow g^* \\
 & G\text{-str}(N) &
 \end{array}$$

Proof By 3.5 we have

$$d(f \circ g)_x = (df)_{g(x)} \circ dg_x.$$

In diagrams, this yields a 3-cell between the pasting of

$$\begin{array}{ccc}
 M & \xrightarrow{\tau_M} & \\
 \uparrow f & \swarrow df & \nearrow \\
 N & \xrightarrow{\tau_N} & \text{BAut}(\mathbb{D}_e) \\
 \uparrow g & \swarrow dg & \nearrow \\
 L & \xrightarrow{\tau_L} &
 \end{array}$$

and

$$\begin{array}{ccc}
 M & \xrightarrow{\tau_M} & \\
 \uparrow f \circ g & \swarrow d(f \circ g) & \nearrow \\
 L & \xrightarrow{\tau_L} & \text{BAut}(\mathbb{D}_e)
 \end{array}$$

This means the 2-cells we paste when applying $(f \circ g)^*$ or $g^* \circ f^*$ are equal, so the functions must be equal, too.

Let M be a fixed formal \mathbb{D}_e -space from now on. The final definition of this article, is that of a torsionfree²² G -structure. The aim is to ask, if a G -structure “looks like the trivial G -structure everywhere on an infinitesimal scale”. The latter means, we restrict a G -structure to the formal disk at a point and compare it to the trivial structure on \mathbb{D}_e . So let us fix a notation for this structure:

Definition 4.26

Let $\xi : G\text{-str}(V)$ be the trivial G -structure from 4.23 and $\iota_e : \mathbb{D}_e \rightarrow V$ the formal disk inclusion. Then

$$\xi_e := \iota_e^* \xi$$

is the *trivial G -structure* on \mathbb{D}_e .

But a priori, we have no means of comparing G -structures on formal disks with this trivial structure, so we need formally étale maps from all formal disks to \mathbb{D}_e . For formal \mathbb{D}_e -spaces we merely have an equivalence from any formal disk to \mathbb{D}_e . More precisely, by 4.9 we have

$$\tau_M : \prod_{x:M} \|\mathbb{D}_x \simeq \mathbb{D}_e\|_{-1}.$$

And by pulling back to the canonical cover $w : W \rightarrow M$ from 4.11 we get

$$\omega_M : \prod_{x:W} \mathbb{D}_{w(x)} \simeq \mathbb{D}_e$$

This is enough to make the indicated comparison.

Definition 4.27

A G -structure Θ on M is *torsionfree*, if

$$\prod_{x:W} \|\omega_{M,w(x)}^* \iota_{w(x)}^* \Theta = \xi_e\|_{-1} \equiv: \text{torsionfree}(\Theta)$$

It turns out, that even for the trivial 1-structure on V torsionfreeness is non-trivial. The following example and its presentation are a result of a discussion with Urs Schreiber. If the trivial 1-structure is *left-invariant* as defined below, it is an example of a torsionfree 1-structure. To match classic notions, we assume that the equivalences of the homogeneous structure are left-translations.

Definition 4.28

The trivial G -structure Θ on V is called *left-invariant*, if the following conditions hold:

$$\begin{aligned} \Theta_e &= \text{id}_{\mathbb{D}_e} \\ \prod_{x:V} \iota_x^* \Theta &= \Theta \end{aligned}$$

²²This matches the classical terminology in the case of a first-order smooth model.

If our homogeneous space V is a Lie-Group, the trivial 1-structure is constructed the same way as the Maurer-Cartan form, which satisfies the equation above. Turning this around, we get the following example:

Theorem 4.29

Let V be a 1-group and its homogeneous structure be given by left translations, then the trivial 1-structure given by this homogeneous structure is left-invariant.

Proof The first equation follows from t_e being the identity.

For the second equation, we need the following equation given by the group structure:

$$t_{t_x(y)} = t_{xy} = t_x \circ t_y$$

Evaluating at e and using the chain rule 3.5 yields:

$$d(t_{t_x(y)})_e = (dt_x)_{t_y(e)} \circ (dt_y)_e = (dt_x)_y \circ (dt_y)_e = (dt_y)_e \bullet (dt_x)_y$$

The latter equality is just moving our equation to $\text{BAut}(\mathbb{D}_e)$.

Now for the trivial 1-structure $\Theta \equiv (_ \mapsto *, y \mapsto (dt_y)_e)$ we can calculate

$$\begin{aligned} t_x^* \Theta &= (_ \mapsto *, y \mapsto (dt_{t_x(y)})_e \bullet (dt_x)_y^{-1}) \\ &= (_ \mapsto *, y \mapsto ((dt_y)_e \bullet (dt_x)_y) \bullet (dt_x)_y^{-1}) \\ &= (_ \mapsto *, y \mapsto (dt_y)_e) \\ &= \Theta \end{aligned}$$

Theorem 4.30

Let V be a 1-group, then the trivial 1-structure on V is torsionfree.

Proof Let t_x be the translation to $x : V$ given by the homogeneous structure on V and $\Theta \equiv (_ \mapsto *, x \mapsto (dt_x)_e)$ the trivial 1-structure on V . Then for all $x : V$ we have a square of formally étale maps:

$$\begin{array}{ccc} \mathbb{D}_x & \xrightarrow{t_x} & V \\ dt_x \uparrow & & \uparrow t_x \\ \mathbb{D}_e & \xrightarrow{t_e} & V \end{array}$$

By 4.25, we get the following formula:

$$\iota_e^* t_x^* \Theta = dt_x^* \iota_x^* \Theta$$

By 4.29 we can simplify the left hand side:

$$\iota_e^* \Theta = dt_x^* \iota_x^* \Theta$$

Let us call the witness of the above equation $c_{e,x,\Theta}$ then Θ is torsionfree by

$$c_{e,y,\Theta}^{-1} \bullet c_{e,x,\Theta} : dt_y^* \iota_y^* \Theta = dt_x^* \iota_x^* \Theta$$

Since torsionfreeness as we defined it is a proposition, the type of torsionfree G -structures is a subtype of the type G -structures. The latter should be distinguished from the *moduli space* of G -structures on M , which is the quotient of the type of G -structures by the action of the automorphism group of M . If M is a 0-type, we could just build this quotient as a higher inductive type, but this is a bit unsatisfactory and not the most pleasant definition to work with. A more promising approach is to use that the quotient of an action given as a dependent type $\rho : \text{BG} \rightarrow \mathcal{U}$ is just $\sum_{x:\text{BG}} \rho(x)$. To make this approach work, the author reformulated a lot of the original theory in [Wel17]. With the present version, we will see that this construction works without considerable effort.

To realize the construction of the moduli space as a dependent sum, we need to note, that the definition of G -structures is actually a dependent type over $\text{BAut}(M)$.

Lemma 4.31

There is a dependent type $G\text{-str} : \text{BAut}(M) \rightarrow \mathcal{U}$ with $G\text{-str}(M')$ being the G -structures on M' .

Proof Since any $M' : \text{BAut}(M)$ is equivalent to M , it is merely a formal \mathbb{D}_e -space. It is very important that we can forget about the “merely” here because being a formal \mathbb{D}_e -space is already a proposition. The latter proposition contains the *data* $\tau_{M'} : M' \rightarrow \text{BAut}(\mathbb{D}_e)$. This means we can just use our former definition of “ $G\text{-str}$ ”.

This means that we can now construct the moduli spaces of G -structures and torsionfree G -structures in a nice way:

Definition 4.32

Let M be a formal \mathbb{D}_e -space and $\iota : \text{BG} \rightarrow \text{BAut}(\mathbb{D}_e)$ a pointed map.

- (a) The *moduli space* of G -structures on M is given as

$$\sum_{M':\text{BAut}(M)} G\text{-str}(M').$$

- (b) The *moduli space of torsionfree G -structures* on M is given as

$$\sum_{M':\text{BAut}(M)} \sum_{\Theta:G\text{-str}(M')} \text{torsionfree}(\Theta).$$

While we did not further discuss this, we expect that the homotopy type theory developed here has interpretation in suitable ∞ -toposes equipped with a fibered idempotent ∞ -monad. Our abstract construction of moduli spaces of torsionfree G -structures should then have a translation to a corresponding construction internal to any of these ∞ -toposes. When written out in terms of traditional higher category theory, say as simplicial sheaves, these objects will look rather complicated and be cumbersome to work with. Our abstract language should hence serve to make the development of Higher Cartan Geometry in ∞ -toposes tractable.

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