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How to Convexify the Intersection of a Second Order Cone and a Nonconvex Quadratic

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Abstract A recent series of papers has examined the extension of disjunctive-programming techniques to mixed-integer second-order-cone programming. For example, it has been shown—by several authors using different techniques—that the convex hull of the intersection of an ellipsoid, $\mathcal{E}$, and a split disjunction, $(l - x_j)(x_j - u) \le 0$ with $l < u$, equals the intersection of $\mathcal{E}$ with an additional second-order-cone representable (SOCr) set. In this paper, we study more general intersections of the form $\mathcal{K} \cap \mathcal{Q}$ and $\mathcal{K} \cap \mathcal{Q} \cap H$, where $\mathcal{K}$ is a SOCr cone, $\mathcal{Q}$ is a nonconvex cone defined by a single homogeneous quadratic, and H is an affine hyperplane. Under several easy-to-verify conditions, we derive simple, computable convex relaxations $\mathcal{K} \cap \mathcal{S}$ and $\mathcal{K} \cap \mathcal{S} \cap H$, where $\mathcal{S}$ is a SOCr cone. Under further conditions, we prove that these two sets capture precisely the corresponding conic/convex hulls. Our approach unifies and extends previous results, and we illustrate its applicability and generality with many examples.

Keywords: convex hull, disjunctive programming, mixed-integer linear programming, mixed-integer nonlinear programming, mixed-integer quadratic programming, nonconvex quadratic programming, second-order-cone programming, trust-region subproblem.

Mathematics Subject Classification: 90C25, 90C10, 90C11, 90C20, 90C26.

1 Introduction

In this paper, we study nonconvex intersections of the form $\mathcal{K} \cap \mathcal{Q}$ and $\mathcal{K} \cap \mathcal{Q} \cap H$, where the cone $\mathcal{K}$ is second-order-cone representable (SOCr), $\mathcal{Q}$ is a nonconvex cone defined by a single homogeneous quadratic, and H is an affine hyperplane. Our goal is to develop tight convex relaxations of these sets and to characterize the conic/convex hulls whenever possible. We are motivated by recent research on Mixed Integer Conic

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Programs (MICPs), though our results here enjoy wider applicability to nonconvex quadratic programs.

Prior to the study of MICPs in recent years, cutting plane theory has been fundamental in the development of efficient and powerful solvers for Mixed Integer Linear Programs (MILPs). In this theory, one considers a convex relaxation of the problem, e.g., its continuous relaxation, and then enforces integrality restrictions to eliminate regions containing no integer feasible points—so-called *lattice-free sets*. The complement of a valid *two-term linear disjunction*, say $x_j \leq l \vee x_j \geq u$, is a simple form of a lattice-free set. The additional inequalities required to describe the convex hull of such a disjunction are known as *disjunctive cuts*. Such a disjunctive point of view was introduced by Balas [6] in the context of MILPs, and it has since been studied extensively in mixed integer linear and nonlinear optimization [7,8,17,18,20,22,33,48, 49], complementarity [29,31,43,51] and other nonconvex optimization problems [11, 17]. In the case of MILPs, several well-known classes of cuts such as *Chvátal-Gomory*, *lift-and-project*, *mixed-integer rounding (MIR)*, *split*, and *intersection cuts* are known to be special types of disjunctive cuts. Stubbs and Mehrotra [50] and Ceria and Soares [20] extended cutting plane theory from MILP to convex mixed integer problems. These works were followed by several papers [15,24,25,33,53] that investigated linear-outer-approximation based approaches, as well as others that extended specific classes of inequalities, such as Chvátal-Gomory cuts [19] for MICPs and MIR cuts [5] for SOC-based MICPs.

Recently there has been growing interest in developing closed-form expressions for convex inequalities that fully describe the convex hull of a disjunctive set involving an SOC. In this vein, Günlük and Linderoth [27] studied a simple set involving an SOC in $\mathbb{R}^3$ and a single binary variable and showed that the resulting convex hull is characterized by adding a single SOCr constraint. For general SOCs in $\mathbb{R}^n$, this line of work was furthered by Dadush et al. [23], who derived cuts for ellipsoids based on parallel two-term disjunctions, that is, *split disjunctions*. Modaresi et al. [40] extended this by studying *intersection cuts* for SOC and all of its cross-sections (i.e., all conic sections), based on split disjunctions as well as a number of other lattice-free sets such as ellipsoids and paraboloids. A theoretical and computational comparison of intersection cuts from [40] with extended formulations and conic MIR inequalities from [5] is given in [39]. Taking a different approach, Andersen and Jensen [2] derived an SOC constraint describing the convex hull of a split disjunction applied to an SOC. Belotti et al. [12] studied families of quadratic surfaces having fixed intersections with two given hyperplanes, and in [13], they identified a procedure for constructing two-term disjunctive cuts when the sets defined by the disjunctions are bounded and disjoint. Kılınç-Karzan [34] introduced and examined minimal valid linear inequalities for general conic sets with a disjunctive structure, and under a mild technical assumption, established that they are sufficient to describe the resulting closed convex hulls. For general two-term disjunctions on regular (closed, convex, pointed with nonempty interior) cones, Kılınç-Karzan and Yıldız [36] studied the structure of tight minimal valid linear inequalities. In the particular case of SOCs, based on conic duality, a class of convex valid inequalities that is sufficient to describe the convex hull were derived in [36] along with the conditions for SOCr representability of these inequalities as well as for the sufficiency of a single inequality from this class. This work was recently extended in Yıldız and Cornuéjols [55] to all cross-sections of SOC that can be covered by the same assumptions of [36]. Bienstock and Michalka [14] studied the characterization and separation of valid linear inequalities that convexify the epigraph of a convex, differentiable function whose domain is restricted to the complement of a convex set defined by linear or convex quadratic inequalities. Although all of these authors take different approaches, their results are comparable, for example, in the case of analyzing split disjunctions of the SOC or its cross-sections. We remark also that these methods convexify in the space of the original variables, i.e., they do not involve lifting. For additional convexification approaches for nonconvex quadratic programming, which convexify in the lifted space of products $x_i x_j$ of variables, we refer the reader to [4, 9, 16, 17, 52], for example.

In this paper, our main contributions can be summarized as follows (see Section 3 and Theorem 1 in particular). First, we derive a simple, computable convex relaxation $\mathcal{K} \cap \mathcal{S}$ of $\mathcal{K} \cap \mathcal{Q}$, where $\mathcal{S}$ is an additional SOCr cone. This also provides the convex relaxation $\mathcal{K} \cap \mathcal{S} \cap H \supseteq \mathcal{K} \cap \mathcal{Q} \cap H$. The derivation relies on several easy-to-verify conditions (see Section 3.2). Second, we identify stronger conditions guaranteeing moreover that $\mathcal{K} \cap \mathcal{S} = \mathrm{cl.\,conic.\,hull}(\mathcal{K} \cap \mathcal{Q})$ and $\mathcal{K} \cap \mathcal{S} \cap H = \mathrm{cl.\,conv.\,hull}(\mathcal{K} \cap \mathcal{Q} \cap H)$, where *cl* indicates the closure, *conic.hull* indicates the conic hull, and *conv.hull* indicates the convex hull. Our approach unifies and significantly extends previous results. In particular, in contrast to the existing literature on cuts based on lattice-free sets, here we allow a general $\mathcal{Q}$ without making an assumption that $\mathbb{R}^n \setminus \mathcal{Q}$ is convex. We illustrate the applicability and generality of our approach with many examples and explicitly contrast our work with the existing literature.

Our approach can be seen as a variation of the following basic, yet general, idea of conic aggregation to generate valid inequalities. Suppose that $f_0 = f_0(x)$ is convex, while $f_1 = f_1(x)$ is nonconvex, and suppose we are interested in the closed convex hull of the set $Q := \{x : f_0 \le 0,\ f_1 \le 0\}$. For any $0 \le t \le 1$, the inequality $f_t := (1-t)f_0 + tf_1 \le 0$ is valid for Q, but f_t is generally nonconvex. Hence, it is natural to seek values of t such that the function f_t is convex for all x. One might even conjecture that some particular convex f_s with $0 \le s \le 1$ guarantees $\mathrm{cl.\,conv.\,hull}(Q) = \{x : f_0 \le 0, f_s \le 0\}$. However, it is known that this approach cannot generally achieve the convex hull even when f_0, f_1 are quadratic functions; see [40]. Such aggregation techniques to obtain convex under-estimators have also been explored in the global-optimization literature, albeit without explicit results on the resulting convex hull descriptions (see [1] for example).

In this paper, we follow a similar approach in spirit, but instead of determining $0 \le t \le 1$ guaranteeing the convexity of f_t for all x, we only require "almost" convexity, that is, the function f_t is required to be convex on $\{x : f_0 \le 0\}$. This weakened requirement is crucial. In particular, it allows us to obtain convex hulls for many cases where $\{x : f_0 \le 0\}$ is SOCr and f_1 is a nonconvex quadratic, and we recover all of the known results regarding two-term disjunctions cited above (see Section 5). We note that using quite different techniques and under completely different assumptions, a similar idea of aggregation for quadratic functions has been explored in [13, 40] as well. Specifically, our weakened requirement is in contrast to the developments in [40], which explicitly requires the function f_t to be convex everywhere. Also, our general $\mathcal{Q}$ allows us to study general nonconvex quadratics f_1 as opposed to the specific ones arising from two-term disjunctions studied in [13]. As a practical and technical matter, instead of working directly with convex functions in this paper, we work in the equivalent realm of convex sets, in particular SOCr cones. Section 2 discusses in detail the features of SOCr cones required for our analysis.

Compared to the previous literature on MICPs, our work here is broader in that we study a general nonconvex cone $\mathcal{Q}$ defined by a single homogeneous quadratic function. As a result, we assume neither the underlying matrix defining the homogeneous quadratic $\mathcal{Q}$ to be of rank at most 2 nor $\mathbb{R}^n \setminus \mathcal{Q}$ to be convex. This is in contrast to a key underlying assumption used in the literature. Specifically, the majority of the earlier literature on MICPs focus on specific lattice-free sets, e.g., all of the works [2, 5, 13, 23, 36, 55] focus on either split or two-term disjunctions on SOCs or its cross-sections. In the case of two-term disjunctions, the matrix defining the homogeneous quadratic for $\mathcal{Q}$ is of rank at most 2; and moreover, the complement of any two-term disjunction is a convex set. Even though, nonconvex quadratics $\mathcal{Q}$ with rank higher than 2 are considered in [40], unlike our general, $\mathcal{Q}$ this is done under the assumption that the complement of the nonconvex quadratic defines a convex set. Our general $\mathcal{Q}$ allows for a *unified framework* and works under weaker assumptions. In Sections 3.3 and 5 and the Online Supplement, we illustrate and highlight these features of our approach and contrast it with the existing literature through a series of examples. Bienstock and Michalka [14] also consider more general $\mathcal{Q}$ under the assumption that $\mathbb{R}^n \setminus \mathcal{Q}$ is convex, but their approach is quite different than ours. Whereas [14] relies on polynomial time procedures for separating and tilting valid linear inequalities, we directly give the convex hull description. In contrast, our study of the general, nonconvex quadratic cone $\mathcal{Q}$ allows its complement $\mathbb{R}^n \setminus \mathcal{Q}$ to be nonconvex as well.

We remark that our convexification tools for general nonconvex quadratics have potential applications beyond MICPs, for example in the nonconvex quadratic programming domain. We also can, for example, characterize: the convex hull of the deletion of an arbitrary ball from another ball; and the convex hull of the deletion of an arbitrary ellipsoid from another ellipsoid sharing the same center. In addition, we can use our results to solve the classical trust region subproblem [21] using SOC optimization, complementing previous approaches relying on nonlinear [26, 42] or semidefinite programming [47]. Section 6 discusses these examples.

Another useful feature of our approach is that we clearly distinguish the conditions guaranteeing validity of our relaxation from those ensuring sufficiency. In [2, 13, 23, 40], validity and sufficiency are intertwined making it difficult to construct convex relaxations when their conditions are only partly satisfied. Furthermore, our derivation of the convex relaxation is efficiently computable and relies on conditions that are easily verifiable. Finally, our conditions regarding the cross-sections (that is, intersection with the affine hyperplane H) are applicable for general cones other than SOCs.

We would like to stress that the inequality describing the SOCr set $\mathcal{S}$ is efficiently computable. In other words, given the sets $\mathcal{K} \cap \mathcal{Q}$ and $\mathcal{K} \cap \mathcal{Q} \cap \mathcal{H}$, one can verify in polynomial time the required conditions and then calculate in polynomial time the inequality for $\mathcal{S}$ to form the relaxations $\mathcal{K} \cap \mathcal{S}$ and $\mathcal{K} \cap \mathcal{S} \cap H$. The core operations include calculating eigenvalues/eigenvectors for several symmetric and non-symmetric matrices and solving a two-constraint semidefinite program. The computation can also be streamlined in cases when any special structure of $\mathcal{K}$ and $\mathcal{Q}$ is known ahead of time.

The paper is structured as follows. Section 2 discusses the details of SOCr cones, and Section 3 states our conditions and main theorem. In Section 3.2, we provide a detailed discussion and pseudocode for verifying our conditions and computing the resulting SOC based relaxation $\mathcal{S}$. Section 3.3 then provides a low-dimensional example with figures and comparisons with existing literature. We provide more examples with corresponding figures and comparisons in the Online Supplement accompanying this article. In Section 4, we prove the main theorem, and then in Sections 5 and 6, we discuss and prove many interesting general examples covered by our theory. Section 7 concludes the paper with a few final remarks. Our notation is mostly standard. We will define any particular notation upon its first use.

2 Second-Order-Cone Representable Sets

Our analysis in this paper is based on the concept of SOCr (second-order-cone representable) cones. In this section, we define and introduce the basic properties of such sets.

A cone $\mathcal{F}^+ \subseteq \mathbb{R}^n$ is said to be *second-order-cone representable* (or *SOCr*) if there exists a matrix $0 \neq B \in \mathbb{R}^{n\times(n-1)}$ and a vector $b \in \mathbb{R}^n$ such that the nonzero columns of B are linearly independent, $b \notin \text{Range}(B)$, and

$$\mathcal{F}^+ = \{x : \|B^T x\| \leq b^T x\}, \tag{1}$$

where $\|\cdot\|$ denotes the usual Euclidean norm. The negative of $\mathcal{F}^+$ is also SOCr:

$$\mathcal{F}^- := -\mathcal{F}^+ = \{x : \|B^T x\| \leq -b^T x\}. \tag{2}$$

Defining $A := BB^T - bb^T$, the union $\mathcal{F}^+ \cup \mathcal{F}^-$ corresponds to the homogeneous quadratic inequality $x^T Ax \leq 0$:

$$\mathcal{F} := \mathcal{F}^+ \cup \mathcal{F}^- = \{x : \|B^T x\|^2 \leq (b^T x)^2\} = \{x : x^T Ax \leq 0\}. \tag{3}$$

We also define

$$\begin{aligned} \text{int}(\mathcal{F}^+) &:= \{x : \|B^T x\| < b^T x\} \\ \text{bd}(\mathcal{F}^+) &:= \{x : \|B^T x\| = b^T x\} \\ \text{apex}(\mathcal{F}^+) &:= \{x : B^T x = 0, b^T x = 0\}. \end{aligned}$$

We next study properties of $\mathcal{F}, \mathcal{F}^+, \mathcal{F}^-$ such as their representations and uniqueness thereof. On a related note, Mahajan and Munson [38] have also studied sets associated with nonconvex quadratics with a single negative eigenvalue but from a more computational point of view. The following proposition establishes some important features of SOCr cones:

Proposition 1 *Let $\mathcal{F}^+$ be SOCr as in (1), and define $A := BB^T - bb^T$. Then* $\text{apex}(\mathcal{F}^+) = \text{null}(A)$*, A has at least one positive eigenvalue, and A has exactly one negative eigenvalue. As a consequence,* $\text{int}(\mathcal{F}^+) \neq \emptyset$*.*

Proof For any x, we have the equation

$$Ax = (BB^T - bb^T)x = B(B^T x) - b(b^T x). \tag{4}$$

So $x \in \text{apex}(\mathcal{F}^+)$ implies $x \in \text{null}(A)$. The converse also holds by (4) because, by definition, the nonzero columns of B are independent and $b \notin \text{Range}(B)$. Hence, $\text{apex}(\mathcal{F}^+) = \text{null}(A)$.

The equation $A = BB^T - bb^T$, with $0 \neq BB^T \succeq 0$ and $bb^T \succeq 0$ rank-1 and $b \notin \text{Range}(B)$, implies that A has at least one positive eigenvalue and at most one negative eigenvalue. Because $b \notin \text{Range}(B)$, we can write $b = x + y$ such that $x \in \text{Range}(B)$, $0 \neq y \in \text{null}(B^T)$, and $x^T y = 0$. Then

$$y^T Ay = y^T (BB^T - bb^T)y = 0 - (b^T y)^2 = -\|y\|^2 < 0,$$

showing that A has exactly one negative eigenvalue, and so $\text{int}(\mathcal{F}^+)$ contains either y or $-y$. $\square$

We define analogous sets $\mathrm{int}(\mathcal{F}^-)$, $\mathrm{bd}(\mathcal{F}^-)$, and $\mathrm{apex}(\mathcal{F}^-)$ for $\mathcal{F}^-$. In addition:

$$\begin{aligned}\mathrm{int}(\mathcal{F}) &:= \{x : x^T Ax < 0\} = \mathrm{int}(\mathcal{F}^+) \cup \mathrm{int}(\mathcal{F}^-)\\ \mathrm{bd}(\mathcal{F}) &:= \{x : x^T Ax = 0\} = \mathrm{bd}(\mathcal{F}^+) \cup \mathrm{bd}(\mathcal{F}^-).\end{aligned}$$

Similarly, we have $\mathrm{apex}(\mathcal{F}^-) = \mathrm{null}(A) = \mathrm{apex}(\mathcal{F}^+)$, and if A has exactly one negative eigenvalue, then $\mathrm{int}(\mathcal{F}^-) \neq \emptyset$ and $\mathrm{int}(\mathcal{F}) \neq \emptyset$.

When considered as a pair of sets $\{\mathcal{F}^+, \mathcal{F}^-\}$, it is possible that another choice $(\bar{B}, \bar{b})$ in place of (B, b) leads to the same pair and hence to the same $\mathcal{F}$. For example, $(\bar{B}, \bar{b}) = (-B, -b)$ simply switches the roles of $\mathcal{F}^+$ and $\mathcal{F}^-$, but $\mathcal{F}$ does not change. However, we prove next that $\mathcal{F}$ is essentially invariant up to positive scaling. As a corollary, any alternative $(\bar{B}, \bar{b})$ yields $A = \rho(\bar{B}\bar{B}^T - \bar{b}\bar{b}^T)$ for some $\rho > 0$, i.e., A is essentially invariant with respect to its (B, b) representation.

Proposition 2 *Let $A, \bar{A}$ be two $n \times n$ symmetric matrices such that $\{x \in \mathbb{R}^n : x^T Ax \leq 0\} = \{x \in \mathbb{R}^n : x^T \bar{A}x \leq 0\}$. Suppose that A satisfies $\lambda_{\min}(A) < 0 < \lambda_{\max}(A)$. Then there exists $\rho > 0$ such that $\bar{A} = \rho A$.*

Proof Since $\lambda_{\min}(A) < 0$, there exists $\bar{x} \in \mathbb{R}^n$ such that $\bar{x}^T A\bar{x} < 0$. Because $x^T Ax \leq 0 \Leftrightarrow x^T \bar{A}x \leq 0$, there exists no x such that $x^T Ax \leq 0$ and $x^T(-\bar{A})x < 0$. Then, by the S-lemma (see Theorem 2.2 in [45], for example), there exists $\lambda_1 \geq 0$ such that $-\bar{A} + \lambda_1 A \succeq 0$. Switching the roles of A and $\bar{A}$, a similar argument implies the existence of $\lambda_2 \geq 0$ such that $-A + \lambda_2 \bar{A} \succeq 0$. Note $\lambda_2 > 0$; otherwise, A would be negative semidefinite, contradicting $\lambda_{\max}(A) > 0$. Likewise, $\lambda_1 > 0$. Hence,

$$A \succeq \frac{1}{\lambda_1}\bar{A} \succeq \frac{1}{\lambda_1\lambda_2}A \iff (1 - \lambda_1\lambda_2)A \succeq 0.$$

Since $\lambda_{\min}(A) < 0 < \lambda_{\max}(A)$, we conclude $\lambda_1\lambda_2 = 1$, which in turn implies $A = \frac{1}{\lambda_1}\bar{A}$, as claimed. $\square$

Corollary 1 *Let $\{\mathcal{F}^+, \mathcal{F}^-\}$ be SOCr sets as in (1) and (2), and define $A := BB^T - bb^T$. Let $(\bar{B}, \bar{b})$ be another choice in place of (B, b) leading to the same pair $\{\mathcal{F}^+, \mathcal{F}^-\}$. Then $A = \rho(\bar{B}\bar{B}^T - \bar{b}\bar{b}^T)$ for some $\rho > 0$.*

We can reverse the discussion thus far to start from a symmetric matrix A with at least one positive eigenvalue and a single negative eigenvalue and define associated SOCr cones $\mathcal{F}^+$ and $\mathcal{F}^-$. Indeed, given such an A, let $Q\,\mathrm{Diag}(\lambda)Q^T$ be a spectral decomposition of A such that $\lambda_1 < 0$. Let q_j be the j-th column of Q, and define

$$B := \left(\lambda_2^{1/2} q_2 \ \cdots \ \lambda_n^{1/2} q_n\right) \in \mathbb{R}^{n\times(n-1)}, \qquad b := (-\lambda_1)^{1/2} q_1 \in \mathbb{R}^n. \tag{5}$$

Note that the nonzero columns of B are linearly independent and $b \notin \mathrm{Range}(B)$. Then $A = BB^T - bb^T$, and $\mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-$ can be defined as in (1)–(3). An important observation is that, as a collection of sets, $\{\mathcal{F}^+, \mathcal{F}^-\}$ is independent of the choice of spectral decomposition.

Proposition 3 *Let A be a given symmetric matrix with at least one positive eigenvalue and a single negative eigenvalue, and let $A = Q\,\mathrm{Diag}(\lambda)Q^T$ be a spectral decomposition such that $\lambda_1 < 0$. Define the SOCr sets $\{\mathcal{F}^+, \mathcal{F}^-\}$ according to (1) and (2), where (B, b) is given by (5). Similarly, let $\{\bar{\mathcal{F}}^+, \bar{\mathcal{F}}^-\}$ be defined by an alternative spectral decomposition $A = \bar{Q}\,\mathrm{Diag}(\bar{\lambda})\bar{Q}^T$. Then $\{\bar{\mathcal{F}}^+, \bar{\mathcal{F}}^-\} = \{\mathcal{F}^+, \mathcal{F}^-\}$.*

Proof Let $(\bar{B}, \bar{b})$ be given by the alternative spectral decomposition. Because A has a single negative eigenvalue, $\bar{b} = b$ or $\bar{b} = -b$. In addition, we claim $\|\bar{B}^T x\| = \|B^T x\|$ for all x. This holds because $\bar{B}\bar{B}^T = BB^T$ is the positive semidefinite part of A. This proves the result. $\square$

To resolve the ambiguity inherent in Proposition 3, one could choose a specific $\bar{x} \in \text{int}(\mathcal{F})$, which exists by Proposition 1, and enforce the convention that, for any spectral decomposition, $\mathcal{F}^+$ is chosen to contain $\bar{x}$. This simply amounts to flipping the sign of b so that $b^T \bar{x} > 0$.

3 The Result and Its Computability

In Section 3.1, we state our main theorem (Theorem 1) and the conditions upon which it is based. The proof of Theorem 1 is delayed until Section 4. In Section 3.2, we discuss computational details related to our conditions and Theorem 1.

3.1 The result

To begin, let A_0 be a symmetric matrix satisfying the following:

Condition 1 *A_0 has at least one positive eigenvalue and exactly one negative eigenvalue.*

As described in Section 2, we may define SOCr cones $\mathcal{F}_0 = \mathcal{F}_0^+ \cup \mathcal{F}_0^-$ based on A_0. We also introduce a symmetric matrix A_1 and define the cone $\mathcal{F}_1 := \{x : x^T A_1 x \leq 0\}$ in analogy with $\mathcal{F}_0$. However, we do *not* assume that A_1 has exactly one negative eigenvalue, so $\mathcal{F}_1$ does not necessarily decompose into two SOCr cones.

We investigate the set $\mathcal{F}_0^+ \cap \mathcal{F}_1$, which has been expressed as $\mathcal{K} \cap \mathcal{Q}$ in the Introduction. In particular, we would like to develop strong convex relaxations of $\mathcal{F}_0^+ \cap \mathcal{F}_1$ and, whenever possible, characterize its closed conic hull. We focus on the full-dimensional case, and so we assume:

Condition 2 *There exists $\bar{x} \in \text{int}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$.*

Note that $\text{int}(\mathcal{F}_0^+ \cap \mathcal{F}_1) = \text{int}(\mathcal{F}_0^+) \cap \text{int}(\mathcal{F}_1)$, and so Condition 2 is equivalent to

$$\bar{x}^T A_0 \bar{x} < 0 \qquad \text{and} \qquad \bar{x}^T A_1 \bar{x} < 0. \tag{6}$$

In particular, this implies A_1 has at least one negative eigenvalue.

The first part of Theorem 1 below establishes that $\text{cl.conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ is contained within the convex intersection of $\mathcal{F}_0^+$ with a second set of the same type, i.e., one that is SOCr. In addition to Conditions 1 and 2, we require the following condition, which handles the singularity of A_0 carefully via several cases:

Condition 3 *Either (i) A_0 is nonsingular, (ii) A_0 is singular and A_1 is positive definite on $\text{null}(A_0)$, or (iii) A_0 is singular and A_1 is negative definite on $\text{null}(A_0)$.*

Conditions 1–3 will ensure (see Proposition 4 in Section 4.1) the existence of a maximal $s \in [0, 1]$ such that

$$A_t := (1 - t)A_0 + tA_1$$

has a single negative eigenvalue for all $t \in [0, s]$, A_t is invertible for all $t \in (0, s)$, and A_s is singular—that is, $\mathrm{null}(A_s)$ is non-trivial. (Actually, A_s may be nonsingular when s equals 1, but this is a small detail.) Indeed, we define s formally as follows. Let $\mathcal{T} := \{t \in \mathbb{R} : A_t \text{ is singular}\}$. Then

$$s := \begin{cases} \min(\mathcal{T} \cap (0, 1]) & \text{under Condition 3(i) or 3(ii)} \\ 0 & \text{under Condition 3(iii).} \end{cases} \tag{7}$$

Sections 3.2 and 4 will clarify the role of Condition 3 in this definition.

With s given by (7), we can then define, for all A_t with $t \in [0, s]$, SOCr sets $\mathcal{F}_t = \mathcal{F}_t^+ \cup \mathcal{F}_t^-$ as described in Section 2. Furthermore, for $\bar{x}$ of Condition 2, noting that $\bar{x}^T A_t \bar{x} = (1 - t)\, \bar{x} A_0 \bar{x} + t\, \bar{x}^T A_1 \bar{x} < 0$ by (6), we can choose without loss of generality that $\bar{x} \in \mathcal{F}_t^+$ for all such t. Then Theorem 1 asserts that $\mathrm{cl.\,conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ is contained in $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$. We remark that while $\mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s$ (no "+" superscript on $\mathcal{F}_s$) follows trivially from the definition of $\mathcal{F}_s$, strengthening the inclusion to $\mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$ (with the "+" superscript) is nontrivial.

The second part of Theorem 1 provides an additional condition under which $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ actually equals the closed conic hull. The required condition is:

Condition 4 *When* $s < 1$, $\mathrm{apex}(\mathcal{F}_s^+) \cap \mathrm{int}(\mathcal{F}_1) \neq \emptyset$.

While Condition 4 may appear quite strong, we will actually show (see Lemma 3 in Section 4) that Conditions 1–3 and the definition of s already ensure $\mathrm{apex}(\mathcal{F}_s^+) \subseteq \mathcal{F}_1$. So Condition 4 is a type of regularity condition guaranteeing that the set $\mathrm{apex}(\mathcal{F}_s^+) = \mathrm{null}(A_s)$ is not restricted to the boundary of $\mathcal{F}_1$.

We also include in Theorem 1 a specialization for the case when $\mathcal{F}_0^+ \cap \mathcal{F}_1$ is intersected with an affine hyperplane H^1, which has been expressed as $\mathcal{K} \cap \mathcal{Q} \cap H$ in the Introduction. For this, let $h \in \mathbb{R}^n$ be given, and define the hyperplanes

$$H^1 := \{x : h^T x = 1\}, \tag{8}$$

$$H^0 := \{x : h^T x = 0\}. \tag{9}$$

We introduce an additional condition related to H^0:

Condition 5 *When* $s < 1$, $\mathrm{apex}(\mathcal{F}_s^+) \cap \mathrm{int}(\mathcal{F}_1) \cap H^0 \neq \emptyset$ *or* $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$.

We now state the main theorem of the paper. See Section 4 for its proof.

Theorem 1 *Suppose Conditions 1–3 are satisfied, and let* s *be defined by (7). Then* $\mathrm{cl.\,conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1) \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$, *and equality holds under Condition 4. Moreover, Conditions 1–5 imply* $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1 = \mathrm{cl.\,conv.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$.

3.2 Computational details

In practice, Theorem 1 can be used to generate a valid convex relaxation $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ of the nonconvex cone $\mathcal{F}_0^+ \cap \mathcal{F}_1$. For the purposes of computation, we assume that $\mathcal{F}_0^+ \cap \mathcal{F}_1$ is described as

$$\mathcal{F}_0^+ \cap \mathcal{F}_1 = \{x \in \mathbb{R}^n : \|B_0^T x\| \leq b_0^T x,\ x^T A_1 x \leq 0\},$$

where B_0 is nonzero, $0 \neq b_0 \notin \mathrm{Range}(B_0)$, and $A_0 = B_0 B_0^T - b_0 b_0^T$ in accordance with (5). In particular, $\mathcal{F}_0^+$ is given in its direct SOC form. Our goal is to calculate $\mathcal{F}_s^+$ in terms of its SOC form $\|B_s^T x\| \leq b_s^T x$, to which we will refer as the SOC cut.

Before one can apply Theorem 1 to generate the cut, Conditions 1–3 must be verified. By construction, Condition 1 is satisfied, and verifying Condition 3(i) is easy. Conditions 3(ii) and 3(iii) are also easy to verify by computing the eigenvalues of $Z_0^T A_1 Z_0$, where Z_0 is a matrix whose columns span $\mathrm{null}(A_0)$. Due to (6) and the fact that $\mathcal{F}_0$ and $\mathcal{F}_1$ are cones, verifying Condition 2 is equivalent to checking the feasibility of the following quadratic equations in the original variables $x \in \mathbb{R}^n$ and the auxiliary "squared slack" variables $s, t \in \mathbb{R}$:

$$x^T A_0 x + s^2 = -1,\ x^T A_1 x + t^2 = -1.$$

Let us define the underlying symmetric $(n+2) \times (n+2)$ matrices for these quadratics as $\hat{A}_0$ and $\hat{A}_1$. Since there are only two quadratic equations with symmetric matrices, by [10, Corollary 13.2], checking Condition 2 is equivalent to checking the feasibility of the following linear semidefinite system, which can be done easily in practice:

$$Y \succeq 0,\ \ \mathrm{trace}(\hat{A}_0 Y) = -1,\ \ \mathrm{trace}(\hat{A}_1 Y) = -1. \tag{10}$$

See also [44] for a similar result.

This equivalence of Condition 2 and the feasibility of system (10) relies on the fact that every extreme point of (10) is a rank-1 matrix, and such extreme points can be calculated in polynomial time [44]. Extreme points can also be generated reliably (albeit heuristically) in practice to calculate an interior point $\bar{x} \in \mathrm{int}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$. One can simply minimize over (10) the objective $\mathrm{trace}((I+R)Y)$, where I is the identity matrix and R is a random matrix, small enough so that $I+R$ remains positive definite. The objective $\mathrm{trace}((I+R)Y)$ is bounded over (10), and hence an optimal solution occurs at an extreme point. The random nature of the objective also makes it highly likely that the optimal solution is unique, in which case the optimal Y^* must be rank-1. Then $\bar{x}$ can easily be extracted from the rank-1 factorization of Y^*. Note that in certain specific cases $\bar{x}$ might be known ahead of time or could be computed right away by some other means.

Once Conditions 1–3 have been verified, we are then ready to calculate s according to its definition (7). If Condition 3(iii) holds, we simply set $s = 0$. For Conditions 3(i) and 3(ii), we need to calculate $\mathcal{T}$, the set of scalars t such that $A_t := (1-t)A_0 + tA_1$ is singular. Let us first consider Condition 3(i), which is the simpler case. The following calculation with $t \neq 0$ shows that the elements of $\mathcal{T}$ are in bijective correspondence with the real eigenvalues of $A_0^{-1} A_1$:

$$\begin{aligned} A_t \text{ is singular} \quad &\Longleftrightarrow \quad \exists\, x \neq 0 \ \text{ s.t. } \ A_t x = 0 \\ &\Longleftrightarrow \quad \exists\, x \neq 0 \ \text{ s.t. } \ A_0^{-1} A_1 x = -\left(\tfrac{1-t}{t}\right) x \\ &\Longleftrightarrow \quad -\left(\tfrac{1-t}{t}\right) \text{ is an eigenvalue of } A_0^{-1} A_1. \end{aligned}$$

So to calculate $\mathcal{T}$, we calculate the real eigenvalues $\mathcal{E}$ of $A_0^{-1}A_1$, and then calculate $\mathcal{T} = \{(1-e)^{-1} : e \in \mathcal{E}\}$, where by convention $0^{-1} = \infty$. In particular, $|\mathcal{T}|$ is finite.

When Condition 3(ii) holds, we calculate $\mathcal{T}$ in a slightly different manner. We will show in Section 4 (see Lemma 1 in particular) that, even though A_0 is singular, A_ϵ is nonsingular for all $\epsilon > 0$ sufficiently small. Such an A_ϵ could be calculated by systematically testing values of ϵ near 0, for example. Then we can apply the procedure of the previous paragraph to calculate the set $\overline{\mathcal{T}}$ of all $\bar{t}$ such that $(1-\bar{t})A_\epsilon + \bar{t}A_1$ is singular. Then one can check that $\mathcal{T}$ is calculated by the following affine transformation: $\mathcal{T} = \{(1-\epsilon)\bar{t} + \epsilon : \bar{t} \in \overline{\mathcal{T}}\}$.

Once $\mathcal{T}$ is computed, we can easily calculate $s = \min(\mathcal{T} \cap (0,1])$ according to (7), and then we construct $A_s := (1-s)A_0 + sA_1$ and calculate (B_s, b_s) according to (5). Then our cut is $\|B_s^T x\| \le b_s^T x$ with only one final provision. We must check the sign of $b_s^T \bar{x}$, where $\bar{x} \in \mathrm{int}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ has been calculated previously. If $b_s^T \bar{x} \ge 0$, then the cut is as stated; if $b_s^T \bar{x} < 0$, then the cut is as stated but b_s is first replaced by $-b_s$.

We summarize the preceding discussion by the pseudocode in Algorithm 1. While this algorithm is quite general, it is also important to point out that it can be streamlined if one already knows the structure of $\|B_0^T x\| \le b_0^T x$ and $x^T A_1 x \le 0$. For example, one may already know that A_0 is invertible, in which case it would be unnecessary to calculate the spectral decomposition of A_0 in Algorithm 1. In addition, for many of the specific cases that we consider in Sections 5 and 6, we can explicitly point out the corresponding value of s without even relying on the computation of the set $\mathcal{T}$. Because of space considerations, we do not include these closed-form expressions for s and the corresponding computations.

Finally, we mention briefly the computability of Conditions 4 and 5, which are not necessary for the validity of the cut but can establish its sufficiency. Given $s < 1$, Condition 4 can be checked by computing $Z_s^T A_1 Z_s$, where Z_s has columns spanning $\mathrm{null}(A_s)$. We know $Z_s^T A_1 Z_s \preceq 0$ because $\mathrm{apex}(\mathcal{F}_s^+) \subseteq \mathcal{F}_1$ (see Lemma 3 in Section 4), and then Condition 4 holds as long as $Z_s^T A_1 Z_s \ne 0$. On the other hand, it seems challenging to verify Condition 5 in general. However, in Sections 5 and 6, we will show that it can be verified in many examples of interest.

3.3 An ellipsoid and a nonconvex quadratic

In $\mathbb{R}^3$, consider the intersection of the unit ball defined by $y_1^2 + y_2^2 + y_3^2 \le 1$ and the nonconvex set defined by the quadratic $-y_1^2 - y_2^2 + \frac{1}{2}y_3^2 \le y_1 + \frac{1}{2}y_2$. By homogenizing via $x = \binom{y}{x_4}$ with $x_4 = 1$, we can represent the intersection as $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ with

$$A_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} -1 & 0 & 0 & -\frac{1}{2} \\ 0 & -1 & 0 & -\frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 0 & 0 \end{pmatrix}, \quad H^1 := \{x : x_4 = 1\}.$$

Conditions 1 and 3(i) are straightforward to verify, and Condition 2 is satisfied with $\bar{x} = (\frac{1}{2}; 0; 0; 1)$, for example. We can also calculate $s = \frac{1}{2}$ from (7). Then

$$A_s = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 6 & 0 \\ -2 & -1 & 0 & -4 \end{pmatrix}, \qquad \mathcal{F}_s = \left\{x : 3x_3^2 \le 2x_1x_4 + x_2x_4 + 2x_4^2\right\}.$$

Algorithm 1 Calculate Cut (see also Section 3.2)

Input: Inequalities $\|B_0^T x\| \le b_0^T x$ and $x^T A_1 x \le 0$.
Output: Valid cut $\|B_s^T x\| \le b_s^T x$.

Calculate $A_0 = B_0 B_0^T - b_0 b_0^T$ and a spectral decomposition $Q_0 \operatorname{Diag}(\lambda_0) Q_0^T$. Let Z_0 be the submatrix of Q_0 of zero eigenvectors (possibly empty).
Minimize trace($(I + R)Y$ over (10). If infeasible, then STOP. Otherwise, extract $\bar{x} \in \operatorname{int}(\mathcal{F}_0^+ \cap \mathcal{F}^1)$ from Y^*.
if Z_0 is empty **then**
  Calculate the set $\mathcal{E}$ of real eigenvalues of $A_0^{-1} A_1$.
  Set $\mathcal{T} = \{(1 - e)^{-1} : e \in \mathcal{E}\}$.
  Set $s = \min(\mathcal{T} \cap (0, 1])$.
else if $Z_0^T A_1 Z_0 \succ 0$ **then**
  Determine $\epsilon > 0$ small such that $A_\epsilon = (1 - \epsilon)A_0 + \epsilon A_1$ is invertible.
  Calculate the set $\overline{\mathcal{E}}$ of real eigenvalues of $A_\epsilon^{-1} A_1$.
  Set $\overline{\mathcal{T}} = \{(1 - \bar{e})^{-1} : \bar{e} \in \overline{\mathcal{E}}\}$.
  Set $\mathcal{T} = \{(1 - \epsilon)\bar{t} + \epsilon : \bar{t} \in \overline{\mathcal{T}}\}$.
  Set $s = \min(\mathcal{T} \cap (0, 1])$.
else if $Z_0^T A_1 Z_0 \prec 0$ **then**
  Set $s = 0$.
else
  STOP.
end if
Calculate $A_s = B_s B_s^T - b_s b_s^T$ and a spectral decomposition $Q_s \operatorname{Diag}(\lambda_s) Q_s^T$. Let (B_s, b_s) be given by (5).
If $b_s^T \bar{x} < 0$, replace b_s by $-b_s$.

The negative eigenvalue of A_s is $\lambda_{s1} := -\frac{5}{8}$ with corresponding eigenvector $q_{s1} := (2; 1; 0; 5)$, and so, in accordance with the Section 2, we have that $\mathcal{F}_s^+$ equals all $x \in \mathcal{F}_s$ satisfying $b_s^T x \ge 0$, where

$$b_s := (-\lambda_{s1})^{1/2} q_{s1} = \sqrt{5/8} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 5 \end{pmatrix}.$$

In other words,

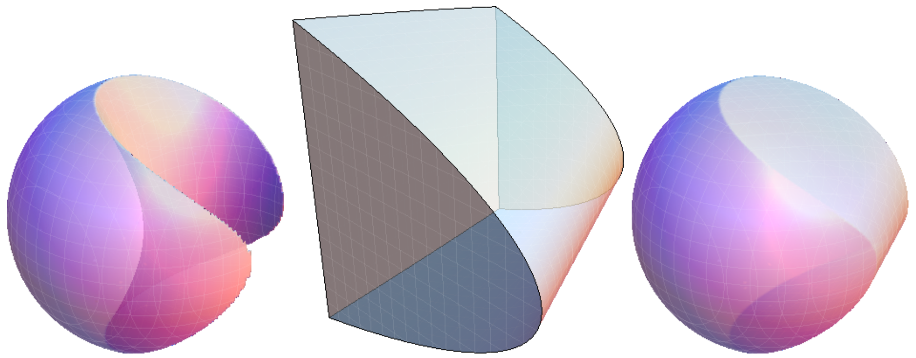
$$\mathcal{F}_s^+ := \left\{ x : \begin{array}{l} 3x_3^2 \le 2x_1 x_4 + x_2 x_4 + 2x_4^2 \\ 2x_1 + x_2 + 5x_4 \ge 0 \end{array} \right\}.$$

Note that $\bar{x} \in \mathcal{F}_s^+$. In addition, $\operatorname{apex}(\mathcal{F}_s^+) = \operatorname{null}(A_s) = \operatorname{span}\{d\}$, where $d = (1; -2; 0; 0)$. Clearly, $d \in H^0$ and $d^T A_1 d < 0$, which verifies Conditions 4 and 5 simultaneously. Setting $x_4 = 1$ and returning to the original variables y, we see

$$\left\{ y : \begin{array}{l} y_1^2 + y_2^2 + y_3^2 \le 1 \\ 3y_3^2 \le 2y_1 + y_2 + 2 \end{array} \right\} = \operatorname{cl.conv.hull} \left\{ y : \begin{array}{l} y_1^2 + y_2^2 + y_3^2 \le 1 \\ -y_1^2 - y_2^2 + \frac{1}{2} y_3^2 \le y_1 + \frac{1}{2} y_2 \end{array} \right\},$$

where the now redundant constraint $2y_1 + y_2 \ge -5$ has been dropped. Figure 1 depicts the original set, $\mathcal{F}_s^+ \cap H^1$, and the closed convex hull.

Of the earlier, related approaches, this example can be handled by [40] only. In particular, [2, 13, 23, 35, 36, 55] cannot handle this example because they deal with only



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ (b) $\mathcal{F}_s^+ \cap H^1$ (c) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$

Fig. 1 An ellipsoid and a nonconvex quadratic

split or two-term disjunctions but cannot cover general nonconvex quadratics. The approach of [14] is based on eliminating a convex region from a convex epigraphical set, but this example removes a nonconvex region (specifically, $\mathbb{R}^n \setminus \mathcal{F}_1$). So [14] cannot handle this example either.

In actuality, the results of [40] do not handle this example explicitly since the authors only state results for: the removal of a paraboloid or an ellipsoid from a paraboloid; or the removal of an ellipsoid (or an ellipsoidal cylinder) from another ellipsoid with a common center. However, in this particular example, the function obtained from the aggregation technique described in [40] is convex on all of $\mathbb{R}^3$. Therefore, their global convexity requirement on the aggregated function is satisfied for this example.

4 The Proof

In this section, we build the proof of Theorem 1, and we provide important insights along the way. The key results are Propositions 5–7, which state

$$\mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \subseteq \text{conic. hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$$
$$\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1 \subseteq \text{conv. hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1),$$

where s is given by (7). In each line here, the first containment depends only on Conditions 1–3, which proves the first part of Theorem 1. On the other hand, the second containments require Condition 4 and Conditions 4–5, respectively. Then the second part of Theorem 1 follows by simply taking the closed conic hull and the closed convex hull, respectively, and noting that $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ and $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$ are already closed and convex.

4.1 The interval $[0, s]$

Our next result, Lemma 1, is quite technical but critically important. For example, it establishes that the line of matrices $\{A_t\}$ contains at least one invertible matrix not equal to A_1. As discussed in Section 3, this proves that the set $\mathcal{T}$ used in the definition

(7) of s is finite and easily computable. The lemma also provides additional insight into the definition of s. Specifically, the lemma clarifies the role of Condition 3 in (7).

Lemma 1 *For $\epsilon > 0$ small, consider A_ϵ and $A_{-\epsilon}$. Relative to Condition 3:*

- *if (i) holds, then A_ϵ and $A_{-\epsilon}$ are each invertible with one negative eigenvalue;*
- *if (ii) holds, then only A_ϵ is invertible with one negative eigenvalue;*
- *if (iii) holds, then only $A_{-\epsilon}$ is invertible with one negative eigenvalue.*

Since the proof of Lemma 1 is involved, we delay it until the end of this subsection.

If Condition 3(i) or 3(ii) holds, then Lemma 1 shows that the interval $(0, \epsilon)$ contains invertible A_t, each with exactly one negative eigenvalue, and (7) takes s to be the largest ϵ with this property. By continuity, A_s is singular (when $s < 1$) but still retains exactly one negative eigenvalue, a necessary condition for defining $\mathcal{F}_s^+$ in Theorem 1. On the other hand, if Condition 3(iii) holds, then A_0 is singular and no $\epsilon > 0$ has the property just mentioned. Yet, $s = 0$ is still the natural "right-hand limit" of invertible $A_{-\epsilon}$, each with exactly one negative eigenvalue. This will be all that is required for Theorem 1.

With Lemma 1 in hand, we can prove the following key result, which sets up the remainder of this section. The proof of Lemma 1 follows afterwards.

Proposition 4 *Suppose Conditions 1–3 hold. For all $t \in [0, s]$, A_t has exactly one negative eigenvalue. In addition, A_t is nonsingular for all $t \in (0, s)$, and if $s < 1$, then A_s is singular.*

Proof Condition 2 implies (6), and so $\bar{x}^T A_t \bar{x} = (1-t)\, \bar{x}^T A_0 \bar{x} + t\, \bar{x}^T A_1 \bar{x} < 0$ for every t. So each A_t has at least one negative eigenvalue. Also, the definition of s ensures that all A_t for $t \in (0, s)$ are nonsingular and that A_s is singular when $s < 1$.

Suppose that some A_t with $t \in [0, s]$ has two negative eigenvalues. Then by Condition 1 and the facts that the entries of A_t are affine functions of t and the eigenvalues depend continuously on the matrix entries [28, Section 2.4.9], there exists some $0 \le r < t \le s$ with at least one zero eigenvalue, i.e., with A_r singular. From the definition of s, we deduce that $r = 0$ and A_ϵ has two negative eigenvalues for $\epsilon > 0$ small. Then Condition 3(ii) holds since $s > 0$. However, we then encounter a contradiction with Lemma 1, which states that A_ϵ has exactly one negative eigenvalue. $\square$

Proof (of Lemma 1) The lemma holds under Condition 3(i) since A_0 is invertible with exactly one negative eigenvalue and the eigenvalues are continuous in ϵ.

Suppose Condition 3(ii) holds. Let V be the subspace spanned by the zero and positive eigenvectors of A_0, and consider

$$\theta := \inf\{x^T A_0 x : x^T (A_0 - A_1) x = 1, x \in V\}.$$

Clearly $\theta \ge 0$, and we claim $\theta > 0$. If $\theta = 0$, then there exists $\{x^k\} \subseteq V$ with $(x^k)^T A_0 x^k \to 0$ and $(x^k)^T (A_0 - A_1) x^k = 1$ for all k. If $\{x^k\}$ is bounded, then passing to a subsequence if necessary, we have $x^k \to \hat{x}$ such that $\hat{x}^T A_0 \hat{x} = 0$ and $\hat{x}^T (A_0 - A_1)\hat{x} = 1$, which implies $\hat{x}^T A_1 \hat{x} = -1$, a contradiction of Condition 3(ii). On the other hand, if $\{x^k\}$ is unbounded, then the sequence $d^k := x^k / \|x^k\|$ is bounded, and passing to a subsequence if necessary, we see that $d^k \to \hat{d}$ with $\|\hat{d}\| = 1$, $\hat{d}^T A_0 \hat{d} = 0$ and $\hat{d}^T (A_0 - A_1)\hat{d} = 0$. This implies $\hat{d}^T A_1 \hat{d} = 0$, violating Condition 3(ii). So $\theta > 0$.

Now choose any $0 < \epsilon \le \theta/2$, and take any nonzero $x \in V$. Note that

$$x^T A_\epsilon x = (1 - \epsilon) x^T A_0 x + \epsilon x^T A_1 x = x^T A_0 x - \epsilon x^T (A_0 - A_1) x. \tag{11}$$

We wish to show $x^T A_\epsilon x > 0$, and so we consider three subcases. First, if $x^T(A_0 - A_1)x = 0$, then it must hold that $x^T A_0 x > 0$. If not, then $x^T A_1 x = 0$ also, violating Condition 3(ii). So $x^T A_\epsilon x = x^T A_0 x > 0$. Second, if $x^T(A_0 - A_1)x < 0$, then because $x \in V$ we have $x^T A_\epsilon x > 0$. Third, if $x^T(A_0 - A_1)x > 0$, then we may assume without loss of generality by scaling that $x^T(A_0 - A_1)x = 1$ in which case $x^T A_\epsilon x \geq \theta - \epsilon > 0$.

So we have shown that A_ϵ is positive definite on a subspace of dimension $n-1$, which implies that A_ϵ has at least $n-1$ positive eigenvalues. In addition, we know that A_ϵ has at least one negative eigenvalue because $\bar{x}^T A_\epsilon \bar{x} < 0$ according to Condition 2 and (6). Hence, A_ϵ is invertible with exactly one negative eigenvalue, as claimed.

By repeating a very similar argument for vectors $x \in W$, the subspace spanned by the negative and zero eigenvectors of A_0 (note that W is at least two-dimensional because Condition 3(ii) holds), and once again using the relation (11), we can show that $A_{-\epsilon}$ has at least two negative eigenvalues, as claimed.

Finally, suppose Condition 3(iii) holds and define

$$\begin{aligned}
\bar{A}_\epsilon &:= \left(\tfrac{1}{1+2\epsilon}\right) A_{-\epsilon} = \left(\tfrac{1}{1+2\epsilon}\right)((1+\epsilon)A_0 - \epsilon A_1) = \left(\tfrac{1+\epsilon}{1+2\epsilon}\right) A_0 + \left(\tfrac{\epsilon}{1+2\epsilon}\right)(-A_1) \\
\bar{A}_{-\epsilon} &:= \left(\tfrac{1}{1-2\epsilon}\right) A_\epsilon = \left(\tfrac{1}{1-2\epsilon}\right)((1-\epsilon)A_0 + \epsilon A_1) = \left(\tfrac{1-\epsilon}{1-2\epsilon}\right) A_0 + \left(\tfrac{-\epsilon}{1-2\epsilon}\right)(-A_1).
\end{aligned}$$

Then $\bar{A}_\epsilon$ and $\bar{A}_{-\epsilon}$ are on the line generated by A_0 and $-A_1$ such that $-A_1$ is positive definite on the null space of A_0. Applying the previous case for Condition 3(ii), we see that only $\bar{A}_\epsilon$ is invertible with a single negative eigenvalue. This proves the result. ☐

4.2 The containment $\mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$

For each $t \in [0, s]$, Proposition 4 allows us to define analogs $\mathcal{F}_t = \mathcal{F}_t^+ \cup \mathcal{F}_t^-$ as described in Section 2 based on any spectral decomposition $A_t = Q_t \operatorname{Diag}(\lambda_t) Q_t^T$.

It is an important technical point, however, that in this paper we require λ_t and Q_t to be defined continuously in t. While it is well known that the vector of eigenvalues λ_t can be defined continuously, it is also known that—if the eigenvalues are ordered, say, such that $[\lambda_t]_1 \leq \cdots \leq [\lambda_t]_n$ for all t—then the corresponding eigenvectors, i.e., the ordered columns of Q_t, cannot be defined continuously in general. On the other hand, if one drops the requirement that the eigenvalues in λ_t stay ordered, then the following result of Rellich [46] (see also [32]) guarantees that λ_t and Q_t can be constructed continuously—in fact, analytically—in t:

Theorem 2 (Rellich [46]) *Because A_t is analytic in the single parameter t, there exist spectral decompositions $A_t = Q_t \operatorname{Diag}(\lambda_t) Q_t^T$ such that λ_t and Q_t are analytic in t.*

So we define $\mathcal{F}_t^+$ and $\mathcal{F}_t^-$ using continuous spectral decompositions provided by Theorem 2:

$$\begin{aligned}
\mathcal{F}_t^+ &:= \{x : \|B_t^T x\| \leq b_t^T x\} \\
\mathcal{F}_t^- &:= \{x : \|B_t^T x\| \leq -b_t^T x\},
\end{aligned}$$

where B_t and b_t such that $A_t = B_t B_t^T - b_t b_t^T$ are derived from the spectral decomposition as described in Section 2. Recall from Proposition 3 that, for each t, a different spectral decomposition could flip the roles of $\mathcal{F}_t^+$ and $\mathcal{F}_t^-$, but we now observe that

Theorem 2 and Condition 2 together guarantee that each $\mathcal{F}_t^+$ contains $\bar{x}$ from Condition 2. In this sense, every $\mathcal{F}_t^+$ has the same "orientation." Our observation is enabled by a lemma that will be independently helpful in subsequent analysis.

Lemma 2 *Suppose Conditions 1–3 hold. Given* $t \in [0, s]$*, suppose some* $x \in \mathcal{F}_t^+$ *satisfies* $b_t^T x = 0$*. Then* $t = 0$ *or* $t = s$*.*

Proof Since $x^T A_t x \le 0$ with $b_t^T x = 0$, we have $0 = (b_t^T x)^2 \ge \|B_t^T x\|^2$ which implies $A_t x = (B_t B_t^T - b_t b_t^T)x = B_t(B_t^T x) - b_t(b_t^T x) = 0$. So A_t is singular. By Proposition 4, this implies $t = 0$ or $t = s$. □

Observation 1 *Suppose Conditions 1–3 hold. Let* $\bar{x} \in \operatorname{int}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$*. Then for all* $t \in [0, s]$*,* $\bar{x} \in \mathcal{F}_t^+$*.*

Proof Condition 2 implies $b_0^T \bar{x} > 0$. Let $t \in (0, s]$ be fixed. Since $\bar{x}^T A_t \bar{x} < 0$ by (6), either $\bar{x} \in \mathcal{F}_t^+$ or $\bar{x} \in \mathcal{F}_t^-$. Suppose for contradiction that $\bar{x} \in \mathcal{F}_t^-$, i.e., $b_t^T \bar{x} < 0$. Then the continuity of b_t by Theorem 2 implies the existence of $r \in (0, t)$ such that $b_r^T \bar{x} = 0$. Because $\bar{x}^T A_r \bar{x} < 0$ as well, $\bar{x} \in \mathcal{F}_r^+$. By Lemma 2, this implies $r = 0$ or $r = s$, a contradiction. □

In particular, Observation 1 implies that our discussion in Section 3 on choosing $\bar{x} \in \mathcal{F}_t^+$ to facilitate the statement of Theorem 1 is indeed consistent with the discussion here.

The primary result of this subsection, $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ is a valid convex relaxation of $\mathcal{F}_0^+ \cap \mathcal{F}_1$, is given below.

Proposition 5 *Suppose Conditions 1–3 hold. Then* $\mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$*.*

Proof If $s = 0$, the result is trivial. So assume $s > 0$. In particular, Condition 3(i) or 3(ii) holds. Let $x \in \mathcal{F}_0^+ \cap \mathcal{F}_1$, that is, $x^T A_0 x \le 0$, $b_0^T x \ge 0$, and $x^T A_1 x \le 0$. We would like to show $x \in \mathcal{F}_0^+ \cap \mathcal{F}_s^+$. So we need $x^T A_s x \le 0$ and $b_s^T x \ge 0$. The first inequality holds because $x^T A_s x = (1 - s)\, x^T A_0 x + s\, x^T A_1 x \le 0$. Now suppose for contradiction that $b_s^T x < 0$. In particular, $x \ne 0$. Then by the continuity of b_t via Theorem 2, there exists $0 \le r < s$ such that $b_r^T x = 0$. Since $x^T A_r x \le 0$ also, $x \in \mathcal{F}_r^+$, and Lemma 2 implies $r = 0$. So Condition 3(ii) holds. However, $x \in \mathcal{F}_1$ also, contradicting that A_1 is positive definite on $\operatorname{null}(A_0)$. □

4.3 The containment $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \subseteq \operatorname{conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$

Proposition 5 in the preceding subsection establishes that $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ is a valid convex relaxation of $\mathcal{F}_0^+ \cap \mathcal{F}_1$ under Conditions 1–3. We now show that, in essence, the reverse inclusion holds under Condition 4 (see Proposition 6). Indeed, when $s = 1$, we clearly have $\mathcal{F}_0^+ \cap \mathcal{F}_1^+ \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \operatorname{conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$. So the true case of interest is $s < 1$, for which Condition 4 is the key ingredient. (However, results are stated to cover the cases $s < 1$ and $s = 1$ simultaneously.)

As mentioned in Section 3, Condition 4 is a type of regularity condition in light of Lemma 3 next. The proof of Proposition 6 also relies on Lemma 3.

Lemma 3 *Suppose Conditions 1–3 hold. Then* $\operatorname{apex}(\mathcal{F}_s^+) \subseteq \mathcal{F}_1$*.*

Proof By Proposition 1, the claimed result is equivalent to $\mathrm{null}(A_s) \subseteq \mathcal{F}_1$. Let $d \in \mathrm{null}(A_s)$. If $s = 1$, then $d^T A_1 d = 0$, i.e., $d \in \mathrm{bd}(\mathcal{F}_1) \subseteq \mathcal{F}_1$, as desired. If $s = 0$, then Condition 3(iii) holds, that is, A_0 is singular and A_1 is negative definite on $\mathrm{null}(A_0)$. Then $d \in \mathrm{null}(A_0)$ implies $d^T A_1 d \leq 0$, as desired.

So assume $s \in (0, 1)$. If $d \notin \mathrm{int}(\mathcal{F}_0)$, that is, $d^T A_0 d \geq 0$, then the equation $0 = (1 - s)\, d^T A_0 d + s\, d^T A_1 d$ implies $d^T A_1 d \leq 0$, as desired.

We have thus reduced to the case $s \in (0, 1)$ and $d \in \mathrm{int}(\mathcal{F}_0)$, and we proceed to derive a contradiction. Without loss of generality, assume that $d \in \mathrm{int}(\mathcal{F}_0^+)$ and $-d \in \mathrm{int}(\mathcal{F}_0^-)$. We know $-d \in \mathrm{null}(A_s) = \mathrm{apex}(\mathcal{F}_s^+) \subseteq \mathcal{F}_s^+$. In total, we have $-d \in \mathcal{F}_s^+ \cap \mathrm{int}(\mathcal{F}_0^-)$. We claim that, in fact, $\mathcal{F}_t^+ \cap \mathrm{int}(\mathcal{F}_0^-) \neq \emptyset$ as $t \to s$.

Note that $\mathcal{F}_t^+$ is a full-dimensional set because $\bar{x}^T A_t \bar{x} < 0$ by (6). Also, $\mathcal{F}_t^+$ is defined by the intersection of a homogeneous quadratic $x^T A_t x \leq 0$ and a linear constraint $b_t^T x \geq 0$ and $(A_t, b_t) \to (A_s, b_s)$ as $t \to s$. Then the boundary of $\mathcal{F}_t^+$ converges to the boundary of $\mathcal{F}_s^+$ as $t \to s$. Since $\mathcal{F}_t^+$ is a full-dimensional, convex set (in fact SOC), $\mathcal{F}_t^+$ then converges as a set to $\mathcal{F}_s^+$ as $t \to s$. So there exists a sequence $y_t \in \mathcal{F}_t^+$ converging to $-d$. In particular, $\mathcal{F}_t^+ \cap \mathrm{int}(\mathcal{F}_0^-) \neq \emptyset$ for $t \to s$.

We can now achieve the desired contradiction. For $t < s$, let $x \in \mathcal{F}_t^+ \cap \mathrm{int}(\mathcal{F}_0^-)$. Then $x^T A_0 x \leq 0,\ b_0^T x < 0$ and $x^T A_t x \leq 0,\ b_t^T x \geq 0$. It follows that $x^T A_r x \leq 0,\ b_r^T x = 0$ for some $0 < r \leq t < s$. Hence, Lemma 2 implies $r = 0$ or $r = s$, a contradiction. $\square$

Proposition 6 *Suppose Conditions 1–4 hold. Then* $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \subseteq \mathrm{conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$.

Proof First, suppose $s = 1$. Then the result follows because $\mathcal{F}_0^+ \cap \mathcal{F}_1^+ \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \mathrm{conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$. So assume $s \in [0, 1)$.

Let $x \in \mathcal{F}_0^+ \cap \mathcal{F}_s^+$, that is, $x^T A_0 x \leq 0$, $b_0^T x \geq 0$ and $x^T A_s x \leq 0$, $b_s^T x \geq 0$. If $x^T A_1 x \leq 0$, we are done. So assume $x^T A_1 x > 0$.

By Condition 4, there exists $d \in \mathrm{null}(A_s)$ such that $d^T A_1 d < 0$. In addition, d is necessarily perpendicular to the negative eigenvector b_s. For all $\epsilon \in \mathbb{R}$, consider the affine line of points given by $x_\varepsilon := x + \epsilon\, d$. We have

$$\left.\begin{aligned} x_\varepsilon^T A_s x_\varepsilon = (x + \epsilon\, d)^T A_s (x + \epsilon\, d) = x^T A_s x \leq 0 \\ b_s^T x_\varepsilon = b_s^T (x + \epsilon\, d) = b_s^T x \geq 0 \end{aligned}\right\} \quad \Longrightarrow \quad x_\epsilon \in \mathcal{F}_s^+.$$

Note that $x_\varepsilon^T A_1 x_\varepsilon = x^T A_1 x + 2\,\epsilon\, d^T A_1 x + \epsilon^2\, d^T A_1 d$. Then $x_\varepsilon^T A_1 x_\varepsilon$ defines a quadratic function of ε and its roots are given by $\varepsilon_\pm = \frac{-d^T A_1 x \pm \sqrt{(d^T A_1 x)^2 - (x^T A_1 x)(d^T A_1 d)}}{d^T A_1 d}$. Since $x^T A_1 x > 0$ and $d^T A_1 d < 0$, the discriminant is greater than $|d^T A_1 x|$. Hence, one of the roots will be positive and the other one will be negative. Then there exist $l := \varepsilon_- < 0 < \varepsilon_+ =: u$ such that $x_l^T A_1 x_l^T = x_u^T A_1 x_u = 0$, i.e., $x_l, x_u \in \mathcal{F}_1$. Then $s < 1$ and $x_l^T A_s x_l \leq 0$ imply $x_l^T A_0 x_l \leq 0$, and hence $x_l \in \mathcal{F}_0$. Similarly, $x_u^T A_0 x_u \leq 0$ leading to $x_u \in \mathcal{F}_0$. We will prove in the next paragraph that both x_l and x_u are in $\mathcal{F}_0^+$, which will establish the result because then $x_l, x_u \in \mathcal{F}_0^+ \cap \mathcal{F}^1$ and x is a convex combination of x_l and x_u.

Suppose that at least one of the two points x_l or x_u is not a member of $\mathcal{F}_0^+$. Without loss of generality, say $x_l \notin \mathcal{F}_0^+$. Then $x_l \in \mathcal{F}_0^-$ with $-b_0^T x_l > 0$. Similar to Proposition 5, we can prove $\mathcal{F}_0^- \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^- \cap \mathcal{F}_s^-$, and so $x_l \in \mathcal{F}_0^- \cap \mathcal{F}_s^-$. Then $x_l \in \mathcal{F}_s^+ \cap \mathcal{F}_s^-$, which implies $b_s^T x_l = 0$ and $B_s^T x_l = 0$, which in turn implies $A_s x_l = 0$, i.e., $x_l \in \mathrm{null}(A_s)$. Then $x + l\, d = x_l \in \mathrm{null}(A_s)$ implies $x \in \mathrm{null}(A_s)$ also. Then $x \in \mathcal{F}_1$ by Lemma 3, but this contradicts the earlier assumption that $x^T A_1 x > 0$. $\square$

4.4 Intersection with an affine hyperplane

As discussed at the beginning of this section, Propositions 5-6 allow us to prove the first two statements of Theorem 1. In this subsection, we prove the last statement of the theorem via Proposition 7 below. Recall that H^1 and H^0 are defined according to (8) and (9), where $h \in \mathbb{R}^n$. Also define

$$H^+ := \{x : h^T x \geq 0\}.$$

Our first task is to prove the analog of Propositions 5–6 under intersection with H^+. Specifically, we wish to show that the inclusions

$$\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+ \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^+ \subseteq \operatorname{conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+) \tag{12}$$

hold under Conditions 1–5. As Condition 5 consists of two parts, we break the proof into two corresponding parts (Lemma 4 and Corollary 2). Note that Condition 5 only applies when $s < 1$, although results are stated covering both $s < 1$ and $s = 1$ simultaneously.

Lemma 4 *Suppose Conditions 1–4 and the first part of Condition 5 hold. Then (12) holds.*

Proof Proposition 5 implies that $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+ \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^+$. Moreover, we can repeat the proof of Proposition 6, intersecting with H^+ along the way. However, we require one key modification in the proof of Proposition 6.

Let $x \in \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^+$ with $x^T A_1 x > 0$. Then, mimicking the proof of Proposition 6 for $s \in [0, 1)$ and $d \in \operatorname{apex}(\mathcal{F}_s^+) \cap \operatorname{int}(\mathcal{F}_1)$ from Condition 4, $x \in \{x_\epsilon := x + \epsilon\, d : \epsilon \in \mathbb{R}\} \subseteq \mathcal{F}_s^+$. Moreover, x is a strict convex combination of points $x_l, x_u \in \mathcal{F}_0^+ \cap \mathcal{F}_1$ where x_l, x_u are as defined in the proof of Proposition 6. Hence, the entire closed interval from x_l to x_u is contained in $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$.

Under the first part of Condition 5, if there exists $d \in \operatorname{apex}(\mathcal{F}_s^+) \cap \operatorname{int}(\mathcal{F}_1) \cap H^0$, then $h^T d = 0$ and this particular d can be used to show that x_l, x_u identified in the proof of Proposition 6 also satisfy $h^T x_l = h^T(x + l\, d) = h^T x \geq 0$ (recall that $x \in H^+$) and $h^T x_u = h^T(x + u\, d) = h^T x \geq 0$, i.e., $x_l, x_u \in \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+$. Then this implies $x \in \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+$, as desired. $\square$

Regarding the second part of Condition 5, we prove Corollary 2 using the following more general lemma involving cones that are not necessarily SOCr:

Lemma 5 *Let $\mathcal{G}_0$, $\mathcal{G}_1$, and $\mathcal{G}_s$ be cones such that $\mathcal{G}_0, \mathcal{G}_s$ are convex, $\mathcal{G}_0 \cap \mathcal{G}_1 \subseteq \mathcal{G}_0 \cap \mathcal{G}_s \subseteq \operatorname{conic.hull}(\mathcal{G}_0 \cap \mathcal{G}_1)$ and $\mathcal{G}_0 \cap \mathcal{G}_s \cap H^0 \subseteq \mathcal{G}_1$. Then*

$$\mathcal{G}_0 \cap \mathcal{G}_1 \cap H^+ \subseteq \mathcal{G}_0 \cap \mathcal{G}_s \cap H^+ \subseteq \operatorname{conic.hull}(\mathcal{G}_0 \cap \mathcal{G}_1 \cap H^+).$$

Proof For notational convenience, define $\mathcal{G}_{01} := \mathcal{G}_0 \cap \mathcal{G}_1$ and $\mathcal{G}_{0s} := \mathcal{G}_0 \cap \mathcal{G}_s$. We clearly have $\mathcal{G}_{01} \cap H^+ \subseteq \mathcal{G}_{0s} \cap H^+ \subseteq \operatorname{conic.hull}(\mathcal{G}_{01}) \cap H^+$. We will show $\mathcal{G}_{0s} \cap H^+ \subseteq \operatorname{conic.hull}(\mathcal{G}_{01} \cap H^+)$. Consider $x \in \mathcal{G}_{0s} \cap H^+$. Either $h^T x = 0$ or $h^T x > 0$.

If $h^T x = 0$, then $x \in \mathcal{G}_{0s} \cap H^0 \subseteq \mathcal{G}_1$ by the premise of the lemma. Thus $x \in \mathcal{G}_{0s} \cap H^+ \cap \mathcal{G}_1 \subseteq \operatorname{conic.hull}(\mathcal{G}_{01} \cap H^+)$, as desired.

When $h^T x > 0$, because $\mathcal{G}_{0s} \subseteq \operatorname{conic.hull}(\mathcal{G}_{01})$, we know that x can be expressed as a finite sum $x = \sum_k \lambda_k x^k$, where each $x^k \in \mathcal{G}_{01} \subseteq \mathcal{G}_{0s}$ and $\lambda_i > 0$. Define $I := \{k : h^T x^k \geq 0\}$ and $J := \{k : h^T x^k < 0\}$. If $J = \emptyset$, then we are done as we have shown

$x \in \operatorname{conic.hull}(\mathcal{G}_{01} \cap H^+)$. If not, then for all $j \in J$, let y^j be a strict conic combination of x and x^j such that $y^j \in H^0$. In particular, there exists $\alpha_j \geq 0$ and $\beta_j > 0$ such that $y^j = \alpha_j x + \beta_j x^j$. Note also that $y^j \in \mathcal{G}_{0s}$ because $\mathcal{G}_{0s}$ is convex and $x, x^j \in \mathcal{G}_{0s}$. Then $y^j \in \mathcal{G}_{0s} \cap H^0 \subseteq \mathcal{G}_1$. As a result, for all $j \in J$, we have $y^j \in \mathcal{G}_{01} \cap H^+$. Rewriting x as

$$x = \sum_{i \in I} \lambda_i x^i + \sum_{j \in J} \frac{\lambda_j}{\beta_j} \left(y^j - \alpha_j x \right) \quad \Longleftrightarrow \quad \left(1 + \sum_{j \in J} \frac{\lambda_j \alpha_j}{\beta_j} \right) x = \sum_{i \in I} \lambda_i x^i + \sum_{j \in J} \frac{\lambda_j}{\beta_j} y^j,$$

we conclude that x is a conic combination of points in $\mathcal{G}_{01} \cap H^+$, as desired. □

Corollary 2 *Suppose Conditions 1–4 and the second part of Condition 5 hold. Then (12) holds.*

Proof Apply Lemma 5 with $\mathcal{G}_0 := \mathcal{F}_0^+$, $\mathcal{G}_1 := \mathcal{F}_1$, and $\mathcal{G}_s := \mathcal{F}_s^+$. Propositions 5–6 and the second part of Condition 5 ensure that the hypotheses of Lemma 5 are met. Then the result follows. □

Even though our goal in this subsection is Proposition 7, which involves intersection with the hyperplane H^1, we remark that Lemmas 4–5 can help us investigate intersections with homogeneous halfspaces H^+ for SOCr cones (Lemma 4) or more general cones (Lemma 5). Further, by iteratively applying Lemmas 4–5, we can consider intersections with multiple halfspaces, say, $H_1^+, \ldots, H_m^+$.

Given Lemma 4 and Corollary 2, we are now ready to prove our main result for this subsection, Proposition 7, which establishes the second part of Theorem 1. It requires the following simple lemmas which are applicable to general sets and cones:

Lemma 6 *Let S be any set, and let* $\operatorname{rec.cone}(S)$ *be its recession cone. Then* $\operatorname{conv.hull}(S) + \operatorname{conic.hull}(\operatorname{rec.cone}(S)) = \operatorname{conv.hull}(S)$.

Proof The containment $\supseteq$ is clear. Now let $x + y$ be in the left-hand side such that

$$\begin{aligned} x &= \sum_k \lambda_k x_k, \quad x_k \in S, \quad \lambda_k > 0, \quad \sum_k \lambda_k = 1, \\ \text{and} \quad y &= \sum_j \rho_j y_j, \quad y_j \in \operatorname{rec.cone}(S), \quad \rho_j > 0. \end{aligned}$$

Without loss of generality, we may assume the number of x_k's equals the number of y_j's by splitting some $\lambda_k x_k$ or some $\rho_j y_j$ as necessary. Then

$$x + y = \sum_k (\lambda_k x_k + \rho_k y_k) = \sum_k \lambda_k (x_k + \lambda_k^{-1} \rho_k y_k) \in \operatorname{conv.hull}(S). \quad \square$$

Lemma 7 *Let $\mathcal{G}_{01}$ and $\mathcal{G}_{0s}$ be cones (not necessarily convex) such that $\mathcal{G}_{01} \cap H^+ \subseteq \mathcal{G}_{0s} \cap H^+ \subseteq$* $\operatorname{conic.hull}(\mathcal{G}_{01} \cap H^+)$*. Then $\mathcal{G}_{01} \cap H^1 \subseteq \mathcal{G}_{0s} \cap H^1 \subseteq$* $\operatorname{conv.hull}(\mathcal{G}_{01} \cap H^1)$.

Proof We have $\mathcal{G}_{01} \cap H^1 \subseteq \mathcal{G}_{0s} \cap H^1 \subseteq \operatorname{conic.hull}(\mathcal{G}_{01} \cap H^+) \cap H^1$. We claim further that

$$\operatorname{conic.hull}(\mathcal{G}_{01} \cap H^+) \cap H^1 \subseteq \operatorname{conic.hull}(\mathcal{G}_{01} \cap H^0) + \operatorname{conv.hull}(\mathcal{G}_{01} \cap H^1). \tag{13}$$

Then applying Lemma 6 with $S := \mathcal{G}_{01} \cap H^1$ and $\operatorname{rec.cone}(S) = \mathcal{G}_{01} \cap H^0$, we see that $\operatorname{conic.hull}(\mathcal{G}_{01} \cap H^+) \cap H^1 \subseteq \operatorname{conv.hull}(\mathcal{G}_{01} \cap H^1)$, which proves the lemma.

To prove the claim (13), let $x \in \text{conic.hull}(\mathcal{G}_{01} \cap H^+) \cap H^1$. Then

$$h^T x = 1 \quad \text{and} \quad x = \sum_k \lambda_k x_k, \quad x_k \in \mathcal{G}_{01} \cap H^+, \quad \lambda_k > 0,$$

which may further be separated as

$$x = \underbrace{\sum_{k\,:\,h^T x_k > 0} \lambda_k x_k}_{:=y} + \underbrace{\sum_{k\,:\,h^T x_k = 0} \lambda_k x_k}_{:=r} = y + r.$$

Note that $r \in \text{conic.hull}(\mathcal{G}_{01} \cap H^0)$, and so it sufficies to show $y \in \text{conv.hull}(\mathcal{G}_{01} \cap H^1)$. Rewrite y as

$$y = \sum_{k\,:\,h^T x_k > 0} \lambda_k x_k = \sum_{k\,:\,h^T x_k > 0} \underbrace{(\lambda_k \cdot h^T x_k)}_{:=\tilde{\lambda}_k} \underbrace{(x_k / h^T x_k)}_{:=\tilde{x}_k} =: \sum_{k\,:\,h^T x_k > 0} \tilde{\lambda}_k \tilde{x}_k.$$

By construction, each $\tilde{x}_k \in \mathcal{G}_{01} \cap H^1$. Moreover, each $\tilde{\lambda}_k$ is positive and

$$\sum_{k\,:\,h^T x_k > 0} \tilde{\lambda}_k = \sum_{k\,:\,h^T x_k > 0} \lambda_k \cdot h^T x_k = h^T y = h^T (x - r) = 1 - 0 = 1,$$

since $x \in H^1$. So $y \in \text{conv.hull}(\mathcal{G}_{01} \cap H^1)$. $\square$

Proposition 7 *Suppose Conditions 1–5 hold. Then* $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1 \subseteq \text{conv.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$.

Proof Define $\mathcal{G}_{01} := \mathcal{F}_0^+ \cap \mathcal{F}_1$ and $\mathcal{G}_{0s} := \mathcal{F}_0^+ \cap \mathcal{F}_s^+$. Lemma 4 and Corollary 2 imply $\mathcal{G}_{01} \cap H^+ \subset \mathcal{G}_{0s} \cap H^+ \subseteq \text{conic.hull}(\mathcal{G}_{01}) \cap H^+$. Then Lemma 7 implies the result. $\square$

As with Lemma 5, we have stated Lemma 7 in terms of general cones, extending beyond just SOCr cones. In particular, in future research, these results may allow the derivation of conic and convex hulls for the intersects with more general cones.

5 Two-term disjunctions on the second-order cone

In this section (specifically Sections 5.1–5.4), we consider the intersection of the canonical second-order cone

$$\mathcal{K} := \{x : \|\tilde{x}\| \le x_n\}, \quad \text{where } \tilde{x} = (x_1; \ldots; x_{n-1}),$$

and a two-term linear disjunction defined by $c_1^T x \ge d_1 \vee c_2^T x \ge d_2$. Without loss of generality, we take $d_1, d_2 \in \{0, \pm 1\}$ with $d_1 \ge d_2$, and we work with the following condition:

Condition 6 *The disjunctive sets* $\mathcal{K}_1 := \mathcal{K} \cap \{x : c_1^T x \ge d_1\}$ *and* $\mathcal{K}_2 := \mathcal{K} \cap \{x : c_2^T x \ge d_2\}$ *are non-intersecting except possibly on their boundaries, e.g.,*

$$\mathcal{K}_1 \cap \mathcal{K}_2 \subseteq \left\{ x \in \mathcal{K} : \begin{array}{l} c_1^T x = d_1 \\ c_2^T x = d_2 \end{array} \right\}.$$

This condition ensures that, on $\mathcal{K}$, the disjunction $c_1^T x \geq d_1 \vee c_2^T x \geq d_2$ is equivalent to the quadratic inequality $(c_1^T x - d_1)(c_2^T x - d_2) \leq 0$. Condition 6 is satisfied, for example, when the disjunction is a proper split, i.e., $c_1 \parallel c_2$ with $c_1^T c_2 < 0$, $\mathcal{K}_1 \cup \mathcal{K}_2 \neq \mathcal{K}$, and $d_1 = d_2$. (In this case of a split disjunction, if $d_1 \neq d_2$, then it can be shown that the closed conic hull of $\mathcal{K}_1 \cup \mathcal{K}_2$ is just $\mathcal{K}$.)

Because $d_1, d_2 \in \{0, \pm 1\}$ with $d_1 \geq d_2$, we can break our analysis into the following three cases with a total of six subcases:

(a) $d_1 = d_2 = 0$, covering subcase $(d_1, d_2) = (0, 0)$;
(b) $d_1 = d_2$ nonzero, covering subcases $(d_1, d_2) \in \{(-1, -1), (1, 1)\}$;
(c) $d_1 > d_2$, covering subcases $(d_1, d_2) \in \{(0, -1), (1, -1), (1, 0)\}$.

Case (a) is the homogeneous case, in which we take $A_0 = J := \mathrm{Diag}(1, \ldots, 1, -1)$ and $A_1 = c_1 c_2^T + c_2 c_1^T$ to match our set of interest $\mathcal{K} \cap \mathcal{F}_1$. Note that $\mathcal{K} = \mathcal{F}_0^+$ in this case. For the non-homogeneous cases (b) and (c), we can homogenize via $y = \binom{x}{x_{n+1}}$ with $h^T y = x_{n+1} = 1$. Defining

$$A_0 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} c_1 c_2^T + c_2 c_1^T & -d_2 c_1 - d_1 c_2 \\ -d_2 c_1^T - d_1 c_2^T & 2 d_1 d_2 \end{pmatrix},$$

we then wish to examine $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$.

In fact, by the results in [36, Section 5.2], case (c) implies that $\mathrm{cl.\,conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ cannot in general be captured by two conic inequalities, making it unlikely that our desired equality $\mathrm{cl.\,conv.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1) = \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$ will hold in general. So we will focus on cases (a) and (b). Nevertheless, we include some comments on case (c) in Section 5.4.

Later on, in Section 5.3, we will also revisit Condition 6 to show that it is unnecessary in some sense. Precisely, even when Condition 6 does not hold, we can derive a related convex valid inequality, which, together with $\mathcal{F}_0^+$, gives the complete convex hull description. This inequality precisely matches the one already described in [36], but it does not have an SOC form.

In contrast to Sections 5.1–5.4, Section 5.5 examines two-term disjunctions on conic sections of $\mathcal{K}$, i.e., intersections of $\mathcal{K}$ with a hyperplane.

5.1 The case (a) of $d_1 = d_2 = 0$

As discussed above, we have $A_0 := J$ and $A_1 := c_1 c_2^T + c_2 c_1^T$. If either $c_i \in \mathcal{K}$, then the corresponding side of the disjunction $\mathcal{K}_i$ simply equals $\mathcal{K}$, so the conic hull is $\mathcal{K}$. In addition, if either $c_i \in \mathrm{int}(-\mathcal{K})$, then $\mathcal{K}_i = \{0\}$, so the conic hull equals the other $\mathcal{K}_j$. Hence, we assume both $c_i \notin \mathcal{K} \cup \mathrm{int}(-\mathcal{K})$, i.e., $\|\tilde{c}_i\| \geq |c_{i,n}|$, where $c_i = \binom{\tilde{c}_i}{c_{i,n}}$. Since the example in Section 4 of the Online Supplement violates Condition 4 with $\|\tilde{c}_2\| = |c_{2,n}|$, we further assume that both $\|\tilde{c}_i\| > |c_{i,n}|$.

Conditions 1 and 3(i) are easily verified. In particular, $s > 0$. Condition 2 describes the full-dimensional case of interest. It remains to verify Condition 4. (Note that Condition 4 is only relevant when $s < 1$ and that Condition 5 is not of interest in this homogeneous case.) So suppose $s < 1$, and given nonzero $z \in \mathrm{null}(A_s)$, we will show

$$z^T A_1 z = 2(c_1^T z)(c_2^T z) < 0,$$

verifying Condition 4. We already know from Lemma 3 that $z^T A_1 z \leq 0$. So it remains to show that both $c_1^T z$ and $c_2^T z$ are nonzero.

Since $z \in \text{null}(A_s)$, we know $\left(\frac{1-s}{s}\right) A_0 z = -A_1 z$, i.e.,

$$\left(\tfrac{1-s}{s}\right) \begin{pmatrix} \tilde{z} \\ -z_n \end{pmatrix} = -c_1(c_2^T z) - c_2(c_1^T z). \tag{14}$$

Note that $c_1^T z = \binom{\tilde{c}_1}{-c_{1,n}}^T \binom{\tilde{z}}{-z_n}$, so multiplying both sides of equation (14) with $\binom{\tilde{c}_1}{-c_{1,n}}^T$ and rearranging terms, we obtain

$$\left[\tfrac{1-s}{s} + \tilde{c}_1^T \tilde{c}_2 - c_{1,n} c_{2,n}\right] (c_1^T z) = \left(c_{1,n}^2 - \|\tilde{c}_1\|_2^2\right) (c_2^T z).$$

Similarly, using $\binom{\tilde{c}_2}{-c_{2,n}}^T$, we obtain:

$$\left[\tfrac{1-s}{s} + \tilde{c}_1^T \tilde{c}_2 - c_{1,n} c_{2,n}\right] (c_2^T z) = \left(c_{2,n}^2 - \|\tilde{c}_2\|_2^2\right) (c_1^T z).$$

The inequalities $\|\tilde{c}_1\| > |c_{1,n}|$ and $\|\tilde{c}_2\| > |c_{2,n}|$ thus imply $c_1^T z \neq 0 \Leftrightarrow c_2^T z \neq 0$. Moreover, $c_1^T z$ and $c_2^T z$ cannot both be 0; otherwise, z would be 0 by (14).

Note that [35,36] give an infinite family of valid inequalities in this setup but do not prove the sufficiency of a single inequality from this family. In this case, the sufficiency proof for a single inequality from this family is given recently in [55]. None of the other papers [2,23,40] are relevant here because they consider only split disjunctions, not general two-term disjunctions. Because of the boundedness assumption used in [13], [13] is not applicable here either. Similar to the example in Section 1 of the Online Supplement, as long as the disjunction can be viewed as removing a convex set, we can try to apply [14] to this case by considering the SOC as the epigraph of the norm $\|\tilde{x}\|$. However, the authors' special conditions for polynomial-time separability such as differentiability or growth rate are not satisfied; see Theorem IV therein.

5.2 The case (b) of nonzero $d_1 = d_2$

In [36], it was shown that $c_1 - c_2 \in \pm\mathcal{K}$ implies one of the sets $\mathcal{K}_i$ defining the disjunction is contained in the other $\mathcal{K}_j$, and thus the desired closed convex hull trivially equals $\mathcal{K}_j$. So we assume $c_1 - c_2 \notin \pm\mathcal{K}$, i.e., $\|\tilde{c}_1 - \tilde{c}_2\|^2 > (c_{1,n} - c_{2,n})^2$, where $c_i = \binom{\tilde{c}_i}{c_{i,n}}$.

Defining $\sigma = d_1 = d_2$, we have

$$A_0 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} c_1 c_2^T + c_2 c_1^T & -\sigma(c_1 + c_2) \\ -\sigma(c_1 + c_2)^T & 2 \end{pmatrix}.$$

Conditions 1 and 3(ii) are easily verified, and Condition 2 describes the full-dimensional case of interest. It remains to verify Conditions 4 and 5. So assume $s < 1$, and note $s > 0$ due to Condition 3(ii).

For any $z^+ \in \mathbb{R}^{n+1}$, write $z^+ = \binom{z}{z_{n+1}}$ and $z = \binom{\tilde{z}}{z_n} \in \mathbb{R}^n$. Suppose $z^+ \neq 0$. Then

$$\begin{aligned} z^+ \in \text{null}(A_s) \quad &\Longleftrightarrow \quad \left(\tfrac{1-s}{s}\right) A_0 z^+ = -A_1 z^+ \\ &\Longleftrightarrow \quad \left(\tfrac{1-s}{s}\right) A_0 z^+ = -\binom{c_1}{-\sigma}\binom{c_2}{-\sigma}^T z^+ - \binom{c_2}{-\sigma}\binom{c_1}{-\sigma}^T z^+ \\ &\qquad\qquad\qquad\qquad\; =: \alpha\binom{c_1}{-\sigma} + \beta\binom{c_2}{-\sigma}. \end{aligned}$$

Since the last component of $A_0 z^+$ is zero, we must have $\beta = -\alpha$. We claim $\alpha \neq 0$. Assume for contradiction that $\alpha = 0$. Then $z = 0$, but $z_{n+1} \neq 0$ as z^+ is nonzero. On the other hand, because $z^+ \in \text{null}(A_s)$, Lemma 3 implies $0 \geq (z^+)^T A_1 z^+ = 2z_{n+1}^2$, a contradiction. So indeed $\alpha \neq 0$.

Because $z^+ \in \text{null}(A_s)$ and $s \in (0,1)$, the equation

$$0 = (z^+)^T A_s z^+ = (1-s)(z^+)^T A_0 z^+ + s(z^+)^T A_1 z^+,$$

implies Condition 4 holds if and only if $(z^+)^T A_0 z^+ > 0$. From the previous paragraph, we have $\left(\frac{1-s}{s}\right) A_0 z^+ = \alpha\binom{c1-c2}{0}$ with $\alpha \neq 0$. Then

$$\begin{aligned}\left(\tfrac{1-s}{s}\right)(z^+)^T A_0 z^+ &= \begin{pmatrix}\alpha(\tilde{c}_1 - \tilde{c}_2)\\ -\alpha(c_{1,n} - c_{2,n})\\ z_{n+1}\end{pmatrix}^T \begin{pmatrix}\alpha(\tilde{c}_1 - \tilde{c}_2)\\ \alpha(c_{1,n} - c_{2,n})\\ 0\end{pmatrix}\\ &= \alpha^2\left(\|\tilde{c}_1 - \tilde{c}_2\|^2 - (c_{1,n} - c_{2,n})^2\right) > 0,\end{aligned}$$

as desired.

However, it seems difficult to verify Condition 5 generally. For example, consider its second part $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$. In the current context, we have $\mathcal{F}_0^+ \cap H^0 = \mathcal{K} \times \{0\}$, and it is unclear if its intersection with $\mathcal{F}_s^+$ would be contained in $\mathcal{F}_1$. Letting $\binom{\hat{h}}{0} \in \mathcal{F}_s^+$ with $\hat{h} \in \mathcal{K}$, we would have to check the following:

$$0 \geq \begin{pmatrix}\hat{h}\\ 0\end{pmatrix}^T A_s \begin{pmatrix}\hat{h}\\ 0\end{pmatrix} = (1-s)\,\hat{h}^T J\hat{h} + 2s\,(c_1^T\hat{h})(c_2^T\hat{h}) \quad\Longrightarrow\quad \begin{pmatrix}\hat{h}\\ 0\end{pmatrix} \in \mathcal{F}_1.$$

If $\hat{h}$ were in the interior of $\mathcal{K}$, then $\hat{h}^T J\hat{h} < 0$ could still allow $(c_1^T\hat{h})(c_2^T\hat{h}) > 0$, so that $\binom{\hat{h}}{0} \in \mathcal{F}_1$ would not be achieved. So it seems Condition 5 will hold under additional conditions only.

One such set of conditions ensuring Condition 5 is as follows: there exists $\beta_1, \beta_2 \geq 0$ such that $\beta_1 c_1 + c_2 \in -\mathcal{K}$ and $\beta_2 c_1 + c_2 \in \mathcal{K}$. These hold, for example, for split disjunctions, i.e., when c_2 is a negative multiple of c_1. To prove Condition 5, take $\hat{h} \in \mathcal{K}$. Then $c_1^T\hat{h} \geq 0$ implies

$$c_2^T\hat{h} = -\beta_1 c_1^T\hat{h} + (\beta_1 c_1 + c_2)^T\hat{h} \leq 0 + 0 = 0,$$

and similarly $c_1^T\hat{h} \leq 0$ implies $c_2^T\hat{h} \geq 0$. Then overall $\hat{h} \in \mathcal{K}$ implies $(c_1^T\hat{h})(c_2^T\hat{h}) \leq 0$. In the context of the previous paragraph, this ensures $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_0^+ \cap H^0 \subseteq \mathcal{F}_1$, thus verifying Condition 5.

Note that [35,36] cover this case. In the case of split disjunctions with $d_1 = d_2 = 1$, these results are also presented in [2,40]. Whenever the boundedness assumption of [13] is satisfied, one can use their result as well, but the papers [23,55] are not relevant here. Similar to the previous subsection, [14] is limited in its application to this case.

5.3 Revisiting Condition 6

For the cases $d_1 = d_2$ of Sections 5.1 and 5.2, we know that $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ is a valid convex relaxation of $\mathcal{F}_0^+ \cap \mathcal{F}_1$ under Conditions 1–3 and 6. The same holds for the cross-sections: $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$ is a relaxation of $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$. Because Condition 3(i) is verified in the case of $d_1 = d_2 = 0$ and Condition 3(ii) is verified in the case of nonzero $d_1 = d_2$, we have $s > 0$. However, when Condition 6 is violated, it may be possible that $\mathcal{F}_s^+$ is invalid for points simultaneously satisfying both sides of the disjunction, i.e., points x with $c_1^T x \geq d_1$ and $c_2^T x \geq d_2$. This is because such points can violate the quadratic $(c_1^T x - d_1)(c_2^T x - d_2) \leq 0$ from which $\mathcal{F}_s^+$ is derived. In such cases, the set $\mathcal{F}_s^+$ should be relaxed somehow.

Recall that, by definition, $\mathcal{F}_s^+ = \{x : x^T A_s x \leq 0,\ b_s^T x \geq 0\}$. Let us examine the inequality $x^T A_s x \leq 0$, which can be rewritten as

$$\begin{aligned}
&0 \geq (1-s)\, x^T J x + 2s\, (c_1^T x - d_1)(c_2^T x - d_2) \\
\iff &0 \geq 2(1-s)\, x^T J x + s\, \Big([(c_1^T x - d_1) + (c_2^T x - d_2)]^2 - [(c_1^T x - d_1) - (c_2^T x - d_2)]^2 \Big) \\
\iff &s\, [(c_1 - c_2)^T x - (d_1 - d_2)]^2 - 2(1-s)\, x^T J x \;\; \geq \;\; s\, [(c_1 + c_2)^T x - (d_1 + d_2)]^2.
\end{aligned}$$

Note that the left hand-side of the third inequality is nonnegative for any $x \in \mathcal{K}$ since $x^T J x \leq 0$. Therefore, $x \in \mathcal{K}$ implies $x^T A_s x \leq 0$ is equivalent to

$$\sqrt{\big[(c_1 - c_2)^T x - (d_1 - d_2)\big]^2 - 2\left(\tfrac{1-s}{s}\right) x^T J x} \;\; \geq \;\; |(c_1 + c_2)^T x - (d_1 + d_2)|. \quad (15)$$

An immediate relaxation of (15) is

$$\sqrt{\big[(c_1 - c_2)^T x - (d_1 - d_2)\big]^2 - 2\left(\tfrac{1-s}{s}\right) x^T J x} \;\; \geq \;\; (d_1 + d_2) - (c_1 + c_2)^T x \quad (16)$$

since $|(c_1 + c_2)^T x - (d_1 + d_2)| \geq (d_1 + d_2) - (c_1 + c_2)^T x$. Note also that (16) is clearly valid for any x satisfying $c_1^T x \geq d_1$ and $c_2^T x \geq d_2$ since the two sides of the inequality have different signs in this case. In total, the set

$$\mathcal{G}_s^+ := \{x : (16) \text{ holds},\ b_s^T x \geq 0\}$$

is a valid relaxation when Condition 6 does not hold. Although not obvious, it follows from [36] that (16) is a convex inequality. In that paper, (16) was encountered from a different viewpoint, and its convexity was established directly, even though it does not admit an SOC representation. So in fact $\mathcal{G}_s^+$ is convex.

Now let us assume that Condition 4 holds as well so that $\mathcal{F}_s^+$ captures the conic hull of the intersection of $\mathcal{F}_0^+$ and $(c_1^T x - d_1)(c_2^T x - d_2) \leq 0$. We claim that $\mathcal{F}_0^+ \cap \mathcal{G}_s^+$ captures the conic hull when Condition 6 does not hold. (A similar claim will also hold when Condition 5 holds for the further intersection with H^1.) So let $\hat{x} \in \mathcal{F}_0^+ \cap \mathcal{G}_s^+$ be given. If (15) happens to hold also, then $\hat{x}^T A_s \hat{x} \leq 0 \Rightarrow \hat{x} \in \mathcal{F}_s^+$. Then $\hat{x}$ is already in the closed convex hull given by $(c_1^T x - d_1)(c_2^T x - d_2) \leq 0$ by assumption. On the other hand, if (15) does not hold, then it must be that $(c_1 + c_2)^T \hat{x} > d_1 + d_2$. So either $c_1^T \hat{x} > d_1$ or $c_2^T \hat{x} > d_2$. Whichever the case, $\hat{x}$ satisfies the disjunction. Therefore $\hat{x}$ is in the closed convex hull, which gives the desired conclusion.

We remark that, despite their different forms, (16) and the inequality defining $\mathcal{F}_s^+$ both originate from $x^T A_s x \leq 0$ and match precisely on the boundary of conic. hull$(\mathcal{F}_0^+ \cap \mathcal{F}_1) \backslash (\mathcal{F}_0^+ \cap \mathcal{F}_1)$, e.g., the points added due to the convexification process. Moreover, (16) can be interpreted as adding all of the recessive directions $\{d \in \mathcal{K} : c_1^T d \geq 0,\ c_2^T d \geq 0\}$ of the disjunction to the set $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$. Finally, the analysis in [36] shows in addition that the linear inequality $b_s^T x \geq 0$ is in fact redundant for $\mathcal{G}_s^+$.

Note that [35,36] cover this case. Because the resulting convex hull is not conic representable [13] is not applicable in this case. The papers [23,55] are not relevant here and none of the other papers [2,40] cover this case because they focus on split disjunctions only. As in the previous two subsections, [14] is limited in its application.

5.4 The case (c) of $d_1 > d_2$

As mentioned above, the results of [36] ensure that $\operatorname{cl.conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ requires more than two conic inequalities, making it highly likely that the closed convex hull of $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ requires more than two also. In other words, our theory would not apply in this case in general. So we ask: which conditions are violated in this case?

Let us first consider when $d_1 d_2 = 0$, which covers two subcases. Then

$$A_0 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} c_1 c_2^T + c_2 c_1^T & -d_2 c_1 - d_1 c_2 \\ -d_2 c_1^T - d_1 c_2^T & 0 \end{pmatrix},$$

and it is clear that Condition 3 is not satisfied.

Now consider the remaining subcase when $(d_1, d_2) = (1, -1)$. Then

$$A_0 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} c_1 c_2^T + c_2 c_1^T & c_1 - c_2 \\ c_1^T - c_2^T & -2 \end{pmatrix}.$$

Condition 1 holds, and Condition 2 is the full-dimensional case of interest. Condition 3(iii) holds as well, so $s = 0$. Then Condition 4 requires $v^T A_1 v < 0$, where $v = (0; \ldots; 0; 1)$, which is true. On the other hand, Condition 5 might fail. In fact, the example in Section 5 of the Online Supplement provides just such an instance. This being said, the same stronger condition discussed in Section 5.2 can be seen to imply Condition 5, that is, when there exists $\beta_1, \beta_2 \geq 0$ such that $\beta_1 c_1 + c_2 \in -\mathcal{K}$ and $\beta_2 c_1 + c_2 \in \mathcal{K}$. This covers the case of split disjunctions, for example.

Of course, even when all conditions do not hold, just Conditions 1-3, which hold when $d_1 d_2 = -1$, are enough to ensure the valid relaxations $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ and $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$. However, these relaxations may not be sufficient to describe the conic and convex hulls.

If necessary, another way to generate valid conic inequalities when $d_1 > d_2$ is as follows. Instead of the original disjunction, consider the weakened disjunction $c_1^T x \geq d_2 \vee c_2^T x \geq d_2$, where d_2 replaces d_1 in the first term. Clearly any point satisfying the original disjunction will also satisfy the new disjunction. Therefore any valid inequality for the new disjunction will also be valid for the original one. In Sections 5.1 and 5.2, we have discussed the conditions under which Conditions 1-5 are satisfied when $d_1 = d_2$. Even if the new disjunction violates Condition 6, as long as the original disjunction satisfies Condition 6, the resulting inequalities from this approach will be valid.

Regarding the existing literature, the conclusions at the end of Section 5.3 also apply here.

5.5 Conic sections

Let $\rho_1^T x \geq d_1 \vee \rho_2^T x \geq d_2$ be a disjunction on a cross-section $\mathcal{K} \cap H^1$ of the second-order cone, where $H^1 = \{x : h^T x = 1\}$. We work with an analogous of Condition 6:

Condition 7 *The disjunctive sets $\mathcal{K}_1 := \mathcal{K} \cap H^1 \cap \{x : \rho_1^T x \geq d_1\}$ and $\mathcal{K}_2 := \mathcal{K} \cap H^1 \cap \{x : \rho_2^T x \geq d_2\}$ are non-intersecting except possibly on their boundaries, e.g.,*

$$\mathcal{K}_1 \cap \mathcal{K}_2 \subseteq \left\{ x \in \mathcal{K} \cap H^1 : \begin{array}{l} \rho_1^T x = d_1 \\ \rho_2^T x = d_2 \end{array} \right\}.$$

We would like to characterize the convex hull of the disjunction, which is the same as the convex hull of the disjunction $(\rho_1 - d_1 h)^T x \geq 0 \vee (\rho_2 - d_2 h)^T x \geq 0$ on $\mathcal{K} \cap H^1$. Defining $c_1 := \rho_1 - d_1 h$, $c_2 := \rho_2 - d_2 h$, $A_0 := J$, and $A_1 := c_1 c_2^T + c_2 c_1^T$, our goal is to characterize $\mathrm{cl.conv.hull}(\mathcal{K} \cap \mathcal{F}_1 \cap H^1)$. This is quite similar to the analysis in Section 5.1 except that here we also must verify Condition 5.

Conditions 1 and 3(i) are easily verified, and Condition 2 describes the full-dimensional case of interest. Following the development in Section 5.1, we can verify Condition 4 when $\|\tilde{\rho}_1 - d_1 \tilde{h}\|_2 > |\rho_{1,n} - d_1 h_n|$ and $\|\tilde{\rho}_2 - d_2 \tilde{h}\|_2 > |\rho_{2,n} - d_2 h_n|$, and otherwise the convex hull is easy to determine. For Condition 5, we consider the cases of ellipsoids, paraboloids, and hyperboloids separately.

Ellipsoids are characterized by $h \in \mathrm{int}(\mathcal{K})$, and so $\mathcal{K} \cap H^0 = \{0\}$. Thus $\mathcal{K} \cap \mathcal{F}_s^+ \cap H^0 = \{0\} \subseteq \mathcal{F}_1$ easily verifying Condition 5. On the other hand, paraboloids are characterized by $0 \neq h \in \mathrm{bd}(\mathcal{K})$, and in this case, $\mathcal{K} \cap H^0 = \mathrm{cone}\{\hat{h}\}$, where $\hat{h} := -Jh = \binom{-\tilde{h}}{h_n}$. Thus, to verify Condition 5, it suffices to show $\hat{h} \in \mathcal{F}_s^+ \Rightarrow \hat{h} \in \mathcal{F}_1$. Indeed $\hat{h} \in \mathcal{F}_s^+$ implies

$$0 \geq \hat{h}^T A_s \hat{h} = (1-s)\, \hat{h}^T J \hat{h} + s\, \hat{h}^T A_1 \hat{h} = s\, \hat{h}^T A_1 \hat{h}$$

because $h \in \mathrm{bd}(\mathcal{K})$ ensures $\hat{h}^T J \hat{h} = 0$. So $\hat{h} \in \mathcal{F}_1$.

It remains only to verify Condition 5 for hyperboloids, which are characterized by $h \notin \pm\mathcal{K}$, i.e., $h = \binom{\tilde{h}}{h_n}$ satisfies $\|\tilde{h}\| > |h_n|$. However, it seems difficult to verify Condition 5 generally. Still, we note that $\hat{h} \in H^0$ implies

$$\hat{h}^T A_1 \hat{h} = 2(c_1^T \hat{h})(c_2^T \hat{h}) = 2(\rho_1^T \hat{h} - d_1 h^T \hat{h})(\rho_2^T \hat{h} - d_2 h^T \hat{h}) = 2(\rho_1^T \hat{h})(\rho_2^T \hat{h}).$$

Then Condition 5 would hold, for example, when ρ_1 and ρ_2 satisfy the following, which is identical to conditions discussed in Sections 5.2 and 5.4: there exists $\beta_1, \beta_2 \geq 0$ such that $\beta_1 \rho_1 + \rho_2 \in -\mathcal{K}$ and $\beta_2 \rho_1 + \rho_2 \in \mathcal{K}$. This covers the case of split disjunctions, for example.

We remark that our analysis in this subsection covers all of the various cases of split disjunctions found in [40] and more. In particular, we handle ellipsoids and paraboloids for all possible general two-term disjunctions (including the non-disjoint ones). On the other hand, the cases we can cover for hyperboloids is a subset of those recently given in [55]. Note that [23] covers only split disjunctions on ellipsoids. [13] covers two-term disjunctions on ellipsoids and certain specific two-term disjunctions on paraboloids and hyperboloids satisfying their disjointness and boundedness assumptions. None of the papers [2,35,36] are relevant here. Finally, when the disjunction correspond to the deletion of a convex set, the paper [14] applies to the cases for ellipsoids and paraboloids because those sets can be viewed as epigraphs of strictly convex quadratics.

6 General Quadratics with Conic Sections

In this section, we examine the case of (nearly) general quadratics intersected with conic sections of the SOC. For simplicity of presentation, we will employ affine transformations of the sets $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ of interest. It is clear that our theory is not affected by affine transformations.

6.1 Ellipsoids

Consider the set
$$\left\{ y \in \mathbb{R}^n \ : \ \begin{array}{c} y^T y \le 1 \\ y^T Q y + 2\, g^T y + f \le 0 \end{array} \right\},$$
where $\lambda_{\min}[Q] < 0$. Note that if $\lambda_{\min}[Q] \ge 0$, then the set is already convex. Allowing an affine transformation, this set models the intersection of any ellipsoid with a general quadratic inequality. We can model this set in our framework by homogenizing $x = \binom{y}{x_{n+1}}$ and taking

$$A_0 := \begin{pmatrix} I & 0 \\ 0^T & -1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} Q & g \\ g^T & f \end{pmatrix}, \quad H^1 := \{x : x_{n+1} = 1\}.$$

We would like to compute $\text{cl.conv.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$.

Conditions 1 and 3(i) are clear, and Condition 2 describes the full-dimensional case of interest. When $s < 1$, Condition 5 is satisfied because, in this case, $\mathcal{F}_0^+ \cap H^0 = \{0\}$ making the containment $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$ trivial. In Sections 6.1.1 and 6.1.2 below, we break the analysis of verifying Condition 4 into two subcases that we are able to handle: (i) when $\lambda_{\min}[Q]$ has multiplicity $k \ge 2$; and (ii) when $\lambda_{\min}[Q] \le f$ and $g = 0$.

Subcase (i) covers, for example, the situation of deleting the interior of an arbitrary ball from the unit ball. Indeed, consider

$$\left\{ x \in \mathbb{R}^n \ : \ \begin{array}{c} x^T x \le 1 \\ (x-c)^T (x-c) \ge r^2 \end{array} \right\},$$

where $c \in \mathbb{R}^n$ and $r > 0$ are the center and radius of the ball to be deleted. Then case (i) holds with $(Q, g, f) = (-I, c, r^2 - c^T c)$. On the other hand, subcase (ii) can handle, for example, the deletion of the interior of an arbitrary ellipsoid from the unit ball—as long as that ellipsoid shares the origin as its center. In other words, the portion to delete is defined by $x^T E x < r^2$, for some $E \succ 0$ and $r > 0$, and we take $(Q, g, f) = (-E, 0, r^2)$. Note that $\lambda_{\min}[Q] \le -f \Leftrightarrow \lambda_{\max}[E] \ge r^2$, which occurs if and only if the deleted ellipsoid contains a point on the boundary of the unit ball. This is the most interesting case because, if the deleted ellipsoid were either completely inside or outside the unit ball, then the convex hull would simply be the unit ball itself. The subcase (ii) was also studied in Corollary 9 of [40] and in [14]. Moreover, none of the other papers [2, 13, 23, 35, 36, 55] can handle this case.

6.1.1 When $\lambda_{\min}[Q]$ has multiplicity $k \ge 2$

Define $B_t := (1-t)I + tQ$ to be the top-left $n \times n$ corner of A_t. Since $\lambda_{\min}[B_1] < 0$ with multiplicity $k \ge 2$, there exists $r \in (0,1)$ such that: (i) $B_r \succeq 0$; (ii) $\lambda_{\min}[B_r] = 0$ with multiplicity k; (ii) $B_t \succ 0$ for all $t < r$. We claim that $s = r$ as a consequence of the interlacing of eigenvalues with respect to A_t and B_t. Indeed, let $\lambda_{n+1}^t := \lambda_{\min}[A_t]$ and λ_n^t denote the two smallest eigenvalues of A_t, and let ρ_n^t and ρ_{n-1}^t denote the analogous eigenvalues of B_t. It is well known that

$$\lambda_{n+1}^t \ \le \ \rho_n^t \ \le \ \lambda_n^t \ \le \ \rho_{n-1}^t.$$

When $t < r$, we have $\lambda_{n+1}^t < 0 < \rho_n^t \le \lambda_n^t$, and when $t = r$, we have $\lambda_{n+1}^r < 0 \le \lambda_n^r \le 0$, which proves $s = r$.

Since $\dim(\text{null}(B_s)) = k \geq 2$ and $\dim(\text{span}\{g\}^{\perp}) = n-1$, there exists $0 \neq z \in \text{null}(B_s)$ such that $g^T z = 0$. We can show that $\binom{z}{0} \in \text{null}(A_s)$:

$$A_s \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} B_s & s\,g \\ s\,g^T & (1-s)(-1)+sf \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} B_s z \\ s\,g^T z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, $\binom{z}{0}^T A_1 \binom{z}{0} = z^T B_1 z = z^T Q z < 0$ because $z \in \text{null}(B_s)$ if and only if z is a eigenvector of $B_1 = Q$ corresponding to $\lambda_{\min}[Q]$. This verifies Condition 4.

6.1.2 When $\lambda_{\min}[Q] \leq -f$ and $g = 0$

The argument is similar to the preceding subcase in Section 6.1.1. Note that

$$A_t = \begin{pmatrix} (1-t)I + tQ & 0 \\ 0 & (1-t)(-1)+tf \end{pmatrix} =: \begin{pmatrix} B_t & 0 \\ 0 & \beta_t \end{pmatrix}$$

is block diagonal, so that the singularity of A_t is determined by the singularity of B_t and β_t. B_t is first singular when $t = 1/(1-\lambda_{\min}[Q])$, while β_t is first singular when $t = 1/(1+f)$ (assuming $f > 0$; if not, then β_t is never singular). Then

$$\frac{1}{1-\lambda_{\min}[Q]} \leq \frac{1}{1+f} \quad \Longleftrightarrow \quad \lambda_{\min}[Q] \leq -f,$$

which holds by assumption. So B_t is singular before β_t, leading to $s = 1/(1-\lambda_{\min}[Q])$. Let $0 \neq z \in \text{null}(B_s)$. Then, we have $Qz = -\frac{1-s}{s} z$, and thus, $\binom{z}{0} \in \text{null}(A_s)$ with $\binom{z}{0}^T A_1 \binom{z}{0} = z^T B_1 z = z^T Q z < 0$. Condition 4 is hence verified.

6.2 The trust-region subproblem

We show in this subsection that our methodology can be used to solve the trust-region subproblem (TRS)

$$\min_{\tilde{y} \in \mathbb{R}^{n-1}} \left\{ \tilde{y}^T \tilde{Q} \tilde{y} + 2\,\tilde{g}^T \tilde{y} : \; \tilde{y}^T \tilde{y} \leq 1 \right\}, \tag{17}$$

where $\lambda_{\min}[\tilde{Q}] < 0$. Without loss of generality, we assume that $\tilde{Q}$ is diagonal with $\tilde{Q}_{(n-1)(n-1)} = \lambda_{\min}[\tilde{Q}]$ after applying an orthogonal transformation that does not change the feasible set.

Our intention is not necessarily to argue that the TRS should be solved numerically with our approach, although this is an interesting question left as future work. Our goal is to illustrate that the well-known problem (17) can be handled by our machinery. We also believe that the corresponding SOCP formulation for the TRS as opposed to its usual SDP formulation is independently interesting. Our transformations to follow require simply two eigenvalue decompositions and the resulting SOCP can be solved by interior point solvers very efficiently. We note that none of the previous papers, in particular, [2, 13, 23, 35, 36, 40, 55] have given a transformation of the TRS into an SOC optimization problem before. We recently became aware that an SOC based reformulation of TRS was also given in Jeyakumar and Li [30]; our approach parallels their developments from a different, convexification based, perspective.

We first argue that (17) is equivalent to a trust-region subproblem

$$\min_{y\in\mathbb{R}^n} \left\{ y^T Q y + 2\, g^T y : \; y^T y \le 1 \right\} \tag{18}$$

in the n-dimensional variable $y := \binom{\tilde{y}}{y_n}$. Indeed, define

$$Q := \begin{pmatrix} \tilde{Q} & 0 \\ 0^T & \lambda_{\min}[\tilde{Q}] \end{pmatrix}, \quad g := \begin{pmatrix} \tilde{g} \\ 0 \end{pmatrix},$$

and note that $\lambda_{\min}[Q]$ has multiplicity at least 2. The following proposition shows that (18) is equivalent to (17).

Proposition 8 *There exists an optimal solution of (18) with $y_n = 0$. In particular, the optimal values of (17) and (18) are equal.*

Proof Let $\bar{y}$ be an optimal solution of (18). Then $(\bar{y}_{n-1}; \bar{y}_n)$ is an optimal solution of the two-dimensional trust-region subproblem

$$\min_{y_{n-1}, y_n} \left\{ |\lambda_{\min}[\tilde{Q}]|(-y_{n-1}^2 - y_n^2) + 2\tilde{g}_{n-1} y_{n-1} : \; y_{n-1}^2 + y_n^2 \le \epsilon \right\}.$$

where $\epsilon := 1 - (\bar{y}_1^2 + \cdots \bar{y}_{n-2}^2)$. Since we are minimizing a concave function over the ellipsoid, at least one optimal solution will be on the boundary of this set. In particular, whenever $\tilde{g}_{n-1} > 0$, the solution $\binom{y_{n-1}}{y_n} = \binom{-\sqrt{\epsilon}}{0}$ is optimal, and when $\tilde{g}_{n-1} \le 0$, the solution $\binom{y_{n-1}}{y_n} = \binom{\sqrt{\epsilon}}{0}$ is optimal. Thus, this problem has at least one optimal solution with $y_n = 0$. Hence, $\bar{y}_n$ can be taken as 0. $\square$

With the proposition in hand, we now focus on the solution of (18).

A typical approach to solve (18) is to introduce an auxiliary variable x_{n+2} (where we reserve the variable x_{n+1} for later homogenization) and to recast the problem as

$$\min \left\{ x_{n+2} : \begin{array}{l} y^T y \le 1 \\ y^T Q y + 2\, g^T y \le x_{n+2} \end{array} \right\}.$$

If one can compute the closed convex hull of this feasible set, then (18) is solvable by simply minimizing x_{n+2} over the convex hull. We can represent this approach in our framework by taking $x = (y; x_{n+1}; x_{n+2})$, homogenizing via $x_{n+1} = 1$, and defining

$$A_0 := \begin{pmatrix} I & 0 & 0 \\ 0^T & -1 & 0 \\ 0^T & 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} Q & g & 0 \\ g^T & 0 & -\frac{1}{2} \\ 0^T & -\frac{1}{2} & 0 \end{pmatrix}, \quad H^1 := \{x \in \mathbb{R}^{n+2} : x_{n+1} = 1\}.$$

Clearly, Conditions 1 and 2 are satisfied. However, no part of Condition 3 is satisfied. So we require a different approach.

Since $x = 0$ is feasible for (18), its optimal value is nonpositive. (In fact, it is negative since Q has a negative eigenvector, so that $x = 0$ is not a local minimizer). Hence, (18) is equivalent to

$$v := \min \left\{ x_{n+2}^2 : \begin{array}{l} y^T y \le 1 \\ y^T Q y + 2\, g^T y \le -x_{n+2}^2 \end{array} \right\}, \tag{19}$$

which can be solved in stages: first, minimize x_{n+2} over the feasible set of (19) (let l be the minimal value); second, separately maximize x_{n+2} over the same (let u be the maximal value); and finally take $v = \min\{-l^2, -u^2\}$. If one can compute the closed convex hull of (19), then l and u can be computed easily.

To represent the feasible set of (19) in our framework, we define $x = (y; x_{n+1}; x_{n+2})$ and take

$$A_0 := \begin{pmatrix} I & 0 & 0 \\ 0^T & -1 & 0 \\ 0^T & 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} Q & g & 0 \\ g^T & 0 & 0 \\ 0^T & 0 & 1 \end{pmatrix}, \quad H^1 := \{x \in \mathbb{R}^{n+2} : \ x_{n+1} = 1\}.$$

Clearly, Conditions 1 and 2 are satisfied, and Condition 3(ii) is now satisfied. For Conditions 4 and 5, we note that A_t has a block structure such that s equals the smallest positive t such that

$$B_t := (1-t)\begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} + t\begin{pmatrix} Q & g \\ g^T & 0 \end{pmatrix}$$

is singular. Using an argument similar to Section 6.1.1 and exploiting the fact that $\lambda_{\min}[Q]$ has multiplicity at least 2, we can compute s such that there exists $0 \neq z \in \text{null}(B_s) \subseteq \mathbb{R}^{n+1}$ with $z^T B_1 z < 0$ and $z_{n+1} = 0$. By appending an extra 0 entry, this z can be easily extended to $z \in \mathbb{R}^{n+2}$ with $z^T A_1 z < 0$ and $z \in H^0$. This simultaneously verifies Conditions 4 and 5.

6.3 Paraboloids

Consider the set

$$\left\{ y = \begin{pmatrix} \tilde{y} \\ y_n \end{pmatrix} \in \mathbb{R}^n \ : \ \begin{array}{c} \tilde{y}^T \tilde{y} \leq y_n \\ \tilde{y}^T \tilde{Q} \tilde{y} + 2\, g^T y + f \leq 0 \end{array} \right\},$$

where $\lambda := \lambda_{\min}[\tilde{Q}] < 0$ and $2g_n \leq -\lambda$. After an affine transformation, this models the intersection of a paraboloid with any quadratic inequality that is strictly linear in y_n, i.e., no quadratic terms involve y_n. Note that if $\lambda_{\min}[Q] \geq 0$, then the set is already convex. The reason for the upper bound on $2g_n$ will become evident shortly.

Writing $g := \binom{\tilde{g}}{g_n}$, we can model this situation with $x = \binom{y}{x_{n+1}}$ and

$$A_0 := \begin{pmatrix} I & 0 & 0 \\ 0^T & 0 & -\frac{1}{2} \\ 0^T & -\frac{1}{2} & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} \tilde{Q} & 0 & \tilde{g} \\ 0^T & 0 & g_n \\ \tilde{g}^T & g_n & f \end{pmatrix}, \quad H^1 := \{x : x_{n+1} = 1\},$$

and we would like to compute $\text{cl.conv.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$. Conditions 1 and 3(i) are clear, and Condition 2 describes the full-dimensional case of interest. So it remains to verify Conditions 4 and 5.

Define

$$B_t := \begin{pmatrix} (1-t)I + t\tilde{Q} & 0 \\ 0 & 0 \end{pmatrix}$$

to be the top-left $n \times n$ corner of A_t, and define $r := 1/(1-\lambda)$. Due to its structure, B_t is positive semidefinite for all $t \leq r$. Moreover, B_t has exactly one zero eigenvalue for $t < r$, and B_r has at least two zero eigenvalues. Those two zero eigenvalues ensure that A_r is singular by the interlacing of eigenvalues of A_t and B_t (similar to Section 6.1.1). So $s \le r$.

We claim that in fact $s = r$. Let $t < r$; and consider the following system for $\mathrm{null}(A_t)$:

$$\begin{pmatrix} (1-t)I + t\tilde{Q} & 0 & t\,\tilde{g} \\ 0^T & 0 & (1-t)(-\frac{1}{2}) + t\,g_n \\ t\,\tilde{g}^T & (1-t)(-\frac{1}{2}) + t\,g_n & tf \end{pmatrix} \begin{pmatrix} \tilde{z} \\ z_n \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that $2g_n \le -\lambda$ and $0 \le t < r$ imply

$$2\left[(1-t)(-\tfrac{1}{2}) + t\,g_n\right] = t(1 + 2\,g_n) - 1 \le t(1-\lambda) - 1 < r(1-\lambda) - 1 = 0, \qquad (20)$$

which implies $z_{n+1} = 0$. This in turn implies $\tilde{z} = 0$ because $(1-t)I + t\tilde{Q} \succ 0$ when $t < r$. Finally, $z_n = 0$ again due to (20). So we conclude that $t < r$ implies $\mathrm{null}(A_t) = \{0\}$. Hence, $s = r$. We next write

$$A_s = \begin{pmatrix} B_s & g_s \\ g_s & sf \end{pmatrix}.$$

Since $\dim(\mathrm{null}(B_s)) \ge 2$ and $\dim(\mathrm{span}\{g_s\}^\perp) = n - 1$, there exists $0 \ne z \in \mathrm{null}(B_s)$ such that $g_s^T z = 0$. From the structure of B_s, we have $z = \binom{\tilde{z}}{z_n}$, where $\tilde{z}$ is a negative eigenvector of $\tilde{Q}$. We claim that $\binom{z}{0} \in \mathrm{null}(A_s)$. Indeed:

$$A_s \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} B_s & g_s \\ g_s^T & sf \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} B_s z \\ g_s^T z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, $\binom{z}{0}^T A_1 \binom{z}{0} = z^T B_1 z = \tilde{z}^T \tilde{Q} \tilde{z} < 0$. This verifies Conditions 4 and 5.

We remark that Corollary 8 in [40] studies the closed convex hull of the set

$$\left\{ y = \begin{pmatrix} \tilde{y} \\ y_n \end{pmatrix} \in \mathbb{R}^n : \ \|\tilde{A}(\tilde{y} - \tilde{c})\|^2 \le y_n,\ \|\tilde{D}(\tilde{y} - \tilde{d})\|^2 \ge -\gamma\, y_n + q \right\},$$

where $\tilde{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ is an invertible matrix, $\tilde{c}, \tilde{d} \in \mathbb{R}^{n-1}$ and $\gamma \ge 0$. This situation is covered by our theory here. The paper [14] also applies to this case, but none of the other papers [2,13,23,35,36,55] are relevant here.

7 Conclusion

This paper provides basic convexity results regarding the intersection of a second-order-cone representable set and a nonconvex quadratic. Although several results have appeared in the prior literature, we unify and extend these by introducing a simple, computable technique for aggregating (with nonnegative weights) the inequalities defining the two intersected sets. The underlying conditions of our theory can be checked easily in many cases of interest.

Beyond the examples detailed in this paper, our technique can be used in other ways. Consider for example, a general quadratically constrained quadratic program, whose objective has been linearized without loss of generality. If the constraints include an ellipsoid constraint, then our techniques can be used to generate valid SOC inequalities for the convex hull of the feasible region by pairing each nonconvex quadratic constraint with the ellipsoid constraint one by one. The theoretical and practical strength of this technique is of interest for future research, and the techniques in [3, 37] could provide a good point of comparison.

In addition, it would be interesting to investigate whether our techniques could be extended to produce valid inequalities or explicit convex hull descriptions for intersections involving multiple second-order cones or multiple nonconvex quadratics. After our initial June 2014 submission of this paper, a similar aggregation idea has been recently explored in [41] in November 2014 by using the results from [54]. We note that as opposed to our emphasis on the computability of SOCr relaxations, these recent results rely on numerical algorithms to compute such relaxations and further topological conditions for verifying their sufficiency.

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Online Supplement: Low-Dimensional Examples

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In this Online Supplement, we illustrate Theorem 1 of the main article with several low-dimensional examples and discuss which of the earlier approaches [2, 13, 14, 23, 35, 36, 40, 55] cannot replicate these examples. Section 5 of the main article is devoted to the important case for which the dimension n is arbitrary, $\mathcal{F}_0^+$ is the second-order cone, and $\mathcal{F}_1$ represents a two-term linear disjunction $c_1^T x \geq d_1 \vee c_2^T x \geq d_2$. Section 6 of the main article investigates cases in which $\mathcal{F}_1$ is given by a (nearly) general quadratic inequality.

1 A proper split of the second-order cone

In $\mathbb{R}^3$, consider the intersection of the canonical second-order cone, defined by $\|(y_1; y_2)\| \leq y_3$, and a specific linear disjunction, defined by $y_1 \leq -1 \vee y_1 \geq 1$, which is a proper split. By homogenizing via $x = \binom{y}{x_4}$ with $x_4 = 1$ and noting that the disjunction is equivalent to $y_1^2 \geq 1 \Leftrightarrow y_1^2 \geq x_4^2$, we can represent the intersection as $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ with

$$A_0 := \mathrm{Diag}(1, 1, -1, 0), \quad A_1 := \mathrm{Diag}(-1, 0, 0, 1), \quad H^1 := \{x : x_4 = 1\}.$$

Note that $A_t = \mathrm{Diag}(1 - 2t, 1 - t, -1 + t, t)$. Conditions 1 and 3(ii) are easily verified, and Condition 2 holds with $\bar{x} := (2; 0; 3; 1)$, for example.

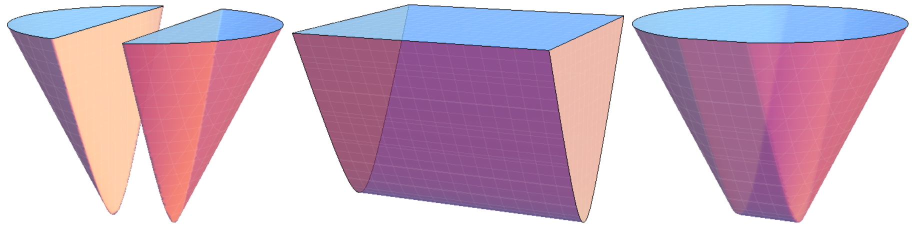
In this case, $s = \frac{1}{2}$, $A_s = \frac{1}{2}\mathrm{Diag}(0, 1, -1, 1)$, $\mathcal{F}_s = \{x : x_2^2 + x_4^2 \leq x_3^2\}$, and $\mathcal{F}_s^+ = \{x : \|(x_2; x_4)\| \leq x_3\}$, which contains $\bar{x}$. Note that $\mathrm{apex}(\mathcal{F}_s^+) = \mathrm{null}(A_s) = \mathrm{span}\{d\}$, where $d := (1; 0; 0; 0)$. It is easy to check that $d \in H^0$ with $d^T A_1 d < 0$, and so Conditions 4 and 5 are simultaneously verified.

So, in the original variable y, the explicit convex hull is given by

$$\left\{ y : \begin{array}{l} \|(y_1; y_2)\| \leq y_3 \\ \|(y_2; 1)\| \leq y_3 \end{array} \right\} = \mathrm{cl.\,conv.\,hull} \left\{ y : \begin{array}{c} \|(y_1; y_2)\| \leq y_3 \\ y_1 \leq -1 \vee y_1 \geq 1 \end{array} \right\}.$$

Figure 1 depicts the original intersection, $\mathcal{F}_s^+ \cap H^1$, and the closed convex hull.

Papers [2, 35, 36, 40] can handle this example, and in fact they can handle all split disjunctions on SOCs. On the other hand, [13] cannot handle this example because of their boundedness assumption on the sides of the disjunction. Because this example concerns a disjunction on SOC itself—not a disjunction on a cross-section of SOC—the papers [23, 55] are not relevant here. In order to apply the results from [14], we need to consider the SOC $\|(y_1; y_2)\| \leq y_3$ as the epigraph of the convex norm $\|(y_1; y_2)\|$. However, this viewpoint does not satisfy the special conditions for polynomial-time separability, such as differentiability or growth rate, in that paper; see Theorem IV therein.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ (b) $\mathcal{F}_s^+ \cap H^1$ (c) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$

Fig. 1 A proper split of the second-order cone

2 A paraboloid and a second-order-cone disjunction

In $\mathbb{R}^3$, consider the intersection of the paraboloid defined by $y_1^2+y_2^2 \le y_3$ and the "two-sided" second-order cone disjunction defined by $y_1^2 + y_3^2 \le y_2^2$. One side has $y_2 \ge 0$, while the other has $y_2 \le 0$. By homogenizing via $x = \binom{y}{x_4}$ with $x_4 = 1$, we can represent the intersection as $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ with

$$A_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H^1 := \{x : x_4 = 1\}.$$

Conditions 1 and 3(i) are straightforward to verify, and Condition 2 is satisfied with $\bar{x} = (0; \frac{1}{\sqrt{2}}; \frac{1}{\sqrt{3}}; 1)$, for example. We can also calculate $s = \frac{1}{2}$ from (7). Then

$$A_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & 0 \end{pmatrix}, \qquad \mathcal{F}_s = \left\{ x : x_1^2 + \tfrac{1}{2}\, x_3^2 \le \tfrac{1}{2}\, x_3 x_4 \right\}.$$

The negative eigenvalue of A_s is $\lambda_{s1} := (1-\sqrt{2})/4$ with corresponding eigenvector $q_{s1} := (0; 0; \sqrt{2}-1; 1)$, and so, in accordance with the Section 2, we have that $\mathcal{F}_s^+$ equals all $x \in \mathcal{F}_s$ satisfying $b_s^T x \ge 0$, where

$$b_s := (-\lambda_{s1})^{1/2} q_{s1} = \frac{\sqrt{\sqrt{2}-1}}{2} \begin{pmatrix} 0 \\ 0 \\ \sqrt{2}-1 \\ 1 \end{pmatrix}.$$

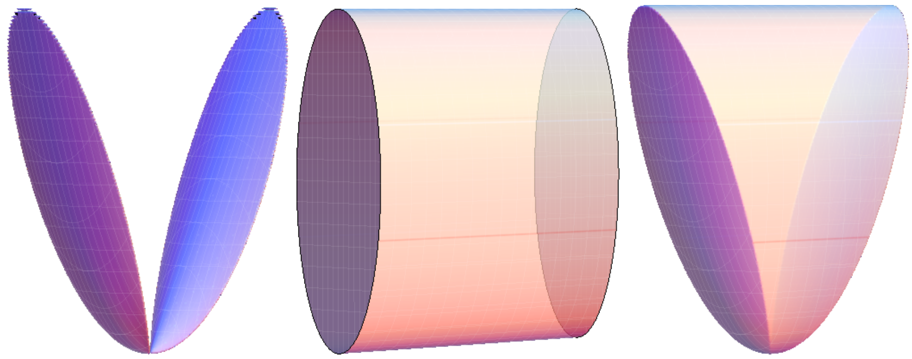
Scaling b_s by a positive constant, we thus have

$$\mathcal{F}_s^+ := \left\{ x : \begin{array}{c} x_1^2 + \frac{1}{2}\, x_3^2 \le \frac{1}{2}\, x_3 x_4 \\ (\sqrt{2}-1) x_3 + x_4 \ge 0 \end{array} \right\}.$$

Note that $\bar{x} \in \mathcal{F}_s^+$. In addition, $\operatorname{apex}(\mathcal{F}_s^+) = \operatorname{null}(A_s) = \operatorname{span}\{d\}$, where $d = (0; 1; 0; 0)$. Clearly, $d \in H^0$ and $d^T A_1 d < 0$, which verifies Conditions 4 and 5 simultaneously. Setting $x_4 = 1$ and returning to the original variable y, we see

$$\left\{ y : \begin{array}{c} y_1^2 + y_2^2 \le y_3 \\ y_1^2 + \frac{1}{2}\, y_3^2 \le \frac{1}{2}\, y_3 \end{array} \right\} = \operatorname{cl.conv.hull} \left\{ y : \begin{array}{c} y_1^2 + y_2^2 \le y_3 \\ y_1^2 + y_3^2 \le y_2^2 \end{array} \right\},$$

where the now redundant constraint $(\sqrt{2}-1)y_3+1 \geq 0$ has been dropped. Figure 2 depicts the original intersection, $\mathcal{F}_s^+ \cap H^1$, and the closed convex hull.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ (b) $\mathcal{F}_s^+ \cap H^1$ (c) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$

Fig. 2 A paraboloid and a second-order-cone disjunction

Of the earlier, related approaches, this example can be handled by [40] only. In particular, [2, 13, 23, 35, 36, 55] cannot handle this example because they deal with only split or two-term disjunctions but cannot cover general nonconvex quadratics. The approach of [14] is based on eliminating a convex region from a convex epigraphical set, but this example removes a nonconvex region (specifically, $\mathbb{R}^n \setminus \mathcal{F}_1$). So [14] cannot handle this example either.

In actuality, the results of [40] do not handle this example explicitly since the authors only state results for: the removal of a paraboloid or an ellipsoid from a paraboloid; or the removal of an ellipsoid (or an ellipsoidal cylinder) from another ellipsoid with a common center. However, in this particular example, the function obtained from the aggregation technique described in [40] is convex on all of $\mathbb{R}^3$. Therefore, their global convexity requirement on the aggregated function is satisfied for this example.

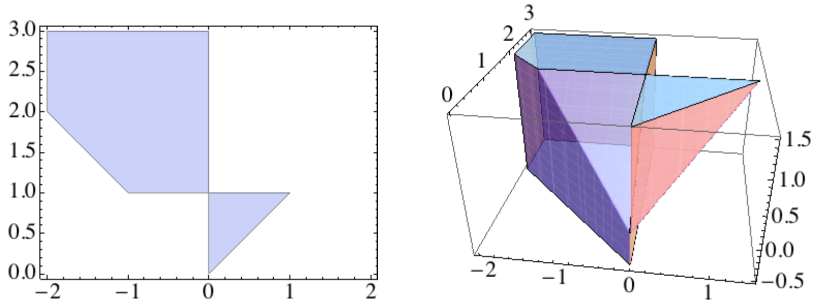
3 An example violating Condition 3

In $\mathbb{R}^2$, consider the intersection of the canonical second-order cone defined by $|y_1| \leq y_2$ and the set defined by the quadratic $y_1(y_2-1) \leq 0$. By homogenizing via $x = \binom{y}{x_3}$ with $x_3 = 1$, we can represent the set as $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ with

$$A_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad H^1 := \{x : x_3 = 1\}.$$

While Conditions 1 and 2 hold, Condition 3 does not hold because A_0 is singular and A_1 is zero on the null space $\operatorname{span}\{(0;0;1)\}$ of A_0. Figure 3 depicts $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ and $\mathcal{F}_0^+ \cap \mathcal{F}_1$.

In this example, even though Condition 3 is violated, we still have the trivial convex relaxation given by $\operatorname{cl.conv.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1) \subseteq \mathcal{F}_0^+ \cap H^1$. Of course, this trivial convex relaxation is not sufficient.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ (b) $\mathcal{F}_0^+ \cap \mathcal{F}_1$

Fig. 3 An example violating Condition 3

The papers [2,13,23,35,36,55] also cannot handle this example because they deal with only split or two-term disjunctions that are not general enough to cover general nonconvex quadratics. Moreover, $\mathbb{R}^2 \setminus \mathcal{F}_1$ defines a nonconvex region, so neither of the approaches from [14,40] related to excluding convex sets is applicable in this case.

4 An example violating Condition 4

In $\mathbb{R}^2$, consider the intersection of the second-order cone defined by $|x_1| \le x_2$ and the two-term linear disjunction defined by $x_1 \le 0 \vee x_2 \le x_1$. Note that, in the second-order cone, $x_2 \le x_1$ implies $x_1 = x_2$. So one side of the disjunction is contained in the boundary of the second-order cone. We also note that—in the second-order cone—the disjunction is equivalent to the quadratic $x_1(x_2 - x_1) \le 0$. Thus, to compute the closed conic hull of the intersection of cone and the disjunction, we define

$$A_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix},$$

and we wish to calculate $\operatorname{cl.conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$.

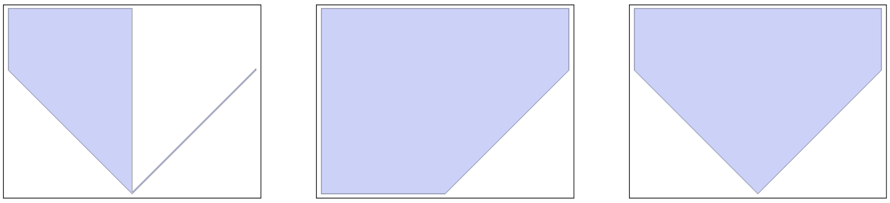
Conditions 1, 2, and 3(i) are easily verified, and the eigenvalues of $A_0^{-1}A_1$ are -1 (with multiplicity 2). This implies $s = \frac{1}{2}$ by (7), and so

$$A_s = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Also, $\operatorname{null}(A_s)$ is spanned by $d = (1;1)$, and yet $d^T A_1 d = 0$, which violates Condition 4.

Note that $A_s = -\frac{1}{2}\binom{1}{-1}\binom{1}{-1}^T$, and so $\mathcal{F}_s^+ = \{x : x_2 \ge x_1\}$. Figure 4 depicts $\mathcal{F}_0^+ \cap \mathcal{F}_1$, $\mathcal{F}_s^+$, and $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$. Since Conditions 1–3 are satisfied, we know that $\operatorname{cl.conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1) \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$, and it is evident from the figures that—in this particular example—equality holds. This simply indicates that the results of Theorem 1 may still hold even when Condition 4 is violated.

The approach [2] can only handle split disjunctions on SOCs and thus is not applicable here. This is also the case for that portion of the approach from [40] associated



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1$ (b) $\mathcal{F}_s^+$ (c) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$

Fig. 4 An example violating Condition 4

with split disjunctions. Moreover, [35, 36] cannot handle this two-term disjunction because of their strict feasibility assumption on both sides of the sets defined by the disjunction. Also, [13] cannot handle this example because of their boundedness assumption on both of the sets defined by the disjunction. In addition, $\mathbb{R}^2 \setminus \mathcal{F}_1$ defines a nonconvex region, therefore neither of the approaches from [14, 40] related to excluding convex sets is applicable in this case. Since this example concerns a disjunction on SOC itself but not on the cross-section of an SOC, [23, 55] are not relevant here.

5 An example violating Condition 5

In $\mathbb{R}^2$, consider the intersection of the second-order cone defined by $|y_1| \le y_2$ and the two-term linear disjunction defined by $y_1 \ge 2 \vee y_2 \le 1$. Note that, in the second-order cone, the disjunction is equivalent to the quadratic $(y_1 - 2)(1 - y_2) \le 0$. Thus, to compute the closed conic hull of the intersection of cone and the disjunction, we define $x = \binom{y}{x_3}$ and

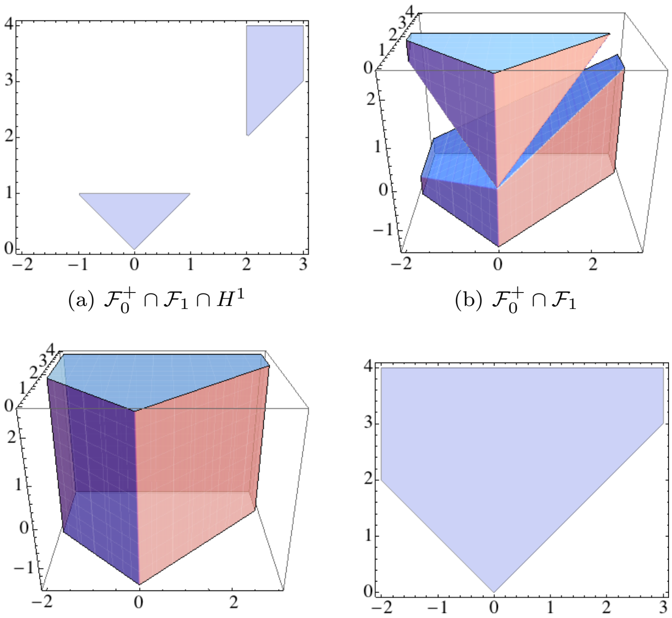
$$A_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 := \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 2 & -4 \end{pmatrix}, \quad H^1 := \{x : x_3 = 1\}$$

and we wish to calculate $\mathrm{cl.\,conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$.

Conditions 1, 2, and 3(iii) are easily verified, and so $s = 0$ with $\mathrm{null}(A_s)$ spanned by $d = (0; 0; 1)$. Then Condition 4 is clearly satisfied. However, $d_3 \ne 0$, and so the first option for Condition 5 is not satisfied. The second option is the containment $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$, which simplifies to $\mathcal{F}_0^+ \cap H^0 \subseteq \mathcal{F}_1$ in this case. This is also *not* true because the point $x = (1; 2; 0) \in \mathcal{F}_0^+ \cap H^0$ but $x \notin \mathcal{F}_1$.

Figure 5 depicts this example. Note that the inequality $y_1 \ge -1$ is valid for the convex hull of $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$. In addition, $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ = \mathrm{cl.\,conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ because Conditions 1–4 are satisfied. However, the projection $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$ is not the desired convex hull since, for example, it violates $y_1 \ge -1$.

Similar to the previous example in Section 4, the papers [2, 13, 14, 23, 40, 55] cannot handle this example. On the other hand, [35, 36] provide the infinite family of convex inequalities describing the closed convex hull of this set, but they do not specifically identify the corresponding finite collection that is necessary and sufficient.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$

(b) $\mathcal{F}_0^+ \cap \mathcal{F}_1$

(c) $\mathcal{F}_s^+ = \mathcal{F}_0^+ \cap \mathcal{F}_s^+$

(d) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$

Fig. 5 An example violating Condition 5. Note that $s = 0$ in this case.