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On Minimal Valid Inequalities for Mixed Integer Conic Programs

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We study disjunctive conic sets involving a general regular (closed, convex, full dimensional, and pointed) cone $\mathcal{K}$ such as the nonnegative orthant, the Lorentz cone, or the positive semidefinite cone. In a unified framework, we introduce $\mathcal{K}$-minimal inequalities and show that, under mild assumptions, these inequalities together with the trivial cone-implied inequalities are sufficient to describe the convex hull. We focus on the properties of $\mathcal{K}$-minimal inequalities by establishing algebraic necessary conditions for an inequality to be $\mathcal{K}$-minimal. This characterization leads to a broader algebraically defined class of $\mathcal{K}$-sublinear inequalities. We demonstrate a close connection between $\mathcal{K}$-sublinear inequalities and the support functions of convex sets with a particular structure. This connection results in practical ways of verifying $\mathcal{K}$-sublinearity and/or $\mathcal{K}$-minimality of inequalities.

Our study generalizes some of the results from the mixed integer linear case. It is well known that the minimal inequalities for mixed integer linear programs are generated by sublinear (positively homogeneous, subadditive, and convex) functions which are also piecewise linear. Our analysis easily recovers this result. However, in the case of general regular cones other than the nonnegative orthant, our study reveals that such a cut-generating function view that treats the data associated with each individual variable independently is far from sufficient.

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1. Introduction

A *Mixed Integer Conic Program* (MICP) is an optimization program of the form

$$\text{Opt} = \inf_{x\in E}\{\langle c,x\rangle : Ax=b,\ x\in\mathcal{K},\ x\in\mathcal{Z}\} \qquad (MICP)$$

where $\mathcal{K}$ is a *regular* (full-dimensional, closed, convex and pointed) cone in a finite dimensional Euclidean space E with an inner product $\langle\cdot,\cdot\rangle$, $c\in E$ is the objective vector, $b\in\mathbb{R}^m$ is the right hand side vector, $A:E\to\mathbb{R}^m$ is a linear map, and $\mathcal{Z}$ is a set imposing certain structural restrictions on the variables x. Examples of regular cones include the nonnegative orthant $\mathbb{R}^n_+ := \{x\in\mathbb{R}^n:\ x_i\geq 0\ \forall i=1,\ldots,n\}$, the Lorentz cone $\mathcal{L}^n := \{x\in\mathbb{R}^n:\ x_n\geq\sqrt{x_1^2+\ldots+x_{n-1}^2}\,\}$, and the positive semidefinite cone $\mathcal{S}^n_+ := \{x\in\mathbb{R}^{n\times n}:\ a^Tx\,a\geq 0\ \forall a\in\mathbb{R}^n,\ x=x^T\}$ and their direct products. When $E=\mathbb{R}^n$, the most common form of structural restrictions is integrality $x_i\in\mathbb{Z}$ for all $i\in I$, where $I\subset\{1,\ldots,n\}$ is the index set of integer variables. We assume that all of the data involved with MICP, i.e., c,b,A are rational.

Mixed Integer Linear Programs (MILPs) arise as a special case of MICPs where $\mathcal{K}$ is the nonnegative orthant. Conic constraints include various specific convex constraints such as linear, convex quadratic, eigenvalue, etc., and hence offer significant representation power over linear constraints (see [18] for a detailed introduction to conic programming and its applications in various domains). Allowing discrete decisions in addition to the conic constraints further enhances the representation power of MICPs. This modeling flexibility of MICPs is essential for a broad range of optimization problems in the decision making under uncertainty domain. For example, robust counterparts of MILPs with ellipsoidal uncertainty sets result in MICPs with Lorentz cone [16]. Likewise, robust counterparts of MICPs with Lorentz cone under ellipsoidal uncertainty lead to MICPs with semidefinite cone. Some application areas in this context include portfolio optimization [36, 55], stochastic joint facility location-inventory models [6], and unit commitment [37]. MICPs are also encountered in statistics [60, 65] and optimal control [38]. Moreover, the most powerful relaxations to many combinatorial optimization problems are based on conic (in particular semidefinite) relaxations (see [39] for a survey of this topic). Reintroducing the integer variables into these relaxations yields exact MICP formulations of these problems with tighter continuous relaxations. Besides, MILPs have been heavily exploited for approximating a vast array of non-convex nonlinear optimization problems across many diverse fields [15]. For a wide range of these problems, MICPs offer tighter relaxations and thus potentially a better overall algorithmic performance.

Because of the increasing considerable interest in MICPs, the theoretical and practical research on MICPs is growing rapidly. This growing demand for solving MICPs has recently led many commercial software packages such as CPLEX [2], Gurobi [1], and MOSEK [3] to expand their features and include the technology to solve MICPs. Nevertheless, the theory and algorithms for solving MICPs are still in their infancy [7]. On one hand, any method for general nonlinear integer programming applies to MICPs as well. A significant body of work has extended known techniques dealing with MILPs to nonlinear integer programs (see [15] for a recent survey). Currently, the most promising approaches to solve MICPs are based on the extension of cutting plane techniques in combination with branch-and-bound based algorithms [7, 8, 19, 20, 21, 22, 24, 26, 27, 35, 51, 61, 63, 66, 67, 68]. Exploiting the conic structure when present, as opposed to general convexity, paves the way for developing algorithms with much better performance. Particularly in the case of MILPs, this has led to very successful results. In addition, efficient interior point methods exist for solving conic optimization problems with $\mathcal{K} = \mathcal{L}^n$ or $\mathcal{K} = \mathcal{S}^n_+$ [18]. As a result, supplying the branch-and-bound tree with the natural continuous conic relaxation at the nodes and deriving cutting planes (or surfaces) to strengthen these relaxations have recently gained considerable interest. In this vein, Çezik and Iyengar [27] extended Chvatal-Gomory integer rounding cuts (valid inequalities) [58] to MICPs with general regular cones. In a recent and fast growing literature, several authors [4, 7, 8, 14, 19, 25, 34, 35, 36, 53, 54, 56, 69, 71] study MICPs involving Lorentz cones, $\mathcal{K} = \mathcal{L}^n$, and suggest cutting planes or surfaces.

While the numerical performance of these techniques is still under investigation, evidence from MILPs indicates that adding a small yet strong set of valid inequalities is the key to the success of such procedures. Selection of effective valid inequalities is particularly important to avoid numerical instability issues. Nevertheless, for MICPs, except very specific and simple cases, there is no formal framework studying the strength (redundancy, domination, etc.) of valid inequalities. This is in sharp contrast to the MILP case, where the related questions have been studied extensively. In particular, the feasible region of an MILP with rational data is a polyhedron and the facial structure of a polyhedron (its faces and facets) is very well understood. Besides, various ways of proving whether or not a given linear inequality is a facet, that is, it is necessary in the closed convex hull description of the feasible set associated with an MILP, are well established [58]. In addition, a new framework to demonstrate minimality and extremality of valid inequalities for certain generic infinite relaxations of MILPs [30] as well as their relations to facets in certain simplified settings [32] are developing rapidly. Thus far, results in this vein are lacking in the MICP context. Consequently, establishing a theoretical framework to measure the necessity and strength of valid inequalities in the MICP context remains a natural and important question. We pursue this question in this paper.

In this paper, given a *disjunctive conic set* –the union of finitely or infinitely many conic sets involving a common cone $\mathcal{K}$– we study the linear inequality description of its closed convex hull in the original space of variables. We are mainly motivated by the facts that general disjunctive conic programming framework encompasses MICPs (cf. section 1.2) and most cutting planes used in MILP can be viewed in this framework. The disjunctive conic sets, specifically, the cone $\mathcal{K}$ plays a central role in our developments. In particular, we use $\mathcal{K}$ to identify an appropriate dominance relation among valid linear inequalities and define our $\mathcal{K}$-minimality notion. In the context of MILPs [47, 49] and the associated infinite dimensional relaxations [30], minimality of a valid inequality has traditionally been defined with respect to the nonnegative orthant $\mathbb{R}^n_+$. Our notion of conic minimality not only significantly extends this notion to disjunctive conic sets but also allows us to encode information from a convex relaxation of the problem. Indeed, we show in section 2.2 that even in the context of MILPs defining minimality with respect to a regular polyhedral cone can be valuable. Despite the extensive literature for $\mathcal{K} = \mathbb{R}^n_+$, to the best of our knowledge there is no literature on this topic in the general conic case with an arbitrary regular cone $\mathcal{K}$. In this regard, we contribute to the literature by introducing $\mathcal{K}$-minimal inequalities for disjunctive conic sets and performing a systemic study of their properties for all regular cones $\mathcal{K}$ in a unified manner. In particular, we establish the sufficiency of $\mathcal{K}$-minimal inequalities for describing the closed convex hulls of disjunctive conic sets. Moreover, we provide necessary conditions, sufficient conditions, and practical tools for testing whether or not a given inequality is $\mathcal{K}$-minimal.

When $\mathcal{K}$ is taken as the nonnegative orthant, our approach relates back to the beautiful works of Jeroslow [47] and Johnson [49] as well as the recent work of Conforti et al. [28]. In this particular case of $\mathcal{K} = \mathbb{R}^n_+$, we show that *every $\mathbb{R}^n_+$-minimal inequality (its coefficient vector, and the corresponding best possible right hand side value) is generated by the support function of a specific closed convex set.* This connection in the case of $\mathcal{K} = \mathbb{R}^n_+$, taken together with the sufficiency of $\mathbb{R}^n_+$-minimal inequalities, highlights the roots of functional strong duality results for MILPs. While we capture some of the earlier results from the MILP literature and demonstrate that they naturally extend to MICPs, our study also exposes some challenges associated with MICPs. Specifically, for general regular cones, we show that not all extreme inequalities can be generated by cut-generating functions[1)] when we straightforwardly extend the definition of cut-generating functions from MILP context to MICPs.

Finally, we note that our derivations are based on finite dimensional problems. This is in contrast to much of the literature on minimal inequalities in the MILP context that relies on infinite relaxations initiated by [41, 42, 48]. Therefore, our study does not rely on and differs substantially from the majority of this literature. In a practical cutting plane procedure for solving MILPs and/or MICPs, one is indeed faced with a problem in a finite dimensional space. Thus, we believe that our finite dimensional focus is not a limitation but rather a contribution to the corresponding MILP literature. Besides, to the best of our knowledge, the extensions of other well-known regular cones such as $\mathcal{L}^n$ and $\mathcal{S}^n_+$ to the infinite dimensional spaces are either not well defined or quite nontrivial. Hence, an infinite relaxation seems to be more meaningful when the associated cone is the nonnegative orthant. Nevertheless, we discuss some connections of our work with cut-generating functions and infinite relaxations in section 4.3.

1.1. Preliminaries and Notation In this paper, given a linear map A from a finite dimensional Euclidean space E to $\mathbb{R}^m$, i.e., $A : E \to \mathbb{R}^m$, a convex cone $\mathcal{K} \subset E$, and a *nonempty* set of right hand side vectors $\mathcal{B} \subset \mathbb{R}^m$, we study the *disjunctive conic set* defined by A, $\mathcal{K}$, and $\mathcal{B}$:

$$\mathcal{S}(A, \mathcal{K}, \mathcal{B}) := \{x \in \mathcal{K} : \ Ax \in \mathcal{B}\}.$$

We are mainly interested in determining the properties of linear valid inequalities that are necessary and sufficient for describing the closed convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ denoted by $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ in the original space of variables.

When $\mathcal{B}$ is a convex set given by finitely many conic inequalities, $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ is convex and is also defined by conic inequalities. That said, our focus is on the general case of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with a non-convex set $\mathcal{B}$. Then the linear inequality description is the most general and flexible form of representation for $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ because when $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ is non-convex the algebraic representation of $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ is not necessarily given by finitely many conic inequalities.

We impose no other structural assumptions on A and $\mathcal{B}$. In particular, A is an arbitrary linear map from E to $\mathbb{R}^m$ and $\mathcal{B}$ is an arbitrary non-convex set of vectors in $\mathbb{R}^m$ that can be finite or infinite, structured such as lattice points, or completely unstructured. On the other hand, in order to identify dominance relations

[1)] Informally, a cut-generating function generates the coefficient of a variable in a cut using only information of the instance pertaining to this variable. See [28] and section 4.3 for an extended discussion.

among valid linear inequalities we assume that the cone $\mathcal{K}$ is *regular* (full-dimensional, closed, convex, and pointed). Given a set of regular cones $\mathcal{K}_i \subseteq E_i$ for $i = 1, \ldots, k$, their direct product $\widetilde{\mathcal{K}} = \mathcal{K}_1 \times \ldots \times \mathcal{K}_k$ is also a regular cone in the Euclidean space $\widetilde{E} = E_1 \times \ldots \times E_k$ with inner product $\langle \cdot, \cdot \rangle_{\widetilde{E}}$, which is the sum of the inner products $\langle \cdot, \cdot \rangle_{E_i}$. Therefore, our focus on a single regular cone $\mathcal{K}$ is without loss of generality.

To avoid trial cases, we assume that $\mathcal{S}(A,\mathcal{K},\mathcal{B}) \neq \mathcal{K}$, in particular $\mathcal{K} \not\subseteq \{x \in E : Ax \in \mathcal{B}\}$, and $\mathcal{S}(A,\mathcal{K},\mathcal{B}) \neq \emptyset$, i.e., there exists $b \in \mathcal{B}$ and $x_b \in \mathcal{K}$ satisfying $Ax_b = b$.

For a given set S, we denote its topological interior with $\text{int}(S)$, its closure with $\overline{S}$, and its boundary with $\partial S = \overline{S} \setminus \text{int}(S)$. We use $\text{conv}(S)$ to denote the convex hull of S, $\overline{\text{conv}}(S)$ for its closed convex hull, and $\text{cone}(S)$ to denote the cone generated by the set S. We denote the kernel of a linear map $A : E \to \mathbb{R}^m$ by $\text{Ker}(A) = \{u \in E : \ Au = 0\}$, and its image by $\text{Im}(A) = \{Au : \ u \in E\}$. We use A^* to denote the conjugate linear map [2)] given by the identity

$$y^T Ax = \langle A^* y, x \rangle \ \ \forall (x \in E, y \in \mathbb{R}^m).$$

We use $\langle \cdot, \cdot \rangle$ notation for the inner product in Euclidean space E, and proceed with usual dot product notation with transpose for the inner product in $\mathbb{R}^m$. We assume all vectors in $\mathbb{R}^m$ are given in column form.

For a given cone $\mathcal{K} \subset E$, we let $\text{Ext}(\mathcal{K})$ denote the set of its extreme rays, and use $\mathcal{K}^*$ to denote its dual cone given by

$$\mathcal{K}^* := \{y \in E : \ \langle x, y \rangle \geq 0 \ \ \forall x \in \mathcal{K}\}.$$

Whenever the cone $\mathcal{K}$ is regular, so is $\mathcal{K}^*$.

Given a regular cone $\mathcal{K}$, a relation $a - b \in \mathcal{K}$ (also denoted by $a \succeq_{\mathcal{K}} b$) is called conic inequality between a and b. Such a relation indeed preserves the major properties of the usual coordinate-wise vector inequality $\geq$. We denote the strict conic inequality by $a \succ_{\mathcal{K}} b$ to indicate that $a - b \in \text{int}(\mathcal{K})$. In the sequel, we refer to a constraint of the form $Ax - b \in \mathcal{K}$ as a conic inequality constraint or simply conic constraint and also use $Ax \succeq_{\mathcal{K}} b$ interchangeably in the same sense.

There are three important regular cones common to most MICPs, namely the nonnegative orthant $\mathbb{R}^n_+$, the Lorentz cone $\mathcal{L}^n$, and the positive semidefinite cone $\mathcal{S}^n_+$. In the first two cases, the corresponding Euclidean space E is just $\mathbb{R}^n$ with dot product as the corresponding inner product. In the last case, E becomes the space of symmetric $n \times n$ matrices with Frobenius inner product $\langle x, y \rangle = \text{Tr}(xy^T)$. These three regular cones are also self-dual, that is, $\mathcal{K}^* = \mathcal{K}$.

Notation e^i is used for the i^{th} unit vector of $\mathbb{R}^n$, and Id for the identity map in E. When $E = \mathbb{R}^n$, Id is just the $n \times n$ identity matrix I_n.

1.2. Motivation and Connections to MICPs While the *disjunctive conic set* $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ can be of interest by itself, here we provide a few examples to expose our naming choice and the significance of this framework. In particular, we show that these sets naturally represent the feasible regions of MICPs as well as some natural relaxations for them.

We start with the following example transformation that generalizes the usual *disjunctive programming* from the polyhedral (linear) case [9, 10, 11, 12, 64] to the one with conic constraints.

2) When we consider the standard Euclidean space $E = \mathbb{R}^n$, a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ is just an $m \times n$ real-valued matrix, and its conjugate is given by its transpose, $A^* = A^T$.

The space of symmetric $n \times n$ matrices $E = \mathcal{S}^n$ is also of interest. We use $\text{Tr}(\cdot)$ to denote the trace of a matrix, i.e., the sum of its diagonal entries. When $E = \mathcal{S}^n$, it is natural to specify a linear map $A : \mathcal{S}^n \to \mathbb{R}^m$ as a collection $\{A^1, \ldots, A^m\}$ of m matrices from $\mathcal{S}^n$ such that

$$AZ = (\text{Tr}(ZA^1); \ldots; \text{Tr}(ZA^m)) : \mathcal{S}^n \to \mathbb{R}^m.$$

In this case, the conjugate linear map $A^* : \mathbb{R}^m \to \mathcal{S}^n$ is given by

$$A^* y = \sum_{j=1}^m y_j A^j, \ \ y = (y_1; \ldots; y_m) \in \mathbb{R}^m.$$

EXAMPLE 1. Suppose that we are given a finite collection of convex sets of the form $C_i = \{x \in \mathcal{K} : A^i x \succeq_{\mathcal{K}_i} b^i\}$ for $i \in \{1,\dots,\ell\}$, where $\mathcal{K} \subset \mathbb{R}^n$ is a regular cone and $\mathcal{K}_i \subset \mathbb{R}^{m_i}$ are cones, A^i are $m_i \times n$ matrices, and $b^i \in \mathbb{R}^{m_i}$. Then $\bigcup_{i\in\{1,\dots,\ell\}} C_i$ can be represented in the form of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ as follows:

$$\left\{ x \in \mathbb{R}^n : \underbrace{\begin{pmatrix} (A^1)^T \\ (A^2)^T \\ \vdots \\ (A^\ell)^T \end{pmatrix} x}_{:=Ax} \in \underbrace{\left\{ \begin{pmatrix} \{b^1\}+\mathcal{K}_1 \\ \mathbb{R}^{m_2} \\ \vdots \\ \mathbb{R}^{m_\ell} \end{pmatrix} \bigcup \begin{pmatrix} \mathbb{R}^{m_1} \\ \{b^2\}+\mathcal{K}_2 \\ \vdots \\ \mathbb{R}^{m_\ell} \end{pmatrix} \bigcup \begin{pmatrix} \mathbb{R}^{m_1} \\ \mathbb{R}^{m_2} \\ \vdots \\ \{b^m\}+\mathcal{K}_m \end{pmatrix} \right\}}_{:=\mathcal{B}},\ x \in \mathcal{K} \right\}.$$

When $\mathcal{K} = \mathbb{R}^n_+$ and $\mathcal{K}_i = \mathbb{R}^{m_i}_+$ for all $i = 1,\dots,\ell$, then $\bigcup_{i\in\{1,\dots,\ell\}} C_i$ is the well-known disjunctive set representing the union of polyhedra.

Moreover, when $\mathcal{K}$ is a general regular cone but $\mathcal{K}_i = \mathbb{R}_+$ for all $i = 1,\dots,\ell$, then the set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ models multi-term disjunctions on the cone $\mathcal{K}$. $\diamondsuit$

In fact, the structure of Example 1 amenable to multi-term disjunctions allows us to model the removal of any polyhedral lattice-free set such as triangle, quadrilateral, or cross disjunction from a regular cone (or its cross-section) by appropriately selecting the cones $\mathcal{K}_i$, the matrices A^i, and the vectors b^i. Besides, every convex set $Q \in E$ can be regarded as the cross-section of a convex cone in $E \times \mathbb{R}$ given by $\mathcal{K}_Q :=$ cone$(\{(x,1) \in E \times \mathbb{R} :\ x \in Q\})$ and the hyperplane $H = \{(x,\lambda) \in E \times \mathbb{R} :\ \lambda = 1\}$. Yet, the resulting cone $\mathcal{K}$ may not be regular in general.

Our next set of examples highlights the connection of disjunctive conic sets $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ with the feasible sets of MICPs and their relaxations.

EXAMPLE 2. Suppose that we are given the following MICP

$$\text{Opt} = \inf_{x\in\mathbb{R}^n} \left\{ c^T x :\ \widetilde{A}x = b,\ x \in \mathcal{K},\ x_i \in \mathbb{Z} \text{ for all } i = 1,\dots,\ell \right\}. \tag{1}$$

Let

$$A = \begin{bmatrix} \widetilde{A} \\ I_n \end{bmatrix}, \quad \text{and} \quad \mathcal{B} = \left\{ \begin{pmatrix} b \\ \mathbb{Z}^\ell \\ \mathbb{R}^{n-\ell} \end{pmatrix} \right\},$$

where I_n is the $n \times n$ identity matrix. Then, Opt $= \inf_{x\in\mathbb{R}^n} \{c^T x :\ Ax \in \mathcal{B},\ x \in \mathcal{K}\}$, that is, we optimize the same linear function $c^T x$ over $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. $\diamondsuit$

EXAMPLE 3. Let us also consider another MICP of the form

$$\text{Opt} := \inf_{y\in\mathbb{R}^n} \left\{ \widetilde{c}^T y :\ \widetilde{A}y - b \in \widetilde{\mathcal{K}},\ y_i \in \mathbb{Z} \text{ for all } i = 1,\dots,\ell \right\}, \tag{2}$$

where $\widetilde{\mathcal{K}}$ is a regular cone in the Euclidean space E. Then, by introducing new variables y^+, y^-, and setting

$$x = \begin{pmatrix} y^+ \\ y^- \end{pmatrix}, \quad \mathcal{K} = \mathbb{R}^n_+ \times \mathbb{R}^n_+, \quad c = \begin{pmatrix} \widetilde{c} \\ -\widetilde{c} \end{pmatrix}, \quad A = \begin{bmatrix} \widetilde{A} & -\widetilde{A} \\ I_n & -I_n \end{bmatrix}, \quad \text{and} \quad \mathcal{B} = \left\{ \begin{pmatrix} b + \widetilde{\mathcal{K}} \\ \mathbb{Z}^\ell \\ \mathbb{R}^{n-\ell} \end{pmatrix} \right\},$$

we can once again precisely represent this problem in disjunctive conic form. $\diamondsuit$

There is an important structural difference between the disjunctive conic representations given in Examples 2 and 3: The cone $\mathcal{K}$ of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ in Example 2 is rather general; in particular, it can be any regular cone. On the other hand, the resulting cone used in Example 3 after the transformation is very specific: it is the nonnegative orthant. There are two important distinctions between a general regular cone and the specific case of nonnegative orthant that will appear in our discussion later on in section 3. First, the nonnegative orthant is decomposable, i.e., it does not introduce correlations among variables; and second, all of its extreme rays are orthogonal to each other.

EXAMPLE 4. Let us revisit Example 3 and investigate the following alternative disjunctive conic form given in a lifted space by a single additional variable $t \in \mathbb{R}$. We define

$$x = \begin{pmatrix} y \\ t \end{pmatrix}, \quad \mathcal{K} = \left\{ (y;t) \in \mathbb{R}^n \times \mathbb{R}_+ : \ \widetilde{A}y - bt \in \widetilde{\mathcal{K}} \right\},$$

together with

$$c = \begin{pmatrix} \widetilde{c} \\ 0 \end{pmatrix}, \quad A = \begin{bmatrix} I_\ell & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathcal{B} = \left\{ \begin{pmatrix} \mathbb{Z}^\ell \\ 1 \end{pmatrix} \right\},$$

where I_ℓ is the $\ell \times \ell$ identity matrix. The resulting optimization problem over this disjunctive conic set is also exactly equivalent to (2).

Analogous transformations are possible for Examples 1 and 2 as well. $\diamondsuit$

REMARK 1. The transformation given in Example 3 may seem more attractive in comparison to that of Example 4 because the final disjunctive conic form $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ in Example 3 possesses the very simple conic structure $\mathcal{K} = \mathbb{R}^{2n}_+$. On the other hand, not only does the transformation used in Example 4 get us to a disjunctive conic form with fewer additional variables, but also the new cone $\mathcal{K}$ encodes important structural information about the problem such as the linear map $\widetilde{A}$ and the vector b.

As we detail in section 2, the cone $\mathcal{K}$ plays a critical role in identifying dominance relations among valid inequalities for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. In particular, our minimality notion is based explicitly on the ordering defined by the dual cone $\mathcal{K}^*$. As a result, any structural information encoded in $\mathcal{K}$ is quite useful in identifying the properties of extremal inequalities. In fact, this opens up new possibilities even for the well-studied case of MILPs, which we discuss further in Remark 7. $\diamondsuit$

In Examples 2-4, we provide disjunctive conic sets $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ that represent the corresponding feasible sets of MICPs exactly. This indicates that the explicit description of the resulting $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ is often not easy to characterize. An alternative use of our disjunctive conic framework in the context of MICPs is to obtain and study disjunctive conic form relaxations that are practical, yet still nontrivial and useful. One possibility for obtaining such relaxations in the form of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ is to iteratively add the integrality requirements by changing $\mathbb{R}$ to $\mathbb{Z}$ in the description of the set $\mathcal{B}$ corresponding to a variable x_i.

Another option for developing relaxations in disjunctive conic form is based on a more practical separation problem. Suppose that in Example 2 we have obtained a feasible solution $\hat{x}$ to the continuous relaxation of the MICP, yet $\hat{x} \notin \overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. We can then exploit the following disjunctive conic set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ to identify valid inequalities that cut off $\hat{x}$. Consider $d \in \mathbb{Z}^n$ and $r_0 \in \mathbb{Z}$ such that $d_i = 0$ for all $i = \ell+1,\ldots,n$ and $r_0 < \sum_{i=1}^{\ell} d_i \hat{x}_i < r_0 + 1$. Then the split disjunction induced by $\sum_{i=1}^{\ell} d_i x_i \leq r_0 \ \vee \ \sum_{i=1}^{\ell} d_i x_i \geq r_0 + 1$ is valid for the feasible set of the optimization problem (1), whereas the current solution $\hat{x}$ violates it. Given such a split disjunction, the question of obtaining cuts separating $\hat{x}$ is equivalent to studying $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ where

$$A = \begin{bmatrix} \widetilde{A} \\ d^T \end{bmatrix}, \quad \text{and} \quad \mathcal{B} = \left\{ \begin{pmatrix} b \\ r_0 - \mathbb{R}_+ \end{pmatrix} \bigcup \begin{pmatrix} b \\ r_0 + 1 + \mathbb{R}_+ \end{pmatrix} \right\}.$$

In particular, the inequality description of this $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ will contain cuts for the original MICP separating $\hat{x}$. The same reasoning also applies in the case of Example 3: such a split disjunction in this case can be represented by defining $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ with

$$x = \begin{pmatrix} y^+ \\ y^- \end{pmatrix}, \ \mathcal{K} = \mathbb{R}^n_+ \times \mathbb{R}^n_+, \ A = \begin{bmatrix} \widetilde{A} & -\widetilde{A} \\ d^T & -d^T \end{bmatrix}, \ \text{and} \ \mathcal{B} = \left\{ \begin{pmatrix} b + \widetilde{\mathcal{K}} \\ r_0 - \mathbb{R}_+ \end{pmatrix} \bigcup \begin{pmatrix} b + \widetilde{\mathcal{K}} \\ r_0 + 1 + \mathbb{R}_+ \end{pmatrix} \right\}.$$

We stress that in our discussion above $\hat{x}$ is not restricted to an extreme point solution. The solution $\hat{x}$ will be obtained by solving a continuous relaxation of MICP usually via interior point methods. Therefore, it will not necessarily be an extreme point solution. Nevertheless, our framework is flexible enough as it allows us to study the separation of an arbitrary point $\hat{x} \notin \overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. In contrast, most of the MILP literature, and almost all of the so-called cut-generating function literature, focuses on separating extreme point solutions. The reason for this main focus on the separation of extreme point solutions in theory and practice of MILPs is that the overwhelming choice for solving the linear programming relaxations is the simplex algorithm, which leads to extreme point solutions $\hat{x}$. In the MILP literature, by translation of the associated point $\hat{x}$ and the feasible set, this separation problem is often cast as separating the origin from the convex hull of a set of points.

Nonetheless, the theoretical framework of disjunctive programming in MILP does provide general techniques to separate non-extreme-point solutions in the manner discussed above. Thus, exact representations and relaxations of the above forms have been studied in a number of other contexts in the specific case of $\mathcal{K} = \mathbb{R}^n_+$. In particular, when we additionally assume that $\mathcal{B}$ is finite, we immediately arrive at the *disjunctive programming* framework of Balas [11]. Furthermore, Johnson [49] has studied the set $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ when $\mathcal{B}$ is a finite list under the name of *linear programs with multiple right hand side choice*. In another closely related recent work, Conforti et al. [28] study $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ with $\mathcal{K} = \mathbb{R}^n_+$ and possibly an infinite set $\mathcal{B}$ such that $\mathcal{B} \neq \emptyset$, is closed, and $0 \notin \mathcal{B}$, and demonstrate that Gomory's *corner polyhedron* [40] as well as some other problems such as *linear programs with complementarity restrictions* [50, 59] can be viewed in this framework. In contrast to [11], Johnson [49] studies the characterizations of minimal inequalities, and Conforti et al. [28] study minimal cut-generating functions. We discuss the connections between these and our study in section 4.3.

As opposed to [11, 28, 49], we study general regular cones $\mathcal{K}$, and we are not making any particular assumption on A and $\mathcal{B}$ beyond the basic ones to avoid trivial cases such as $\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \emptyset$ or $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) = \mathcal{K}$. Because $\mathcal{B}$ can be completely arbitrary, the set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ offers great flexibility, which can go far beyond the relaxations/representations related to MICPs. Specifically, a solid understanding of disjunctive conic sets will be particularly relevant to conic complementarity problems.

1.3. Classes of Valid Inequalities and Our Goal We are interested in the closed convex hull characterization of the disjunctive conic set

$$\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \{x \in \mathcal{K} : \ Ax \in \mathcal{B}\}$$

via linear valid inequalities. Without loss of generality, we assume that all of the linear *valid inequalities* for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ are of the form

$$\langle \mu, x \rangle \geq \eta_0,$$

where $\mu \in E$ and $\eta_0 \in \mathbb{R}$. We denote the resulting inequality with $(\mu;\eta_0)$ for shorthand notation. For any $\mu \in E$, we define

$$\vartheta(\mu) := \inf_x \{\langle \mu, x \rangle : \ x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})\} \tag{3}$$

as the best possible right hand side value η_0 for an inequality $(\mu;\eta_0)$ defined by μ to be valid for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. We say that a valid inequality $(\mu;\eta_0)$ is *tight* if $\eta_0 = \vartheta(\mu)$. If both $(\mu;\eta_0)$ and $(-\mu;-\eta_0)$ are valid inequalities, then $\langle \mu, x \rangle = \eta_0$ holds for all $x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$, and in this case, we refer to $(\mu;\eta_0)$ as a *valid equation* for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. We let $\Pi(A,\mathcal{K},\mathcal{B}) \subset E$ be the set of all nonzero vectors $\mu \in E$ leading to nontrivial valid inequalities for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. That is, $\Pi(A,\mathcal{K},\mathcal{B}) = \{\mu \in E : \ \mu \neq 0, \ \vartheta(\mu) \in \mathbb{R}\}$. We denote the convex cone of all valid inequalities given by $(\mu;\eta_0)$ by $C(A,\mathcal{K},\mathcal{B}) \subset E \times \mathbb{R}$. Identifying linear valid inequalities that are necessary in the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ is equivalent to studying $C(A,\mathcal{K},\mathcal{B})$ and its generators.

Note that any convex cone K can be written as the sum of a linear subspace L of $E \times \mathbb{R}$ and a pointed cone C, i.e., $K = L + C$. Let L denote the largest linear subspace contained in K; and define $L^\perp$ as the orthogonal complement of L. Then a unique representation for the pointed cone C in $K = L + C$ is given by $C = K \cap L^{\perp}$. A *generating set* (G_L, G_C) for a cone K is a minimal set of elements from K such that $G_L \subseteq L$, $G_C \subseteq C$, and

$$K = \left\{ \sum_{w \in G_L} \alpha_w w + \sum_{v \in G_C} \lambda_v v : \ \lambda_v \geq 0 \right\}.$$

REMARK 2. Based on the definition of a generating set, we can always select a generating set (G_L, G_C) of any convex cone K, where each vector from G_C is orthogonal to every vector in G_L, and all vectors in G_L are orthogonal to each other. Also, because L is a linear subspace, for any $0 \neq w \in L$, we can always find a generating set (G_L, G_C) where $w \in G_L$ holds by appropriately selecting the other vectors in G_L. $\diamondsuit$

Given $C(A, \mathcal{K}, \mathcal{B})$ is a convex cone in $E \times \mathbb{R}$, our study of $C(A, \mathcal{K}, \mathcal{B})$ will be based on characterizing the properties of the elements $(\mu; \eta_0)$ of its generating sets (G_L, G_C). We will refer to the vectors in G_L as *generating equalities* and the vectors in G_C as *generating inequalities* of $C(A, \mathcal{K}, \mathcal{B})$. An inequality $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$ is called an *extreme inequality* of $C(A, \mathcal{K}, \mathcal{B})$ if there exists a generating set for $C(A, \mathcal{K}, \mathcal{B})$ including $(\mu; \eta_0)$ as a generating inequality either in G_L or in G_C. When the cone $C(A, \mathcal{K}, \mathcal{B})$ is pointed, G_L is trivial and G_C is uniquely defined up to positive scalings. Then, our definition of extreme inequalities based on generating inequalities matches precisely with the usual definition of extreme inequalities stated as "an inequality is extreme if it cannot be written as the average of two other distinct valid inequalities." Note also that any nontight valid inequality $(\mu; \eta_0)$ with $\eta_0 < \vartheta(\mu)$ does not belong to a generating set of $C(A, \mathcal{K}, \mathcal{B})$.

Clearly, the inequalities in a generating set (G_L, G_C) of the cone $C(A, \mathcal{K}, \mathcal{B})$ are of great importance; they are necessary and sufficient for the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$. In such a represetation G_L is finite because a basis of the subspace L can be taken as G_L. For nonpolyhedral (nonlinear) cones such as $\mathcal{K} = \mathcal{L}^n$ with $n \geq 3$, G_C need not be finite.

1.4. Outline The main body of this paper is organized as follows. In section 2, we introduce the class of $\mathcal{K}$-minimal inequalities and show that under a mild assumption, this class of inequalities together with the constraint $x \in \mathcal{K}$ is sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$. We follow this by discussing the importance of the cone $\mathcal{K}$ for defining minimality and establishing a number of necessary conditions for $\mathcal{K}$-minimality. In particular, we show that $\mathcal{K}$-minimal inequalities are tight in many cases. Nonetheless, we highlight that depending on the structure of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$, $\mathcal{K}$-minimality does not necessarily imply tightness of the inequality. We provide algebraic necessary conditions for $\mathcal{K}$-minimality, one of which leads us to our next class of valid inequalities, $\mathcal{K}$-sublinear inequalities. We study $\mathcal{K}$-sublinear inequalities in section 3 and establish a precise relation between $\mathcal{K}$-sublinearity and $\mathcal{K}$-minimality: the set of extreme inequalities in the cone of $\mathcal{K}$-sublinear inequalities contains all of the extreme inequalities from the cone of $\mathcal{K}$-minimal inequalities. In section 4, we show that every $\mathcal{K}$-sublinear inequality is associated with a convex set of particular structure, which we refer to as a *cut-generating set*. Through this connection with cut-generating sets, we provide necessary conditions for $\mathcal{K}$-sublinearity, as well as sufficient conditions for a valid inequality to be $\mathcal{K}$-sublinear and $\mathcal{K}$-minimal. In the case of $\mathcal{K} = \mathbb{R}^n_+$, our necessary condition and sufficient condition for $\mathcal{K}$-sublinearity match precisely and thus results in a strong relation between $\mathcal{K}$-sublinear inequalities and the support functions of cut-generating sets. In particular, we show that for every $\mathbb{R}^n_+$-sublinear inequality, the cut coefficient of any variable and the cut right hand side value are generated by the support function of the associated cut-generating set. This relation provides nice connections with the existing literature, which we highlight in section 4.3. We close section 4 by examining the conic mixed integer rounding inequality from [7] in our framework. We provide some characterizations of the lineality space of $C(A, \mathcal{K}, \mathcal{B})$ in section 5, and finish by stating a few further research questions in section 6.

2. $\mathcal{K}$-Minimal Inequalities In this section, based on the ordering induced by the regular cone $\mathcal{K}^*$, we first state a domination relation among valid linear inequalities for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$. This domination concept immediately leads to a relatively small class of valid linear inequalities, namely *$\mathcal{K}$-minimal inequalities*.

We show that the class of $\mathcal{K}$-minimal inequalities is nonempty under a mild technical assumption, which is satisfied, for example, when $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ is full dimensional. Under this assumption, we establish that $\mathcal{K}$-minimal inequalities along with the constraint $x\in\mathcal{K}$ are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. We then study the properties of inequalities from this class.

We start by pointing out a trivial class of valid linear inequalities for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, which stem from the observation that $\mathcal{S}(A,\mathcal{K},\mathcal{B})\subseteq\mathcal{K}$. The definition of dual cone immediately implies that for any $\delta\in\mathcal{K}^*$, the inequality $\langle\delta,x\rangle\geq 0$ is valid for $\mathcal{K}$, and thus, it is also valid for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. Therefore, $(\delta;0)\in C(A,\mathcal{K},\mathcal{B})$ for any $\delta\in\mathcal{K}^*$. We refer to these inequalities as *cone-implied inequalities*. Note that all cone-implied inequalities are readily captured by the constraint $x\in\mathcal{K}$. Hence, they are not of great interest. In particular, unless $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))=\mathcal{K}$, the family of cone-implied inequalities is not sufficient to fully describe $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. Given our assumption $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))\neq\mathcal{K}$, from now on, we focus on the characterization of valid linear inequalities that are non-cone-implied and are needed to obtain a complete description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. This leads us to our definition of $\mathcal{K}$-minimal inequalities.

DEFINITION 1. A valid linear inequality $(\mu;\eta_0)$ with $\mu\neq 0$ and $\eta_0\in\mathbb{R}$ is $\mathcal{K}$-*minimal* (for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$) if for all valid inequalities $(\rho;\rho_0)$ for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ satisfying $\rho\neq\mu$, and $\rho\preceq_{\mathcal{K}^*}\mu$, we have $\rho_0<\eta_0$.

REMARK 3. In the case of MILP, $\mathcal{K}=\mathbb{R}^n_+$, a *minimal inequality* is defined as a valid linear inequality $(\mu;\eta_0)$ such that if $\rho\leq\mu$ (where the $\leq$ is interpreted in the component-wise sense) and $\rho\neq\mu$, then $(\rho;\eta_0)$ is not valid, i.e., reducing any μ_i for $i\in\{1,\dots,n\}$ will lead to a strict reduction in the right hand side value of the inequality (cf. [47, 49]). Because $\mathbb{R}^n_+$ is a regular and also self-dual cone, $\mathcal{K}$-minimality definition is indeed a natural extension of the *minimality* definition of valid inequalities studied in the context of MILPs to more general disjunctive conic sets with regular cones $\mathcal{K}$. $\diamondsuit$

We next observe that the cone $\mathcal{K}$ indeed induces a natural dominance relation among the valid linear inequalities, and the $\mathcal{K}$-minimality definition is a result of this dominance relation. Let us consider a valid inequality $(\mu;\eta_0)$ that is not $\mathcal{K}$-minimal. Thus, there exists another valid inequality $(\rho;\rho_0)$ such that $\rho\neq\mu$, $\rho\preceq_{\mathcal{K}^*}\mu$, and $\rho_0\geq\eta_0$. But, then

$$\langle\mu,x\rangle=\langle\rho+(\mu-\rho),x\rangle=\underbrace{\langle\rho,x\rangle}_{\geq\rho_0}+\underbrace{\langle\mu-\rho,x\rangle}_{\geq 0}\geq\rho_0\geq\eta_0,$$

where the first inequality follows from $x\in\mathcal{K}$ and $\mu-\rho\in\mathcal{K}^*$. Thus, the inequality $(\rho;\rho_0)$ together with the constraint $x\in\mathcal{K}$ implies the inequality $(\mu;\eta_0)$. Then, when the constraint $x\in\mathcal{K}$ and the linear inequality $(\rho;\rho_0)$ are included, the non-$\mathcal{K}$-minimal inequality $(\mu;\eta_0)$ is not necessary in the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. The definition of $\mathcal{K}$-minimality simply requires an inequality not to be dominated in this fashion: a $\mathcal{K}$-minimal inequality $(\mu;\eta_0)$ cannot be dominated by another inequality, which is the sum of a cone-implied inequality and another valid inequality for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$.

REMARK 4. None of the cone-implied inequalities $(\mu;\eta_0)=(\delta;0)$ with $\delta\in\mathcal{K}^*\setminus\{0\}$ is $\mathcal{K}$-minimal because we can always write them as the sum of a valid inequality $(\rho;\rho_0)=(\frac{1}{2}\delta;0)$ with $\rho_0=0=\eta_0$ and a cone-implied inequality $(\frac{1}{2}\delta;0)$. Nevertheless, a cone-implied inequality can be extreme,[3)] and thus necessary in the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. $\diamondsuit$

REMARK 5. Whether a valid inequality is necessary for the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ depends on $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, and it can very well be independent of the choice of $A,\mathcal{B}$, and $\mathcal{K}$. In particular, if there exist $A',\mathcal{B}'$, and $\mathcal{K}'$ such that $\overline{\text{conv}}(\mathcal{S}(A',\mathcal{K}',\mathcal{B}'))=\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, then the extreme inequalities for these will be the same. Additionally, as long as the set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ remains the same, the $\mathcal{K}$-minimality definition is independent of A and $\mathcal{B}$ but depends on $\mathcal{K}$ explicitly. That is, the $\mathcal{K}$-minimal inequalities for both $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ and $\mathcal{S}(A',\mathcal{K},\mathcal{B}')$ are the same as long as $\overline{\text{conv}}(\mathcal{S}(A',\mathcal{K},\mathcal{B}'))=\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. However, when $\mathcal{K}'\neq\mathcal{K}$, the $\mathcal{K}$-minimal inequalities for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ might differ from the $\mathcal{K}'$-minimal inequalities for $\mathcal{S}(A',\mathcal{K}',\mathcal{B}')$

3) See section 1.3 and the definition of extreme inequalities based on generating inequalities.

even when $\overline{\text{conv}}(\mathcal{S}(A',\mathcal{K}',\mathcal{B}')) = \overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. We comment more on the choice of the cone $\mathcal{K}$ and its impact on identifying dominance relations and our $\mathcal{K}$-minimality definition in Remark 7. ◇

In the light of this remark, from now on we will emphasize the classification of valid inequalities based explicitly on the cone $\mathcal{K}$.

We let $C_m(A,\mathcal{K},\mathcal{B})$ denote the set of $\mathcal{K}$-minimal valid inequalities for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. Note that $C_m(A,\mathcal{K},\mathcal{B})$ is closed under positive scalar multiplication and is thus a cone (but it is not necessarily a convex cone).

In general, there are $\mathcal{K}$-minimal inequalities that are not extreme. In particular, the definition of $\mathcal{K}$-minimality allows for a $\mathcal{K}$-minimal inequality to be implied by the sum of two other non-cone-implied valid inequalities. That said, under a technical assumption, we will show that all non-cone-implied extreme inequalities are $\mathcal{K}$-minimal. Because characterization of extreme inequalities in general is known to be a much more difficult task, in this paper, we limit our focus to the characterization of $\mathcal{K}$-minimal inequalities.

2.1. Existence and Sufficiency of $\mathcal{K}$-Minimal Inequalities We start with the following simple example demonstrating a set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ where the $\mathcal{K}$-minimal inequalities along with the original conic constraint $x\in\mathcal{K}$ are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$.

EXAMPLE 5. Consider the disjunctive conic set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ defined by $\mathcal{K}=\mathcal{L}^3=\mathcal{K}^*$, $A=[-1,0,1]$ [4)] and $\mathcal{B}=\{0,2\}$. That is,

$$\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \{x\in\mathcal{K}:\ -x_1+x_3=0\}\bigcup\{x\in\mathcal{K}:\ -x_1+x_3=2\}.$$

Then, we easily see that

$$\begin{aligned}\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) &= \{x\in\mathbb{R}^3:\ x\in\mathcal{K},\ 0\le -x_1+x_3\le 2\}\\ &= \{x\in\mathbb{R}^3:\ \langle x,\delta\rangle\ge 0\ \forall\delta\in\text{Ext}(\mathcal{K}^*),\ x_1-x_3\ge -2\},\end{aligned}$$

and $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ is closed. Thus, the cone of valid inequalities is given by

$$C(A,\mathcal{K},\mathcal{B}) = \text{cone}\left(\mathcal{K}^*\times\{0\},([1;0;-1];-2)\right).$$

The only non-cone-implied extreme inequality in this description is given by $\mu=[1;0;-1]$ with $\eta_0 = -2=\vartheta(\mu)$. It is easy to see that this inequality is valid and also necessary for the description of the convex hull. To verify that it is in fact $\mathcal{K}$-minimal, consider any $\delta\in\mathcal{K}^*\setminus\{0\}$, and set $\rho=\mu-\delta$. Then the best possible right hand side value ρ_0 for which $\langle\rho,x\rangle\ge\rho_0$ is valid is given by

$$\begin{aligned}\rho_0 &:= \inf_x\{\langle\rho,x\rangle:\ x\in\mathcal{S}(A,\mathcal{K},\mathcal{B})\}\\ &\le \inf_x\{\langle\rho,x\rangle:\ x\in\mathcal{K},\ -x_1+x_3=2\}\\ &= \inf_x\{x_1-x_3-\langle\delta,x\rangle:\ x\in\mathcal{K},\ -x_1+x_3=2\}\\ &= \inf_x\{-2-\langle\delta,x\rangle:\ x\in\mathcal{K},\ -x_1+x_3=2\}\\ &= -2-\sup_x\{\langle\delta,x\rangle:\ x\in\mathcal{K},\ -x_1+x_3=2\}\\ &< -2=\vartheta(\mu),\end{aligned}$$

where the strict inequality follows from the fact that $u=[0;1;2]$ is in the interior of $\mathcal{K}$ and satisfies $-u_1+u_3=2$ (and thus is feasible to the last optimization problem in the above chain), and also for any $\delta\in\mathcal{K}^*\setminus\{0\}$, $\langle\delta,u\rangle>0$. Thus, $(\mu;\eta_0)$ is $\mathcal{K}$-minimal. Finally, all of the other inequalities involved in the description of $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ are of the form $\langle\delta,x\rangle\ge 0$ with $\delta\in\text{Ext}(\mathcal{K}^*)$, and hence, are not $\mathcal{K}$-minimal. ◇

[4)] Throughout this paper, we use Matlab notation with brackets $[\cdot]$ to denote explicit vectors and matrices.

Our goal is to generalize Example 5 and establish that $\mathcal{K}$-minimal inequalities along with the constraint $x \in \mathcal{K}$ are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. However, we need a structural assumption for this result. This assumption is a result of the important fact that there can be situations where none of the inequalities describing $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ is $\mathcal{K}$-minimal even when $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) \subsetneq \mathcal{K}$. To emphasize this technical difficulty and motivate our assumption, let us consider a slightly modified version of Example 5 with a different set $\mathcal{B}$:

EXAMPLE 6. Let $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ be defined with $\mathcal{K} = \mathcal{L}^3$, $A = [-1,0,1]$ and $\mathcal{B} = \{0\}$. Then

$$\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) = \{x \in \mathbb{R}^3 :\ x \in \mathcal{K},\ -x_1 + x_3 = 0\} = \{x \in \mathbb{R}^3 :\ x_1 = x_3,\ x_2 = 0,\ x_1, x_3 \geq 0\}.$$

For this example, we prove that none of the inequalities in the description of $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ is $\mathcal{K}$-minimal. To observe this, let us fix a particular generating set (G_L, G_C) for the cone $C(A,\mathcal{K},\mathcal{B})$. Based on the above representation of $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, we can take for example $G_C = \mathcal{L}^3 \times \{0\}$ and $G_L = (\mu; 0)$ where $\mu = [-1;0;1]$ with $\eta_0 = 0 = \vartheta(\mu)$. Note that all of the inequalities in G_C as well as one side of the valid equation given by $(\mu;0)$ are cone-implied (because $\mu \in \mathcal{L}^3$), and thus are not $\mathcal{K}$-minimal. Moreover, the inequality given by $(-\mu;0)$, e.g., the other side of the valid equation also cannot be $\mathcal{K}$-minimal because $\rho = [1.5;0;-1.5]$ satisfies $\delta = -\mu - \rho = [-0.5;0;0.5] \in \text{Ext}(\mathcal{K}^*)$ and $(\rho;\eta_0)$ is also valid. In fact, for any valid inequality $(\mu;\eta_0)$ that is in the description of $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, there exists $\tau > 0$ such that we can subtract the vector $\delta = \tau[-1;0;1] \in \text{Ext}(\mathcal{K}^*)$ from μ, and still obtain $(\mu - \delta;\eta_0)$ as a valid inequality. Note that the generators of $C(A,\mathcal{K},\mathcal{B})$ are uniquely defined up to shifts by the vector $(\mu;0)$ defining the valid equation; and these shifts do not change the $\mathcal{K}$-minimality properties of the inequalities. $\diamondsuit$

The peculiar situation of Example 6 is a result of the fact that $\mathcal{S}(A,\mathcal{K},\mathcal{B}) \subset \{x \in \mathcal{K} : -x_1 + x_3 = 0\}$, i.e., $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ is contained in a subspace defined by a cone-implied valid equation. The next proposition formally states that this is precisely the situation in which none of the valid linear inequalities, including the extreme ones, is $\mathcal{K}$-minimal.

PROPOSITION 1. *Suppose that there exists $\delta \in \mathcal{K}^* \setminus \{0\}$ such that $\langle \delta, x\rangle = 0$ for all $x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$, i.e., $(\delta;0)$ is a valid equation. Then $C_m(A,\mathcal{K},\mathcal{B}) = \emptyset$.*

Proof. Let $\delta \in \mathcal{K}^* \setminus \{0\}$ be such that $(\delta;0)$ is a valid equation. Consider any valid inequality $(\mu;\eta_0)$. Because $(-\delta;0)$ is also valid, the inequality $(\mu - \delta;\eta_0)$ is valid as well. But then $(\mu;\eta_0)$ is not $\mathcal{K}$-minimal because $\delta \in \mathcal{K}^* \setminus \{0\}$. Given that $(\mu;\eta_0)$ was arbitrary, this implies that there is no $\mathcal{K}$-minimal valid inequality under the hypothesis of the proposition. $\square$

Based on Proposition 1, we make the following assumption when working with $\mathcal{K}$-minimal inequalities in the remainder of this paper:

Assumption 1: For each $\delta \in \mathcal{K}^* \setminus \{0\}$, there exists some $x_\delta \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ such that $\langle \delta, x_\delta\rangle > 0$.
Assumption 1 is indeed not very restrictive and is trivially satisfied, for example, when $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) \neq \mathcal{K}$ and is full dimensional, e.g., when $\text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset$ (see Proposition 4).

Our main result in this section shows that under **Assumption 1**, $\mathcal{K}$-minimal inequalities, along with the constraint $x \in \mathcal{K}$, are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. In particular, we prove that under **Assumption 1**, all extreme inequalities are $\mathcal{K}$-minimal. Given the previous discussion on the dominance relation among linear inequalities and $\mathcal{K}$-minimality, this result is expected. However, to formalize this, we need the following definition: Given two vectors, $u, v \in C$ where C is a cone with lineality space L, u is said to be an *L-multiple* of v if $u = \tau v + \ell$ for some $\tau > 0$, and $\ell \in L$. From this definition, it is clear that if u is an L-multiple of v, then v is also an L-multiple of u. Also, we need the following lemma from [49]:

LEMMA 1. *Suppose v is in a generating set for cone C and there exist $v^1, v^2 \in C$ such that $v = v^1 + v^2$, then v^1, v^2 are L-multiples of v.*

Let (G_L, G_C) be a generating set for the cone $C(A,\mathcal{K},\mathcal{B})$. Recall that whenever the lineality space L of the cone $C(A,\mathcal{K},\mathcal{B})$ is nontrivial, the generating valid inequalities are defined uniquely only up to the L-multiples. We define G_C^+ to be the vectors from G_C that are not L-multiples of any cone-implied inequality $(\delta;0)$ with $\delta \in \mathcal{K}^* \setminus \{0\}$. Then G_C^+ is again uniquely defined only up to L-multiples.

The following result is a straightforward extension of the associated result from [49] given in the linear case to our conic case.

PROPOSITION 2. *Let (G_L, G_C) be a generating set for the cone $C(A,\mathcal{K},\mathcal{B})$. Under* **Assumption 1**, *every valid equation in G_L and every generating valid inequality in G_C^+ is $\mathcal{K}$-minimal.*

Proof. Suppose $(\mu;\eta_0) \in G_L \cup G_C^+$ is not $\mathcal{K}$-minimal. Then there exists a nonzero $\delta \in \mathcal{K}^*$ such that $(\mu - \delta;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$. Because $(\delta;0) \in C(A,\mathcal{K},\mathcal{B})$ and $C(A,\mathcal{K},\mathcal{B})$ is a convex cone, $(\mu+\delta;\eta_0)$ is valid as well. Then Lemma 1 implies that $(\delta;0)$ is an L-multiple of $(\mu;\eta_0)$. Using the definition of G_C^+, we get $(\mu;\eta_0) \in G_L$. Given that $(\delta;0)$ is an L-multiple of $(\mu;\eta_0)$ and G_L is uniquely defined up to L-multiples, we get that $(\delta;0) \in G_L$. Hence, $\langle \delta, x\rangle = 0$ is a valid equation, which contradicts to **Assumption 1**. □

Based on Proposition 2, **Assumption 1** ensures that $C_m(A,\mathcal{K},\mathcal{B}) \neq \emptyset$. Moreover, Proposition 2 along with Remark 2 immediately implies the following results.

COROLLARY 1. *Suppose that* ***Assumption 1*** *holds. Then, any valid equation $(\mu;\vartheta(\mu))$ is $\mathcal{K}$-minimal.*

COROLLARY 2. *Suppose that* ***Assumption 1*** *holds. Then, for any generating set (G_L, G_C) of $C(A,\mathcal{K},\mathcal{B})$, (G_L, G_C^+) generates $C_m(A,\mathcal{K},\mathcal{B})$. In particular, all non-cone-implied extreme inequalities are $\mathcal{K}$-minimal. Thus, the $\mathcal{K}$-minimal inequalities from (G_L, G_C^+) along with the original conic constraint $x \in \mathcal{K}$ are sufficient to describe $\overline{\mathrm{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$.*

Under **Assumption 1**, in the light of Proposition 2 and Corollary 2, we arrive at

$$\begin{aligned}\overline{\mathrm{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) &= \{x \in E:\ x \in \mathcal{K},\ \langle \mu, x\rangle = \eta_0\ \forall(\mu;\eta_0) \in G_L,\ \langle \mu, x\rangle \geq \eta_0\ \forall(\mu;\eta_0) \in G_C^+\}\\ &= \{x \in E:\ x \in \mathcal{K},\ \langle \mu, x\rangle \geq \eta_0\ \forall(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})\}.\end{aligned}$$

REMARK 6. As a result of Corollary 2, under **Assumption 1**, any valid inequality $(\mu;\eta_0)$ for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ is dominated by a set of $\mathcal{K}$-minimal inequalities $(\mu^i;\eta_0^i)$ where $i \in I$ is a set of indices and a cone-implied inequality $(\delta;0)$ with $\delta \in \mathcal{K}^*$ (note that the cone of cone-implied inequalities is convex). That is, $\mu = \sum_{i\in I}\mu^i + \delta$ and $\eta_0 \leq \sum_{i\in I}\eta_0^i$. When $C_m(A,\mathcal{K},\mathcal{B})$ is convex, the set of indices I can be taken as a singleton. ◇

2.2. On the Choice of the Cone $\mathcal{K}$ in Identifying Dominance Relations Next, we scrutinize the importance of the cone $\mathcal{K}$ in establishing dominance relations and in our $\mathcal{K}$-minimality definition.

REMARK 7. Based on Remark 5 and our $\mathcal{K}$-minimality definition, the structural information encoded in the cone $\mathcal{K}$ is rather important in giving a more refined characterization of extreme inequalities, i.e., identifying smaller classes of valid inequalities that are sufficient to describe the closed convex hulls of disjunctive conic sets.

To demonstrate this, let us consider a situation where we are given two disjunctive conic representations of the same set $\mathcal{S}(A_1,\mathcal{K}_1,\mathcal{B}_1) = \mathcal{S}(A_2,\mathcal{K}_2,\mathcal{B}_2) = \mathcal{S}(A,\mathcal{K},\mathcal{B})$ such that $\mathcal{K}_1 \subset \mathcal{K}_2$. Then $C(A,\mathcal{K},\mathcal{B}) = C(A_1,\mathcal{K}_1,\mathcal{B}_1) = C(A_2,\mathcal{K}_2,\mathcal{B}_2)$. From Remark 5, we observe that as long as $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ remains the same the choice of $A,\mathcal{B}$ in our representation does not affect $\mathcal{K}$-minimality definition. In such a case, we argue that the smaller cone $\mathcal{K}_1$ encodes the structural information of the disjunctive conic set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ better than $\mathcal{K}_2$. To avoid technical difficulties, let us assume $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ satisfies **Assumption 1** with respect to $\mathcal{K}_1$ and $\overline{\mathrm{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) \neq \mathcal{K}_1$. Thus $C_m(A_1,\mathcal{K}_1,\mathcal{B}_1)$ is nonempty. The definition of $\mathcal{K}$-minimality together with the relation $\mathcal{K}_1^* \supset \mathcal{K}_2^*$ automatically implies that $\mathcal{K}_1$-minimal inequalities are also $\mathcal{K}_2$-minimal. But the reverse does not necessarily hold because $\mathcal{K}_1^* \neq \mathcal{K}_2^*$. Therefore, $C_m(A_1,\mathcal{K}_1,\mathcal{B}_1) \subsetneq C_m(A_2,\mathcal{K}_2,\mathcal{B}_2)$. Let (G_L, G_C) be a generating set for $C(A,\mathcal{K},\mathcal{B})$. Let us define $G_C^{1;+}$ to be the vectors from G_C that are not

L-multiples of any cone-implied inequality $(\delta;0)$ with $\delta\in\mathcal{K}_1^*\setminus\{0\}$, and $G_C^{2,+}$ analogously with respect to $\mathcal{K}_2^*$. Then Corollary 2 states

$$\begin{aligned}\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) &= \{x\in E:\ x\in\mathcal{K}_1,\ \langle\mu,x\rangle=\eta_0\ \forall(\mu;\eta_0)\in G_L,\ \langle\mu,x\rangle\geq\eta_0\ \forall(\mu;\eta_0)\in G_C^{1,+}\}\\ &= \{x\in E:\ x\in\mathcal{K}_2,\ \langle\mu,x\rangle=\eta_0\ \forall(\mu;\eta_0)\in G_L,\ \langle\mu,x\rangle\geq\eta_0\ \forall(\mu;\eta_0)\in G_C^{2,+}\}.\end{aligned}$$

Moreover, $(G_L,G_C^{1,+})$ generates $C_m(A_1,\mathcal{K}_1,\mathcal{B}_1)$ and $(G_L,G_C^{2,+})$ generates $C_m(A_2,\mathcal{K}_2,\mathcal{B}_2)$ by Corollary 2. Because $G_C^{1,+}\subset G_C$, $G_C^{2,+}\subset G_C$ and $C_m(A_1,\mathcal{K}_1,\mathcal{B}_1)\subsetneq C_m(A_2,\mathcal{K}_2,\mathcal{B}_2)$, we have $G_C^{1,+}\subsetneq G_C^{2,+}$. Thus, all extreme $\mathcal{K}_2$-minimal inequalities are also $\mathcal{K}_1$-minimal, but some $\mathcal{K}_2$-minimal inequalities, namely $G_C^{2,+}\setminus G_C^{1,+}$, are not extreme. Hence, we conclude that whenever we have a choice between two different cones $\mathcal{K}_1\subset\mathcal{K}_2$ representing the same disjunctive conic set, minimality defined with respect to the smaller cone $\mathcal{K}_1$ results in a stronger dominance relation among valid linear inequalities defining $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. $\diamondsuit$

We illustrate the importance of encoding structural information in $\mathcal{K}$ as much as possible, e.g., Remark 7 via the following example: Consider the mixed integer conic set given by

$$\{x\in\mathbb{R}^n_+:\ Ax-b\in\mathcal{K}, x_i\in\mathbb{Z}\ \forall i=1,\ldots,\ell\}.$$

For this set, from the transformations presented in Examples 4 and 3, we obtain two different disjunctive conic representations respectively:

$$\begin{aligned}\mathcal{S}(A_1,\mathcal{K}_1,\mathcal{B}_1) &= \{z=[t;x]\in\mathbb{R}^{n+1}:\ A_1z\in\mathcal{B}_1, z\in\mathcal{K}_1\}\\ \text{with}\quad & A_1=I_{\ell+1},\ \mathcal{B}_1=\left\{\begin{pmatrix}1\\ \mathbb{Z}^\ell\end{pmatrix}\right\},\ \text{and}\ \mathcal{K}_1=\left\{[t;x]\in\mathbb{R}^{n+1}_+:\ Ax-b\,t\in\mathcal{K}\right\},\\ \text{and}\quad \mathcal{S}(A_2,\mathcal{K}_2,\mathcal{B}_2) &= \{z=[t;x]\in\mathbb{R}^{n+1}:\ A_2z\in\mathcal{B}_2, z\in\mathcal{K}_2\}\\ \text{with}\quad & A_2=\begin{bmatrix}0 & A\\ 1 & 0^T\\ 0 & I_\ell\end{bmatrix},\ \mathcal{B}_2=\left\{\begin{pmatrix}b+\mathcal{K}\\ 1\\ \mathbb{Z}^\ell\end{pmatrix}\right\},\ \text{and}\ \mathcal{K}_2=\mathbb{R}^{n+1}_+.\end{aligned}$$

Clearly, $\mathcal{S}(A_1,\mathcal{K}_1,\mathcal{B}_1)=\mathcal{S}(A_2,\mathcal{K}_2,\mathcal{B}_2)$. Moreover, using Remark 5 we note that $A_1,\mathcal{B}_1$ and $A_2,\mathcal{B}_2$ do not affect our definition for $\mathcal{K}_1$-minimality and $\mathcal{K}_2$-minimality respectively.[5)] Because $\mathcal{K}_1\subset\mathcal{K}_2$, by Remark 7, we conclude that $\mathcal{K}_1$-minimality leads to a stronger dominance relation among valid inequalities and results in a more refined characterization of extreme inequalities. Thus we conclude that among these two different choices of disjunctive conic representations of the same set, $\mathcal{S}(A_1,\mathcal{K}_1,\mathcal{B}_1)$ is superior. This is so even when the cone $\mathcal{K}$ is as simple as $\mathbb{R}^m_+$. The following simple numerical example is instrumental to illustrate the shortcoming of defining minimality with respect to $\mathbb{R}^n_+$. In the preceding setup, let us select $A=I_2$, $b=[1.5;1]$, $\mathcal{K}=\mathbb{R}^2_+$, and $\ell=n=2$. Then $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))=\{[t;x_1;x_2]\in\mathbb{R}^3:\ t=1,\ x_1\geq 2,\ x_2\geq 1\}$. Therefore, we can take $G_L=\{([1;0;0];1)\}$ and $G_C=\{([0;1;0];2),([0;0;1];1)\}$ as a generating set for $C(A,\mathcal{K},\mathcal{B})$. Note that $\mathcal{K}_1=\{[t;x_1;x_2]\in\mathbb{R}^3_+:\ -1.5t+x_1\geq 0,\ -t+x_2\geq 0\}$ is a regular cone and $\mathcal{K}_2=\mathbb{R}^3_+$. By Corollary 1, all of the vectors from G_L define valid equations that are both $\mathcal{K}_1$- and $\mathbb{R}^3_+$-minimal. Let us examine the extreme valid inequality given by $x_2\geq 1$, i.e., $(\mu;\eta_0)=([0;0;1];1)$ from G_C. Note that $x_2\geq 1$ is already included in the description of a natural continuous relaxation of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, therefore when our domination concept is strong, we would expect it not to be identified as a minimal inequality. In fact, because $\delta=[-1;0;1]\in\mathcal{K}_1^*$ and $(\mu;\eta_0)-(\delta;0)=([1;0;0];1)$ is a valid inequality for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ we conclude that $(\mu;\eta_0)=([0;0;1];1)\notin\mathcal{C}_m(A,\mathcal{K}_1,\mathcal{B})$, that is, $x_2\geq 1$ is not $\mathcal{K}_1$-minimal. On the other hand, $x_2\geq 1$ is a $\mathbb{R}^3_+$-minimal inequality because it is in G_C and it is not an L-multiple of any $(\delta;0)$ with $\delta\in\mathbb{R}^3_+\setminus\{0\}$.

Therefore, even in the case of MILPs, whenever such structural information, e.g., a polyhedral relaxation, is present, there is a benefit in defining the minimality of an inequality based on a cone $\mathcal{K}$ given in a lifted

[5)] For simplicity, we assume here that $\mathcal{K}_1$ is a regular cone.

space as described above (or in Example 4) as opposed to the usual choice of the nonnegative orthant from the MILP literature. We note that a weaker notion of minimality that only incorporates information from the set $\mathcal{B}$ is also recently studied in Yıldız and Cornuéjols [70] albeit in the context of cut-generating functions for an infinite relaxation.

Our results on $\mathcal{K}$-minimal inequalities, in particular, their importance in identifying dominance relations and their sufficiency motivate us to further study the properties of $\mathcal{K}$-minimal inequalities in the next section.

2.3. Necessary Conditions for $\mathcal{K}$-Minimality Our first proposition states that certain $\mathcal{K}$-minimal inequalities are always tight. This also gives us our first necessary condition for $\mathcal{K}$-minimality.

PROPOSITION 3. *Suppose **Assumption 1** holds; and consider a $\mathcal{K}$-minimal inequality $(\mu;\eta_0)$ with either $\mu \in \mathcal{K}^*$ or $\mu \in -\mathcal{K}^*$. Then, $(\mu;\eta_0)$ is tight, i.e., $\eta_0 = \vartheta(\mu)$ (cf. (3)); and furthermore, $\mu \in \mathcal{K}^*$ (respectively $\mu \in -\mathcal{K}^*$) implies $\vartheta(\mu) > 0$ (respectively $\vartheta(\mu) < 0$).*

Proof. Consider $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$. Then $\mu \neq 0$, because $\mu = 0$ leads to trivial or cone-implied valid inequalities. The validity of $(\mu;\eta_0)$ immediately implies $\eta_0 \leq \vartheta(\mu)$. Assume for contradiction that $\eta_0 < \vartheta(\mu)$. We will consider two cases of $\mu \in \mathcal{K}^* \setminus \{0\}$ and $\mu \in -\mathcal{K}^* \setminus \{0\}$ separately:

(i) $\mu \in \mathcal{K}^* \setminus \{0\}$: Then $\vartheta(\mu) \geq \eta_0 > 0$, because otherwise $(\mu;\eta_0)$ is either a cone-implied inequality or is dominated by a cone-implied inequality, neither of which is possible. Let $\beta := \frac{\eta_0}{\vartheta(\mu)} \in (0,1]$, and consider $\rho = \beta \cdot \mu$. Then $(\rho;\eta_0)$ is a valid inequality because $0 < \beta$, $(\mu;\vartheta(\mu)) \in C(A,\mathcal{K},\mathcal{B})$ and $C(A,\mathcal{K},\mathcal{B})$ is a cone. Also, since $\mu \neq 0$, $\mu - \rho = (1-\beta)\mu \in \mathcal{K}^* \setminus \{0\}$ holds for all $\beta < 1$. Therefore, $(\mu;\eta_0)$ is not $\mathcal{K}$-minimal unless $\beta = 1$, and thus $\eta_0 = \vartheta(\mu) > 0$.

(ii) $-\mu \in \mathcal{K}^* \setminus \{0\}$: Then $(-\mu;0)$ is a cone-implied inequality. Because $\mathcal{S}(A,\mathcal{K},\mathcal{B}) \neq \emptyset$, we cannot satisfy both $(-\mu;0)$ and $(\mu;\vartheta(\mu))$ when $\vartheta(\mu) > 0$. Thus, $\vartheta(\mu) \leq 0$. Moreover, because $\vartheta(\mu) = 0$ implies $\mathcal{S}(A,\mathcal{K},\mathcal{B}) \subset \{x \in \mathcal{K} :\ \langle \mu, x\rangle = 0\}$ contradicting **Assumption 1**, we have $\eta_0 \leq \vartheta(\mu) < 0$. Once again let $\beta := \frac{\eta_0}{\vartheta(\mu)} \in [1,\infty)$, and consider $\rho = \beta \cdot \mu$. Then $(\rho;\eta_0)$ is a valid inequality since $\beta \geq 1$, $(\mu;\vartheta(\mu)) \in C(A,\mathcal{K},\mathcal{B})$ and $C(A,\mathcal{K},\mathcal{B})$ is a cone. Given $\mu \in -\mathcal{K}^* \setminus \{0\}$, $\mu - \rho = (1-\beta)\mu \in \mathcal{K}^* \setminus \{0\}$ for all $\beta > 1$. Thus, the $\mathcal{K}$-minimality of $(\mu;\eta_0)$ implies $\beta = 1$ and hence $\eta_0 = \vartheta(\mu) < 0$. □

Clearly, Proposition 3 does not cover all possible cases for μ. As a matter of fact, it is possible for $\mu \not\in \pm\mathcal{K}^*$ to lead to a $\mathcal{K}$-minimal inequality. While one is naturally inclined to believe that a $\mathcal{K}$-minimal inequality $(\mu;\eta_0)$ is always tight, i.e., $\eta_0 = \vartheta(\mu)$, we have the following counterexample.

EXAMPLE 7. Consider the disjunctive conic set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ defined with $A = [-1,1]$, $\mathcal{B} = \{-2,1\}$ and $\mathcal{K} = \mathbb{R}^2_+$. First, note that **Assumption 1** holds because $\{[0;1],[2;0]\} \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$, and $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) = \overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) \neq \mathbb{R}^2_+$. Thus, $\mathcal{K}$-minimal inequalities exist, and together with nonnegativity restrictions they are sufficient to describe $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. In fact,

$$\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) = \{x \in \mathbb{R}^2 :\ -x_1 + x_2 \geq -2,\ x_1 - x_2 \geq -1,\ x_1 + 2x_2 \geq 2,\ x_1, x_2 \geq 0\},$$

and one can easily show that each of the non-cone-implied inequalities in this description is in fact $\mathcal{K}$-minimal. Note that, in this example, $\mathcal{K}$-minimality is the same as the minimality definition used in the usual MILP literature.

Now, let us consider the valid inequality given by $(\mu;\eta_0) = ([1;-1];-2)$. Then $\vartheta(\mu) = -1$; therefore, $(\mu;\eta_0)$ is not tight and is dominated by the valid inequality $x_1 - x_2 \geq -1$. We will show that $(\mu;\eta_0)$ is $\mathcal{K}$-minimal regardless of the fact that it is not tight.

Suppose that $(\mu;\eta_0)$ is not $\mathcal{K}$-minimal. Then there exists $\rho = \mu - \delta$ with $0 \neq \delta \in \mathcal{K}^* = \mathbb{R}^2_+$ such that $(\rho;\eta_0)$ is a valid inequality. This implies

$$\begin{aligned}
-2 = \eta_0 &\leq \inf_x\{\langle \rho, x\rangle :\ x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})\} = \min_x\{\langle \rho, x\rangle :\ x \in \text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))\} \\
&= \min_x\{\langle \rho, x\rangle :\ -x_1 + x_2 \geq -2,\ x_1 - x_2 \geq -1,\ x_1 + 2x_2 \geq 2,\ x_1, x_2 \geq 0\} \\
&= \max_\lambda\{-2\lambda_1 - \lambda_2 + 2\lambda_3 :\ -\lambda_1 + \lambda_2 + \lambda_3 \leq \rho_1,\ \lambda_1 - \lambda_2 + 2\lambda_3 \leq \rho_2,\ \lambda \in \mathbb{R}^3_+\} \\
&= \max_\lambda\{-2\lambda_1 - \lambda_2 + 2\lambda_3 :\ -\lambda_1 + \lambda_2 + \lambda_3 \leq 1 - \delta_1,\ \lambda_1 - \lambda_2 + 2\lambda_3 \leq -1 - \delta_2,\ \lambda \in \mathbb{R}^3_+\},
\end{aligned}$$

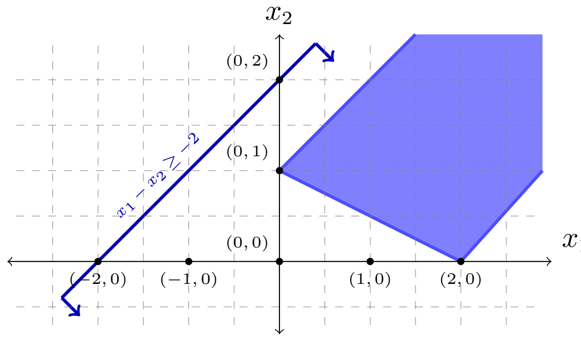

FIGURE 1. Convex hull of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ for Example 7

where the third equation follows from strong duality because the primal problem is feasible, and the last equation follows from the definition of $\rho = \mu - \delta$. On the other hand, the following system

$$\begin{aligned} \lambda &\geq 0 \\ \lambda_1 - \lambda_2 - \lambda_3 &\geq \delta_1 - 1 \\ -\lambda_1 + \lambda_2 - 2\lambda_3 &\geq 1 + \delta_2, \end{aligned}$$

implies that $0 \geq -3\lambda_3 \geq \delta_1 + \delta_2$. Considering that $\delta \in \mathbb{R}^2_+$, this leads to $\delta_1 = \delta_2 = 0$, which contradicts $\delta \neq 0$. Therefore, $(\mu;\eta_0) = ([1;-1];-2) \in C_m(A,\mathcal{K},\mathcal{B})$ yet $\eta_0 < \vartheta(\mu)$. ♢

REMARK 8. This issue of the nontightness of some $\mathcal{K}$-minimal inequalities is independent of whether the $\mathcal{K}$-minimal inequality separates the origin. When we consider a variation of Example 7 given by $A = [-1,1]$, $\mathcal{B} = \{-2,-1\}$ and $\mathcal{K} = \mathbb{R}^2_+$, the inequality given by $(\mu;\eta_0) = ([1;-1];\frac{1}{2})$ is valid and separates the origin from the closed convex hull. Moreover, following the same reasoning of Example 7, we can show that this inequality $(\mu;\eta_0)$ is $\mathcal{K}$-minimal, whereas it has $\vartheta(\mu) = 1$ and hence it is not tight. ♢

The pathology illustrated by Example 7 is because of the facts that $\text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset$ and $\mu \in \text{Im}(A^*)$. We have the following proposition handling such cases in general.

PROPOSITION 4. *Suppose* $\text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset$*. Then, for any* $\mu \in \text{Im}(A^*)$ *and any* $-\infty < \eta_0 \leq \vartheta(\mu)$*, we have* $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$.

Proof. Consider $d \in \text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset$; then $d \neq 0$. For any $b \in \mathcal{B}$, define the set $\mathcal{S}_b := \{x \in E : \ Ax = b, \ x \in \mathcal{K}\}$, and let $\widehat{\mathcal{B}} := \{b \in \mathcal{B} : \ \mathcal{S}_b \neq \emptyset\}$. Because $\mathcal{S}(A,\mathcal{K},\mathcal{B}) \neq \emptyset$, we have $\widehat{\mathcal{B}} \neq \emptyset$. For any $b \in \widehat{\mathcal{B}}$, let $x_b \in \mathcal{S}_b$. Then, for any $b \in \widehat{\mathcal{B}}$, $P_b := \{x_b + \tau d : \ \tau \geq 0\} \subseteq \mathcal{S}_b$ and $P_b \cap \text{int}(\mathcal{K}) \neq \emptyset$. Thus, **Assumption 1** holds.

Assume for contradiction that there exists $\mu \in \text{Im}(A^*)$ together with $\eta_0 \leq \vartheta(\mu)$ such that $(\mu;\eta_0) \notin C_m(A,\mathcal{K},\mathcal{B})$. Then there exists $\delta \in \mathcal{K}^* \setminus \{0\}$ such that $(\mu - \delta;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$, which implies

$$\begin{aligned} -\infty < \eta_0 &\leq \inf_x \{\langle \mu - \delta, x\rangle : \ x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})\} \\ &\leq \inf_{b \in \mathcal{B}} \inf_x \{\langle \mu - \delta, x\rangle : \ Ax = b, \ x \in \mathcal{K}\} \\ &\leq \inf_{b \in \widehat{\mathcal{B}}} \inf_x \{\langle \mu - \delta, x\rangle : \ x \in P_b\} \\ &= \inf_{b \in \widehat{\mathcal{B}}} \left[\underbrace{\langle \mu - \delta, x_b\rangle}_{\in \mathbb{R}} + \inf_\tau \{\langle \mu - \delta, d\rangle \tau : \ \tau \geq 0\} \right]. \end{aligned}$$

When $\langle \mu - \delta, d\rangle < 0$, we have $\inf_\tau \{\langle \mu - \delta, d\rangle \tau : \ \tau \geq 0\} = -\infty$ implying $-\infty < \eta_0 \leq -\infty$, which is impossible. Therefore, $\langle \mu - \delta, d\rangle \geq 0$.

Moreover, because $\mu \in \text{Im}(A^*)$, there exists λ such that $\mu = A^*\lambda$. This together with $d \in \text{Ker}(A)$ implies

$$0 \leq \langle \mu - \delta, d \rangle = \langle A^*\lambda, d \rangle - \langle \delta, d \rangle = \lambda^T \underbrace{(Ad)}_{=0} - \langle \delta, d \rangle = -\langle \delta, d \rangle.$$

But, this contradicts $\langle \delta, d \rangle > 0$, which holds since $d \in \text{int}(\mathcal{K})$ and $\delta \in \mathcal{K}^* \setminus \{0\}$. □

Proposition 4 as well as its demonstration in Example 7 indicate a weakness in the $\mathcal{K}$-minimality definition. To address this, we should focus on only *tight $\mathcal{K}$-minimal* inequalities, that is, $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$ where $\eta_0 = \vartheta(\mu)$ (η_0 cannot be increased without making the current inequality invalid). While we can include a tightness requirement in our $\mathcal{K}$-minimality definition, we note that tightness has a direct characterization through $\vartheta(\mu)$. Also, to remain consistent with the original minimality definition for $\mathcal{K} = \mathbb{R}^n_+$, we opt to work with our original $\mathcal{K}$-minimality definition. As will be clear from the rest of the paper, tightness considerations require minimal change in our analysis.

We next state a proposition that identifies a key necessary condition for $\mathcal{K}$-minimality via a certain non-expansiveness property. The following set of linear maps will be of importance for this result.

$$\mathcal{F}_{\mathcal{K}} := \{(Z : E \to E) : \ Z \text{ is a linear map, and } Z^*v \in \mathcal{K} \ \forall v \in \mathcal{K}\},$$

where Z^* denotes the conjugate linear map of Z.[6)]

PROPOSITION 5. *Let $(\mu;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$ and suppose that there exists a linear map $Z \in \mathcal{F}_{\mathcal{K}}$ such that $AZ^* = A$, and $\mu - Z\mu \in \mathcal{K}^* \setminus \{0\}$. Then $(\mu;\eta_0) \notin C_m(A,\mathcal{K},\mathcal{B})$.*

Proof. Let $(\mu;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$ and Z be a linear map as described in the proposition. Since $Z \in \mathcal{F}_{\mathcal{K}}$, for any $x \in \mathcal{K}$, we have $Z^*x \in \mathcal{K}$. Moreover, $AZ^*x = Ax$ due to $AZ^* = A$, and thus for any $x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$, $AZ^*x = Ax \in \mathcal{B}$. Therefore, we have $Z^*x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ for any $x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$. Now, let $\delta = \mu - Z\mu$, then $\delta \in \mathcal{K}^* \setminus \{0\}$ by the premise of the proposition. Define $\rho := \mu - \delta$, then for any $x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ we have

$$\langle \rho, x \rangle = \langle \mu - \delta, x \rangle = \langle Z\mu, x \rangle = \langle \mu, Z^*x \rangle \geq \eta_0,$$

where the last inequality follows from the fact that $Z^*x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ and $(\mu;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$. Hence, we get

$$\inf_x \{\langle \rho, x \rangle : \ Ax \in \mathcal{B}, \ x \in \mathcal{K}\} \geq \eta_0,$$

which implies that $(\mu;\eta_0) \notin C_m(A,\mathcal{K},\mathcal{B})$ because $(\mu;\eta_0) = (\rho;\eta_0) + (\delta;0)$ with $(\rho;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$ and $0 \neq \delta \in \mathcal{K}^*$. □

Proposition 5 states an involved necessary condition for a valid inequality to be $\mathcal{K}$-minimal. It states that $(\mu;\eta_0)$ is a $\mathcal{K}$-minimal inequality only if the following holds:

$$(\text{Id} - Z)\mu \notin \mathcal{K}^* \setminus \{0\} \ \ \forall Z \in \mathcal{F}_{\mathcal{K}} \text{ such that } AZ^* = A.$$

Based on this result, the set $\mathcal{F}_{\mathcal{K}}$ has certain importance. In fact $\mathcal{F}_{\mathcal{K}}$ is the cone of $\mathcal{K}^* - \mathcal{K}^*$ positive maps, which also appear in applications of robust optimization and quantum physics (cf. [17]). When $\mathcal{K} = \mathbb{R}^n_+$, $\mathcal{F}_{\mathcal{K}} = \{Z \in \mathbb{R}^{n\times n} : \ Z_{ij} \geq 0 \ \forall i,j\}$. However, in general, the description of $\mathcal{F}_{\mathcal{K}}$ can be rather nontrivial for different cones $\mathcal{K}$. In fact, Ben-Tal and Nemirovski [17] has shown that deciding whether a given linear map takes $\mathcal{S}^n_+$ to itself is an NP-Hard optimization problem. In another case of interest, when $\mathcal{K} = \mathcal{L}^n$, a quite nontrivial explicit description of $\mathcal{F}_{\mathcal{K}}$ via linear matrix inequalities is given by Hildebrand in [44, 45]. Given the general difficulty of characterizing $\mathcal{F}_{\mathcal{K}}$, and thus, testing the necessary condition of $\mathcal{K}$-minimality given in Proposition 5, in the next section, we study a relaxed version of the condition from Proposition 5. This leads to a larger class of valid inequalities, namely $\mathcal{K}$-*sublinear inequalities*, which subsumes the class of $\mathcal{K}$-minimal inequalities.

6) Given a linear map $Z : E \to E$, we use Z^* to denote its conjugate map given by the identity

$$\langle x, Zv \rangle = \langle Z^*x, v \rangle \ \ \forall (x \in E, v \in E).$$

3. $\mathcal{K}$-Sublinear Inequalities

DEFINITION 2. An inequality $(\mu;\eta_0)$ with $\mu \neq 0$ and $\eta_0 \in \mathbb{R}$ is $\mathcal{K}$-*sublinear* (for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$) if it satisfies the conditions (**A.1**(α)) for all $\alpha \in \text{Ext}(\mathcal{K}^*)$ and (**A.2**) where

$$\begin{aligned} (\mathbf{A.1}(\alpha)) \quad & 0 \leq \langle \mu, u\rangle \text{ for all } u \in E \text{ s.t. } Au = 0 \text{ and } \langle \alpha, v\rangle u + v \in \mathcal{K} \;\; \forall v \in \text{Ext}(\mathcal{K}), \\ (\mathbf{A.2}) \quad & \mu_0 \leq \langle \mu, x\rangle \text{ for all } x \in \mathcal{S}(A,\mathcal{K},\mathcal{B}). \end{aligned}$$

When an inequality satisfies (**A.1**(α)) for all $\alpha \in \text{Ext}(\mathcal{K}^*)$ we say that it satisfies condition (**A.1**).

It can be easily verified that the set of inequalities $(\mu;\eta_0)$ satisfying conditions (**A.1**)-(**A.2**) leads to a convex cone in the space $E \times \mathbb{R}$. We denote this cone of $\mathcal{K}$-sublinear inequalities with $C_s(A,\mathcal{K},\mathcal{B})$.

Condition (**A.2**) simply ensures the validity of a given inequality, and thus, it is satisfied by every valid inequality. On the other hand, condition (**A.1**) is not very intuitive. The main role of condition (**A.1**) is to ensure the necessary non-expansivity condition for $\mathcal{K}$-minimality established in Proposition 5.

A particular and simple case of (**A.1**) is of interest and deserves a separate treatment:

Let $(\mu;\eta_0)$ satisfy (**A.1**). Then $(\mu;\eta_0)$ also satisfies the following condition:

$$(\mathbf{A.0}) \quad 0 \leq \langle \mu, u\rangle \text{ for all } u \in \mathcal{K} \cap \text{Ker}(A).$$

To see that (**A.0**) is in fact a special case of (**A.1**), consider any $u \in \mathcal{K} \cap \text{Ker}(A)$. Also, for any $\alpha \in \text{Ext}(\mathcal{K}^*)$, we have $\langle \alpha, v\rangle \geq 0$ for all $v \in \text{Ext}(\mathcal{K})$. Then, because $u \in \mathcal{K}$ and $\mathcal{K}$ is a cone, the requirement of condition (**A.1**) on u, is automatically satisfied for any $u \in \mathcal{K} \cap \text{Ker}(A)$.

Besides, condition (**A.0**) is precisely equivalent to

$$(\mathbf{A.0}) \quad \mu \in (\mathcal{K} \cap \text{Ker}(A))^* = \mathcal{K}^* + (\text{Ker}(A))^* = \mathcal{K}^* + \text{Im}(A^*),$$

where the last equation follows from the facts that $\text{Ker}(A)^* = \text{Ker}(A)^\perp = \text{Im}(A^*)$ and $\mathcal{K}^* + \text{Im}(A^*)$ is closed whenever $\mathcal{K}$ is closed [62, Corollary 16.4.2].

While condition (**A.1**) immediately implies (**A.0**), treating (**A.0**) separately seems to be handy as some of our results depend solely on conditions (**A.0**) and (**A.2**). We next show that condition (**A.0**) is necessary for any nontrivial valid inequality. Recall that we denote the set of nontrivial valid inequalities by $\Pi(A,\mathcal{K},\mathcal{B}) = \{\mu \in E : \; \mu \neq 0, \; \vartheta(\mu) \in \mathbb{R}\}$.

PROPOSITION 6. *Suppose $\mu \in \Pi(A,\mathcal{K},\mathcal{B})$. Then μ satisfies condition (**A.0**).*

Proof. Suppose condition (**A.0**) is violated by some $\mu \in \Pi(A,\mathcal{K},\mathcal{B})$. That is, there exists $u \in \mathcal{K} \cap \text{Ker}(A)$ such that $\langle \mu, u\rangle < 0$. Then, for any $\beta > 0$ and $x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$, $x + \beta u \in \mathcal{K}$ and $A(x+\beta u) = Ax \in \mathcal{B}$. Hence $x + \beta u \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ for all $\beta > 0$. On the other hand, the term,

$$\langle \mu, x + \beta u\rangle = \langle \mu, x\rangle + \beta\langle \mu, u\rangle,$$

can be made arbitrarily small by increasing β, which implies $\vartheta(\mu) = -\infty$ where $\vartheta(\mu)$ is as defined in (3), contradicting $\mu \in \Pi(A,\mathcal{K},\mathcal{B})$. $\square$

As a consequence of Proposition 6, we conclude that to obtain $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ one is required to add only an appropriate subset of valid inequalities $(\mu;\vartheta(\mu))$ with $\mu \in \mathcal{K}^* + \text{Im}(A^*)$.

Our next theorem states that every $\mathcal{K}$-minimal inequality is also $\mathcal{K}$-sublinear.

THEOREM 1. *If $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$, then $(\mu;\eta_0) \in C_s(A,\mathcal{K},\mathcal{B})$.*

Proof. Consider any $\mathcal{K}$-minimal inequality $(\mu;\eta_0)$. Because $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$, $(\mu;\eta_0)$ is valid for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, and hence, condition (**A.2**) is automatically satisfied.

Assume for contradiction that $(\mu;\eta_0)$ violates condition (**A.1**(α)) for some $\alpha \in \text{Ext}(\mathcal{K}^*)$. That is, there exists u such that $\langle \mu, u\rangle < 0$, $Au = 0$, and $\langle \alpha, v\rangle u + v \in \mathcal{K} \;\; \forall v \in \text{Ext}(\mathcal{K})$. Based on u and α, let us define a linear map $Z : E \to E$ as

$$Zx = \langle x, u\rangle \alpha + x \;\text{ for any } x \in E.$$

Note that $A : E \to \mathbb{R}^m$ and thus its conjugate $A^* : \mathbb{R}^m \to E$. We let $A^* e^i =: A^i \in E$ for $i = 1, \ldots, m$, where e^i is the i^{th} unit vector in $\mathbb{R}^m$. This way, we have $ZA^* e^i = \langle A^i, u\rangle \alpha + A^i = A^i$ for all $i = 1, \ldots, m$ because $u \in \text{Ker}(A)$ implies $\langle A^i, u\rangle = 0$. Therefore, $ZA^* = A^*$. Also, since $A : E \to \mathbb{R}^m$ and $Z : E \to E$ are linear maps, ZA^* is a linear map and its conjugate is given by $AZ^* = A$ as desired.

Moreover, for all $w \in \mathcal{K}^*$ and $v \in \text{Ext}(\mathcal{K})$,

$$\langle Zw, v\rangle = \langle (\langle w, u\rangle \alpha + w), v\rangle = \langle w, u\rangle\langle \alpha, v\rangle + \langle w, v\rangle = \langle w, \underbrace{\langle \alpha, v\rangle u + v}_{\in \mathcal{K}}\rangle \geq 0.$$

Because any $v \in \mathcal{K}$ can be written as a convex combination of points from $\text{Ext}(\mathcal{K})$, we conclude that $Z \in \mathcal{F}_{\mathcal{K}}$. Finally, by recalling that $\alpha \in \mathcal{K}^*$ and $\alpha \neq 0$, we get

$$\mu - Z\mu = -\underbrace{\langle \mu, u\rangle}_{<0} \alpha \in \mathcal{K}^* \setminus \{0\},$$

which is a contradiction with the necessary condition for $\mathcal{K}$-minimality given in Proposition 5. □

The proof of Theorem 1 reveals the importance of condition (**A.1**) and its implications in terms of $\mathcal{K}$-minimality. Next, we show that condition (**A.1**) further simplifies in the case of $\mathcal{K} = \mathbb{R}^n_+$, and conditions (**A.0**)-(**A.2**) underlie the *subadditive inequalities* defined for MILPs in [49].

REMARK 9. When the cone $\mathcal{K}$ has a simple structure, in particular, when it has finitely many extreme rays that are orthogonal to each other, the interesting cases of condition (**A.1**) that are not covered by condition (**A.0**) can be simplified. When in addition the cone $\mathcal{K}$ is assumed to be regular, without loss of generality, we can assume that $\mathcal{K} = \mathbb{R}^n_+$.

Suppose $\mathcal{K} = \mathbb{R}^n_+$. Then the extreme rays of $\mathcal{K}$ as well as $\mathcal{K}^*$ are just the unit vectors, e^i. Let us consider (**A.1**(α)) for the case of $\alpha = e^i$. Then the vectors u considered in the condition (**A.1**(e^i)) are required to satisfy

$$v_i u + v \in \mathcal{K} \;\; \forall v \in \text{Ext}(\mathcal{K}) = \{e^1, \ldots, e^n\}.$$

Because all of the extreme rays of $\mathcal{K}$ are unit vectors, this requirement affects only the extreme rays v with a nonzero v_i value, which is just the case of $v = e^i$. Hence, for $i = 1, \ldots, n$, we can equivalently rewrite condition (**A.1**(e^i)) as follows:

$$(\textbf{A.1i}) \quad 0 \leq \langle \mu, u\rangle \text{ for all } u \text{ such that } Au = 0 \text{ and } u + e^i \in \mathbb{R}^n_+.$$

Let a^i denote the i^{th} column of the matrix A. By a change of variables, this requirement is equivalent to the following relation:

$$(\textbf{A.1i}) \quad \mu_i \leq \langle \mu, w\rangle \text{ for all } w \in \mathbb{R}^n_+ \text{ such that } \; Aw = a^i.$$

When $\mathcal{K} = \mathbb{R}^n_+$ and $\mathcal{B}$ is a finite set, Johnson [49] has defined the class of so-called *subadditive valid inequalities* precisely as those inequalities that satisfy the collection of conditions (**A.1i**) for $i = 1, \ldots, n$, along with the conditions (**A.0**) and (**A.2**). In this specific setup, Johnson [49] has further shown that $\mathbb{R}^n_+$-sublinearity of an inequality can be verified by checking requirements (**A.0**), (**A.1i**) for $i = 1, \ldots, n$, and (**A.2**) on only a finite set of vectors (those satisfying a minimal linear dependence condition).

Moreover, let us for a moment assume that there exists a function $\sigma(\cdot)$ underlying the $\mathcal{K}$-sublinear inequality $(\mu; \eta_0)$. That is, for all $i = 1, \ldots, n$, given the data associated with variable x_i, namely a^i, $\sigma(\cdot)$ generates the corresponding coefficient in the valid inequality $\sigma(a^i) = \mu_i$. Then, condition (**A.1i**) above precisely represents the subadditivity property of the function $\sigma(\cdot)$ over the columns of A. In fact, in section 4 for general disjunctive conic sets $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with $\mathcal{K} = \mathbb{R}^n_+$ without making any assumptions on A or $\mathcal{B}$, we show that for every $\mathcal{K}$-sublinear inequality $(\mu; \eta_0)$, such a function $\sigma(\cdot)$ generating μ always exists. In the specific case of $\mathcal{K} = \mathbb{R}^n_+$ and a finite set $\mathcal{B}$, this connection was previously established in [49]. Thus, our result generalizes Johnson's work [49] by removing his assumption that $\mathcal{B}$ is a finite set. We discuss the implications of these with regard to existing MILP literature in detail in section 4.3. ◇

Under **Assumption 1**, there is a precise relation between the generators of the cones of $\mathcal{K}$-sublinear inequalities and $\mathcal{K}$-minimal inequalities. We state this below in Theorem 2, which is a generalization of the corresponding result from [49] to the conic case. For completeness, we include the following proof, which simultaneously simplifies and generalizes the approach of [49].

THEOREM 2. *Suppose that* ***Assumption 1*** *holds. Then, any generating set of $C_s(A,\mathcal{K},\mathcal{B})$ is of the form (G_L,G_s) where $G_s \supseteq G_C^+$ and (G_L,G_C) is a generating set of $C(A,\mathcal{K},\mathcal{B})$. Moreover, if $(\mu;\eta_0)\in G_s\setminus G_C^+$, then $(\mu;\eta_0)$ is not $\mathcal{K}$-minimal.*

Proof. Based on Remark 2, let (G_L,G_C) be a generating set of $C(A,\mathcal{K},\mathcal{B})$ such that each vector in G_C is orthogonal to every vector in G_L, and all vectors in G_L are orthogonal to each other. Let (G_ℓ,G_s) be a generating set of $C_s(A,\mathcal{K},\mathcal{B})$ in which each vector in G_s is orthogonal to every vector in G_ℓ. Note that by Theorem 1, we have $C_m(A,\mathcal{K},\mathcal{B})\subseteq C_s(A,\mathcal{K},\mathcal{B})\subseteq C(A,\mathcal{K},\mathcal{B})$.

Under **Assumption 1**, using Corollary 2, $C_m(A,\mathcal{K},\mathcal{B})$ has a generating set of the form (G_L,G_C^+). Hence, the subspace spanned by G_ℓ both simultaneously contains, and is contained in, the subspace generated by G_L. Therefore, $G_\ell=G_L$.

Let Q be the orthogonal complement to the subspace generated by G_L and define $C'=C(A,\mathcal{K},\mathcal{B})\cap Q$, $C'_m=C_m(A,\mathcal{K},\mathcal{B})\cap Q$ and $C'_s=C_s(A,\mathcal{K},\mathcal{B})\cap Q$. Then $C'=\text{cone}(G_C)$, and under **Assumption 1**, $C'_m=\text{cone}(G_C^+)$. Also, C', C'_m, and C'_s are pointed cones and satisfy $C'_m\subseteq C'_s\subseteq C'$. Given that the elements of G_C^+ are extreme in both C' and C'_m, they remain extreme in C'_s as well. Therefore, $G_C^+\subseteq G_s$.

Finally, consider any $(\mu;\eta_0)\in G_s\setminus G_C^+$. We need to show that $(\mu;\eta_0)\notin C_m(A,\mathcal{K},\mathcal{B})$. Suppose not; then $(\mu;\eta_0)\in C_m(A,\mathcal{K},\mathcal{B})$ but not in G_C^+, which implies that $(\mu;\eta_0)$ is not extreme in $C_m(A,\mathcal{K},\mathcal{B})$. Noting $C_m(A,\mathcal{K},\mathcal{B})\subseteq C_s(A,\mathcal{K},\mathcal{B})$, we conclude that $(\mu;\eta_0)$ is not extreme in $C_s(A,\mathcal{K},\mathcal{B})$ as well. But this contradicts the facts that $(\mu;\eta_0)\in G_s$ and (G_L,G_s) is a generating set for $C_s(A,\mathcal{K},\mathcal{B})$. Therefore, for any $(\mu;\eta_0)\in G_s\setminus G_C^+$, $(\mu;\eta_0)\notin C_m(A,\mathcal{K},\mathcal{B})$. ☐

Theorem 2 implicitly describes a way of obtaining all of the nontrivial extreme valid inequalities of $C(A,\mathcal{K},\mathcal{B})$: first identify a generating set (G_L,G_s) for $C_s(A,\mathcal{K},\mathcal{B})$ and then test its elements for $\mathcal{K}$-minimality to identify G_C^+. On one hand, this is good news, as we seem to have a better algebraic handle on $C_s(A,\mathcal{K},\mathcal{B})$ via the conditions given by **(A.0)-(A.2)**. On the other hand, testing these conditions as stated in **(A.0)-(A.2)** is quite nontrivial. Therefore, simpler conditions for the verification of $\mathcal{K}$-sublinearity and $\mathcal{K}$-minimality are desirable. This task is tackled in the next section.

4. Relations to Support Functions and Cut-Generating Sets In this section, we first relate $\mathcal{K}$-sublinear inequalities to the support functions of sets with certain structure. Recall that a *support function* of a nonempty set $D\subseteq\mathbb{R}^m$ is defined as

$$\sigma_D(z):=\sup_\lambda\{z^T\lambda:\ \lambda\in D\}\quad\text{for any } z\in\mathbb{R}^m.$$

For any nonempty set D, it is well known that its support function $\sigma_D(\cdot)$ satisfies the following properties:
(S.1) $\sigma_D(0)=0$,
(S.2) $\sigma_D(z^1+z^2)\le\sigma_D(z^1)+\sigma_D(z^2)$ (subadditive),
(S.3) $\sigma_D(\beta z)=\beta\sigma_D(z)\ \ \forall\beta>0$ and for all $z\in\mathbb{R}^m$ (positively homogeneous).
In particular, support functions are positively homogeneous and subadditive, and thus sublinear and convex. We refer the reader to [46, 62] for an extended discussion of the topic.

Given Proposition 6, every nontrivial valid linear inequality $(\mu;\eta_0)\in\Pi(A,\mathcal{K},\mathcal{B})$ satisfies $\mu\in\text{Im}(A^*)+\mathcal{K}^*$. Thus, any μ such that $\mu\notin\text{Im}(A^*)+\mathcal{K}^*$ is redundant in the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. As a result of this, nontrivial valid inequalities are closely related to support functions of convex sets with a certain structure. This connection leads to a number of insights into the right hand side values of the valid inequalities as well as necessary conditions for $\mathcal{K}$-sublinearity. We state this connection in a series of results as follows:

THEOREM 3. *Consider any $\mu \in E$ satisfying condition (**A.0**), and define*

$$D_\mu = \{\lambda \in \mathbb{R}^m : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\}. \tag{4}$$

Then, $D_\mu \neq \emptyset$, $\sigma_{D_\mu}(0) = 0$ and $\sigma_{D_\mu}(Az) \leq \langle \mu, z\rangle$ for all $z \in \mathcal{K}$.

Proof. Since μ satisfies condition (**A.0**), $\mu \in \mathcal{K}^* + \mathrm{Im}(A^*)$, which trivially implies the nonemptiness of D_μ. Also, $\sigma_{D_\mu}(0) = 0$ because $\sigma_{D_\mu}(\cdot)$ is the support function of D_μ and $D_\mu \neq \emptyset$. Furthermore, for any $z \in \mathcal{K}$, we have

$$\begin{aligned}\sigma_{D_\mu}(Az) &= \sup_\lambda\{\lambda^T Az : \ \lambda \in D_\mu\} = \sup_\lambda\{\langle z, A^*\lambda\rangle : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\} \\ &\leq \sup_\lambda\{\langle z, \mu\rangle : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\} = \langle z, \mu\rangle,\end{aligned}$$

where the last inequality follows from the fact that $z \in \mathcal{K}$ and for any $\lambda \in D_\mu$, we have $\mu - A^*\lambda \in \mathcal{K}^*$, implying $\langle \mu - A^*\lambda, z\rangle \geq 0$. Therefore, $\sigma_{D_\mu}(Az) \leq \langle \mu, z\rangle$. □

Based on Theorem 3, given a vector $\mu \in \mathrm{Im}(A^*) + \mathcal{K}^*$, we can use the support function of the corresponding set D_μ and easily establish a condition on the right hand side value, η_0, that will ensure the validity of the inequality $(\mu; \eta_0)$.

PROPOSITION 7. *Suppose $\mu \in E$ satisfies condition (**A.0**). Then, $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \leq \vartheta(\mu)$, and thus, any inequality given by $(\mu; \eta_0)$ with $\eta_0 \leq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$ is valid for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$.*

Proof. Given μ satisfying condition (**A.0**), Theorem 3 implies $D_\mu \neq \emptyset$ and $\sigma_{D_\mu}(Az) \leq \langle \mu, z\rangle$. Let $\widehat{\mathcal{B}} := \{b \in \mathcal{B} : \ \exists x \text{ s.t. } Ax = b, \ x \in \mathcal{K}\}$. Then

$$\begin{aligned}\eta_0 \leq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \leq \inf_{b\in\widehat{\mathcal{B}}} \sigma_{D_\mu}(b) &= \inf_{b\in\mathbb{R}^m, x\in E}\left\{\sigma_{D_\mu}(Ax) : \ Ax = b, \ b \in \widehat{\mathcal{B}}\right\} \\ &\leq \inf_x \left\{\sigma_{D_\mu}(Ax) : \ x \in \mathcal{K}, \ Ax \in \widehat{\mathcal{B}}\right\} \\ &\leq \inf_x \left\{\langle \mu, x\rangle : \ x \in \mathcal{K}, \ Ax \in \widehat{\mathcal{B}}\right\} \\ &= \inf_x \{\langle \mu, x\rangle : \ x \in \mathcal{K}, \ Ax \in \mathcal{B}\} = \vartheta(\mu),\end{aligned}$$

where the last inequality follows from the fact that for all $z \in \mathcal{K}$, we have $\sigma_{D_\mu}(Az) \leq \langle \mu, z\rangle$, and the last two equations follow from $\widehat{\mathcal{B}} \subseteq B$ and the definition of $\vartheta(\mu)$ (cf. (3)). Then the validity of the inequality $(\mu; \eta_0)$ with $\eta_0 \leq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$ follows immediately because $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \leq \vartheta(\mu)$. □

Proposition 7 indicates that $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \leq \vartheta(\mu)$ when μ satisfies condition (**A.0**). In certain cases, a much more precise relation between $\vartheta(\mu)$ and $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$ exists as described next.

PROPOSITION 8. *Consider $\mu \in E$ satisfying condition (**A.0**). Suppose at least one of the following conditions holds*

- *$\mathcal{K}$ is polyhedral,*
- $\mathrm{Ker}(A) \cap \mathrm{int}(\mathcal{K}) \neq \emptyset$,
- $\mu \in \mathrm{int}(\mathcal{K}^*) + \mathrm{Im}(A^*)$.

Then, we have $\vartheta(\mu) = \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$.

Proof. By Proposition 7, we already have $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \leq \vartheta(\mu)$. Moreover,

$$\begin{aligned}\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) &= \inf_{b\in\mathcal{B}} \sup_{\lambda\in\mathbb{R}^m} \{b^T\lambda : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\} \\ &= \inf_{b\in\mathcal{B}} \underbrace{\inf_x\{\langle \mu, x\rangle : \ x \in \mathcal{K}, \ Ax = b\}}_{\geq\vartheta(\mu)} \geq \vartheta(\mu),\end{aligned}$$

where the last equation follows from the feasibility of the primal problem and linear programming duality whenever $\mathcal{K}$ is polyhedral and from strong conic duality whenever $\text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset$ or $\mu \in \text{int}(\mathcal{K}^*) + \text{Im}(A^*)$ holds, and the last inequality follows from $b \in \mathcal{B}$, and the definition of $\vartheta(\mu)$ in (3). Thus, we obtain $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) = \vartheta(\mu)$. □

Given μ satisfying condition (**A.0**), there is a unique set D_μ associated with it. Propositions 7 and 8 highlight that one can use the support functions $\sigma_{D_\mu}(\cdot)$ of these sets D_μ to obtain a right hand side value η_0 ensuring the validity of the inequality $(\mu;\eta_0)$. These sets D_μ have a particular importance in our discussion in section 4.3. Because of their common structure, we refer to the sets of this form as *cut-generating sets*. We point out that it is possible to have two distinct vectors $\mu' \neq \mu$ such that $D_\mu = D_{\mu'}$ (cf. Example 8).

4.1. Necessary Conditions for $\mathcal{K}$-Sublinearity We next establish a number of necessary conditions for $\mathcal{K}$-sublinearity via cut-generating sets and their support functions.

LEMMA 2. *For any given $z \in \mathcal{K}$, define*

$$\perp_z := \{\gamma \in \mathcal{K}^* : \langle \gamma, z\rangle = 0\}. \tag{5}$$

*Suppose $\mu \in E$ satisfies condition (**A.0**). Then, for all $z \in \mathcal{K}$ such that $\perp_z \cap(\mu - \text{Im}(A^*)) \neq \emptyset$, we have $\sigma_{D_\mu}(Az) = \langle \mu, z\rangle$ where D_μ is defined by (4).*

Proof. Consider any $z \in \mathcal{K}$ satisfying the premise of the lemma. Then

$$\begin{aligned}\sigma_{D_\mu}(Az) &= \sup_{\lambda\in\mathbb{R}^m} \{\lambda^T Az : \lambda \in D_\mu\}\\ &= \sup_{\gamma\in E,\ \lambda\in\mathbb{R}^m} \{\langle z, A^*\lambda\rangle : A^*\lambda = \mu - \gamma,\ \gamma \in \mathcal{K}^*\}\\ &= \langle z,\mu\rangle - \inf_{\gamma\in E}\{\langle z,\gamma\rangle : \gamma \in \mu - \text{Im}(A^*),\ \gamma\in\mathcal{K}^*\} = \langle z,\mu\rangle,\end{aligned}$$

where the last equation follows because $\langle z,\gamma\rangle \geq 0$ for all $z \in \mathcal{K}$ and $\gamma \in \mathcal{K}^*$, and from the premise of the lemma, there exists $\bar\gamma \in \perp_z \cap(\mu - \text{Im}(A^*))$, that is, $\bar\gamma \in \mathcal{K}^* \cap (\mu - \text{Im}(A^*))$ and $\langle \mu, \bar\gamma\rangle = 0$. □

Whenever $\mu \in \partial\mathcal{K}^* + \text{Im}(A^*)$, we immediately have $\partial\mathcal{K}^* \cap (\mu - \text{Im}(A^*)) \neq \emptyset$; and thus, there exists $z \in \partial\mathcal{K}$ such that $\perp_z \cap(\mu - \text{Im}(A^*)) \neq \emptyset$. In particular, for $\mu \in \text{Im}(A^*)$, $0 \in \mathcal{K}^* \cap (\mu - \text{Im}(A^*))$. Therefore, taking into account condition (**A.0**) and Theorem 3, we have the following corollary:

COROLLARY 3. *For any $\mu \in \partial\mathcal{K}^* + \text{Im}(A^*)$, $D_\mu \neq \emptyset$ and $\sigma_{D_\mu}(Az) = \langle \mu, z\rangle$ holds for at least one $z \in \text{Ext}(\mathcal{K})$ where D_μ is defined as in (4). Moreover, for any $\mu \in \text{Im}(A^*)$, $\sigma_{D_\mu}(Az) = \langle \mu, z\rangle$ for all $z \in \mathcal{K}$.*

In the case of $\mathcal{K} = \mathbb{R}^n_+$, using Remark 9, the relationship between $\mathcal{K}$-sublinearity and the support functions of cut-generating sets can be further enhanced.

THEOREM 4. *Consider a disjunctive conic set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ with $\mathcal{K} = \mathbb{R}^n_+$, and a $\mathcal{K}$-sublinear inequality $(\mu;\eta_0)$ for it. Then, $\perp_{e^i} \cap(\mu - \text{Im}(A^*)) \neq \emptyset$, and thus, $\sigma_{D_\mu}(a^i) = \mu_i$ for all $i = 1,\ldots,n$ where a^i is the i^{th} column of the matrix A. Moreover, $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) = \vartheta(\mu)$.*

Proof. Because $(\mu;\eta_0)$ is $\mathcal{K}$-sublinear where $\mathcal{K} = \mathbb{R}^n_+$, $\mu \in E = \mathbb{R}^n$ satisfies conditions **(A.0)**-**(A.1i)** for all $i = 1,\ldots,n$, and $\eta_0 \leq \vartheta(\mu)$. Assume for contradiction that the statement is not true. Then there exists i such that $\perp_{e^i} \cap(\mu - \text{Im}(A^*)) = \emptyset$. Note that $\perp_{e^i} = \{\gamma \in \mathbb{R}^n_+ : \gamma_i = 0\} = \text{cone}\{e^1,\ldots,e^{i-1},e^{i+1},\ldots,e^n\}$. Therefore, we arrive at the following system of linear inequalities in γ,λ being infeasible:

$$\begin{aligned}\gamma + A^*\lambda &= \mu,\\ \gamma_j &\geq 0 \quad \forall j \neq i,\\ \gamma_i &= 0.\end{aligned}$$

Using Farkas' Lemma, we conclude that $\exists u, v$ such that $u + v = 0$, $v_j \geq 0$ for all $j \neq i$, $Au = 0$ and $\langle u,\mu\rangle \geq 1$. By eliminating u, this implies that $\exists v$ such that $v_j \geq 0$ for all $j \neq i$, $Av = 0$ and $\langle v,\mu\rangle \leq -1$.

Hence, if $v_i < -1$, we can scale v so that $v_i \geq -1$, and arrive at the conclusion that there exists v such that $v + e^i \in \mathbb{R}^n_+ = \mathcal{K}$, $Av = 0$ and $\langle v, \mu \rangle < 0$, which contradicts the condition **(A.1i)**.

Because the conditions **(A.0)-(A.1i)** are necessary for the $\mathcal{K}$-sublinearity of $(\mu;\eta_0)$, we conclude that $\perp_{e^i} \cap (\mu - \text{Im}(A^*)) \neq \emptyset$ for all $i = 1, \ldots, n$. Then Lemma 2 implies $\sigma_{D_\mu}(a^i) = \mu_i$ for all $i = 1, \ldots, n$.

Finally, note that $\mathcal{K} = \mathbb{R}^n_+$ is a polyhedral cone, and thus Proposition 8 implies $\inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b) = \vartheta(\mu)$. $\square$

Theorem 4 has an important consequence, which we point out next.

REMARK 10. Let $(\mu;\eta_0)$ be a $\mathbb{R}^n_+$-sublinear inequality for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Given a linear map $A \in \mathbb{R}^{m \times n}$, let a^i denote the i^{th} column of A. Then Theorem 4 guarantees that for all $i = 1, \ldots, n$ the value of the support function $\sigma_{D_\mu}(\cdot)$ evaluated at the vector a^i, namely the data corresponding to the variable x_i, precisely matches with the corresponding coefficient of x_i in the inequality $(\mu;\eta_0)$, i.e., $\mu_i = \sigma_{D_\mu}(a^i)$. Besides, $\sigma_{D_\mu}(\cdot)$ generates the tightest possible right hand side value for any valid inequality $(\mu;\eta_0)$ defined by the vector μ because $\inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b) = \vartheta(\mu) \geq \eta_0$. Another way to state this is that *every tight $\mathbb{R}^n_+$-sublinear inequality (its coefficient vector, and the corresponding best possible right hand side value) is generated by the support function $\sigma_{D_\mu}(\cdot)$*, a very specific sublinear function. Moreover, when $\mathcal{K} = \mathbb{R}^n_+$, the cut-generating sets D_μ defined in (4) are polyhedral. Precisely, they are of the form

$$D_\mu = \{\lambda \in \mathbb{R}^m : \ A^*\lambda \leq \mu\}.$$

Thus, the support functions of these sets are automatically sublinear (subadditive and positively homogeneous), and in fact piecewise linear and convex. This relates nicely with the literature on lattice-free sets and cut-generating functions. We discuss these in detail in section 4.3.

Furthermore, given any valid inequality $(\mu;\eta_0)$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, if it is not $\mathbb{R}^n_+$-sublinear, using the support function $\sigma_{D_\mu}(\cdot)$, one can immediately obtain an $\mathbb{R}^n_+$-sublinear inequality dominating it (cf. [52, Proposition 3]. $\diamondsuit$

Motivated by the positive result of Theorem 4 given in the specific case of $\mathcal{K} = \mathbb{R}^n_+$, one wonders whether a similar result holds for general regular cones $\mathcal{K}$. We address this question in Proposition 9, and prove that in the case of general regular cones $\mathcal{K}$, for any $\mathcal{K}$-sublinear inequality $(\mu;\eta_0)$, there exists at least one $z \in \text{Ext}(\mathcal{K})$ such that $\sigma_{D_\mu}(Az) = \langle \mu, z \rangle$. Unfortunately, in the case of general regular cones $\mathcal{K}$, the result of Proposition 9 is not as strong as that of Theorem 4. Before we proceed with Proposition 9, we need a few technical lemmas.

LEMMA 3. *For any two sets U and V that are independent of each other, we have*

$$\inf_{u \in U} \inf_{v \in V} \langle u, v \rangle = \inf_{v \in V} \inf_{u \in U} \langle u, v \rangle.$$

Proof. Let us consider a given $\bar{u} \in U$. Then for any $v \in V$, we have $\inf_{u \in U} \langle u, v \rangle \leq \langle \bar{u}, v \rangle$, and by taking the infimum of both sides of this last inequality over $v \in V$, we obtain $\inf_{v \in V} \inf_{u \in U} \langle u, v \rangle \leq \inf_{v \in V} \langle \bar{u}, v \rangle$ holds for any $\bar{u} \in U$. Now, by taking the infimum of this inequality over $\bar{u} \in U$, and noting that the left hand side is simply a constant, we arrive at $\inf_{v \in V} \inf_{u \in U} \langle u, v \rangle \leq \inf_{\bar{u} \in U} \inf_{v \in V} \langle \bar{u}, v \rangle = \inf_{u \in U} \inf_{v \in V} \langle u, v \rangle$. To see that the reverse inequality also holds, we can start by considering a given $\bar{v} \in V$, and repeat the same reasoning by interchanging the roles of u and v. $\square$

LEMMA 4. *Suppose that $\mu \in E$ satisfies condition **(A.0)**, and $\perp_z \cap (\mu - \text{Im}(A^*)) = \emptyset$ holds for all $z \in \text{Ext}(\mathcal{K})$ where $\perp_z$ is as defined by (5). Then, $\mu \in \text{int}(\mathcal{K}^*) + \text{Im}(A^*)$ and $\inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b) = \vartheta(\mu)$.*

Proof. First, note that because μ satisfies condition **(A.0)**, by Theorem 3, $D_\mu \neq \emptyset$; and hence $\{\gamma \in E : \ \exists \lambda \in \mathbb{R}^m \text{ s.t. } \gamma + A^*\lambda = \mu, \ \gamma \in \mathcal{K}^*\} \neq \emptyset$.

In addition, because $0 \in \bigcap_{z \in \text{Ext}(\mathcal{K})} \perp_z$, using the premise of the lemma that $\perp_z \cap (\mu - \text{Im}(A^*)) = \emptyset$, we conclude $0 \notin \mu - \text{Im}(A^*)$. Moreover, by rephrasing the premise of the lemma and the definition of $\perp_z$, we get

$$\begin{aligned} 0 &< \inf_{z \in \text{Ext}(\mathcal{K})} \inf_{\gamma \in E, \ \lambda \in \mathbb{R}^m} \{\langle \gamma, z \rangle : \ \gamma + A^*\lambda = \mu, \ \gamma \in \mathcal{K}^*\} \\ &= \inf_{\gamma \in E, \ \lambda \in \mathbb{R}^m} \left\{ \inf_z \{\langle \gamma, z \rangle : \ z \in \text{Ext}(\mathcal{K})\} : \ \gamma + A^*\lambda = \mu, \ \gamma \in \mathcal{K}^* \right\}, \end{aligned}$$

where in the last equation we have used Lemma 3 with $U = \mathrm{Ext}(\mathcal{K}) \times 0 \subseteq E \times \mathbb{R}^m$ and $V = \{(\gamma, \lambda) \in E \times \mathbb{R}^m : \gamma + A^*\lambda = \mu, \gamma \in \mathcal{K}^*\}$.

Now assume for contradiction that the set $\{\gamma : \exists \lambda \in \mathbb{R}^m \text{ s.t. } \gamma + A^*\lambda = \mu, \gamma \in \mathcal{K}^*\} \subseteq \partial\mathcal{K}^*$. This together with the above inequality implies that there exists $\bar{\gamma} \in \partial\mathcal{K}^*$ such that $\langle \bar{\gamma}, z\rangle > 0$ for all $z \in \mathrm{Ext}(\mathcal{K})$. Hence, $\langle \bar{\gamma}, z\rangle > 0$ for all $z \in \mathcal{K} \setminus \{0\}$. Since $\mathcal{K}^*$ is a closed convex cone, $\langle \bar{\gamma}, z\rangle > 0$ for all $z \in \mathcal{K} \setminus \{0\}$ only if $\bar{\gamma} \in \mathrm{int}(\mathcal{K}^*)$, which is a contradiction. Thus, there exists $\bar{\gamma} \neq 0$ such that $\bar{\gamma} \in \mathrm{int}(\mathcal{K}^*) \cap (\mu - \mathrm{Im}(A^*))$. To finish the proof, note that $\bar{\gamma} \in \mathrm{int}(\mathcal{K}^*) \cap (\mu - \mathrm{Im}(A^*))$ implies $\mu \in \mathrm{int}(\mathcal{K}^*) + \mathrm{Im}(A^*)$. Then using Proposition 8, we arrive at $\vartheta(\mu) = \inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b)$. ☐

We are now ready to state and prove Proposition 9.

PROPOSITION 9. *Suppose that $\mu \in E$ satisfies condition **(A.0)**, and $\perp_z \cap (\mu - \mathrm{Im}(A^*)) = \emptyset$ holds for all $z \in \mathrm{Ext}(\mathcal{K})$ where $\perp_z$ is as defined by (5). Then, there exists at least one $z \in \mathrm{Ext}(\mathcal{K})$ such that $\sigma_{D_\mu}(Az) = \langle \mu, z\rangle$.*

Proof. Assume for contradiction that $\sigma_{D_\mu}(Az) < \langle \mu, z\rangle$ for all $z \in \mathrm{Ext}(\mathcal{K})$. Then by Lemma 4, $\mu \in \mathrm{int}(\mathcal{K}^*) + \mathrm{Im}(A^*)$ and $\inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b) = \vartheta(\mu)$. Because of weak conic duality and $\mu \in \mathrm{int}(\mathcal{K}^*) + \mathrm{Im}(A^*)$, we have for all b

$$\inf_x \{\langle \mu, x\rangle : \ Ax = b, \ x \in \mathcal{K}\} \geq \sigma_{D_\mu}(b) = \sup_{\lambda \in \mathbb{R}^m} \{b^T\lambda : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\} > -\infty.$$

For any $b \in \mathcal{B}$, define $\mathcal{S}_b := \{x \in \mathcal{K} : \ Ax = b\}$, and let $\widehat{B} := \{b \in \mathcal{B} : \ \mathcal{S}_b \neq \emptyset\}$. Because $\mathcal{S}(A, \mathcal{K}, \mathcal{B}) \neq \emptyset$, $\widehat{\mathcal{B}} \neq \emptyset$. Then for any $b \in \widehat{\mathcal{B}}$, $x_b \in \mathcal{S}_b$ leads to an upper bound on $\sigma_{D_\mu}(b)$, i.e., $\sigma_{D_\mu}(b) \leq \langle \mu, x_b\rangle$. Therefore, for any $b \in \widehat{\mathcal{B}}$, the conic optimization problem defining $\sigma_{D_\mu}(b)$ is bounded above and is strictly feasible, and so strong conic duality holds and the dual problem given by the $\inf_x$ above is solvable. Consider any $b \in \widehat{\mathcal{B}}$, and let $\bar{x}_b$ be the corresponding optimal solution, i.e., $\bar{x}_b \in \mathcal{S}_b$ and $\langle \mu, \bar{x}_b\rangle = \sigma_{D_\mu}(b)$. Because $\bar{x}_b \in \mathcal{K}$, there exists $z^1, \ldots, z^\ell \in \mathrm{Ext}(\mathcal{K})$ with $\ell \leq n$ such that $\bar{x}_b = \sum_{i=1}^{\ell} z^i$, which leads to

$$\langle \mu, \bar{x}_b\rangle = \sigma_{D_\mu}(b) = \sigma_{D_\mu}(A\bar{x}_b) \underbrace{\leq}_{(*)} \sum_{i=1}^{\ell} \sigma_{D_\mu}(Az^i) \underbrace{<}_{(**)} \sum_{i=1}^{\ell} \langle \mu, z^i\rangle = \langle \mu, \bar{x}_b\rangle,$$

where the inequality $(*)$ follows because $\sigma_{D_\mu}(\cdot)$ is a support function, and thus is subadditive, and $(**)$ follows from the assumption that $\sigma_{D_\mu}(Az) < \langle \mu, z\rangle$ for all $z \in \mathrm{Ext}(\mathcal{K})$. But this is a contradiction. Thus, there exists $z \in \mathrm{Ext}(\mathcal{K})$ such that $\sigma_{D_\mu}(Az) = \langle \mu, z\rangle$. ☐

To summarize whenever $\mu \in \Pi(A, \mathcal{K}, \mathcal{B})$, Corollary 3, Lemma 4, and Proposition 9 together cover all possible cases and indicate that for a $\mathcal{K}$-sublinear inequality, there exists at least one $z \in \mathrm{Ext}(\mathcal{K})$ such that $\sigma_{D_\mu}(Az) = \langle \mu, z\rangle$.

We illustrate the necessary conditions for $\mathcal{K}$-sublinearity established so far via the following example.

EXAMPLE 8. Consider the set $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with $\mathcal{K} = \mathcal{L}^3$, $A = [1, 0, 0]$ and $\mathcal{B} = \{-1, 1\}$. In this case, $\mathrm{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) = \{x \in \mathbb{R}^3 : \ x \in \mathcal{K}, \ x_3 \geq \sqrt{1 + x_2^2}, \ -1 \leq x_1 \leq 1\}$ (see Figure 2).

Note that this description of $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ involves the following non-cone-implied inequalities:

(a) $\mu^{(+)} = [1; 0; 0]$ with $\eta_0^{(+)} = -1$ and $\mu^{(-)} = [-1; 0; 0]$ with $\eta_0^{(-)} = -1$;

(b) $\mu^{(t)} = [0; t; \sqrt{t^2 + 1}]$ with $\eta_0^{(t)} = 1$ for all $t \in \mathbb{R}$.

Here, we show that these inequalities satisfy the necessary conditions for $\mathcal{K}$-sublinearity; later on we will show that all of these inequalities are in fact $\mathcal{K}$-minimal.

In case (a), it is easy to see that the corresponding sets associated with these inequalities $\mu^{(+)}, \mu^{(-)}$ are given by

$$\begin{aligned} D_{\mu^{(+)}} &= \{\lambda : \ \exists \gamma \in \mathcal{K}^* \text{ s.t. } \lambda + \gamma_1 = 1; \ \gamma_2 = 0; \ \gamma_3 = 0\} = \{\lambda : \ \lambda = 1\}, \\ D_{\mu^{(-)}} &= \{\lambda : \ \lambda = -1\}. \end{aligned}$$

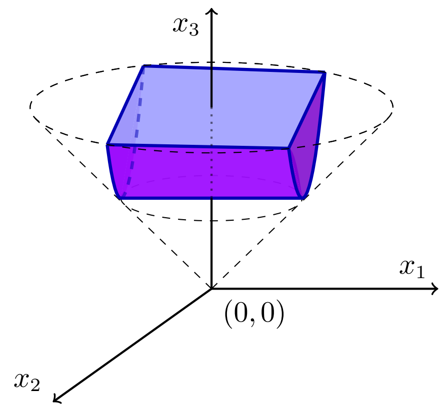

FIGURE 2. Convex hull of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ corresponding to Example 8

Also, both $\mu^{(+)},\mu^{(-)}\in \text{Im}(A^*)$, and thus, by Corollary 3 $\sigma_{D_{\mu^{(i)}}}(Az)=\sigma_{D_{\mu^{(i)}}}(z_1)=\langle\mu^{(i)},z\rangle$ for all $z\in\mathcal{K}$ for $i\in\{+,-\}$. In addition to this, $\inf_{b\in\mathcal{B}}\sigma_{D_{\mu^{(i)}}}(b)=-1=\eta_0^{(i)}$ for $i\in\{+,-\}$.

In case (b), for any given $t\in\mathbb{R}$, we have the associated sets $D_{\mu^{(t)}}$ given by

$$D_{\mu^{(t)}}=\{\lambda:\ \exists\gamma\in\mathcal{K}^* \text{ s.t. } \lambda+\gamma_1=0;\ \gamma_2=t;\ \gamma_3=\sqrt{t^2+1}\}=\{\lambda:\ -1\leq\lambda\leq 1\}.$$

Moreover, for all t, by considering $z^{(t)}\in\{[1;-t;\sqrt{t^2+1}],[-1;-t;\sqrt{t^2+1}]\}\subset \text{Ext}(\mathcal{K})$, we have $\langle\mu^{(t)},z^{(t)}\rangle=1$ and $\sigma_{D_{\mu^{(t)}}}(Az^{(t)})=\sigma_{D_{\mu^{(t)}}}(z_1^{(t)})=\sigma_{D_{\mu^{(t)}}}(1)=1$, proving $\langle\mu^{(t)},z^{(t)}\rangle=\sigma_{D_{\mu^{(t)}}}(Az^{(t)})$. Additionally, $\sigma_{D_{\mu^{(t)}}}(1)=1=\sigma_{D_{\mu^{(t)}}}(-1)$ implying $\inf_{b\in\mathcal{B}}\sigma_{D_{\mu^{(t)}}}(b)=1=\eta_0^{(t)}$ for all t.

We highlight that $D_{\mu^{(t)}}$ is common for all distinct vectors $\mu^{(t)}$ corresponding to the valid inequalities $(\mu^{(t)};1)$. Nevertheless, each of these inequalities $(\mu^{(t)};1)$ is required for the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. This highlights a situation where several vectors μ lead to the same cut-generating set $D=D_\mu$; and we need to consider not only one but a significant number of (in this case infinitely many) such vectors μ associated with a unique cut-generating set to generate valid linear inequalities completely describing $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$.

Let us also consider another valid inequality $(\nu;\nu_0)$ given by $\nu=[0;1;2]$ and $\nu_0=1$. Then the associated set D_ν is given by

$$D_\nu=\left\{\lambda:\ -\sqrt{3}\leq\lambda\leq\sqrt{3}\right\}.$$

Furthermore, for any $z^\nu\in\left\{[\frac{1}{\sqrt{3}};-\frac{1}{3};\frac{2}{3}],[-\frac{1}{\sqrt{3}};-\frac{1}{3};\frac{2}{3}]\right\}\subset\text{Ext}(\mathcal{K})$ we have $\sigma_{D_\nu}(Az^\nu)=\sigma_{D_\nu}(\pm\frac{1}{\sqrt{3}})=1=\langle\nu,z^\nu\rangle$. Also, $\inf_{b\in\mathcal{B}}\sigma_{D_\nu}(b)=\sqrt{3}>1=\nu_0$. Therefore, in terms of the necessary conditions established so far for $\mathcal{K}$-sublinearity, there seems to be no difference between $(\nu;\nu_0)$ and the inequalities $(\mu^{(t)};\eta_0^{(t)})$. When we revisit this example in the next section, we will show that $(\nu;\nu_0)$ is $\mathcal{K}$-sublinear. But, $(\nu;\nu_0)$ is not $\mathcal{K}$-minimal because it is dominated by $\mu^{(1)}=[0;1;\sqrt{2}]$ (note $\delta=\nu-\mu^{(1)}=[0;0;2-\sqrt{2}]\in\mathcal{K}^*\setminus\{0\}$) and $\eta^{(1)}=1$. $\diamondsuit$

4.2. Sufficient Conditions for $\mathcal{K}$-Sublinearity and $\mathcal{K}$-Minimality Given any valid inequality $(\mu;\eta_0)$ satisfying condition (**A.0**), we can easily test $(\mu;\eta_0)$ for $\mathcal{K}$-sublinearity via the following proposition.

PROPOSITION 10. *Let $(\mu;\eta_0)$ be such that μ satisfies condition (**A.0**) and $\eta_0\leq\inf_{b\in\mathcal{B}}\sigma_{D_\mu}(b)$ (or it is known that $(\mu;\eta_0)\in C(A,\mathcal{K},\mathcal{B})$). Then, whenever there exists a collection $i\in I$ of vectors $x^i\in\text{Ext}(\mathcal{K})$ such that $\sigma_{D_\mu}(Ax^i)=\langle\mu,x^i\rangle$ for all $i\in I$ and $\sum_{i\in I}x^i\in\text{int}(\mathcal{K})$, the inequality $(\mu;\eta_0)$ is $\mathcal{K}$-sublinear.*

Proof. When $\eta_0\leq\inf_{b\in\mathcal{B}}\sigma_{D_\mu}(b)$ holds, using Proposition 7, we have $(\mu;\eta_0)\in C(A,\mathcal{K},\mathcal{B})$, which automatically implies that condition (**A.2**) is satisfied.

Next, given any $\alpha \in \text{Ext}(\mathcal{K}^*)$, we verify condition (**A.1**(α)). Consider any u such that $Au = 0$ and $\langle \alpha, v\rangle u + v \in \mathcal{K}\ \forall v \in \text{Ext}(\mathcal{K})$. Let $\mathcal{V}_\alpha = \{v \in \text{Ext}(\mathcal{K}) :\ \langle \alpha, v\rangle = 1\}$. Then $\langle u+v, \gamma\rangle \geq 0$ holds for all $v \in \mathcal{V}_\alpha$ and $\gamma \in \mathcal{K}^*$. Also, there exists $\bar\lambda$ and $\bar\gamma \in \mathcal{K}^*$ satisfying $A^*\bar\lambda + \bar\gamma = \mu$ because μ satisfies condition (**A.0**); that is, $\mu \in \mathcal{K}^* + \text{Im}(A^*)$. In fact, for any such $\bar\lambda, \bar\gamma$, we have

$$\langle \mu, u\rangle = \langle A^*\bar\lambda + \bar\gamma, u\rangle = \langle \bar\lambda, \underbrace{Au}_{=0}\rangle + \langle \bar\gamma, u\rangle \geq \langle \bar\gamma, -v\rangle \quad \forall v \in \mathcal{V}_\alpha.$$

Note that $\langle \gamma, -v\rangle \leq 0$ for all $\gamma \in \mathcal{K}^*$ and $v \in \mathcal{V}_\alpha \subset \mathcal{K}$. To finish the proof, all we need to show is that there exists $\bar v \in \mathcal{V}_\alpha$ such that $\langle \bar\gamma, \bar v\rangle = 0$. Clearly, when $\mu \in \text{Im}(A^*)$, we can take $\bar\gamma = 0$, and thus conclude that $\langle \mu, u\rangle \geq -\langle \bar\gamma, \bar v\rangle = 0$ holds for all such u. In general, we have

$$\begin{aligned}
&\inf_{\gamma,\lambda}\left\{\inf_v\{\langle \gamma, v\rangle :\ v \in \mathcal{V}_\alpha\} :\ A^*\lambda + \gamma = \mu,\ \gamma \in \mathcal{K}^*\right\}\\
&= \inf_v\left\{\inf_{\gamma,\lambda}\{\langle \mu - A^*\lambda, v\rangle :\ A^*\lambda + \gamma = \mu,\ \gamma \in \mathcal{K}^*\} :\ v \in \mathcal{V}_\alpha\right\}\\
&= \inf_v\left\{\langle \mu, v\rangle - \underbrace{\sup_{\gamma,\lambda}\{\lambda^T(Av) :\ A^*\lambda + \gamma = \mu,\ \gamma \in \mathcal{K}^*\}}_{=\sigma_{D_\mu}(Av)} :\ v \in \mathcal{V}_\alpha\right\}
\end{aligned}$$

Because there exists $x^i \in \text{Ext}(\mathcal{K})$ such that $\sigma_{D_\mu}(Ax^i) = \langle \mu, x^i\rangle$ for all $i \in I$ and $\sum_{i\in I} x^i \in \text{int}(\mathcal{K})$, for any $\alpha \in \text{Ext}(\mathcal{K}^*)$, at least one of these x^i's will be in $\mathcal{V}_\alpha$. Otherwise, we have $\langle \alpha, x^i\rangle = 0$ for all $i \in I$, and thus $\langle \alpha, \sum_{i\in I} x^i\rangle = 0$, which is not possible because $\sum_{i\in I} x^i \in \text{int}(\mathcal{K})$ and $\alpha \in \text{Ext}(\mathcal{K}^*)$. Thus, we conclude that the above infimum is zero. This gives us the desired conclusion that $\langle \mu, u\rangle \geq 0$, and hence condition (**A.1**(α)) is satisfied for all $\alpha \in \text{Ext}(\mathcal{K}^*)$. □

REMARK 11. When $\mathcal{K} = \mathbb{R}^n_+$, Theorem 4 together with Proposition 10 implies that the conditions stated in Proposition 10 are necessary and sufficient for $\mathcal{K}$-sublinearity. ◇

For general regular cones $\mathcal{K}$, based on the results from Corollary 3, Lemma 4, and Proposition 9, we conclude that the conditions stated in Proposition 10 are almost necessary. This is up to the fact that for any $\mathcal{K}$-sublinear inequality $(\mu;\eta_0)$, we can prove the existence of at least one $x \in \text{Ext}(\mathcal{K})$ satisfying $\sigma_{D_\mu}(Ax) = \langle \mu, x\rangle$ (cf. Proposition 9), yet the sufficient condition in Proposition 10 requires a number of such extreme rays summing up to an interior point of $\mathcal{K}$. We next provide an example highlighting that for general regular cones $\mathcal{K}$ other than the nonnegative orthant, we cannot close this gap between the sufficient condition and the necessary conditions, i.e., there exists $\mathcal{K}$-sublinear inequalities that satisfy only the necessary condition from Proposition 9 but not the sufficient condition of Proposition 10.

EXAMPLE 9. Consider the disjunctive conic set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ with $\mathcal{K} = \mathcal{L}^3$, $A = [0,1,1]$ and $\mathcal{B} = \{-1,1\}$. Then $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) = \{x \in \mathcal{L}^3 :\ x_2 + x_3 = 1\}$. Let us examine the valid inequality $(\mu;\eta_0)$ given by $\mu = [0;0;1]$ and $\eta_0 = \vartheta(\mu) = \frac{1}{2}$. We will show that there is precisely a single ray $z \in \text{Ext}(\mathcal{K})$ such that $\sigma_{D_\mu}(Az) = \langle \mu, z\rangle$; yet the inequality $(\mu;\eta_0)$ is a $\mathcal{K}$-sublinear inequality.

The cut-generating set associated with μ is $D_\mu = \{\lambda \in \mathbb{R} :\ |\lambda| + \lambda \leq 1\}$. Consider any $z \in \text{Ext}(\mathcal{K}) = \text{Ext}(\mathcal{L}^3)$, and without loss of generality let us assume that z is normalized to have $z_3 = 1$. Then

$$\langle \mu, z\rangle = \sigma_{D_\mu}(Az) \quad \Leftrightarrow \quad z_3 = \sup_{\lambda\in\mathbb{R}}\{\underbrace{(z_2+z_3)}_{\geq 0 \text{ since } z\in\mathcal{L}^3}\lambda :\ |\lambda| + \lambda \leq 1\} \quad \Leftrightarrow \quad z_3 = \frac{1}{2}(z_2+z_3).$$

Therefore, $z_2 = z_3 = 1$. Because $z \in \text{Ext}(\mathcal{L}^3)$, we also have $z_1 = 0$. Thus, there is a unique extreme ray of $\mathcal{L}^3$, in particular $z = [0;1;1]$ that satisfies $\langle \mu, z\rangle = \sigma_{D_\mu}(Az)$.

Let us now prove that $(\mu;\eta_0)$ is indeed $\mathcal{K}$-sublinear. The conditions (**A.0**) and (**A.2**) are easily verified. We need to verify condition (**A.1**(α)) for all $\alpha \in \text{Ext}(\mathcal{K}^*)$. Let $\alpha \in \text{Ext}(\mathcal{K}^*)$ be given. For any $v \in \text{Ext}(\mathcal{K})$, if $\langle \alpha, v\rangle = 0$, we automatically have $\langle \alpha, v\rangle u + v \in \mathcal{K}$. And if $\langle \alpha, v\rangle \geq 0$, we can normalize v to assume that $\langle \alpha, v\rangle = 1$. So, by defining $\mathcal{V}_\alpha := \{v \in \text{Ext}(\mathcal{K}) : \langle \alpha, v\rangle = 1\}$, we can state condition (**A.1**(α)) as

$$0 \leq \langle \mu, u\rangle \text{ for all } u \in E \text{ such that } Au = 0 \text{ and } u + v \in \mathcal{K} \;\; \forall v \in \mathcal{V}_\alpha,$$

which, in our particular case, becomes

$$0 \leq u_3 \text{ for all } u \in \mathbb{R}^3 \text{ such that } u_3 = -u_2 \text{ and } u + v \in \mathcal{L}^3 \;\; \forall v \in \mathcal{V}_\alpha.$$

Because $u_3 = -u_2$, $u + v \in \mathcal{L}^3$ and $v \in \text{Ext}(\mathcal{L}^3)$, we have $u_3 + v_3 \geq 0$ and $u_3^2 + (v_1^2 + v_2^2) + 2u_3v_3 \geq u_3^2 + v_2^2 - 2u_3v_2 + u_1^2 + v_1^2 + 2u_1v_1$, which is equivalent to $2u_3(v_2 + v_3) \geq u_1^2 + 2u_1v_1$. Now suppose that $\alpha_1 = 0$, then $\bar{v} = [\frac{1}{\alpha_3}; 0; \frac{1}{\alpha_3}] \in \mathcal{V}_\alpha$ and $\widetilde{v} = [\frac{-1}{\alpha_3}; 0; \frac{1}{\alpha_3}] \in \mathcal{V}_\alpha$. In this case, using these particular $\bar{v}$ and $\widetilde{v}$, we conclude $u_3 \geq \max\left\{\frac{u_1^2 + 2u_1\bar{v}_1}{2(\bar{v}_2 + \bar{v}_3)}, \frac{u_1^2 + 2u_1\widetilde{v}_1}{2(\widetilde{v}_2 + \widetilde{v}_3)}\right\} = \frac{u_1^2 + 2|u_1\bar{v}_1|}{2\bar{v}_3} \geq 0$. Moreover, when $\alpha_1 \neq 0$, we have $\alpha_2 + \alpha_3 > 0$ because $\alpha \in \text{Ext}(\mathcal{L}^3)$. Then by considering $\hat{v} = \left[0; \frac{1}{2(\alpha_2 + \alpha_3)}; \frac{1}{2(\alpha_2 + \alpha_3)}\right] \in \mathcal{V}_\alpha$, we once again conclude that $u_3 \geq 0$. Note that this is precisely what was needed to prove that $(\mu;\eta_0)$ is $\mathcal{K}$-sublinear. $\diamondsuit$

An immediate implication of Proposition 10 and Corollary 3 is as follows:

COROLLARY 4. *For any $\mu \in \text{Im}(A^*)$ and $\eta_0 \leq \vartheta(\mu)$, the inequality $(\mu;\eta_0)$ is $\mathcal{K}$-sublinear.*

We have already seen in Proposition 4 that when $\text{Ker}(A) \cap \text{int}(K) \neq \emptyset$, then any $\mu \in \text{Im}(A^*)$ and any $-\infty < \eta_0 \leq \vartheta(\mu)$ leads to a $\mathcal{K}$-minimal inequality $(\mu;\eta_0)$. Corollary 4 complements this result by showing that valid inequalities $(\mu;\eta_0)$ with $\mu \in \text{Im}(A^*)$ are always $\mathcal{K}$-sublinear regardless of the requirement $\text{Ker}(A) \cap \text{int}(K) \neq \emptyset$.

In addition to Proposition 10, under **Assumption 1**, we can state a sufficient condition for $\mathcal{K}$-minimality as follows:

PROPOSITION 11. *Suppose that **Assumption 1** holds and we are given a valid inequality $(\mu;\eta_0)$. Then, if there exists $b^i \in \mathcal{B}$ and $x^i \in \mathcal{K}$ for $i \in I$ such that $\sum_{i\in I} x^i \in \text{int}(\mathcal{K})$, $Ax^i = b^i$ and $\langle \mu, x^i\rangle = \eta_0$, then $(\mu;\eta_0)$ is $\mathcal{K}$-minimal.*

Proof. Consider any $(\mu;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$. Assume for contradiction that $(\mu;\eta_0) \notin C_m(A,\mathcal{K},\mathcal{B})$, i.e., $\exists \delta \in \mathcal{K}^* \setminus \{0\}$ such that $(\mu - \delta;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$.

Suppose the premise of the proposition holds for some $b^i \in \mathcal{B}$ and $x^i \in \mathcal{K}$ such that $\sum_{i\in I} x^i \in \text{int}(\mathcal{K})$, $Ax^i = b^i$ and $\langle \mu, x^i\rangle = \eta_0$. Note that for $\beta_i > 0$ with $\sum_{i\in I} \beta_i = 1$, we have $\bar{x} := \sum_{i\in I} \beta_i x^i \in \text{int}(\mathcal{K})$. Moreover, the definition of $\bar{x}$ implies $\bar{x} \in \text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ and $\langle \mu, \bar{x}\rangle = \eta_0$. Because any valid inequality for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, in particular $(\mu - \delta;\eta_0)$, is valid for $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ and thus $\bar{x}$ as well, we arrive at the contradiction

$$\eta_0 \leq \langle \mu - \delta, \bar{x}\rangle < \eta_0,$$

where the last inequality follows from $\langle \mu, \bar{x}\rangle = \eta_0$, $\bar{x} \in \text{int}(\mathcal{K})$ and $\delta \in \mathcal{K}^* \setminus \{0\}$ implying $\langle \delta, \bar{x}\rangle > 0$. $\square$

Proposition 11, in particular, states that a valid inequality is $\mathcal{K}$-minimal whenever the inequality is tight at a point at the intersection of $\text{int}(\mathcal{K})$ and $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. In the MILP case, this resembles a sufficient condition for an inequality to be facet defining. Nonetheless, our minimality notion in general is much weaker. In the MILP case, all of the facets are necessary and sufficient for the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. However, in general, one does not need all of the $\mathcal{K}$-minimal inequalities; only a generating set for $C_m(A,\mathcal{K},\mathcal{B})$ along with the constraint $x \in \mathcal{K}$ is needed.

PROPOSITION 12. *Let $(\mu;\eta_0)$ be a $\mathcal{K}$-minimal inequality such that $\mu \in \text{int}(\mathcal{K}^*)$. Then $\eta_0 = \vartheta(\mu) = \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$.*

Proof. Because $(\mu;\eta_0)\in C_m(A,\mathcal{K},\mathcal{B})$ and $\mu\in\mathcal{K}^*$, by Proposition 3, we have

$$\eta_0=\vartheta(\mu)=\inf_x\{\langle\mu,x\rangle:\ x\in\mathcal{S}(A,\mathcal{K},\mathcal{B})\}.$$

Moreover, because $(\mu;\eta_0)$ is $\mathcal{K}$-minimal, it is also $\mathcal{K}$-sublinear. Then $\mu\in\text{int}(\mathcal{K}^*)$ implies $\inf_{b\in\mathcal{B}}\sigma_{D_\mu}(b)=\vartheta(\mu)$ by Proposition 8. □

Let us demonstrate the uses of Propositions 10, 11 and 12 on Example 8.

Example 8 (cont.) First note that the convex hull of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ is full dimensional. To see this, one can demonstrate the existence of $n+1$ affinely independent points from $\mathcal{S}(A,\mathcal{K},\mathcal{B})\subseteq\mathbb{R}^n$ where $n=3$. Thus, there is no valid equation for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ implying that the lineality space of $C(A,\mathcal{K},\mathcal{B})$ is just the zero vector. Moreover, $\hat{z}=[1;0;2]\in\text{int}(\mathcal{K})\cap\mathcal{S}(A,\mathcal{K},\mathcal{B})$ and hence **Assumption 1** is satisfied.

We claim that

(a) $\mu^{(+)}=[1;0;0]$ with $\eta_0^{(+)}=-1$ and $\mu^{(-)}=[-1;0;0]$ with $\eta_0^{(-)}=-1$;

(b) $\mu^{(t)}=[0;t;\sqrt{t^2+1}]$ with $\eta_0^{(t)}=1$ for all $t\in\mathbb{R}$.

are all $\mathcal{K}$-minimal inequalities. For $i\in\{+,-\}\cup\mathbb{R}$, we have already seen that the associated sets $D_{\mu^{(i)}}$ are nonempty, $\inf_{b\in\mathcal{B}}\sigma_{D_{\mu^{(i)}}}(b)=\eta_0^{(i)}$ holds, and for these inequalities, there are vectors $z^{(i)}\in\text{Ext}(\mathcal{K})$ satisfying the premise of Proposition 10. Hence, by Proposition 10 all of these inequalities are $\mathcal{K}$-sublinear. Moreover, in case (a), by considering the points $z^{(+)}=[1;0;2]\in\text{int}(\mathcal{K})\cap\mathcal{S}(A,\mathcal{K},\mathcal{B})$ and $z^{(-)}=[-1;0;2]\in\text{int}(\mathcal{K})\cap\mathcal{S}(A,\mathcal{K},\mathcal{B})$, we get $\langle\mu^{(i)},z^{(i)}\rangle=\eta_0^{(i)}$ holds for all $i\in\{+,-\}$. Therefore, using Proposition 11, we conclude that these inequalities are also $\mathcal{K}$-minimal. In case (b), for any $t\in\mathbb{R}$, consider $z_+^{(t)}=[1;-t;\sqrt{t^2+1}]\in\mathcal{K}\cap\mathcal{S}(A,\mathcal{K},\mathcal{B})$ and $z_-^{(t)}=[-1;-t;\sqrt{t^2+1}]\in\mathcal{K}\cap\mathcal{S}(A,\mathcal{K},\mathcal{B})$. Note that $\langle\mu^{(t)},z_+^{(t)}\rangle=\eta_0^{(t)}=\langle\mu^{(t)},z_-^{(t)}\rangle$ for all $t\in\mathbb{R}$, and hence $z^{(t)}:=\frac{1}{2}(z_+^{(t)}+z_-^{(t)})=[0;-t;\sqrt{t^2+1}]\in\text{int}(\mathcal{K})\cap\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. Thus, by Proposition 11, we conclude that $(\mu^{(t)};\eta_0^{(t)})\in C_m(A,\mathcal{K},\mathcal{B})$ for all $t\in\mathbb{R}$.

We proceed by showing that the system of infinitely many linear inequalities corresponding to $(\mu^{(t)};\eta_0^{(t)})=([0;t;\sqrt{t^2+1}];1)$ for all $t\in\mathbb{R}$, which leads to the same cut-generating set, indeed has a compact conic representation. Because all of these inequalities are valid for all $x\in\mathcal{S}(A,\mathcal{K},\mathcal{B})$, we have

$$\begin{aligned}
& 1\le 0x_1+tx_2+\sqrt{t^2+1}x_3 \quad \forall t\in\mathbb{R}\\
\iff\ & 1\le\inf_t\{0x_1+tx_2+\sqrt{t^2+1}x_3:\ t\in\mathbb{R}\}\\
\iff\ & 1\le\inf_{t,\tau}\{tx_2+\tau x_3:\ t\in\mathbb{R},\ \tau\ge\sqrt{t^2+1}\}\\
\iff\ & 1\le\inf_{t,\tau}\{tx_2+\tau x_3:\ t\in\mathbb{R},\ (1;t;\tau)\in\mathcal{L}^3\}\\
\iff\ & 1\le\sup_\alpha\{-\alpha_1:\ \alpha_2=x_2,\ \alpha_3=x_3,\ [\alpha_1;\alpha_2;\alpha_3]\in\mathcal{L}^3\} \qquad \text{due to } (*)\\
\iff\ & [-1;x_2;x_3]\in\mathcal{L}^3,
\end{aligned}$$

where $(*)$ follows from the fact that the primal conic optimization problem is strictly feasible, and hence, strong duality applies here. Note that the constraint $x_3\ge\sqrt{1+x_2^2}$ is a cylinder in $\mathbb{R}^3$ and it is the same as the conic quadratic inequality $[1;x_2;x_3]\in\mathcal{L}^3$. The validity of $x_3\ge\sqrt{1+x_2^2}$ for all $x\in\mathcal{S}(A,\mathcal{K},\mathcal{B})$ follows from its derivation. Moreover, this conic quadratic inequality exactly implies all of the $\mathcal{K}$-minimal inequalities $(\mu^{(t)};\eta_0^{(t)})$ for all $t\in\mathbb{R}$. Thus, in this example, the conic constraint $x_3\ge\sqrt{1+x_2^2}$ along with the constraints $-1\le x_1\le 1$ and $x\in\mathcal{L}^3$, completely describes $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$.

Finally, recall that the valid inequality $(\nu;\nu_0)$ given by $\nu=[0;1;2]$ and $\nu_0=1$ has an associated cut-generating set $D_\nu\neq\emptyset$ and $\inf_{b\in\mathcal{B}}\sigma_{D_\nu}(b)=\sqrt{3}>1=\nu_0$. Also, there are points $z^\nu\in\left\{[\frac{1}{\sqrt{3}};-\frac{1}{3};\frac{2}{3}],[-\frac{1}{\sqrt{3}};-\frac{1}{3};\frac{2}{3}]\right\}\subset\text{Ext}(\mathcal{K})$ satisfying the requirement of Proposition 10 for $(\nu;\nu_0)$. Hence, by Proposition 10 $(\nu;\nu_0)$ is $\mathcal{K}$-sublinear. While $\sigma_{D_\nu}(Az^\nu)=\langle\nu,z^\nu\rangle=\nu_0=1$ holds for any (and only) $z^\nu\subset\text{Ext}(\mathcal{K})$, none of these points from z^ν satisfy $Az\in\mathcal{B}$. Thus, the sufficiency condition for $\mathcal{K}$-minimality stated in Proposition 11 fails. In fact, $\nu\in\text{int}(\mathcal{K}^*)$ and $(\nu;\nu_0)$ fails the necessary condition for $\mathcal{K}$-minimality given in Proposition 12, that is, $\inf_{b\in\mathcal{B}} \sigma_{D_\nu}(b) = \sigma_{D_\nu}(1) = \sigma_{D_\nu}(-1) = \sqrt{3} > 1 = \nu_0$. Hence, we conclude that $(\nu;\nu_0)$ is not $\mathcal{K}$-minimal. $\diamondsuit$

Example 8 also suggests a technique to derive closed form expressions for convex valid inequalities by grouping all of the tight $\mathcal{K}$-minimal inequalities associated with the same cut-generating set. This approach is further exploited in [53, 54] in analyzing specific disjunctive conic sets obtained from a two-term disjunction on a regular cone $\mathcal{K}$. In particular, a characterization of tight $\mathcal{K}$-minimal inequalities for this specific disjunctive conic set is given in [53, 54]. Additionally, in the case of $\mathcal{K} = \mathcal{L}^n$, using conic duality, it is shown in [53, 54] that these tight $\mathcal{K}$-minimal inequalities can be grouped appropriately leading to a class of convex inequalities.

4.3. Connections to Lattice-free Sets and Cut-Generating Functions In this section, we relate our results to the literature on lattice-free sets and cut-generating functions studied extensively for $\mathcal{K} = \mathbb{R}^n_+$ and associated infinite relaxations and discuss some implications of our results for general cones $\mathcal{K}$.

In the case of $\mathcal{K} = \mathbb{R}^n_+$, Theorem 4 and Remark 10 together with the basic facts on support functions conjoin nicely and connect to the views based on cut-generating functions and lattice-free sets. To summarize, we have shown that for disjunctive conic sets $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, all tight $\mathbb{R}^n_+$-sublinear inequalities $(\mu; \vartheta(\mu))$ are generated by the support functions $\sigma_{D_\mu}(\cdot)$ of cut-generating sets $D_\mu = \{\lambda \in \mathbb{R}^m : \ A^*\lambda \leq \mu\}$. That is, $\sigma_{D_\mu}(\cdot)$ take as input a^i, the i^{th} column of the linear map A, compute the corresponding cut coefficient of the variable x_i, $\mu_i = \sigma_{D_\mu}(a^i)$ for all $i = 1, \ldots, n$, and the best possible right hand side value $\vartheta(\mu) = \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$. Note that these support functions are automatically sublinear (subadditive and positively homogeneous), and in fact convex and piecewise linear because D_μ is polyhedral. Moreover, under **Assumption 1**, using the sufficiency of $\mathcal{K}$-minimal inequalities (Proposition 2) and Theorem 1, we conclude that all non-cone-implied inequalities for disjunctive conic sets $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ are generated by piecewise-linear, subadditive, positively homogeneous and convex functions. In addition to this, it is recently shown in [52, Proposition 3] that without making any assumptions such as **Assumption 1**, $\mathbb{R}^n_+$-sublinear inequalities always exist, and along with the nonnegativity restrictions $x \in \mathbb{R}^n_+$, they are always sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ for arbitrary A and $\mathcal{B}$.

These observations on the structure and sufficiency of $\mathbb{R}^n_+$-sublinear inequalities for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ provide a simple and intuitive explanation of the well-known strong functional dual for MILPs, e.g., all cutting planes for MILPs are generated by subadditive functions (cf. [58]).

The literature on cutting plane theory for MILP is extensive; we refer the reader to the recent survey [31]. A particular stream of research in this literature initiated by Gomory and Johnson [41, 42] and followed up by Johnson [48], studies an infinite relaxation of an MILP obtained from a simplex tableau corresponding to a fractional solution. The interest in these infinite models originates from deriving cuts from multiple rows of a simplex tableau [5]. Such infinite relaxations have been investigated extensively; we refer the reader to the survey [30] and references therein for further details. A general form of the infinite model is given by:

$$
\begin{aligned}
&\sum_{a\in\mathbb{R}^m} a\, x_a \in -f + S \ \text{ with } f \notin S,\\
&x_a \in \mathbb{R}_+, \ \forall a \in \mathbb{R}^m,\\
&x_a \text{ have finite support,}
\end{aligned}
$$

where an infinite dimensional vector is said to have *finite support* if it has a finite number of nonzero entries. Our disjunctive conic set $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ is a finite form of this model with data A, the cone $\mathcal{K} = \mathbb{R}^n_+$, and the set $\mathcal{B} = -f + S$. On the other hand, the infinite model is specified entirely by the given set S and the vector $f \notin S$ and is completely independent of the data A (defining constraint coefficients in the simplex tableau) of the actual problem. Variants of this infinite model are obtained by imposing further structural restrictions on the set S. Gomory and Johnson [41, 42] studied the case with $S = \mathbb{Z}^m$ and introduced the concept of *cut-generating functions*, that is, functions $\psi : \mathbb{R}^m \to \mathbb{R}$ such that the linear inequality

$$
\sum_{a\in\mathbb{R}^m} \psi(a) x_a \geq 1
$$

holds for all feasible solutions x of the infinite model specified by f and S. In this framework, *extreme functions* and *minimal functions* are used as convenient ways to create a hierarchy of cut-generating functions. A valid function ψ is said to be *extreme* if there are no two distinct valid functions ψ_1, ψ_2 such that $\psi = \frac{1}{2}\psi_1 + \frac{1}{2}\psi_2$. A valid function ψ is *minimal* if there is no valid function ψ' distinct from ψ such that $\psi' \leq \psi$ (the inequality relation between functions is stated as a pointwise relation).

This literature is closely connected to the S*-free (lattice-free) cutting plane* theory for MILPs. An S*-free* convex set is a convex set that does not contain any point from the given set S in its interior. When $S = \mathbb{Z}^m$ an S-free set is called a *lattice-free* set. Usually, one is interested in finding an S-free set to generate a valid inequality that cuts off a given point $f \notin S$. Thus, one seeks an S-free convex set that contains f in its interior. It is well known that extreme functions are sufficient to generate all valid inequalities that separate f from S under further structural assumptions, e.g., when $S = \mathbb{Z}^m$ and $f \notin S$, and all such extreme functions are minimal. Several papers in this literature [5, 23, 29, 30] establish an intimate connection between minimal functions and maximal (with respect to inclusion) S-free convex sets for different models of S. In many cases, e.g., when the nonnegative cut-generating functions are sufficient, for every minimal cut-generating function $\psi(\cdot)$, the corresponding set $\{r \in \mathbb{R}^m : \ \psi(r) \leq 1\}$ is a maximal lattice-free set, and vice-versa. For example, Borozan and Cornuéjols [23] show that minimal functions for the infinite relaxation with $S = \mathbb{Z}^m$ with $f \notin \mathbb{Z}^m$ are precisely the gauge functions of maximal lattice-free convex sets, and thus, they are nonnegative, piecewise linear, positively homogeneous, and convex functions. We refer the interested reader to [28, 29, 30] for further details and recent results.

In the finite dimensional setup, these results are particularly related to the *intersection cuts* of Balas [9, 10]. In his seminal work [9, 10], Balas initiated the use of *gauge functions of lattice-free sets* to generate cuts. This view continues to attract a lot of attention in the MILP context because the gauge functions have the advantage that they can be evaluated using simpler formulas in comparison to the generic cut-generating functions from Gomory-Johnson's infinite relaxation.

For finite dimensional problem instances $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, our study provides an alternative view on the same topic based on support functions. Before we discuss this connection, we underline that, for MILPs, the finite dimensional setup is indeed more relevant in obtaining strong cuts from the simplex tableau because it does not further relax the problem to an infinite model. Besides, it is well known that not all extreme inequalities in an infinite model remain extreme in the underlying finite dimensional model (cf. [32]). That said, whenever the support functions we study here are finite valued, they can be used as cut-generating functions for instances with arbitrary problem data A and dimension n but for the given set $\mathcal{B} = -f + S$. Besides, in contrast to much of the literature on variants of infinite models, our results and also the ones from [52] do not require structural assumptions on $\mathcal{B}$ such as $\mathcal{B} = -f + \mathbb{Z}^m$ with $f \notin \mathbb{Z}^m$ or $\mathcal{B}$ is a nonempty closed set with $0 \notin \mathcal{B}$.

We first identify a $\mathcal{B}$-free set based on the support functions $\sigma_{D_\mu}(\cdot)$. Consider a given $\mathbb{R}^n_+$-sublinear inequality $(\mu; \eta_0)$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Without loss of generality, we can scale μ and assume that $\eta_0 \in \{0, \pm 1\}$. Based on the given $\mathbb{R}^n_+$-sublinear inequality $(\mu; \eta_0)$, we define the set

$$V_\mu := \{r \in \mathbb{R}^m : \sigma_{D_\mu}(r) \leq \eta_0\}.$$

Because $\sigma_{D_\mu}(\cdot)$ is a sublinear function, V_μ is a closed convex set. Also, Theorem 4 together with $\mathcal{K} = \mathbb{R}^n_+$ implies $\vartheta(\mu) = \inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b) \geq \eta_0$. Thus, $\mathcal{B} \cap \operatorname{int}(V_\mu) = \emptyset$ (in fact, we have something slightly stronger: the relative interior of V_μ does not contain any points from $\mathcal{B}$). Then, V_μ is a closed, convex, and $\mathcal{B}$-free set.

The sets underlying gauge functions and support functions are nicely related via polarity. Next, we relate V_μ to the polar set of D_μ defined as

$$D_\mu^o := \{r \in \mathbb{R}^m : \ \lambda^T r \leq 1 \ \forall \lambda \in D_\mu\}.$$

Then, D_μ^o is a closed convex set containing the origin, and the (Minkowski) gauge function of D_μ^o, $\gamma_{D_\mu^o}(\cdot)$, is defined as

$$\gamma_{D_\mu^o}(r) := \inf_t \{t > 0 : \ r \in t\, D_\mu^o\} \quad \text{for all } r \in \mathbb{R}^m.$$

The gauge function $\gamma_{D_\mu^o}(\cdot)$ by definition is nonnegative, closed, and sublinear, and when $0 \notin \text{int}(D_\mu^o)$, $\gamma_{D_\mu^o}(\cdot)$ can take the value of $+\infty$. Moreover, from [46, Theorem C.1.2.5], we have

$$D_\mu^o = \{r \in \mathbb{R}^m : \gamma_{D_\mu^o}(r) \le 1\}.$$

That is, $\gamma_{D_\mu^o}(\cdot)$ represents D_μ^o.[7)] It is well known [46, Corollary C.3.2.5] that whenever Q is a closed convex set containing the origin, the support function of Q is precisely the gauge function of its polar γ_{Q^o}. Hence, whenever D_μ is closed, convex, and $0 \in D_\mu$, the support function of D_μ studied here is precisely the gauge function of the polar set D_μ^o, i.e., $\sigma_{D_\mu} = \gamma_{D_\mu^o}$. Then, when we additionally assume $\eta_0 = 1$, we immediately observe that $V_\mu = D_\mu^o$ and arrive at the following result:

PROPOSITION 13. *Suppose $(\mu;1)$ with $0 \in D_\mu$ is an $\mathbb{R}_+^n$-sublinear inequality for $\mathcal{S}(A, \mathbb{R}_+^n, \mathcal{B})$. Then, the support function $\sigma_{D_\mu}(\cdot)$ of D_μ is exactly the gauge function of its polar D_μ^o, that is, $\sigma_{D_\mu} = \gamma_{D_\mu^o}$; and moreover, D_μ^o is a closed convex and $\mathcal{B}$-free set.*

Proposition 13 calls attention to valid inequalities $(\mu;\eta_0)$ with $\eta_0 = 1$, i.e., the ones separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}_+^n, \mathcal{B}))$. Such valid inequalities have attracted specific attention in the MILP literature and recently for disjunctive sets of form $\mathcal{S}(A, \mathbb{R}_+^n, \mathcal{B})$. Particularly, Conforti et al. [28] consider disjunctive sets of the form $\mathcal{S}(A, \mathbb{R}_+^n, \mathcal{B})$ with $\mathcal{K} = \mathbb{R}_+^n$ for an arbitrary dimension n (and thus A is also arbitrary), but under the additional assumption that $\mathcal{B}$ is a given nonempty, closed set satisfying $0 \notin \mathcal{B}$. In their study, the main focus is on cuts $\mu^T x \ge 1$ that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}_+^n, \mathcal{B}))$, and the properties of cut-generating functions, that is $\psi : \mathbb{R}^m \to \mathbb{R}$, which takes as input a^i, the data pertaining to the variable x_i, and maps it to the corresponding cut coefficient μ_i. Starting from a dominance relation among such functions, [28] establishes a minimality notion for cut-generating functions and studies various structural properties of minimal *finite valued* cut-generating functions and their relations with $\mathcal{B}$-free sets.

Let us examine the connection between our results and those from [28] by assuming that the dimension n is fixed in advance in [28]. Under the assumption $0 \notin \mathcal{B}$ of [28], it is easily seen that $0 \notin \mathcal{S}(A, \mathbb{R}_+^n, \mathcal{B})$ (see [28, Lemma 2.1]); and therefore, without loss of generality, we can assume that the cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}_+^n, \mathcal{B}))$ have the form $(\mu;1)$, i.e., their right hand side value is 1. When the dimension n is fixed, the main set of interest in [28] is exactly our set $\mathcal{S}(A, \mathbb{R}_+^n, \mathcal{B})$, and the corresponding cuts separating the origin are a subset of inequalities from $C(A, \mathbb{R}_+^n, \mathcal{B})$. Furthermore, because these cuts have positive right hand sides, they are non-cone-implied, and thus are all $\mathbb{R}_+^n$-sublinear [52, Proposition 3 and Corollary 2]. Hence, the support functions $\sigma_{D_\mu}(\cdot)$ we examine here do have a direct relation with the corresponding cut-generating functions of interest from [28]. We discuss this next.

The cut-generating function point of view is based on the following motivation: For a fixed set $\mathcal{B}$, a cut-generating function can be used to generate a valid inequality for *any* data matrix A. Note that whenever the function $\sigma(\cdot)$ used to generate a valid inequality is *finite valued everywhere*, it can be used for *any* data matrix A. Furthermore, once the function $\sigma(\cdot)$ is fixed, the right hand side of the inequality depends only on the set $\mathcal{B}$ because the value $\vartheta(\mu) = \inf_{b \in \mathcal{B}} \sigma(b)$ is independent of A. On the other hand, the support functions $\sigma_{D_\mu}(\cdot)$ associated with $\mathbb{R}_+^n$-sublinear inequalities are not always guaranteed to be finite valued. This indicates a distinction between our results and the ones from [28]. Note that it is not necessary to require a function to be finite valued everywhere in order to use it to generate cuts for a given problem instance with data matrix A. In particular, the functions that are not finite valued everywhere, such as the support functions we are considering here, can still be meaningful and interesting in terms of generating valid inequalities for the problem instance at hand. Furthermore, given a problem instance A, $\mathcal{B}$, and $\mathcal{K} = \mathbb{R}_+^n$, under further assumptions on A and $\mathcal{B}$, it may be possible to obtain an appropriate, nonempty, bounded set $\emptyset \ne \widetilde{D} \subseteq D_\mu$ ensuring $\inf_{b \in \mathcal{B}} \sigma_{\widetilde{D}}(b) \ge \eta_0 = 1$ and $\sigma_{\widetilde{D}}(a^i) = \mu_i$ for all $i = 1, \ldots, n$. That is, the support function of $\widetilde{D}$ is finite valued everywhere and generates the same inequality $(\mu;\eta_0)$. Thus, under further

[7)] We say that a sublinear function $\psi(\cdot)$ *represents* a convex set Q when the relation $Q = \{r \in \mathbb{R}^m : \psi(r) \le 1\}$ holds.

technical assumptions we can in addition ensure the finite valuedness of the support functions $\sigma_{D_\mu}(\cdot)$. Then, these support functions will lead to valid inequalities for an arbitrary selection of the columns a^i. That is, they will indeed be cut-generating functions for the given set $\mathcal{B}$. Such a condition ensuring finite valuedness of these support functions is for example studied in [52]. Let us consider Example 6.1 of [28] in the light of this discussion.

EXAMPLE 10. Suppose A is the 2×2 identity matrix, $\mathcal{B} = \{[0;1]\} \cup \{\mathbb{Z};-1\}$ and $\mathcal{K} = \mathbb{R}^2_+$, which leads to $\mathcal{S}(A,\mathbb{R}^2_+,\mathcal{B}) = \text{conv}(\mathcal{S}(A,\mathbb{R}^2_+,\mathcal{B})) = \{[0;1]\}$. This particular disjunctive conic set violates our **Assumption 1**, and therefore, none of the valid inequalities is $\mathbb{R}^2_+$-minimal. Nevertheless, the sufficiency of $\mathbb{R}^2_+$-sublinear inequalities is not based on **Assumption 1** [52, Proposition 3]. Indeed, we next show that the particular inequality $(\mu;\eta_0) = ([-1;1];1)$ considered in [28] is $\mathbb{R}^2_+$-sublinear. It is easy to see that the sufficiency conditions for $\mathcal{K}$-sublinearity established in Proposition 10 are satisfied for this inequality. Actually, $D_\mu = \{(\lambda_1;\lambda_2)\in\mathbb{R}^2 : \lambda_1 \le -1,\ \lambda_2 \le 1\}$, $\sigma_{D_\mu}(A\,e^1) = \sigma_{D_\mu}([1;0]) = -1 = \mu_1 = \mu^T e^1$, $\sigma_{D_\mu}(A\,e^2) = \sigma_{D_\mu}([0;1]) = 1 = \mu_2 = \mu^T e^2$, and clearly $e^1 + e^2 \in \text{int}(\mathbb{R}^2_+)$. Furthermore, $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) = 1 = \eta_0$, proving that $(\mu;\eta_0) = ([-1;1];1)$ is a tight $\mathbb{R}^2_+$-sublinear inequality for this particular $\text{conv}(\mathcal{S}(A,\mathbb{R}^2_+,\mathcal{B}))$. On the other hand, the support function corresponding to this inequality is not finite valued everywhere. As a matter of fact, when we try to bound D_μ to obtain $\widetilde{D} \subseteq D_\mu$ and use $\widetilde{D}$ to generate a valid inequality, we cannot ensure $\sigma_{\widetilde{D}}(A\,e^i) = \mu^T e^i = \mu_i$ for $i = 1,2$, and $\vartheta(\mu) = \inf_{\mathcal{B}} \sigma_{\widetilde{D}}(b) = 1$ simultaneously. $\diamondsuit$

It was conjectured in [28] and later on proved in [33] that, in addition to their earlier assumption $0 \notin \mathcal{B}$, if we further suppose the following "containment" assumption, $\text{cone}(\{a^1,\ldots,a^n\}) \supseteq \mathcal{B}$, we can ensure the existence of finite valued cut-generating functions corresponding to every extreme inequality separating the origin from $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$. Furthermore, it is shown in [52, Proposition 5] that in the same setup and under the same containment assumption of [28, 33], one can ensure that the support functions associated with all $\mathbb{R}^n_+$-sublinear inequalities are finite valued. Actually, there is an explicit duality relation between the support functions studied in this paper (and also [49]) and the value functions studied in [47] and also used in the sufficiency proof of cut-generating functions in [33]. We finish our discussion by examining a slight variant of Example 10 obtained from setting $\bar{\mathcal{B}} = \{[0;1]\} \cup \{(\mathbb{Z}^-;-1)\}$. Note that in this variant we still have $\mathcal{S}(A,\mathbb{R}^2_+,\mathcal{B}) = \mathcal{S}(A,\mathbb{R}^2_+,\bar{\mathcal{B}})$, and $\mathcal{S}(A,\mathbb{R}^2_+,\bar{\mathcal{B}})$ still violates the containment assumption of [28, 33]. Nevertheless, we can show that $(\mu;\eta_0) = ([-1;1];1)$ is generated by a finite valued cut-generating function. Indeed, one can easily check that the support function of the set $\widetilde{D} := \{(\lambda_1;\lambda_2)\in\mathbb{R}^2 : \lambda_1 = -1,\ -1 \le \lambda_2 \le 1\}$ obtained from bounding D_μ will do the job. This indicates the possibility for weakening the containment assumption of [28, 33].

We next have a few comments on $\mathbb{R}^n_+$-sublinear inequalities and the associated sets V_μ, D^o_μ and functions $\sigma_{D_\mu}, \gamma_{D^o_\mu}$.

For $\mathbb{R}^n_+$-sublinear inequalities $(\mu;1)$ that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}))$, under the assumption that $0\in D_\mu$, we have $D^o_\mu = V_\mu$ implying D^o_μ is a closed convex and $\mathcal{B}$-free set and $\gamma_{D^o_\mu} = \sigma_{D_\mu}$ (cf. Proposition 13). Given $\mu \in \Pi(A,\mathcal{K},\mathcal{B})$, the set D_μ is always closed and convex. Yet, $0\in D_\mu$ is not always guaranteed. When $\mu\in\mathcal{K}^*$, we have $0\in D_\mu$; but the other cases of $\mu \in \text{Im}(A^*) + \mathcal{K}^*$ are also of interest. In such cases, by taking the polar of D^o_μ, we obtain $D^{oo}_\mu := (D^o_\mu)^o$, a closed convex set containing the origin. In addition, we always have $D_\mu \subseteq D^{oo}_\mu$; and so, for all $r\in\mathbb{R}^m$ we have $\sigma_{D_\mu}(r) \le \sigma_{D^{oo}_\mu}(r) = \gamma_{D^o_\mu}(r)$, where the equation follows from [46, Proposition C.3.2.4]. Because $\gamma_{D^o_\mu}$ and σ_{D_μ} respectively represent D^o_μ and V_μ, and $\sigma_{D_\mu}(r) \le \gamma_{D^o_\mu}(r)$ holds for all $r\in\mathbb{R}^m$, we have $D^o_\mu \subseteq V_\mu$. Therefore, D^o_μ is also a closed convex and $\mathcal{B}$-free set regardless of $0\in D_\mu$.

In general $\sigma_{D_\mu}(\cdot)$ and $\gamma_{D^o_\mu}(\cdot)$ may differ quite significantly, i.e., a support function can take negative values while a gauge function cannot. Therefore, we expect V_μ to differ significantly from D^o_μ when $0\notin D_\mu$ (cf. Proposition 13). Given that V_μ is a closed convex and $\mathcal{B}$-free set such that $0\in\text{int}(V_\mu)$ and $V_\mu \supseteq D^o_\mu$, we will focus on generating cuts based on V_μ in the following.

For a given valid inequality $(\mu;1)$, recall that $\sigma_{D_\mu}(\cdot)$ represents V_μ, i.e., $V_\mu = \{r\in\mathbb{R}^m :\ \sigma_{D_\mu}(r) \le 1\}$ and we can immediately use $\sigma_{D_\mu}(\cdot)$ to generate cuts. For a given sublinear function there is a unique set associated with it in this manner. However, there can be other sublinear functions $\psi(\cdot)$ representing the same set V_μ, i.e., $V_\mu = \{r \in \mathbb{R}^m : \psi(r) \le 1\}$. To obtain strong cuts via sublinear functions $\psi(\cdot)$, we are interested in the pointwise smallest possible such sublinear function $\psi(\cdot)$ representing V_μ. Because sublinear functions are positively homogeneous, for any sublinear function $\psi(\cdot)$ representing V_μ, we have $\sigma_{D_\mu}(r) = \psi(r)$ for every r satisfying $\psi(r) > 0$. In order to find the smallest sublinear function $\psi(\cdot)$ representing the set V_μ, Basu et al. [13] considers the following subset of the relative boundary of V_μ^o:

$$\widehat{V}_\mu^o := \{\lambda \in V_\mu^o : \exists r \in V_\mu \text{ s.t. } \lambda^T r = 1\}.$$

Under the assumption $0 \in \text{int}(V_\mu)$ (which immediately holds in our setup), it was shown in [13] that among the sublinear functions $\psi(\cdot)$ representing V_μ, we have the following relation $\sigma_{\widehat{V}_\mu^o}(r) \le \psi(r) \le \gamma_{V_\mu}(r)$ for all $r \in \mathbb{R}^m$. Then, $\sigma_{\widehat{V}_\mu^o}(r) \le \sigma_{D_\mu}(r) \le \gamma_{V_\mu}(r)$ holds for all $r \in \mathbb{R}^m$ because $\sigma_{D_\mu}(\cdot)$ represents V_μ. Since V_μ is a closed convex set containing the origin, we also have $\gamma_{V_\mu} = \sigma_{V_\mu^o}$ [46, Proposition C.3.2.4]. Thus, $\sigma_{\widehat{V}_\mu^o}(r) \le \sigma_{D_\mu}(r) \le \sigma_{V_\mu^o}(r)$ holds for all $r \in \mathbb{R}^m$. Hence, studying the cases when we have $\sigma_{D_\mu} = \sigma_{\widehat{V}_\mu^o}$ is of independent interest for understanding the strength, e.g., minimality, of these support functions $\sigma_{D_\mu}(\cdot)$.

REMARK 12. In the case of $\mathcal{K} = \mathbb{R}^n_+$, as discussed above, there are strong connections between our $\mathcal{K}$-sublinear inequalities, the cut-generating functions [30], and the strong functional dual for MILPs [58].

Moving forward, one may be interested in extending the definition of a cut-generating function from MILPs to MICPs. However, the situation seems to be much more complex for general regular cones $\mathcal{K}$ other than the nonnegative orthant. In the MILP context, one of the main properties of a cut-generating function is that the function acts *locally* on each variable. Namely, the cut-generating function takes as input solely the data associated with an individual variable x_i, i.e., the corresponding column a^i, and based on this input, it generates the individual cut coefficient μ_i associated with x_i. Imposing such a local view on cut-generating functions is acceptable in the case of the nonnegative orthant because such cut-generating functions are sufficient in the case of $\mathcal{K} = \mathbb{R}^n_+$ when we drop the finite valuedness restriction on the function or under reasonable structural assumptions [52]. This, we believe, is strongly correlated with the fact that the underlying cone $\mathcal{K} = \mathbb{R}^n_+$ is decomposable in terms of individual variables. On the other hand, for general regular cones $\mathcal{K}$, imposing the same local view requirement on cut-generating functions turns out to be problematic, especially when the cone $\mathcal{K}$ encodes nontrivial dependencies among variables.

In particular, Example 8 reveals an important fact in this discussion: Unlike the case where $\mathcal{K} = \mathbb{R}^n_+$, unless we make further structural assumptions, for general $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with a regular cone $\mathcal{K}$, even when the cone $\mathcal{K}$ is as simple as $\mathcal{L}^3$, not all extreme (and also tight, $\mathcal{K}$-minimal) valid linear inequalities can be generated by functions acting locally on individual variables. Specifically, in Example 8, the linear map is given by $A = [1, 0, 0]$, and the class of valid inequalities $(\mu^{(t)}; \eta_0^{(t)}) = ([0; t; \sqrt{t^2+1}]; 1)$ parametrized by $t \in \mathbb{R}$ are all extreme and thus necessary in the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$. If one considers cut-generating functions of the form that take as input the individual columns of A and output the corresponding cut coefficient, then no such function $\psi(\cdot)$ will precisely generate the vector defining the inequality $(\mu^{(t)}; \eta_0^{(t)}) = ([0; t; \sqrt{t^2+1}]; 1)$ for any $t \in \mathbb{R}$. This is because such a function $\psi(\cdot)$ will inevitably need to satisfy $t = \mu_2^{(t)} = \psi(a^2) = \psi(0) = \psi(a^3) = \mu_3^{(t)} = \sqrt{t^2+1}$, which is impossible.

Therefore, Example 8 demonstrates that for regular cones $\mathcal{K}$ other than the nonnegative orthant, if we were to straightforwardly extend the definition of cut-generating functions based on a local view from the MILP literature and rely only on such functions, we may completely miss large classes of nontrivial extreme inequalities necessary for the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$. On the other hand, it may be possible to introduce and study *cut-generating maps* $\Gamma(\cdot)$, which take a *global view* and consider the *entire data* A to generate the cut coefficient vector μ at once, i.e., $\mu = \Gamma(A)$. We leave the questions surrounding such cut-generating maps, such as their existence, structural properties, sufficiency, etc., for future work.

On a positive note, for specific MICPs of form (2) discussed in Example 3, Moran et al. [57] show that a strong functional dual exists under a technical condition. Existence of strong MICP duals for these specific MICPs is equivalent to the sufficiency of (indeed, very specific classes of) finite valued functions that generate the cut coefficients of all cuts for these sets. In fact, these functions from [57] indeed act locally on each individual variable, and thus, naturally extend the standard cut-generating function framework used in the MILP literature to the specific MICPs of form (2). Thus, in spite of the fact that Moran et al. [57] do not refer to these functions as cut-generating functions, they are indeed so. However, we highlight that the natural disjunctive conic representation $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ for the specific class of MICPs from [57] discussed in Example 3 imposes further structure. In particular, the underlying cone $\mathcal{K}$ in the resulting equivalent disjunctive conic form representation $\mathcal{S}(A,\mathbb{R}^{2n}_+,\mathcal{B})$ of MICP given in (2) is simply $\mathbb{R}^{2n}_+$. On the other hand, the cone involved in Example 8 is $\mathcal{L}^3$. On a related note, we do not know of the existence of a similar strong functional MICP dual result for MICPs of form (1) discussed in Example 2. Example 8 suggests that such a result is not likely. $\diamondsuit$

4.4. Connections to Conic Mixed Integer Rounding Cuts Atamtürk and Narayanan [7] introduced conic mixed integer rounding cuts for the following simple mixed integer set

$$\mathcal{S}_0 := \{(x,y,w,t)\in\mathbb{Z}\times\mathbb{R}^3_+ :\ |x+y-w-b|\le t\}. \tag{6}$$

They have shown that when $b=\lfloor b\rfloor+f$ with $f\in(0,1)$, the valid inequality given by

$$(1-2f)(x-\lfloor b\rfloor)+f\le t+y+w, \tag{7}$$

when added to the description of $\mathcal{S}_0$ gives $\overline{\text{conv}}(\mathcal{S}_0)$.

We can represent $\mathcal{S}_0$ in disjunctive conic form via the following set:

$$\mathcal{S} := \left\{(y,w,t,\gamma)\in\mathbb{R}^3_+\times\mathcal{L}^2 :\ \begin{bmatrix} y-w\\ t\end{bmatrix}-\gamma=\begin{bmatrix} b-x\\ 0\end{bmatrix}\right\}, \tag{8}$$

which leads to a regular cone $\mathcal{K}=\mathbb{R}^3_+\times\mathcal{L}^2$, and

$$A=\begin{bmatrix} 1 & -1 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & -1\end{bmatrix} \text{ and } \mathcal{B}=\left\{\underbrace{\begin{bmatrix} f\\ 0\end{bmatrix}}_{:=b_1^+},\underbrace{\begin{bmatrix} 1+f\\ 0\end{bmatrix}}_{:=b_2^+},\ldots,\underbrace{\begin{bmatrix} f-1\\ 0\end{bmatrix}}_{:=b_1^-},\underbrace{\begin{bmatrix} f-2\\ 0\end{bmatrix}}_{:=b_2^-},\ldots\right\},$$

where $x\in\mathbb{Z}$ is used to define the set $\mathcal{B}$. The first equation in (8) follows from $y-w-\gamma_1=b-x$, and together with $b=\lfloor b\rfloor+f$ it implies $x-\lfloor b\rfloor=-y+w+\gamma_1+f$. By substituting $x-\lfloor b\rfloor$ with $-y+w+\gamma_1+f$ in (7), we rewrite (7) in terms of the variables in our representation as follows:

$$\begin{array}{l}(1-2f)(-y+w+\gamma_1+f)+f\le t+y+w\\ (2-2f)y+2fw+t+(2f-1)\gamma_1+0\gamma_2\ge f(2-2f).\end{array}$$

Then $\eta_0=f(2-2f)$, $\mu_1=2-2f$, $\mu_2=2f$, $\mu_3=1$, $\mu_4=2f-1$ and $\mu_5=0$ in our usual notation.

We are now ready to demonstrate how the results from section 4 can be used to analyze this the inequality $(\mu;\eta_0)$ for $\mathcal{S}$. In particular, without the knowledge of explicit description for $\overline{\text{conv}}(\mathcal{S})$, we will derive the best possible right hand side value for μ, i.e., $\vartheta(\mu)$, and show that $\vartheta(\mu)=\eta_0$ proving the validity and tightness of this inequality (7). We will then verify that (7) is $\mathcal{K}$-minimal as well.

For the given vector μ, we have

$$\begin{aligned} D_\mu &= \{\lambda\in\mathbb{R}^2 :\ A^*\lambda\preceq_{\mathcal{K}^*}\mu\}=\left\{\lambda\in\mathbb{R}^2 :\ \lambda_1\le\mu_1,\ -\lambda_1\le\mu_2,\ \lambda_2\le\mu_3,\ \begin{bmatrix}-\lambda_1\\ -\lambda_2\end{bmatrix}\preceq_{\mathcal{L}^2}\begin{bmatrix}\mu_4\\ \mu_5\end{bmatrix}\right\}\\ &=\left\{\lambda\in\mathbb{R}^2 :\ \lambda_1\le 2-2f,\ -\lambda_1\le 2f,\ \lambda_2\le 1,\ |2f-1+\lambda_1|\le\lambda_2\right\}. \end{aligned}$$

Because $f\in(0,1)$, $D_\mu\ne\emptyset$. The set D_μ for $f=0.25$ is plotted in Figure 3.

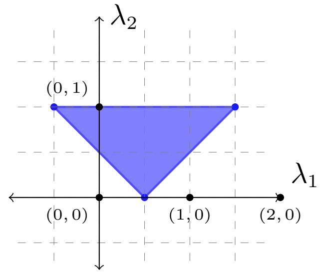

FIGURE 3. Feasible region corresponding to D_μ for $f = 0.25$ in conic mixed integer rounding cut of [7].

Because $D_\mu \neq$ when $f \in (0,1)$, we have $\mu \in \mathcal{K}^* + \text{Im}(A^*)$. Thus, μ satisfies condition **(A.0)**. Let e^i stand for the i^{th} unit vector in $\mathbb{R}^5$. Then $\sigma_{D_\mu}(b_1^+) = f \cdot \sigma_{D_\mu}(e^1) = f(2-2f)$, and for $i = 1, 2, \ldots$, we have $\sigma_{D_\mu}(b_{i+1}^+) = (f+i)(2-2f) = (2-2f)i + 2f - 2f^2$. Therefore, $\sigma_{D_\mu}(b_1^+) < \sigma_{D_\mu}(b_2^+) < \ldots$ holds because $f \in (0,1)$. Similarly, $\sigma_{D_\mu}(b_1^-) = (1-f)\sigma_{D_\mu}(-e^1) = (1-f)(-2f) = 2f(f-1)$, and $\sigma_{D_\mu}(b_i^-) = (f-i)(-2f) = 2fi - 2f^2$ for $i = 1, 2, \ldots$. Thus, $\sigma_{D_\mu}(b_1^-) < \sigma_{D_\mu}(b_2^-) < \ldots$ implying

$$\inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b) = \min\left\{\sigma_{D_\mu}(b_1^+),\ \sigma_{D_\mu}(b_1^-)\right\} = f(2-2f) = \eta_0.$$

By Proposition 7, this proves the validity of the inequality $(\mu; \eta_0)$. Furthermore, because the underlying cone $\mathcal{K} = \mathbb{R}^3_+ \times \mathcal{L}^2$ is polyhedral, by Proposition 8, we have $\vartheta(\mu) = \inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b) = \eta_0$, proving the tightness of $(\mu; \eta_0)$.

Before studying the $\mathcal{K}$-minimality of $(\mu; \eta_0)$, we first verify **Assumption 1**. Because for any $\epsilon_1, \epsilon_2 > 0$, $(y; w; t; \gamma_1; \gamma_2) = (f + \epsilon_1; \epsilon_1; \epsilon_2; 0; \epsilon_2) \in \text{int}(\mathcal{K}) \cap \mathcal{S}$, **Assumption 1** is satisfied for $\mathcal{S}$. Therefore, $\mathcal{K}$-minimal inequalities exist. However, $\mathcal{S}$ is not full dimensional since $t - \gamma_2 = 0$ is a valid equation. By Corollary 1, we immediately have this valid equation is $\mathcal{K}$-minimal.

Finally, consider the following set of points

$$\left\{z^1 := [f;0;0;0;0],\, z^2 := [0;1-f;0;0;0],\, z^3 := [0;0;f;-f;f],\, z^4 := [0;0;1-f;1-f;1-f]\right\}.$$

Given $f \in (0,1)$, one can easily see that for $i = 1, \ldots, 4$, we have $z^i \in \mathcal{S}$ and $\langle \mu, z^i \rangle = \eta_0 = 2f - 2f^2$. Moreover, $\bar{z} := \frac{1}{4}\sum_{i=1}^4 z^i$ is in the interior of $\mathcal{K} = \mathbb{R}^3_+ \times \mathcal{L}^2$. Therefore, using Proposition 11, we conclude that the valid inequality given by $(\mu; \eta_0) = ([2-2f; 2f; 1; 2f-1; 0]; 2f - 2f^2)$ is a $\mathcal{K}$-minimal inequality.

5. Characterization of Valid Equations

Our results with regard to the existence of $\mathcal{K}$-minimal inequalities are based on **Assumption 1**, i.e., we assume that for all $\delta \in \mathcal{K}^* \setminus \{0\}$, there exists $z_\delta \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$ such that $\langle \delta, z_\delta \rangle > 0$. Under a stronger assumption stated below, we can show that all valid equations $(\mu; \eta_0)$ satisfy $\mu \in \text{Im}(A^*)$.

Assumption 2: There exists $\hat{z} \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$ such that $\hat{z} \in \text{int}(\mathcal{K})$ and $A\hat{z} = \hat{b}$ for some $\hat{b} \in \mathcal{B}$.
In this section, we use $\widehat{\mathcal{B}} := \{b \in \mathcal{B} :\ \exists x \in \mathcal{K} \text{ s.t. } Ax = b\}$ to denote the set right hand side choices from $\mathcal{B}$ that are achievable. Note that $\mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \mathcal{S}(A, \mathcal{K}, \widehat{\mathcal{B}})$. Then $\widehat{\mathcal{B}} \neq \emptyset$ because $\mathcal{S}(A, \mathcal{K}, \mathcal{B}) \neq \emptyset$.

THEOREM 5. *Suppose that **Assumption 2** holds. Then $(\mu; \eta_0)$ is a valid equation if and only if there exists some $\lambda_\mu \in \mathbb{R}^m$ such that*

$$A^*\lambda_\mu = \mu \quad \textit{and} \quad b^T\lambda_\mu = \eta_0 = \vartheta(\mu) \quad \textit{for all } b \in \widehat{\mathcal{B}}.$$

Proof.
$(\Leftarrow)$ It is easy to see that the condition in Theorem 5 is sufficient. Existsence of $\lambda_\mu \in \mathbb{R}^m$ satisfying $A^*\lambda_\mu = \mu$ and $b^T\lambda_\mu = \eta_0$ for all $b \in \widehat{\mathcal{B}}$, implies for any $z \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$

$$\langle \mu, z \rangle = \langle A^*\lambda_\mu, z \rangle = \lambda_\mu^T A z = \lambda_\mu^T b = \eta_0 = \vartheta(\mu),$$

where the third equation follows because $z \in \mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \mathcal{S}(A, \mathcal{K}, \widehat{\mathcal{B}})$, and hence $Ax = b \in \widehat{\mathcal{B}}$. Thus, $(\mu; \eta_0)$ is a valid equation.

($\Rightarrow$) To prove the necessity of the condition, suppose that $(\mu;\eta_0)$ is a valid equation. Then, clearly $\eta_0 = \vartheta(\mu)$. Let $\hat{b}$ and $\hat{z}$ be as described in **Assumption 2** preceding the theorem and consider

$$\inf_z \{\langle \mu, z\rangle : \ Az = \hat{b}, \ z \in \mathcal{K}\}.$$

The solution set of this problem is contained in $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. Because $(\mu;\vartheta(\mu))$ is a valid equation, the optimum value of this optimization problem is equal to $\vartheta(\mu)$. Moreover, this problem is strictly feasible because there exists $\hat{z} \in \text{int}(\mathcal{K})$ satisfying $A\hat{z} = \hat{b}$. Then, strong conic duality implies

$$\vartheta(\mu) = \max_{\lambda \in \mathbb{R}^m} \{\hat{b}^T \lambda : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\}.$$

Thus, there exists an optimal solution λ_μ satisfying $A^*\lambda_\mu \preceq_{\mathcal{K}^*} \mu$ and $\hat{b}^T\lambda_\mu = \vartheta(\mu)$. Note that any feasible solution to the primal problem including the strictly feasible solution $\hat{z}$ is optimal. Therefore, using the complementary slackness condition, we have

$$\langle \hat{z}, \mu - A^*\lambda_\mu\rangle = 0.$$

Because $\hat{z} \in \text{int}(\mathcal{K})$, the above equation is possible if and only if $A^*\lambda_\mu = \mu$. Hence, there exists λ_μ such that $A^*\lambda_\mu = \mu$ and $\hat{b}^T\lambda_\mu = \vartheta(\mu)$. Then, for any $b \in \widehat{\mathcal{B}}$ and $z_b \in \mathcal{K}$ satisfying $Az_b = b$, we have

$$\vartheta(\mu) = \langle \mu, z_b\rangle \ \geq \ \inf_z\{\langle \mu, z\rangle : \ Az = b, \ z \in \mathcal{K}\} \ \geq \ \sup_{\lambda \in \mathbb{R}^m} \{b^T\lambda : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\} \geq b^T\lambda_\mu, \tag{9}$$

and

$$-\vartheta(\mu) = \langle -\mu, z_b\rangle \ \geq \ \inf_z\{\langle -\mu, z\rangle : \ Az = b, \ z \in \mathcal{K}\} \ \geq \ \sup_{\lambda \in \mathbb{R}^m} \{b^T\lambda : \ A^*\lambda \preceq_{\mathcal{K}^*} -\mu\} \geq -b^T\lambda_\mu, \tag{10}$$

where in both (9) and (10), the second inequality follows from weak duality, the last inequality follows because λ_μ is a feasible solution to the dual in (9) and $-\lambda_\mu$ is feasible in (10). Then, (9) and (10) together lead to $\vartheta(\mu) = b^T\lambda_\mu$.

□

In addition to the characterization of Theorem 5, each valid equation $(\mu;\vartheta(\mu))$ is related to its corresponding cut-generating set D_μ as follows:

COROLLARY 5. *Suppose that* ***Assumption 2*** *holds. Then, for any valid equation* $(\mu;\vartheta(\mu))$, *there exists* λ_μ *satisfying* $D_\mu = \{\lambda : A^*\lambda \preceq_{\mathcal{K}^*} \mu\} = \lambda_\mu + \{\lambda : \ A^*\lambda \preceq_{\mathcal{K}^*} 0\}$ *and* $\vartheta(\mu) = \inf_{b \in \widehat{\mathcal{B}}} \sigma_{D_\mu}(b) = \sup_{b \in \widehat{\mathcal{B}}} \sigma_{D_\mu}(b)$.

Proof. Suppose $(\mu;\vartheta(\mu))$ is a valid equation. Then by Theorem 5, there exists λ_μ such that $\mu = A^*\lambda_\mu$ and $\vartheta(\mu) = b^T\lambda_\mu$ for all $b \in \widehat{\mathcal{B}}$. Thus, we have

$$D_\mu = \{\lambda : \ A^*\lambda \preceq_{\mathcal{K}^*} A^*\lambda_\mu\} = \{\lambda_\mu + \lambda : \ A^*\lambda \preceq_{\mathcal{K}^*} 0\},$$

and

$$\inf_{b \in \widehat{\mathcal{B}}} \sigma_{D_\mu}(b) = \inf_{b \in \widehat{\mathcal{B}}} \sup_{\lambda \in \mathbb{R}^m} \{b^T(\lambda_\mu + \lambda) : \ A^*\lambda \preceq_{\mathcal{K}^*} 0\} = \inf_{b \in \widehat{\mathcal{B}}} \left[\underbrace{b^T\lambda_\mu}_{=\vartheta(\mu)} + \underbrace{\sup_\lambda \{b^T\lambda : \ A^*\lambda \preceq_{\mathcal{K}^*} 0\}}_{\in\{0,+\infty\}} \right] = \vartheta(\mu),$$

where the last equation follows because $\vartheta(\mu) \in \mathbb{R}$. We can show that $\vartheta(\mu) = \sup_{b \in \widehat{\mathcal{B}}} \sigma_{D_\mu}(b)$ in a similar manner. □

When $\mathcal{K} = \mathbb{R}^n_+$ (or any regular cone where each pair of its extreme rays is orthogonal), under **Assumption 2**, Corollary 5 gives a complete characterization of valid equations.

6. Conclusions and Further Research We introduce the class of $\mathcal{K}$-minimal valid inequalities in the general disjunctive conic programming context that naturally arises in solution set representations and relaxations for MICPs. We show that $\mathcal{K}$-minimality concept captures the dominance relations among valid inequalities induced by the cone $\mathcal{K}$. In particular, under a mild technical assumption, we establish that the class of $\mathcal{K}$-minimal inequalities together with the original constraint $x \in \mathcal{K}$ is sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. This prompts an interest in $\mathcal{K}$-minimal inequalities, suggesting that an efficient cutting plane procedure for solving MICPs should at the least aim at separating inequalities from this class. Nevertheless, the definition of $\mathcal{K}$-minimality reveals little about the structure of $\mathcal{K}$-minimal inequalities. Specifically, testing $\mathcal{K}$-minimality based on its definition is a nontrivial task. To address this, we show that the class of $\mathcal{K}$-minimal inequalities is contained in a slightly larger class of so-called $\mathcal{K}$-sublinear inequalities defined by algebraic conditions. We establish a close connection between $\mathcal{K}$-sublinear inequalities for disjunctive conic sets and the support functions of cut-generating sets. Using this connection, we show that when $\mathcal{K} = \mathbb{R}^n_+$, all $\mathcal{K}$-sublinear inequalities are generated by sublinear (positively homogeneous, subadditive, and convex) functions that are also piecewise linear. Thus, our results naturally capture some of the earlier results from the MILP setup and generalize them to the conic case. Furthermore, this connection with the support functions has led to practical ways of showing $\mathcal{K}$-minimality and/or $\mathcal{K}$-sublinearity properties of inequalities. To the best of our knowledge, these sufficient conditions for $\mathcal{K}$-minimality and/or $\mathcal{K}$-sublinearity of the valid inequalities are new even in the MILP setup.

Our work has shed some light on the structure of $\mathcal{K}$-minimal and $\mathcal{K}$-sublinear inequalities for disjunctive conic sets $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ involving a regular cone $\mathcal{K}$. However, many questions remain open when we start considering regular cones other than $\mathbb{R}^n_+$. In particular, the following questions are of interest:

- *[Characterization of extreme valid inequalities]* Under a mild technical assumption, e.g., Assumption 1, we have shown that all extreme inequalities are $\mathcal{K}$-minimal. However, not every $\mathcal{K}$-minimal inequality is extreme (see e.g., Example 7 and Proposition 4). Further characterizations of extreme inequalities for disjunctive conic sets beyond $\mathcal{K}$-minimality are of great interest and importance.
- *[Finiteness of $\mathcal{K}$-minimal conic inequalities]* When $\mathcal{K} = \mathbb{R}^n_+$ and $\mathcal{B}$ is finite, Johnson [49] proved that the cone of $\mathcal{K}$-minimal inequalities is finitely generated, i.e., G_C is finite. Note that G_L is always finite. For non-polyhedral regular cones, e.g., $\mathcal{L}^n, \mathcal{S}^n_+$, in general, expecting $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ to be given by finitely many linear inequalities is too much and is against the inherent nonlinear nature of these cones. Example 8 shows that this is not possible even for $\mathcal{L}^3$; the resulting $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ requires infinitely many extreme linear inequalities. On the other hand, in that example, it is clear that the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ involves only two linear inequalities and two conic inequalities involving $\mathcal{L}^3$. While the $\mathcal{K}$-minimality notion is seemingly defined for linear inequalities, we can immediately extend it to a conic inequality by saying that a conic quadratic inequality is $\mathcal{K}$-minimal if the associated (possibly infinite) set of linear inequalities are all $\mathcal{K}$-minimal. We believe that instead of focusing on the finiteness of linear inequalities describing $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, it is more natural and relevant to focus on the finiteness of conic inequalities (of the same type of $\mathcal{K}$) describing $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. Therefore, we wonder what can be said in terms of the number of $\mathcal{K}$-minimal conic inequalities required in the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. Is it a finite number when $\mathcal{B}$ is finite? Is it finite regardless of the size of $\mathcal{B}$? Or, can we at least identify the cases where it is finite? In the very specific case of disjunctive conic sets arising from two-term disjunctions on $\mathcal{L}^n$, the recent work in [53, 54] provides partial answers to some of these questions.
- *[Relations with other structured non-convex sets]* In section 4.4 we examined conic MIR inequalities from [7] in our framework. Moreover, in a recent series of papers [53, 54], the characterization of tight $\mathcal{K}$-minimal inequalities has played a critical role in the derivation of explicit expressions for convex valid inequalities for disjunctive conic sets associated with a two-term disjunction on $\mathcal{L}^n$. These derivations relate nicely to other recently developed valid inequalities for MICPs based on split or disjunctive arguments in [4, 14, 19, 34, 56]. The sets obtained from split or general two-term disjunctions are inherently linked to the ones defined by non-convex quadratics. Therefore, convexification techniques for such non-convex

quadratic sets form the next step in this line of research; and these are attracting more attention in the literature [19, 25, 56] lately. Thus, establishing an explicit connection between our disjunctive conic framework and such sets involving non-convex quadratics is compelling.

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