Disjunctive Conic Sets, Conic Minimal Inequalities, and Cut-Generating Functions

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This article summarizes the paper [15] and some recent developments which are concerned with the *disjunctive conic sets* of form

$$\mathcal{S}(A, \mathcal{K}, \mathcal{B}) := \{x \in \mathbb{E} : \ Ax \in \mathcal{B}, \ x \in \mathcal{K}\},$$

where $\mathbb{E}$ is a finite dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$, $A : \mathbb{E} \to \mathbb{R}^m$ is a linear map, $\emptyset \neq \mathcal{B} \subset \mathbb{R}^m$ is a set of right hand side vectors, and $\mathcal{K} \subset \mathbb{E}$ is a *regular* (closed, convex, full-dimensional, and pointed) cone. We restrict our attention to the interesting cases where $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ is nonempty and nonconvex. Thus, we assume $\mathcal{B} \neq \emptyset$ but make no other assumptions on $\mathcal{B}$; in particular, $\mathcal{B}$ may be either finite or infinite. Examples of regular cones include the nonnegative orthant $\mathbb{R}^n_+$, the second-order (Lorentz) cone $\mathbb{L}^n$, and the positive semidefinite cone $\mathbb{S}^n_+$.

Disjunctive conic sets arise naturally in the solution set representations of Mixed Integer Conic Programs (MICPs) where nonlinear convex relations among variables are captured in the conic constraint $x \in \mathcal{K}$ and integrality restrictions are encoded in A and $\mathcal{B}$ by an appropriate selection. These sets also form the basis of fundamental structured relaxations used in generating cutting planes/surfaces for MICPs. For example, a disjunctive conic sets $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ can represent multi-term (or split) disjunctions on regular cones and their cross-sections. Besides, the separation of a fractional solution from the feasible set of a Mixed Integer Linear Program (MILP) can be encoded as a set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with a closed set $\mathcal{B}$ satisfying $0 \notin \mathcal{B}$ [5, 12, 14]. Moreover, the flexibility in the choice of $\mathcal{B}$ makes these sets a relevant model for conic optimization problems with complementarity constraints. See [15, Sec 1.2] for illustrative examples.

The set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ has compelled significant attention. When $\mathcal{B}$ is a finite set, $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ is nothing but a disjunctive set such as those introduced and studied by Balas [2]. Johnson [14] characterized minimal valid linear inequalities for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ through support functions of certain sets. Jeroslow [12] and Blair [5] presented similar characterizations via the value functions of MILPs with bounded feasible sets (in the former) and with rational data (in the latter). This body of work has strong connections to the strong duality theory for MILPs [1, 11].

In this paper, we generalize earlier results on classification and characterization of strong valid linear inequalities for the convex hull description of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ to the case where $\mathcal{K}$ is a general regular cone without relying on the prior assumptions such as the finiteness of $\mathcal{B}$, etc. In order to capture dominance relations among valid linear inequalities, we introduce the notion of *conic minimality* of an inequality. This definition exposes a shortcoming in the usual minimality definition and offers a potential remedy via using $\mathcal{K}$ to encode structural information on the problem. We perform a systemic study of conic minimal inequalities in terms of their existence, sufficiency, strength, necessary conditions and sufficient conditions for their characterization, and establish connections with functions that generate these inequalities.

Introducing some notation

For a set $Q \subset \mathbb{R}^n$, we denote its topological interior by $\operatorname{int}(Q)$ and its closed convex hull by $\overline{\operatorname{conv}}(S)$. The support function of a set $Q \subset \mathbb{R}^n$ is defined as $\sigma_Q(z) := \sup_{q \in \mathbb{R}^n} \{z^\top q : \ q \in Q\}$. Support functions are sublinear (positively homogeneous, subadditive, and thus convex); and when $Q \neq \emptyset$, we also have $\sigma_Q(0) = 0$.

Given two Euclidean spaces $\mathbb{E}, \mathbb{F}$, we define the kernel of a linear map $A : \mathbb{E} \to \mathbb{F}$ as $\mathrm{Ker}(A) := \{u \in \mathbb{E} : \ Au = 0\}$ and its image as $\mathrm{Im}(A) := \{Au : \ u \in \mathbb{E}\}$. We use A^* to denote the conjugate linear map given by the identity $\langle y, Ax\rangle_{\mathbb{F}} = \langle A^* y, x\rangle_{\mathbb{E}} \ \forall (x \in \mathbb{E}, y \in \mathbb{F})$. When the Euclidean space $\mathbb{E}$ is just $\mathbb{R}^n$, we use the dot product as the corresponding inner product.

For a given cone $\mathcal{K} \subset \mathbb{E}$, we let $\mathcal{K}^*$ denote its dual cone given by $\mathcal{K}^* := \{y \in \mathbb{E} : \ \langle x, y\rangle \geq 0 \ \forall x \in \mathcal{K}\}$ and $\mathrm{Ext}(\mathcal{K})$ denote the set of the extreme rays of $\mathcal{K}$. We let $[n] := \{1, \ldots, n\}$ for any positive integer n.

A hierarchy on valid linear inequalities

We pursue a principled study of the structure of valid linear inequalities defining the closed convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$. Given any vector $\mu \in \mathbb{E}$ and a number $\mu_0 \leq \vartheta(\mu)$ where $\vartheta(\mu)$ is defined as

$$\vartheta(\mu) := \inf_{x \in \mathbb{E}} \{\langle \mu, x\rangle : \ x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})\},$$

the linear inequality of the form $\langle \mu, x\rangle \geq \mu_0$ is *valid inequality* for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$. We refer to a valid inequality $\langle \mu, x\rangle \geq \mu_0$ as *trivial* if $\mu_0 = -\infty$, and as *tight* if $\mu_0 = \vartheta(\mu)$. We say that a valid linear inequality for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ is *extreme* if it is a valid equation or if it cannot be written as the sum of two distinct valid linear inequalities (sums of valid equations are excluded here). While extreme inequalities are necessary and sufficient for a complete description of $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$, their identification or algebraic characterization is often quite complicated. We compromise on this by examining the structure of slightly larger classes of inequalities—*minimal* and *sublinear* inequalities defined with respect to the cone $\mathcal{K}$.

Let us start by pointing out a simple class of valid inequalities. From the definition of $\mathcal{K}^*$, any inequality $\langle \delta, x\rangle \geq 0$ with $\delta \in \mathcal{K}^*$ is valid for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ since $\mathcal{S}(A, \mathcal{K}, \mathcal{B}) \subseteq \mathcal{K}$. We refer to these as *cone-implied inequalities*. Cone-implied inequalities may be extreme in certain cases; even so, they are not interesting because the constraint $x \in \mathcal{K}$ captures all of them.

Cone $\mathcal{K}$ in the description of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ plays a critical role in identifying dominance relations among valid linear inequalities. Consider two valid inequalities for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ given by $\langle \mu, x\rangle \geq \mu_0$ and $\langle \rho, x\rangle \geq \rho_0$. We say that $\langle \rho, x\rangle \geq \rho_0$ *dominates* $\langle \mu, x\rangle \geq \mu_0$ with respect to the cone $\mathcal{K}$ whenever $\mu - \rho \in \mathcal{K}^* \setminus \{0\}$ and $\rho_0 \geq \mu_0$. In fact, when $\langle \rho, x\rangle \geq \rho_0$ dominates $\langle \mu, x\rangle \geq \mu_0$, we have

$$\langle \mu, x\rangle = \underbrace{\langle \rho, x\rangle}_{\geq \rho_0} + \underbrace{\langle \mu - \rho, x\rangle}_{\geq 0} \geq \rho_0 \geq \mu_0,$$

where the first inequality follows from $x \in \mathcal{K}$ and $\mu - \rho \in \mathcal{K}^*$. Then in such a case, $\langle \mu, x\rangle \geq \mu_0$ is a consequence of the inequality $\langle \rho, x\rangle \geq \rho_0$ and the conic constraint $x \in \mathcal{K}$. This motivates our definition of *conic minimal* inequalities:

Definition 1 *A valid inequality $\langle \mu, x\rangle \geq \mu_0$ for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ is called $\mathcal{K}$-*minimal *if for all inequalities $\langle \rho, x\rangle \geq \rho_0$ valid for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with $\mu - \rho \in \mathcal{K}^* \setminus \{0\}$, we have $\rho_0 < \mu_0$.*

Conic minimality definition specifically restricts our attention to the class of valid inequalities that cannot be written as the sum of another valid inequality and a cone-implied inequality. Thus, none of the cone-implied inequalities is $\mathcal{K}$-minimal. However, some $\mathcal{K}$-minimal inequalities can be expressed as the sum of two other non-cone-implied valid inequalities. Hence, not all $\mathcal{K}$-minimal inequalities are extreme.

In finite and infinite relaxations associated with MILPs, minimality of a valid inequality is traditionally defined with respect to the nonnegative orthant, i.e.,$\mathcal{K} = \mathbb{R}^n_+$. That is, a valid inequality $\langle \mu, x\rangle \geq \mu_0$ is $\mathbb{R}^n_+$-minimal if reducing any coefficient μ_i for $i \in [n]$ leads to a strict reduction in the right hand side value μ_0 (see [14]). Therefore, our conic minimality concept for disjunctive conic sets is a natural generalization of $\mathbb{R}^n_+$-minimality.

Extending earlier results from [11, 14] given in the case of $\mathcal{K} = \mathbb{R}^n_+$ to general regular cones $\mathcal{K}$, we can easily see that $\mathcal{K}$-minimal inequalities exist only if the following assumption holds (see [15, Prop 1]):

Assumption 1 *For each $\delta \in \mathcal{K}^* \setminus \{0\}$, there exists some $x_\delta \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$ such that $\langle \delta, x_\delta\rangle > 0$.*

When, for example, $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ is full dimensional, Assumption 1 is satisfied and hence $\mathcal{K}$-minimal inequalities exist. Also, under Assumption 1, all non-cone-implied, extreme inequalities are $\mathcal{K}$-minimal.

Proposition 1 ([15, Prop 2 and Cor 2]) *Under Assumption 1, $\mathcal{K}$-minimal inequalities together with the conic constraint $x \in \mathcal{K}$ are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$.*

This prompts an interest in $\mathcal{K}$-minimal inequalities and suggests that in an efficient cutting plane procedure we should at the least aim at separating inequalities from this class.

On the selection of cone $\mathcal{K}$ in disjunctive conic representations

In all of the previous literature, minimality of an inequality is defined with respect to the nonnegative orthant. We next expose a shortcoming of this and illustrate how encoding structural information in the cone $\mathcal{K}$ is rather pivotal in providing a more refined characterization of extreme inequalities. This point is important even in the case of a disjunctive set associated with an MILP; yet it has been completely overlooked in the literature.

First note that we are essentially interested in the closed convex hull characterizations of disjunctive conic sets and because of our flexibility in selecting $\mathcal{B}$ and $\mathcal{K}$, we may have a choice among several different representations $\mathcal{S}(A_1, \mathcal{K}_1, \mathcal{B}_1)$, $\mathcal{S}(A_2, \mathcal{K}_2, \mathcal{B}_2)$, etc. Moreover, whether a valid inequality is necessary for the convex hull description, i.e., extreme, depends on only the closed convex hull and is independent of the choice of $A, \mathcal{B}$, and $\mathcal{K}$ used in the representation. Besides, as long as the closed convex hull remains the same, $\mathcal{K}$-minimality definition is independent of A and $\mathcal{B}$ used in the representation but depends on only $\mathcal{K}$. That said, when $\mathcal{K}_1 \neq \mathcal{K}_2$, $\mathcal{K}_1$-minimal inequalities might differ significantly from $\mathcal{K}_2$-minimal inequalities even when $\overline{\text{conv}}(\mathcal{S}(A_1, \mathcal{K}_1, \mathcal{B}_1)) = \overline{\text{conv}}(\mathcal{S}(A_2, \mathcal{K}_2, \mathcal{B}_2))$. For example, suppose $\mathcal{K}_1 \subset \mathcal{K}_2$ as well as $\overline{\text{conv}}(\mathcal{S}(A_1, \mathcal{K}_1, \mathcal{B}_1)) = \overline{\text{conv}}(\mathcal{S}(A_2, \mathcal{K}_2, \mathcal{B}_2))$; then all $\mathcal{K}_1$-minimal inequalities are also $\mathcal{K}_2$-minimal but not vice versa. This, in the light of Proposition 1, demonstrates how the selection of cone $\mathcal{K}$ in disjunctive conic representations is critical in identifying more refined dominance relations among valid inequalities. We consequently deduce that minimality should be defined with respect to the smallest cone $\mathcal{K}$ as it encodes the largest amount of structural information. See [15, Rem 1, 5, and 7 and Sec 2.2] as well.

Usually, additional structural information of a problem is available in the form of a convex or polyhedral relaxation; and such information can be encoded in a cone $\mathcal{K}$ in a lifted space by a single additional variable through homogenization as described in [15, Ex 4].

$\mathcal{K}$-minimality and tightness

A first and foremost desirable feature of a strong valid inequality $\langle \mu, x\rangle \geq \mu_0$ is its tightness, i.e., $\mu_0 = \vartheta(\mu)$. The concepts of tightness and $\mathcal{K}$-minimality are intrinsically different. Still, for certain vectors $\mu \in \mathbb{E}$, $\mathcal{K}$-minimality not only immediately implies tightness of the inequality but also determines the sign of $\vartheta(\mu)$.

Proposition 2 ([15, Prop 3]) *Let $\langle \mu, x\rangle \geq \mu_0$ with $\mu \in \pm\mathcal{K}^*$ be a $\mathcal{K}$-minimal inequality. Then $\mu_0 = \vartheta(\mu)$; and furthermore, $\mu \in \mathcal{K}^*$ (resp. $\mu \in -\mathcal{K}^*$) implies $\vartheta(\mu) > 0$ (resp. $\vartheta(\mu) < 0$).*

However, there are $\mathcal{K}$-minimal inequalities with $\mu \notin \pm\mathcal{K}^*$ that are not tight. In fact, a pathology occurs when $\operatorname{Ker}(A) \cap \operatorname{int}(\mathcal{K}) \neq \emptyset$ and $\mu \in \operatorname{Im}(A^*)$.

Proposition 3 ([15, Prop 4]) *Suppose $\operatorname{Ker}(A) \cap \operatorname{int}(\mathcal{K}) \neq \emptyset$. Then, for any $\mu \in \operatorname{Im}(A^*)$, the inequality $\langle \mu, x\rangle \geq \mu_0$ with any $\mu_0 \in (-\infty, \vartheta(\mu)]$ is $\mathcal{K}$-minimal; yet only one of these is tight.*

Because tightness has a direct characterization through $\vartheta(\mu)$, we keep it as a separate consideration.

Algebraic necessary conditions

$\mathcal{K}$-minimality concept has a number of algebraic implications.

Any nontrivial valid inequality $\langle \mu, x\rangle \geq \mu_0$ with $\mu_0 \in \mathbb{R}$ has to satisfy $\mu \in \mathcal{K}^* + \operatorname{Im}(A^*)$ (see [15, Prop 6]). Based on this, we can then associate with such an inequality the following nonempty set

$$D_\mu := \{\lambda \in \mathbb{R}^m : \ \mu - A^*\lambda \in \mathcal{K}^*\}.$$

Because of their structure and relation to cut-generating functions, we refer to these sets D_μ as *cut-generating sets*. Given a nontrivial valid inequality, there is a unique set D_μ associated with it. Yet, it is possible to have two distinct vectors μ' and μ yielding the same set $D_\mu = D_{\mu'}$ (see [15, Ex 8]).

The support function σ_{D_μ} plays an important role in our analysis. First of all, given $\mu \in \mathcal{K}^* + \operatorname{Im}(A^*)$, σ_{D_μ} is helpful in determining a lower bound on $\vartheta(\mu)$, i.e., $\vartheta(\mu) \geq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$ and thus ensuring the validity of $\langle \mu, x\rangle \geq \mu_0$ for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$. For $\mathcal{K} = \mathbb{R}^n_+$, this result was first proven in [14, Thm 9]. Below, we provide its refinement and generalization for arbitrary regular cones $\mathcal{K}$.

Proposition 4 ([15, Prop 7 and 8]) *For any $\mu \in \mathcal{K}^* + \operatorname{Im}(A^*)$, $\vartheta(\mu) \geq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$. Moreover, when at least one of the following conditions holds: (i) $\mathcal{K}$ is polyhedral, (ii) $\operatorname{Ker}(A) \cap \operatorname{int}(\mathcal{K}) \neq \emptyset$, (iii) $\mu \in \operatorname{int}(\mathcal{K}^*) + \operatorname{Im}(A^*)$, we have $\vartheta(\mu) = \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$.*

For any nontrivial valid inequality $\langle \mu, x\rangle \geq \mu_0$, there exists at least one $z \in \operatorname{Ext}(\mathcal{K})$ such that $\sigma_{D_\mu}(Az) = \langle \mu, z\rangle$ (see [15, Lem 2, Cor 3, and Prop 9]). Further, there is a much more elegant connection between $\mathbb{R}^n_+$-sublinear inequalities and the support functions of cut-generating sets D_μ. This has striking consequences that we will comment more on later.

A key necessary condition for $\mathcal{K}$-minimality is based on a certain non-expansiveness property. For this, we introduce the cone of $\mathcal{K}^*{-}\mathcal{K}^*$ positive linear maps given by $\mathcal{F}_\mathcal{K} := \{(Z : \mathbb{E} \to \mathbb{E}) : \ Z \text{ is a linear map, and } Z^*v \in \mathcal{K} \ \forall v \in \mathcal{K}\}$, where Z^* denotes the conjugate linear map of Z.

Proposition 5 ([15, Prop 5]) *A valid inequality $\langle \mu, x\rangle \geq \mu_0$ is $\mathcal{K}$-minimal only if $\mu - Z\mu \notin \mathcal{K}^* \setminus \{0\}$ for all $Z \in \mathcal{F}_\mathcal{K}$ such that $AZ^* = A$.*

Description of $\mathcal{F}_\mathcal{K}$, unfortunately, can be rather nontrivial. For example, deciding whether a given linear map takes $\mathbb{S}^n_+$ to itself is an NP-Hard optimization problem [4]. Because of the general difficulty of working with $\mathcal{F}_\mathcal{K}$ and thereby verifying the necessary condition for $\mathcal{K}$-minimality stated above, we next consider an appropriate relaxation of this condition and introduce the class of $\mathcal{K}$-*sublinear inequalities*.

Definition 2 *Given $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$, a valid inequality $\langle \mu, x\rangle \geq \mu_0$ is* $\mathcal{K}$-sublinear *if for all $\alpha \in \operatorname{Ext}(\mathcal{K}^*)$ it satisfies $0 \leq \langle \mu, u\rangle$ for all u such that $Au = 0$ and $\langle \alpha, v\rangle u + v \in \mathcal{K}\ \forall v \in \operatorname{Ext}(\mathcal{K})$.*

Every $\mathcal{K}$-minimal inequality is also $\mathcal{K}$-sublinear [15, Thm 1]. Without any technical assumptions such as Assumption 1, the existence, sufficiency, properties of $\mathcal{K}$-sublinear inequalities, and their connection with CGFs are pursued further in [16].

Sufficient conditions

The following sufficient conditions complement our necessary conditions and also suggest practical ways of verifying $\mathcal{K}$-sublinearity and/or $\mathcal{K}$-minimality of inequalities.

Proposition 6 ([15, Prop 10]) *Let $\langle \mu, x\rangle \geq \mu_0$ be a nontrivial valid inequality. If there exists a collection I of vectors $x^i \in \operatorname{Ext}(\mathcal{K})$ such that $\sigma_{D_\mu}(Ax^i) = \langle \mu, x^i\rangle$ for all $i \in I$ and $\sum_{i\in I} x^i \in \operatorname{int}(\mathcal{K})$, then $\langle \mu, x\rangle \geq \mu_0$ is $\mathcal{K}$-sublinear.*

Proposition 7 ([15, Prop 11]) *Suppose Assumption 1 holds. Consider a valid inequality $\langle \mu, x\rangle \geq \mu_0$. If there exists a collection I of vectors $x^i \in \mathcal{K}$ such that $\sum_{i\in I} x^i \in \operatorname{int}(\mathcal{K})$, $Ax^i \in \mathcal{B}$ and $\langle \mu, x^i\rangle = \mu_0$, then $\langle \mu, x\rangle \geq \mu_0$ is $\mathcal{K}$-minimal.*

Proposition 7 in particular states that a valid inequality is $\mathcal{K}$-minimal whenever the inequality is satisfied as equality at a point at the intersection of $\operatorname{int}(\mathcal{K})$ and $\operatorname{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$. For MILP problems, this resembles a sufficient condition for an inequality to be facet defining. Nonetheless, conic minimality notion is much weaker than extremality.

Cut generating functions

Given a nonconvex set $\mathcal{B} \subset \mathbb{R}^m$, an important class of problems is defined by the infinite family of sets of form $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ given by any realization of $n \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}$. This family of sets is characterized by solely $\mathcal{B}$ which, in its most general form, is assumed to be a closed set satisfying $0 \notin \mathcal{B}$. Then $0 \notin \overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ follows easily [7, Lem 2.1]. This motivates the definition of *cut-generating functions* (CGFs)—a priori formulas to generate cuts that separate the origin from the convex hull of any instance of $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ determined by n and A:

Definition 3 *Given a nonempty and closed set $\mathcal{B} \subset \mathbb{R}^m$ satisfying $0 \notin \mathcal{B}$, a* cut-generating function *for $\mathcal{B}$ is a function $\psi : \mathbb{R}^m \to \mathbb{R}$ such that the inequality given by $\sum_{i=1}^n \psi(A_i)x_i \geq 1$ is valid for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ where A_i is the i-th column of the matrix A, for any natural number $n \in \mathbb{N}$ and any matrix $A \in \mathbb{R}^{m\times n}$.*

This framework has its roots in Gomory functions [9] and Gomory and Johnson's infinite group relaxations studied in the MILP context [10, 13, 1]. Recent work has focused on a variety of structural assumptions on $\mathcal{B}$ such as $\mathcal{B}$ is a general lattice [6], $\mathcal{B}$ is composed of lattice points contained in a rational polyhedron [8, 3], and $\mathcal{B}$ is a closed set [7], and demonstrated strong connections between $\mathbb{R}^n_+$-minimal inequalities and CGFs obtained from the gauge functions of maximal lattice-free sets.

This framework and CGFs are intimately connected to our results on $\mathbb{R}^n_+$-sublinear inequalities and their relation with support functions of cut-generating sets. We discuss this next; see [15, Sec 4.3] for a detailed account.

Decades ago, Johnson [14] considered $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with $\mathcal{K} = \mathbb{R}^n_+$ and introduced *subadditive inequalities*. These inequalities are equivalent to the $\mathbb{R}^n_+$-*sublinear inequalities* (see e.g., [15, Rem 9]). We restate their definition below:

Definition 4 *Given $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, a valid inequality $\langle \mu, x\rangle \geq \mu_0$ is* $\mathbb{R}^n_+$-sublinear *if for all $i \in [n]$, $\langle \mu, u\rangle \geq 0$ holds for all u such that $Au = 0$ and $u + e_i \in \mathbb{R}^n_+$ where e_i denotes the i^{th} unit vector in $\mathbb{R}^n$.*

A fundamental result of Johnson [14, Thm 10] asserts that the cut coefficient vector of any $\mathbb{R}^n_+$-sublinear inequality is generated by its support function σ_{D_μ}, which is also piecewise linear. Our Proposition 6 complements this result and proves that the conditions of Proposition 6 are necessary and sufficient for $\mathbb{R}^n_+$-sublinearity. That is, a valid inequality $\langle \mu, x\rangle \geq \mu_0$ is $\mathbb{R}^n_+$-sublinear and tight if and only if its support function σ_{D_μ} generates its coefficient vector μ and its right hand side value μ_0. The following theorem summarizes these results [14, Thm 10] and [15, Props 6, 8, and 10, and Thm 4] for $\mathcal{K} = \mathbb{R}^n_+$; see also [15, Rem 10 and 11].

Theorem 1 *Consider $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Then any nontrivial valid inequality $\langle \mu, x\rangle \geq \mu_0$ satisfies $\mu \in \mathbb{R}^n_+ + \mathrm{Im}(A^\top)$ and $\vartheta(\mu) = \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \geq \mu_0 > -\infty$. Moreover, $\langle \mu, x\rangle \geq \mu_0$ is $\mathbb{R}^n_+$-sublinear if and only if it is valid ($\vartheta(\mu) \geq \mu_0$) and $\mu_i = \sigma_{D_\mu}(A_i)$ for all $i \in [n]$ where A_i denotes the i-th column of the matrix A.*

More recently, Kılınç-Karzan and Steffy [16] noted that support function σ_{D_μ} associated with any nontrivial valid inequality $\langle \mu, x\rangle \geq \mu_0$ can be utilized in obtaining a stronger and $\mathbb{R}^n_+$-sublinear inequality.

Proposition 8 ([16, Prop 3]) *Any nontrivial valid inequality $\langle \mu, x\rangle \geq \mu_0$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ is equivalent to or dominated by an $\mathbb{R}^n_+$-sublinear inequality given by $\sum_{i=1}^n \sigma_{D_\mu}(A_i)x_i \geq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \geq \mu_0$ where the domination is with respect to the cone $\mathcal{K} = \mathbb{R}^n_+$.*

Thus, $\mathbb{R}^n_+$-sublinear inequalities are *always sufficient* to describe $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Proposition 8 also inspired the following definition of *relaxed* CGFs as the support functions of nonempty sets D in [16]:

Definition 5 *Given $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ and a set $\emptyset \neq D \subset \mathbb{R}^m$, we say that the support function $\sigma_D : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ of D is a* relaxed cut-generating function *for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$.*

Clearly, the support functions associated with $\mathbb{R}^n_+$-sublinear inequalities are relaxed CGFs. Although the relaxed CGFs such as σ_{D_μ} are seemingly tied to a particular set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ defined by fixed n, A, and $\mathcal{B}$, the subadditivity of these support functions permits us at once to generate valid inequalities for any instance $\mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$ with data $A' \in \mathbb{R}^{m\times n'}$, i.e., varying n and A, as long as the set $\mathcal{B}$ is kept the same.

Proposition 9 ([16, Prop 4]) *Suppose $\mathcal{B} \subset \mathbb{R}^m$ is given. Let $\sigma_D(\cdot)$ be a relaxed CGF for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ associated with a nonempty set $D \subset \mathbb{R}^m$. Then, the inequality $\sum_{i=1}^{n'} \sigma_D(A'_i)x_i \geq \inf_{b\in\mathcal{B}} \sigma_D(b)$ is valid for* any *$\mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$ where the dimension n' and the matrix $A' \in \mathbb{R}^{m\times n'}$ are arbitrary, and A'_i denotes the i-th column of the matrix A' for all $i \in [n]$.*

When $\mathcal{B}$ is a closed set satisfying $0 \notin \mathcal{B}$, Proposition 9 essentially binds together relaxed CGFs and regular CGFs. For a relaxed CGF σ_D to be a regular CGF, we need to ensure: (i) $\inf_{b\in\mathcal{B}} \sigma_D(b) \geq 1$ and (ii) σ_D is finite valued. All $\mathbb{R}^n_+$-sublinear inequalities of form $\langle \mu, x\rangle \geq 1$ immediately have $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \geq 1$. Then we infer from Theorem 1 and Propositions 8 and 9 that without any structural or technical assumptions, the relaxed CGFs, specifically the ones associated with the sets D_μ of $\mathbb{R}^n_+$-sublinear inequalities, are sufficient to generate all necessary inequalities for the description of $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ for all choices of n and A. When the set $\mathcal{B}$ is composed of lattice points, a classical result [6, Thm 1.2] states that all $\mathbb{R}^n_+$-minimal inequalities are generated by sublinear functions which are also piecewise linear. Johnson's [14] analysis along with ours easily recovers this. Sufficiency of regular CGFs for generating all cuts separating the origin in the case of general $\mathcal{B}$ relies on additional structural assumptions [7, Ex 6.1 and Thm 6.3]. This is in contrast to the sufficiency of relaxed CGFs for any $\mathcal{B}$. In this respect, the main challenge in transforming a relaxed CGF σ_D into a regular CGF resides in ensuring finite valuedness of σ_D while maintaining

$\inf_{b \in \mathcal{B}} \sigma_D(b) \geq 1$. Whenever σ_D is not finite valued, i.e., D is unbounded, under certain assumptions, the relaxed CGFs obtained from bounded sets $\widehat{D} \subset D$ offer a solution for this challenge.

The sufficiency of CGFs for describing the convex hulls of disjunctive conic sets is intrinsically related to the strong duality theory for integer programs. Morán et al. [19, Thm 2.4] has extended the strong duality theory for MILPs to MICPs of a specific form. Under technical assumptions, these theorems assert that for every integer programming instance, there is a dual problem achieving zero duality gap where the 'dual variables' are finite-valued subadditive functions that are nondecreasing with respect to the underlying cone. These functions indeed act locally on each variable x_i and produce cut coefficient μ_i by considering only the data A_i associated with x_i; therefore, they are simply CGFs. Then the sufficiency of CGFs for generating all cuts of the form $\langle \mu, x \rangle \geq 1$ follows from strong MICP duality theorem. Nevertheless, not only the strong duality results for MILPs and MICPs rely on some technical assumptions but also the sets $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ representing MILPs and the specific form of MICPs from [19] impose a specific structure on $\mathcal{B}$ (see [15, Ex 3]). Additional discussion relating [19] to CGFs is given in [15, Rem 12] and [16, Rem 2].

Our results naturally capture some of the earlier results from the MILP setup and generalize them to the cases with arbitrary nonconvex sets $\mathcal{B}$. That said, our study also reveals some problems associated with such a CGF based view that treats the data associated with each individual variable independently in the case of general regular cones other than the nonnegative orthant. Namely, [15, Ex 8 and Rem 12] features an extreme inequality for a set $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with $\mathcal{K} = \mathbb{L}^3$ that cannot be generated by any CGF or relaxed CGF.

Final remarks

In the context of disjunctive conic sets, characterization of $\mathcal{K}$-minimal and tight inequalities has underlied the development of structured convex (or conic representable) cuts for two-term linear disjunctions applied to a second-order cone (see [18]). The flexible representation structure offered by disjunctive conic sets can easily allow us to pursue a similar principled study of other simple, yet fundamental, nonconvex sets defined by multi-term disjunctions or quadratics on regular cones. In this regard, characterizations of extreme inequalities beyond $\mathcal{K}$-minimality are very appealing as well.

We also hope that the understanding and connections we built on CGFs and relaxed CGFs will be instrumental in understanding when minimal or extreme CGFs will produce strong linear inequalities such as facets for given problem instances. On a related note, the sufficiency of CGFs to generate all valid inequalities for the convex hull description of disjunctive sets or all cuts that separate the origin from the convex hull of disjunctive sets is an indispensable question for the justification of this research focus on CGFs. Along these lines, our results have recently contributed to the foundation of the most general conditions guaranteeing the sufficiency of CGFs for general $\mathcal{B}$ [17].

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