A Second-Order Cone Based Approach for Solving the Trust Region Subproblem and Its Variants

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Abstract

We study the trust region subproblem (TRS) of minimizing a nonconvex quadratic function over the unit ball with additional conic constraints. Despite having a nonconvex objective, it is known that the TRS and a number of its variants are polynomial-time solvable. In this paper, we follow a second-order cone based approach to derive an exact convex formulation of the TRS, and under slightly stronger conditions, give a low-complexity characterization of the convex hull of its epigraph without any additional variables. As a result, our study highlights an explicit connection between the nonconvex TRS and smooth convex quadratic minimization, which allows for the application of cheap iterative methods to the TRS. We also explore the inclusion of additional hollow constraints to the domain of the TRS, and convexification of the associated epigraph.

1 Introduction

In this paper, we study the polynomial-time solvable variants of the *trust region subproblem* (TRS) [15] given by

$$\mathrm{Opt}_h := \min_{y\in\mathbb{R}^n} \left\{ h(y) := y^\top Q y + 2\, g^\top y : \begin{array}{rcl} \|y\| & \leq & 1 \\ Ay - b & \in & \mathcal{K} \end{array} \right\}, \tag{1}$$

where $\|y\|$ denotes the Euclidean norm of y, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $\mathcal{K} \subseteq \mathbb{R}^m$ is a closed convex cone. We assume that the minimum eigenvalue of Q, $\lambda_{\min}(Q)$, is negative. This problem (1) is equivalent to the classical TRS when there are no additional conic constraints, i.e., $A = I_n$, $b = 0$, and $\mathcal{K} = \mathbb{R}^m$.

The classical trust region subproblem is an essential ingredient of the trust region methods that are commonly used to solve continuous nonconvex optimization problems (see [15, 30, 32] and references therein). In each iteration of a trust region method, a quadratic approximation of the objective function is built and then optimized over a ball, called trust region, (or intersection of a ball with linear constraints) to find the new search point. Trust region subproblems are also encountered in the context of robust optimization under matrix norm or polyhedral uncertainty (see [4, 6] and references therein), nonlinear optimization problems with discrete variables [10, 12], least-squares problems [40], constrained eigenvalue problems [20], and more.

As stated above, the optimization problem in (1) is nonlinear and nonconvex when $\lambda_{\min}(Q) < 0$. Nevertheless, it is well-known that classical TRS and a number of its variants can be solved in polynomial time via semidefinite programming [33] or using specialized nonlinear algorithms [22, 27]. Specifically, the semidefinite programming (SDP) relaxation for the classical TRS is known to be exact [33].

Several variants of the TRS that enforce additional constraints on the trust region have been proposed. Among these the most commonly studied is the case when $\mathcal{K}$ is taken to be a nonnegative orthant, i.e., additional linear constraints are modeled via the polyhedral set $P := \{y \in \mathbb{R}^n : Ay - b \in \mathcal{K}\}$ intersected with the unit ball. TRS with additional linear inequalities arises in nonlinear programming and robust optimization (see [11, 25] and references therein) and is studied in [9, 11, 12, 14, 25, 37, 39] under a variety of assumptions. Specifically, [12, 37] give a tight semidefinite formulation when there is a single linear constraint $a^\top y \leq b$ based on an additional constraint derived from second-order reformularization linerazitation technique (SOC-RLT). This approach was extended to two linear constraints in [12, 39] and the tightness of the SDP relaxation is shown when the linear constraints are parallel. More recently, Burer and Yang [14] give a tight SDP relaxation with additional SOC-RLT constraints for an arbitrary number of linear constraints, under the condition that these additional linear inequalities do not intersect on the interior of the unit ball. We refer the readers to Burer [11] for a recent survey and related references for the results on tight SDP relaxations associated with these variants. Recently, Jeyakumar and Li [25] prove convexity of the joint numerical range, exactness of the SDP relaxation, and strong Lagrangian duality for the TRS with additional linear and SOC constraints. A key tool in their analysis is to recast the TRS as a convex quadratic minimization problem under a dimensionality condition. Following a different approach and without relying on a convex reformulation of the problem, Bienstock and Michalka [9] show that TRS with linear inequality constraints is polynomial-time solvable under the milder condition that the number of faces of the linear constraints P intersecting with the unit ball is polynomially bounded. Hollow constraints defined by a single ellipsoid [5, 8, 32, 36, 39] or several ellipsoids [9, 38] or arbitrary quadratics constraints [7] have also attracted some attention in the literature. These approaches are once again either SDP based convexification schemes or customized algorithms.

While the SDP reformulations of the TRS and its variants can be solved using interior-point methods in polynomial time [1, 29], this approach is not practical because the worst-case complexity of these methods for solving SDPs is a relatively large polynomial and these methods do not exploit the sparsity of the data. That said, the classical TRS is closely connected to eigenvalue problems. In the specific case of classical TRS where the objective is convex, i.e., when Q is positive semidefinite, this problem becomes simply the minimization of a smooth convex function over the Euclidean ball; and thus it can be solved efficiently via iterative methods such as Nesterov's accelerated gradient descent algorithm [28]. Moreover, in the nonconvex case with $\lambda_{\min}(Q) < 0$, when the problem is purely quadratic, i.e., when $g = 0$ as well, the TRS reduces to finding the minimum eigenvalue of Q. This can be approximated in linear time by the well-known Power iteration method [21, Chapter 8.2] and can be found efficiently via the Lanczos method [21, Chapter 10.1] in practice. When $g \neq 0$, even though the problem is no longer equivalent to an eigenvalue problem and these methods cannot be applied directly, this observation has lead to the development of efficient, matrix-free algorithms that are based solely on matrix-vector products. The dual-based algorithms of [27], [33] and [35], the generalized Lanczos trust-region method of [22], and the more recent developments of [17, 18, 23, 24, 34] are examples of such iterative algorithms. In most cases, these algorithms are presented together with their convergence proofs. Nevertheless, to the best of our knowledge, the theoretical runtime evaluation of these algorithms lacks formal guarantees with the exception of [24]. In addition, in most of these iterative methods, numerical difficulties are reported in the so-called the "hard case" [27]. The hard case occurs when the linear component vector g is nearly orthogonal to the eigenspace of the smallest eigenvalue of Q. In many cases, the lack of provable worst-case convergence bounds for the TRS is attributed to the hard case. As a result, most research on specific algorithms for the TRS thus far focuses on addressing this issue.

Recently, Hazan and Koren [24] suggested a linear-time algorithm for approximately solving the TRS within a given tolerance ϵ. Their approach relies on an efficient, linear-time solver for a specific SDP relaxation of a feasibility version of the TRS and reduces the TRS into a series of eigenvalue computations. In their approach, they exploit the special structure of the dual problem, a one-dimensional problem for which bisection techniques can be applied, to avoid using interior-point solvers. Each dual step of their algorithm requires a single approximate maximal eigenvalue computation which takes $O\left(\frac{N}{\sqrt{\epsilon}}\log\left(\frac{n}{\delta}\right)\right)$ time, where N is the number of nonzero entries in Q and δ is the desired accuracy to which the maximal eigenvalue is computed. Their overall algorithm converges in $O\left(\log\left(\frac{\Gamma}{\epsilon}\right)\right)$ iterations, where $\Gamma := \max\{2(\|Q\|+\|b\|), 1\}$ and $\|Q\|$ stands for the spectral norm of the matrix Q, i.e. the maximum absolute eigenvalue. Then a primal solution is recovered by solving a small linear program formed by the dual iterates. Finally, they provide an efficient and accurate rounding procedure for converting the SDP solution into a feasible solution to the TRS. As a consequence, their approach does not require the use of interior-point SDP solvers, bypasses the difficulties noted for the hard-case of the TRS, and can exploit data sparsity (runs in time linear in the number of nonzero entries of the input). The overall complexity of their algorithm is $O\left(\frac{N}{\sqrt{\epsilon}}\log\left(\frac{n}{\delta}\log\left(\frac{\Gamma}{\epsilon}\right)\right)\log\left(\frac{\Gamma}{\epsilon}\right)\right)$.

In this paper, as opposed to the previous specialized algorithms or the SDP based approaches, we suggest a second-order cone (SOC) based approach to solve the trust region subproblem and its various variants. That is, under easy-to-verify conditions, we derive tight SOC-based convex reformulations and convex hull characterizations of sets associated with the TRS with additional constraints. Our results on convex hull representations are based on the recent developments of Burer and Kılınç-Karzan [13] for representing certain nonconvex sets defined by the intersection of SOC constraints and constraints involving general nonconvex quadratics. In particular, [13] studies nonconvex sets obtained as the intersections of the form $\mathcal{F}^+\cap\mathcal{Q}$ and $\mathcal{F}^+\cap\mathcal{Q}\cap H$ where the cone $\mathcal{F}^+$ is second-order-cone representable (SOCr), $\mathcal{Q}$ is a nonconvex cone defined by a single homogeneous quadratic, and H is an affine hyperplane. For such sets, under several easy-to-verify conditions, [13] suggests a simple, computable convex relaxation $\mathcal{F}^+\cap\mathcal{S}$ of $\mathcal{F}^+\cap\mathcal{Q}$, where $\mathcal{S}$ is an additional SOC representable cone, and identifies several stronger conditions guaranteeing the tightness of these relaxations, i.e., $\mathcal{F}^+\cap\mathcal{S}=\overline{\text{cone}}(\mathcal{F}^+\cap\mathcal{Q})$ and $\mathcal{F}^+\cap\mathcal{S}\cap H=\overline{\text{conv}}(\mathcal{F}^+\cap\mathcal{Q}\cap H)$, where $\overline{\text{cone}}$ indicates the closed conic hull, and $\overline{\text{conv}}$ indicates the closed convex hull. These conditions have been further verified in many specific cases; and it was shown in [13] that the classical TRS can be solved via the optimization of two second-order conic programs. Similar convex hull descriptions of SOCs intersected with general nonconvex quadratics are also studied recently in [26] under different assumptions.

Our contributions in this paper can be summarized as follows.

(i) We give a tight SOC-based low-complexity convex relaxation of the TRS with additional conic constraints (1). Under an easily checkable and general structural condition on the conic constraints, our Theorem 2.4 states that the convex relaxation of problem (1) obtained by simply replacing the nonconvex objective function $h(y)$ in (1) with the convex objective

$f(y) := y^\top (Q - \lambda_{\min}(Q) I_n) y + 2 g^\top y + \lambda_{\min}(Q)$ is tight.

(ii) Consequently, we show that our convex reformulation of the classical TRS can be solved via a single minimum eigenvalue computation and then minimizing a smooth convex quadratic over the unit ball. Thus, the overall complexity of solving the classical TRS using first-order methods is

$$O\left(N\left(\log\left(\frac{1}{\delta}\right) + \frac{\lambda_{\max}(Q) - \lambda_{\min}(Q)}{\sqrt{\epsilon}}\right)\right),$$

where $\lambda_{\max}(Q)$ is the maximum eigenvalue of the matrix Q. In addition, we discuss implications of approximating the minimum eigenvalue computation in this scheme.

(iii) Finally, we study the convex hull of the epigraph of the TRS given by

$$X := \left\{ \begin{bmatrix} y \\ t \end{bmatrix} \in \mathbb{R}^{n+1} : \begin{array}{rcl} \|y\| & \leq & 1 \\ Ay - b & \in & \mathcal{K} \\ h(y) & \leq & t \end{array} \right\}.$$

In Theorem 3.3, under a slightly stronger condition, we provide an explicit characterization of $\operatorname{conv}(X)$ in the space of original variables. We also examine the inclusion of additional hollow constraints $y \in \mathcal{H} = \mathbb{R}^n \setminus \mathcal{P}$ to the TRS; and in Theorem 3.8 we give a more general condition on the hollow set $\mathcal{H}$ than the existing ones from the literature for which we can derive the convex hull.

The papers [25] and [13] are closely related to our approach due to their SOC based approaches. In particular, our convex reformulation result of the TRS is a generalization of a result in [25]. We show that the conditions from [25] imply our condition and we provide an example where our condition is satisfied but the ones in [25] are not. We also provide an example to demonstrate the necessity of our condition. In the context of the classical TRS with no additional constraints, our structural assumption is immediately satisfied. We note that [13] also give a scheme to solve the classical TRS via SOC programming. The scheme suggested in [13] is in a lifted space with one additional variable and requires solving two related SOC optimization problems. In contrast, our convex reformulation is in the space of original variables and requires only a single minimization of a smooth convex quadratic function over the unit ball.

On the algorithmic side, our transformation of the TRS requires mainly the computation of a minimum eigenvalue of Q which can be done in linear time with $O(N)$ arithmetic operations per iteration, for instance by using the Power iteration method [21, Chapter 8.2]. Due to the fact that $f(x)$ is a convex quadratic function, our convex reformulation for the classical TRS can simply be cast as a conic optimization problem. Specifically, when there are no additional constraints, this problem becomes minimizing a smooth convex function over the Euclidean ball; and thus it is readily amenable to efficient first-order methods. In particular, for this class of problems, given a desired accuracy level of $\epsilon > 0$, Nesterov's accelerated gradient descent algorithm [28] involves only elementary operations such as addition, multiplication, and matrix-vector product computations and achieves the optimal iteration complexity of $O\left(\frac{\lambda_{\max}(Q) - \lambda_{\min}(Q)}{\sqrt{\epsilon}}\right)$. In the specific case where the problem is convex (i.e., when Q is positive semidefinite), the same complexity guarantees can be obtained by applying Nesterov's accelerated gradient descent [28] to the problem. Thus, our approach can be seen as an analog of the latter algorithm to the general nonconvex case. To the best of our knowledge, this connection with iterative methods in the context of the classical TRS has not been made before.

Moreover, convexification based approaches bypass the so-called 'hard case' because they work directly with a convex formulations. Among these the ones more amenable to iterative methods are [5, 25, 24] and ours. This is so for our approach, as well as that of Jeyakumar and Li [25], because we work with an SOC based reformulation of the problem and it only requires the computation of a minimum eigenvalue. To the best of our knowledge iterative algorithms for SDP based relaxations of the TRS have not been studied in the literature with the exception of Hazan and Koren [24]. As compared to the approach in [24], we believe our approach is straightforward and easy to implement while it achieves a slightly better convergence guarantee in the worst case. In particular, our approach directly solves the TRS, as opposed to only solving a feasibility version of the TRS; thus we save an extra logarithmic factor. Also, the convex reformulation given by Ben-Tal and Teboulle [5] requires a full eigenvalue decomposition which is more expensive, i.e., $O(n^3)$ time, and due to its specific structure can be solved at a slower rate by iterative methods.

Our convex hull results on the epigraph of the TRS is inspired by the recent work of Burer and Kılınç-Karzan [13] on convex hulls of general quadratic cones. While the convex hull results in [13] are applicable to many problems, including a set associated with the classical TRS, we present a much more direct analysis for the TRS. There are two main benefits of our approach. Firstly, the convex hull result for the classical TRS in [13] requires the assumption that the optimal value is nonpositive. While this is not an issue for the classical TRS since the optimal value is always negative, with the existence of additional constraints, this may no longer be true. In contrast, our analysis does not require any nonpositivity assumptions; and hence we are able to include additional conic constraints in our convex hull results under only minor conditions. Secondly, our direct analysis of the TRS allows us to bypass verifying several conditions from [13] and to work directly with a single condition which is always satisfied in the case of the classical TRS.

Several papers [2, 3, 25] exploit convexity results on the joint numerical range of quadratic mappings to explore strong duality properties of the TRS and its variants. These convexity results are based on Yakubovich's $\mathcal{S}$-lemma [19] and Dines [16], see also the survey by Pólik and Terlaky [31] for a more detailed discussion. While these results as well as ours both analyze sets associated with the TRS, the actual sets in question are quite different. In the context of the TRS, the joint numerical range is a set of the form

$$\left\{[h(y);\ \|y\|^2;\ Ay-b] : y \in \mathbb{R}^n\right\} \subseteq \mathbb{R}^{m+2}.$$

Under certain conditions, this set is shown to be convex. In contrast, we study the epigraphical set X, which is nonconvex if $h(y)$ is, and give its convex hull description in the original space of variables.

Our paper is structured as follows. Section 2 details the derivation of our convex reformulation of the TRS with additional conic constraints under a structural assumption on the conic constraints. We then discuss the complexity of solving our reformulation of the classical TRS with iterative methods in Section 2.2, as well as the implications of working with an approximate minimum eigenvalue in our reformulation. In addition to our convex reformulation of the TRS with additional conic constraints, we study exact and explicit low-complexity convex hull results for its epigraph obtained by new SOC constraints in Section 3. Finally, we discuss an extension of our SOC based approach to the TRS to handle additional specific convex constraints and nonconvex constraints given by hollows in Section 3.1.

We use Matlab notation to denote vectors and matrices. Furthermore, we let I_n be the $n \times n$ identify matrix and denote the minimum eigenvalue of a symmetric matrix Q as $\lambda_Q := \lambda_{\min}(Q)$. Our notation is mostly standard; we will define any particular notation upon its first use.

2 Tight Low-Complexity Convex Reformulation of the TRS

In this section, we present an exact convex, SOC based reformulation of the trust region subproblem with additional conic constraints. We then explore algorithmic aspects of the classical TRS implied by our SOC formulation.

2.1 Deriving the Reformulation

In this section, we present an exact convex reformulation the trust region subproblem with additional conic constraints:

$$\mathrm{Opt}_h := \min_{y \in \mathbb{R}^n} \left\{ h(y) := y^\top Q y + 2g^\top y : \begin{array}{rcl} \|y\| & \leq & 1 \\ Ay - b & \in & \mathcal{K} \end{array} \right\} \tag{2}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $\mathcal{K} \subset \mathbb{R}^m$ is a closed convex cone.

We start with the following simple observation about the optimal solutions of (2).

Lemma 2.1. *Any optimal solution y^* of (2) must be on the boundary of the domain $\{y : \|y\| \leq 1,\ Ay - b \in \mathcal{K}\}$.*

Proof. Suppose $y^* \in \mathrm{int}(\{y : \|y\| \leq 1,\ Ay - b \in \mathcal{K}\})$. Consider $d \neq 0$ such that $Qd = \lambda_Q d$ and $(g + \lambda_Q y^*)^\top d \leq 0$ (one such d always exists since if $(g + \lambda_Q y^*)^\top d > 0$, we can take the negative instead). Because y^* is in the interior of the domain, there exists a small $\epsilon > 0$ such that $y^* + \epsilon d$ remains feasible. However, the objective function at the point $y^* + \epsilon d$ satisfies

$$h(y^* + \epsilon d) = (y^*)^\top Q (y^*) + 2g^\top y^* + 2(g + \lambda_Q y^*)^\top d\epsilon + \lambda_Q \|d\|^2 \epsilon^2 < h(y^*)$$

because $\lambda_Q < 0$ and d is chosen so that $(g + \lambda_Q y^*)^\top d \leq 0$. This contradicts the optimality of y^*; and hence any optimal solution to (2) must be on the boundary. $\square$

For nonconvex quadratic objective functions, Lemma 2.1 points out the important role of the boundary of our feasible set $\{y : \|y\| \leq 1,\ Ay - b \in \mathcal{K}\}$. In particular, let us examine the situation when there exists a convex function $f(y)$ such that

(i) $f(y) \leq h(y)$ for all $y \in \{y : \|y\| \leq 1,\ Ay - b \in \mathcal{K}\}$,

(ii) $f(y) = h(y)$ for all $y \in \mathrm{bd}(\{y : \|y\| \leq 1,\ Ay - b \in \mathcal{K}\})$, and

(iii) $f(y)$ has a minimizer on $\mathrm{bd}(\{y : \|y\| \leq 1,\ Ay - b \in \mathcal{K}\})$.

Then the convex relaxation given by

$$\mathrm{Opt}_f := \min_{y} \left\{ f(y) : \begin{array}{rcl} \|y\| & \leq & 1 \\ Ay - b & \in & \mathcal{K} \end{array} \right\}$$

is tight. Given these observations, we ideally would like to find a convex function $f(y) \leq h(y)$ satisfying all of the above properties. In general, this is not possible. An natural alternative is to instead consider

$$f(y) := h(y) + \lambda_Q (1 - \|y\|^2) = y^\top (Q - \lambda_Q I_n) y + 2g^\top y + \lambda_Q. \tag{3}$$

Because $Q - \lambda_Q I_n \succeq 0$, that is, $Q - \lambda_Q I_n$ is positive semidefinite, the function $f(y)$ is convex by construction. Also, $f(y) = h(y)$ if and only if $\|y\| = 1$. However, note that $f(y)$ may not be equal to $h(y)$ on all of $\mathrm{bd}(\{y : \|y\| \le 1,\ Ay - b \in \mathcal{K}\})$. More precisely, we will have $f(y) < h(y)$ for $y \in \mathrm{bd}(\{y : \|y\| \le 1,\ Ay - b \in \mathcal{K}\}) \cap \{y : \|y\| < 1\}$. Despite this obstacle, we will present conditions for which the convex relaxation with objective $f(y)$ will still be tight.

The function $f(y)$ given in (3) exemplifies a specific way of obtaining convex functions that underestimates $h(y)$ on the domain via aggregation of $h(y)$ with the convex constraints defining our domain. This type of aggregation based convex functions can be utilized in building convex relaxations of nonconvex optimization problems. In certain cases, such convex relaxations turn out to be tight. We demonstrate these in the following result.

Lemma 2.2. *Let C be a given set and $c_j(x)$ for $j = 1, \ldots, m$ be given functions. Suppose $h(x)$ is a given nonconvex function and $f_j(x)$ is a convex function on the domain $\mathcal{C} := \{x : \ c_j(x) \le 0,\ \forall j = 1, \ldots, m;\ x \in C\}$ such that $f_j(x) = h(x) + \alpha_j\, c_j(x)$ for some $\alpha_j > 0$. Let $F(x) := \max_{j=1,\ldots,m} f_j(x)$. Then*

$$\mathrm{Opt}_h := \min_x \{h(x) : \ x \in \mathcal{C}\} \ge \min_x \{F(x) : \ x \in \mathcal{C}\} =: \mathrm{Opt}_f\,.$$

Moreover, the relation between the optimal values of these optimization problems above holds as equality if and only if there exists an optimal solution x^* to the problem on the right-hand-side satisfying $c_j(x^*) = 0$ for some $j \in \{1, \ldots, m\}$.

Proof. First, we note that for any $x \in \mathcal{C}$, we have $c_j(x) \le 0$ and thus for all $j \in \{1, \ldots, m\}$, $f_j(x) = g(x) + \alpha_j\, c_j(x) \le g(x)$ because $\alpha_j > 0$. This establishes $\mathrm{Opt}_g \ge \mathrm{Opt}_f$.

Let x^* be an optimal solution to $\min_x \{F(x) : \ x \in \mathcal{C}\}$. When $c_j(x^*) = 0$ for some j, we have $f_j(x^*) = g(x^*)$ which implies that x^* is also optimal to the first problem. Now consider the case where every optimal solution $x^* \in \arg\min_x \{F(x) : \ x \in \mathcal{C}\}$ satisfies $c_j(x^*) < 0$ for all j. Note that for any $x \in \mathcal{C}$ satisfying $c_j(x) < 0$ for all j, we have $F(x) < g(x)$. Let $\bar{x} \in \arg\min_x \{g(x) : \ x \in \mathcal{C}\}$. If $c_j(\bar{x}) < 0$ for all j, then we have $F(\bar{x}) < g(\bar{x})$ implying $\mathrm{Opt}_f < \mathrm{Opt}_g$. If $c_j(\bar{x}) = 0$ for some j, then $\bar{x}$ is not optimal for the second problem; and hence $\mathrm{Opt}_g = g(\bar{x}) = f_j(\bar{x}) > F(x^*) = \mathrm{Opt}_f$ holds for any $x^* \in \arg\min_x \{F(x) : \ x \in \mathcal{C}\}$. $\square$

Lemma 2.2 gives us a precise characterization for when the convex relaxation using $f(y)$ is tight.

Corollary 2.3. *Consider the convex relaxation for problem (2) given by*

$$\mathrm{Opt}_f = \min_y \left\{ f(y) : \begin{array}{rcl} \|y\| & \le & 1 \\ Ay - b & \in & \mathcal{K} \end{array} \right\}, \tag{4}$$

where $f(y)$ is defined in (3). This convex relaxation is tight if and only if there exists an optimal solution y^* to (4) such that $\|y^*\| = 1$.

Based on Corollary 2.3, we next provide a condition that guarantees us find minimizers of the convex function $f(y)$ defined in (3) on the boundary of the unit ball.

Condition 2.1. There exists a vector $d \neq 0$ such that $Qd = \lambda_Q d$, $Ad \in \mathcal{K}$ and $g^\top d \le 0$.

Note that Condition 2.1 can be checked by solving the conic program

$$\min_d \left\{ g^\top d : \ (Q - \lambda_Q I_n)d = 0,\ Ad \in \mathcal{K} \right\}.$$

Remark 2.1. From the definition of λ_Q, Condition 2.1 is immediately satisfied when $A = I_n$, $b = 0$, and $\mathcal{K} = \mathbb{R}^n$, i.e., for the classical TRS without additional constraints. $\diamondsuit$

Moreover, Condition 2.1 has an immediate application in giving tight convex relations as follows.

Theorem 2.4. *Suppose that Condition 2.1 holds for the TRS given in (2). Then the convex relaxation given by (4) is tight.*

Proof. Let y^* be an optimum solution for (4). If $\|y^*\| = 1$, then from Corollary 2.3, the result follows immediately. Hence, we assume $\|y^*\| < 1$.

Let $d \neq 0$ be the vector from Condition 2.1, thus $Qd = \lambda_Q d$, $Ad \in \mathcal{K}$ and $g^\top d \leq 0$. Then for any $\epsilon > 0$, $A(y^* + \epsilon d) - b = (Ay^* - b) + Ad\epsilon \in \mathcal{K}$ because $\mathcal{K}$ is a convex cone and $Ad' \in \mathcal{K}$ by assumption. Because $\|y^*\| < 1$, we may increase ϵ until $\|y^* + \epsilon d\| = 1$ and the vector $y^* + \epsilon d'$ is still feasible. Note $(Q - \lambda_Q I_n)d = 0$, so for any $\epsilon > 0$,

$$f(y^* + \epsilon d) = f(y^*) + 2(g^\top d)\epsilon \leq f(y^*).$$

If $g^\top d < 0$, this violates optimality of y^* since $\epsilon > 0$, thus $g^\top d = 0$. Then the vector $y^* + \epsilon d$ is an alternative optimum solution to (4) satisfying $\|y^* + \epsilon d\| = 1$. Hence, the tightness of the relaxation (4) follows from Corollary 2.3. $\square$

We note that a similar result was implicitly proven by Jeyakumar and Li [25] under a dimensionality condition for the case of linear and conic quadratic constraints. We state the linear version of their condition below; the conic quadratic one is very similar.

Condition 2.2. Consider the case of nonnegative orthant, i.e., $\mathcal{K} = \mathbb{R}^m_+$. Suppose that the system of linear inequalities, i.e., the constraint $Ay - b \in \mathcal{K}$ satisfies the requirement that $\dim(\mathrm{Null}(Q - \lambda_Q I_n)) \geq n - \dim(\mathrm{Null}(A)) + 1$.

Observation 2.5. *Condition 2.1 generalizes the dimensionality Condition 2.2 stated for linear and conic quadratic constraints.*

Proof. Suppose Condition 2.2 holds. Then $\dim(\mathrm{Null}(A)) + \dim(\mathrm{Null}(Q - \lambda_Q I_n)) \geq n+1$; thus, there must exist $d \neq 0$ which is in the intersection. This means that $Qd = \lambda_Q d$ and $Ad = 0 \in \mathbb{R}^m_+ = \mathcal{K}$. If $g^\top d \leq 0$, then Condition 2.1 holds with the vector d. If $g^\top d > 0$, then Condition 2.1 holds with the vector $d' = -d$. $\square$

Jeyakumar and Li [25] demonstrates that Condition 2.2 is satisfied in a number of cases related to the robust least squares and robust SOC programming problems. As a consequence of Observation 2.5, our Condition 2.1 is satisfied in these cases as well.

Remark 2.2. In contrast to the results given in [25], Theorem 2.4 holds for general conic constraints when Condition 2.1 holds. Note that such general conic constraints can be a variety of convex restrictions, and in particular, include positive semidefiniteness requirements. In addition, for problem data given by

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \quad b = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathcal{K} = \mathbb{R}^2_+,$$

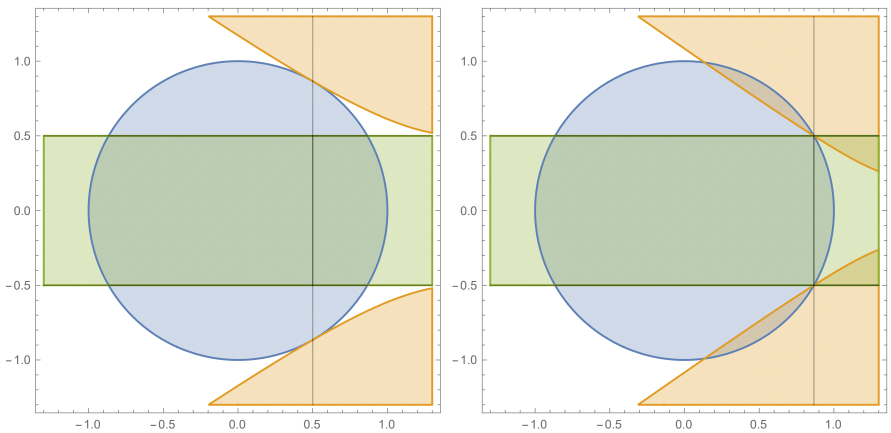
Condition 2.1 is satisfied with $d = [0; -1]$; but Condition 2.2 is not. $\diamondsuit$

We next discuss the necessity of Condition 2.1 via the following example.

Example 2.6. *Suppose we are given the problem data:*

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad g = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad b = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathcal{K} = \mathbb{R}^2_+.$$

Condition 2.1 is violated. To see this, note that any d satisfying $Qd = \lambda_Q d$ is of the form $d = [0; d_2]$. However, $Ad = [d_2; -d_2]$, so if $d_2 \neq 0$, $Ad \notin \mathcal{K} = \mathbb{R}^2_+$. For this problem data, $h(y) = y_1^2 - 2y_2^2 - 3y_1$ and $f(y) = 3y_1^2 - 3y_1 - 2$. It is easy to compute the minimizers of $f(y)$ to be the line $y_1 = 1/2$, with value $\mathrm{Opt}_f = -11/4$. The constraints $Ay - b \in \mathcal{K}$ are equivalent to $-1/2 \le y_2 \le 1/2$.



(a) $h(y) \le -11/4 = \mathrm{Opt}_f$ (b) $h(y) \le (1 - 6\sqrt{3})/2 = \mathrm{Opt}_h$

Figure 1: Contour plots of $h(y)$ over the feasible set.

Figure 1 shows that the minimizers of $h(y)$ over the unit ball $\|y\| \le 1$ lie on the boundary at $y = [1/2; \pm\sqrt{3}/2]$. Due to the linear constraints $-1/2 \le y_2 \le 1/2$, these points are cut off from the feasible region. As a result, any minimizer of $f(y)$ inside the feasible region has norm strictly less than 1. By Corollary 2.3, the relaxation (4) is not tight. We note that while our relaxation is not tight for this example, the SDP relaxation of [39, 12] strengthened with additional SOC-RLT inequalities is tight. $\diamondsuit$

A generalization of Condition 2.1 is instrumental in giving exact convex hull characterization of the sets associated with the TRS. We discuss these further in Section 3.

2.2 Complexity of Solving the Classical TRS

We note that all of the variants of the TRS that we can provide a convex reformulation are solvable via interior point methods and standard software as long as the cone $\mathcal{K}$ has an explicit barrier function. In this section, we explore the classical TRS given by

$$\mathrm{Opt}_h = \min_y \left\{ h(y) := y^\top Q y + 2g^\top y : \ \|y\| \le 1 \right\} \tag{5}$$

and its solution via iterative methods. The problem (5) is equivalent to the TRS with conic constraints given in (2) where $A = I_n$, $b = 0$, and $\mathcal{K} = \mathbb{R}^m$. Then by Remark 2.1, Condition 2.1

is satisfied immediately. Consequently, we have the following theorem as a simple corollary of Theorem 2.4.

Theorem 2.7. *A tight convex relaxation of* (5) *is given by*

$$\mathrm{Opt}_f = \min_y \left\{ f(y) := y^\top (Q - \lambda_Q I_n) y + 2g^\top y + \lambda_Q : \ \|y\| \le 1 \right\}. \tag{6}$$

The reformulation (6) is a second-order conic program and can easily be solved whenever we have λ_Q. However, computing the minimum eigenvalue of Q, λ_Q, itself is a TRS with no linear term because

$$\lambda_Q = \min_y \left\{ y^\top Q y : \ \|y\| \le 1 \right\}.$$

This computation of λ_Q can be done in linear time via the power iteration method [21, Chapter 8.2]. Each iteration of Power method involves a matrix-vector multiplication and requires $O(N)$ time, where N is the number of nonzero entries in Q. Thus, computing λ_Q can be done in $O\left(N \log\left(\frac{1}{\delta}\right)\right)$ time.

Given λ_Q, the problem

$$\min_y \{ f(y) : \ \|y\| \le 1 \}$$

is simply minimizing a smooth convex quadratic function over the unit ball. This can be done efficiently using Nesterov's accelerated gradient descent algorithm [28]. For this class of problems, Nesterov's accelerated gradient descent algorithm achieves the optimal convergence rate of $O\left(\frac{\lambda_{\max}(Q) - \lambda_Q}{\sqrt{\epsilon}}\right)$. Nesterov's algorithm is a classical first-order method; and the major computational burden in each iteration is the evaluation of the gradient of objective function. For a quadratic function, this involves simply a matrix-vector product; and hence each iteration costs $O(N)$ time. The only other main operation in each iteration of Nesterov's algorithm applied to this problem is the projection onto the Euclidean ball; and this can be done in $O(n)$ time. Consequently, Nesterov's algorithm runs in time $O\left(\frac{N(\lambda_{\max}(Q) - \lambda_Q)}{\sqrt{\epsilon}}\right)$. Thus, the total complexity of solving (6) is

$$O\left(N\left(\log\left(\frac{1}{\delta}\right) + \frac{\lambda_{\max}(Q) - \lambda_Q}{\sqrt{\epsilon}}\right)\right).$$

Remark 2.3. Theorem 2.7 shows that the classical TRS decomposes into two special TRS problems: one without a linear term, i.e., $g = 0$, making it a pure minimum eigenvalue problem; and the other one with a convex quadratic objective function. This once again highlights the connection between the TRS and eigenvalue problems. Furthermore, our analysis demonstrates that the notorious 'hard case' reported in much of the existing TRS literature, i.e., when g is orthogonal to the eigenspace of λ_Q, is no longer an issue for our exact reformulation. $\diamondsuit$

2.2.1 Working with Approximate Eigenvalues

In practice, we will actually form the objective $y^\top (Q - \gamma I_n) y + 2g^\top y + \gamma$ where $\gamma \approx \lambda_Q$ is an approximation. Due to this imprecision, we must ensure that the objective remains convex. To do this, suppose that we solve the minimum eigenvalue problem of Q to within δ-accuracy, and obtain an approximate solution $\lambda_Q - \delta < \lambda < \lambda_Q + \delta$. Subtracting δ from the inequality, we obtain $\lambda_Q - 2\delta < \lambda - \delta < \lambda_Q$. To ensure the convexity of the objective, we set $\gamma := \lambda - \delta < \lambda_Q$ which is an underestimate of λ_Q, ensuring that $Q - \gamma I_n \succ 0$. Let $\eta := \lambda_Q - \gamma$ which satisfies $0 < \eta < 2\delta$, and

$$f_\eta(y) := y^\top (Q - \gamma I_n) y + 2g^\top y = y^\top (Q - (\lambda_Q - \eta) I_n) y + 2g^\top y = f(y) + \eta \|y\|^2.$$

Based on this scheme, we next explore the effects of solving

$$\min_{y} \{f_\eta(y) : \ \|y\| \leq 1\} \tag{7}$$

instead of (6). Let y^* be an optimal solution to the true convex reformulation (6). Let y^η be an optimal solution to (7) and $\bar{y}^\eta$ be an approximate optimal solution. Then we can bound the objective value $f(\bar{y}^\eta)$ as

$$f(\bar{y}^\eta) - f(y^*) = f_\eta(\bar{y}^\eta) - f_\eta(y^*) + \eta(\|y^*\|^2 - \|\bar{y}^\eta\|^2) \leq f_\eta(\bar{y}^\eta) - f_\eta(y^\eta) + \eta,$$

where the last inequality follows from $\|y^*\| \leq 1$ and $\|\bar{y}^\eta\| \leq 1$. Thus, the convergence rate of $\bar{y}^\eta$ to the optimum of (6) is controlled by the size of η and the convergence rate for solving (7).

We can also control the distance between y^η and y^*. Because $f_\eta(y)$ is a strongly convex function with parameter 2η, we have

$$\begin{aligned}
\eta \|y^* - y^\eta\|^2 &\leq f_\eta(y^*) - f_\eta(y^\eta) + \nabla f_\eta(y^\eta)^\top (y^\eta - y^*) \\
&= f(y^*) - f(y^\eta) + \nabla f_\eta(y^\eta)^\top (y^\eta - y^*) + \eta(\|y^*\|^2 - \|y^\eta\|^2) \\
&\leq \eta(\|y^*\|^2 - \|y^\eta\|^2),
\end{aligned}$$

where the last inequality follows from the optimality of y^η for the problem (7), i.e., $\nabla f_\eta(y^\eta)^\top (y^\eta - y^*) \leq 0$, and the optimality of y^* for the problem (6). Then $\|y^\eta\| \leq \|y^*\|$. Also, from $\|y^*\| \leq 1$, we deduce that if $\|y^\eta\| = 1$, then $y^* = y^\eta$. When $\|y^\eta\| < 1$, the only constraint in our domain is inactive; and thus we conclude that y^η is also optimum for the unconstrained minimization problem. Then the optimality conditions leads to $\nabla f_\eta(y^\eta) = 0$. This implies that $y^\eta = -(Q + (\eta - \lambda_Q)I_n)^{-1} g$. Moreover, y^* satisfies the optimality condition $\nabla f(y^*)^\top (y^* - y) \leq 0$ for all y such that $\|y\| \leq 1$. Since our domain is the unit ball, this is true if and only if $\nabla f(y^*) = -\alpha y^*$, for some $\alpha \geq 0$. Therefore, $y^* = -(Q + (\alpha - \lambda_Q)I_n)^\dagger g$, where $A^\dagger$ denotes the pseudo-inverse of a matrix A. If we denote the ordered eigenvalues of Q by q_i and their corresponding orthonormal eigenvectors by u_i, we obtain

$$\|y^\eta\|^2 = \sum_{i=1}^{n} \frac{(u_i^\top g)^2}{(q_i - q_n + \eta)^2} \quad \text{and} \quad \|y^*\|^2 = \sum_{i=1}^{n} \frac{(u_i^\top g)^2}{(q_i - q_n + \alpha)^2}.$$

Note that it is possible to have $\alpha = 0$ and $q_i - q_n = 0$. However, this happens only when $u_i^\top g = 0$, so we follow the convention $\frac{0}{0} = 0$. After some simple algebra, we have the equality

$$\begin{aligned}
\|y^*\|^2 - \|y^\eta\|^2 &= \sum_{i=1}^{n} \frac{(u_i^\top g)^2}{(q_i - q_n + \alpha)^2} - \sum_{i=1}^{n} \frac{(u_i^\top g)^2}{(q_i - q_n + \eta)^2} \\
&= (\eta - \alpha) \sum_{i=1}^{n} (u_i^\top g)^2 \frac{2q_i - 2q_n + \eta + \alpha}{(q_i - q_n + \alpha)^2 (q_i - q_n + \eta)^2}.
\end{aligned}$$

Since $\|y^*\| \geq \|y^\eta\|$ and $\eta > 0$, we must have $\eta \geq \alpha$. Also, $\eta \leq \alpha$ is possible only if $y^\eta = y^*$. Hence, we have

$$
\begin{aligned}
\|y^*\|^2 - \|y^\eta\|^2 &= (\eta-\alpha)_+ \sum_{i=1}^n (u_i^\top g)^2 \frac{2q_i - 2q_n + (\eta-\alpha)_+ + 2\alpha}{(q_i - q_n + \alpha)^2 (q_i - q_n + (\eta-\alpha)_+ + \alpha)^2} \\
&\le (\eta-\alpha)_+ \sum_{i=1}^n (u_i^\top g)^2 \frac{2q_i - 2q_n + (\eta-\alpha)_+ + 2\alpha}{(q_i - q_n + \alpha)^4} \\
&= 2(\eta-\alpha)_+ \sum_{i=1}^n \frac{(u_i^\top g)^2}{(q_i - q_n + \alpha)^3} + (\eta-\alpha)_+^2 \sum_{i=1}^n \frac{(u_i^\top g)^2}{(q_i - q_n + \alpha)^4}.
\end{aligned}
$$

This shows that $\|y^*\|^2 - \|y^\eta\|^2 \le C\eta + o(\eta)$, where $C = 2(y^*)^\top (Q + (\alpha - \lambda_Q) I_n)^\dagger y^*$. Therefore,

$$
\|y^\eta - y^*\|^2 \le \|y^*\|^2 - \|y^\eta\|^2 \le C\eta + o(\eta).
$$

3 Convexification of the Epigraph of the TRS

As opposed to the convex reformulation of the TRS and its variants discussed in Section 2.1, in this section, we employ an SOC based approach to explore stronger results on exact convex hull representation results for the epigraph of the TRS and its variants.

By defining a new variable x_{n+2}, and moving the nonconvex function from the objective to the constraints, we can equivalently write (2) as minimizing x_{n+2} over its epigraph

$$
\mathrm{Opt}_h = \min_{y, x_{n+1}, x_{n+2}} \left\{ x_{n+2} : \begin{array}{rcl} \|y\| & \le & x_{n+1} \\ x_{n+1} & = & 1 \\ Ay - b & \in & \mathcal{K} \\ h(y) = y^\top Q y + 2g^\top y & \le & x_{n+2} \end{array} \right\}. \tag{8}
$$

Since the objective x_{n+2} is linear, optimizing over the epigraph is equivalent to optimizing over its convex hull. In this section, under slightly stronger assumptions than in Section 2.1, we give an explicit characterization of the convex hull of the following epigraphical set:

$$
X := \left\{ x = [y; x_{n+1}; x_{n+2}] \in \mathbb{R}^{n+2} : \begin{array}{rcl} \|y\| & \le & x_{n+1} \\ x_{n+1} & = & 1 \\ Ay - b & \in & \mathcal{K} \\ y^\top Q y + 2g^\top y & \le & x_{n+2} \end{array} \right\}. \tag{9}
$$

We follow a second-order cone based approach. That is, we focus mainly on the quadratic parts of the TRS, namely the nonconvex quadratic $y^\top Q y + 2g^\top y$ and the SOC unit ball constraint $\|y\| \le 1$ and provide the convexification of this set via a single new SOC constraint. Our approach is an adaptation of the one from Burer and Kılınç-Karzan [13]. We start with a number of relevant definitions and conditions and then present their main result.

A cone $\mathcal{F}^+ \subseteq \mathbb{R}^n$ is said to be *second-order-cone representable* (or *SOCr*) if there exists a matrix $0 \ne R \in \mathbb{R}^{n \times (n-1)}$ and a vector $r \in \mathbb{R}^n$ such that the nonzero columns of R are linearly independent, $r \notin \mathrm{Range}(R)$, and

$$
\mathcal{F}^+ = \left\{ x : \|R^\top x\| \le r^\top x \right\}, \tag{10}
$$

where $\|\cdot\|$ denotes the usual Euclidean norm.

Given an SOCr cone $\mathcal{F}^+$, the cone $\mathcal{F}^- := -\mathcal{F}^+$ is also SOCr. Based on $\mathcal{F}^+$ from (10), we define $A := RR^\top - rr^\top$ and consider the union $\mathcal{F}^+ \cup (\mathcal{F}^-) = \mathcal{F}^+ \cup (-\mathcal{F}^+) =: \mathcal{F}$. Note that $\mathcal{F}$ corresponds to a nonconvex cone defined by the homogeneous quadratic inequality $x^\top A x \le 0$:

$$\mathcal{F} := \mathcal{F}^+ \cup (\mathcal{F}^-) = \left\{x : \|R^\top x\|^2 \le (r^\top x)^2\right\} = \left\{x : x^\top A x \le 0\right\},$$

and $\operatorname{apex}(\mathcal{F}^+) = \operatorname{apex}(\mathcal{F}^-) = \operatorname{apex}(\mathcal{F}) = \operatorname{Null}(A)$ where $\operatorname{Null}(A)$ denotes the nullspace of the matrix A.

The analysis of [13] is mainly based on two cones (not necessarily convex) $\mathcal{F}_0, \mathcal{F}_1$ defined by homogeneous quadratic inequalities given by the symmetric matrices A_0, A_1 and relies on the following conditions:

Condition 3.1. A_0 has at least one positive eigenvalue and exactly one negative eigenvalue.

Condition 3.2. There exists $\bar{x}$ such that $\bar{x}^\top A_0 \bar{x} < 0$ and $\bar{x}^\top A_1 \bar{x} < 0$.

Condition 3.3. Either (i) A_0 is nonsingular, (ii) A_0 is singular and A_1 is positive definite on $\operatorname{Null}(A_0)$, or (iii) A_0 is singular and A_1 is negative definite on $\operatorname{Null}(A_0)$.

The method of [13] can be roughly described as follows. For any $t \in [0,1]$, define $A_t := (1-t)A_0 + tA_1$. Then the set $\mathcal{F}_t = \left\{x : x^\top A_t x \le 0\right\}$ is obtained by linearly aggregating the quadratic constraints from the matrices A_0 and A_1, and immediately gives us a relaxation

$$\mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_t.$$

Conditions 3.1–3.3 ensure the existence of a maximal $s \in [0,1]$ such that A_t has a single negative eigenvalue for all $t \in [0,s]$, A_t is invertible for all $t \in (0,s)$, and A_s is singular—that is, $\operatorname{Null}(A_s)$ is nontrivial whenever $s < 1$. Then, for all A_t with $t \in [0,s]$, SOCr sets $\mathcal{F}_t = \mathcal{F}_t^+ \cup \mathcal{F}_t^-$ can be explicitly characterized (see [13, Section 5.1]). Furthermore, for $\bar{x}$ of Condition 3.2, noting that $\bar{x}^\top A_t \bar{x} = (1-t)\,\bar{x} A_0 \bar{x} + t\,\bar{x}^\top A_1 \bar{x} < 0$, we can choose $\bar{x} \in \mathcal{F}_t^+$ for all such t without loss of generality. Then [13, Theorem 1] asserts that the closed conic hull of $\mathcal{F}_0^+ \cap \mathcal{F}_1$, that is, $\overline{\operatorname{cone}}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$, is contained in the convex cone $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ under Conditions 3.1–3.3. Let $\operatorname{int}(\cdot)$ stand for the interior of a given set. Furthermore, [13, Theorem 1] establishes $\overline{\operatorname{cone}}(\mathcal{F}_0^+ \cap \mathcal{F}_1) = \mathcal{F}_0^+ \cap \mathcal{F}_s^+$ under the following condition:

Condition 3.4. When $s < 1$, $\operatorname{apex}(\mathcal{F}_s^+) \cap \operatorname{int}(\mathcal{F}_1) \neq \emptyset$.

Because the TRS deals with the cross-section of an SOC, the reformulation of the TRS requires a specialization of [13, Theorem 1] for the case when $\mathcal{F}_0^+ \cap \mathcal{F}_1$ is intersected with an affine hyperplane. For this, let $h \in \mathbb{R}^n$ be given; and define the hyperplanes

$$H^0 := \{x : h^\top x = 0\}, \tag{11}$$

$$H^1 := \{x : h^\top x = 1\}. \tag{12}$$

In order to study the closed convex hull of $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$, that is, $\overline{\operatorname{conv}}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$, the following additional condition related to H_0 is introduced in [13]:

Condition 3.5. When $s < 1$, $\operatorname{apex}(\mathcal{F}_s^+) \cap \operatorname{int}(\mathcal{F}_1) \cap H^0 \neq \emptyset$ or $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$.

Conditions 3.1–3.5 are all that is needed to state the main result of [13]. Here, we state [13, Theorem 1] for completeness.

Theorem 3.1 ([13, Theorem 1]). *Suppose Conditions 3.1–3.3 are satisfied; and let s be the maximal $s \in [0,1]$ such that $A_t := (1-t)A_0 + tA_1$ has a single negative eigenvalue for all $t \in [0,s]$. Then $\overline{\text{cone}}(\mathcal{F}_0^+ \cap \mathcal{F}_1) \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$, and equality holds under Condition 3.4. Moreover, Conditions 3.1–3.5 imply $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1 = \overline{\text{conv}}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$.*

These convexification results were also applied to the classical TRS (5) in [13]. In particular, it is shown in [13, Section 7.2] that the classical TRS (5) can be reformulated in the form of

$$\text{Opt}_h = \min_{y, x_{n+2}} \left\{ -x_{n+2}^2 : \begin{array}{rcl} \|y\| & \leq & 1 \\ y^\top Q y + 2g^\top y & \leq & -x_{n+2}^2 \end{array} \right\};$$

the nonconvex feasible set can be convexified; and then this problem can be solved efficiently in two stages. Specifically, [13] define a new variable $x = [y; x_{n+1}; x_{n+2}]$ and the matrices

$$A_0 = \begin{bmatrix} I_n & 0 & 0 \\ 0^\top & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} Q & g & 0 \\ g^\top & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

which then gives

$$\left\{ [y; 1; x_{n+2}] : \begin{array}{rcl} \|y\| & \leq & 1 \\ y^\top Q y + 2g^\top y & \leq & -x_{n+1}^2 \end{array} \right\} = \left\{ x = [y; x_{n+1}; x_{n+2}] : \begin{array}{rcl} x^\top A_0 x & \leq & 0 \\ x^\top A_1 x & \leq & 0 \\ x_{n+1} & = & 1 \end{array} \right\}$$
$$= \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \{x : x_{n+1} = 1\}.$$

It is then proved in [13] that there exists some $s \in (0,1)$ such that

$$\overline{\text{conv}}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \{x : x_{n+1} = 1\}) = \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap \{x : x_{n+1} = 1\}. \tag{14}$$

While the precise value of s is not given in [13], one can show that it is in fact $s = \frac{1}{1-\lambda_Q}$.

Remark 3.1. The reformulation (14) implicitly requires that $\text{Opt}_h \leq 0$ because of the constraint $h(y) = y^\top Q y + 2g^\top y \leq -x_{n+2}^2 \leq 0$. For the classical TRS without additional constraints, this is not an additional limitation because $y = 0$ will always be a feasible solution with objective value 0 and thus the optimum solution will have a nonpositive objective value. However, this becomes a limitation when we want to extend such arguments for the TRS with additional conic constraints $Ay - b \in \mathcal{K}$ because Opt_h may no longer be nonpositive. ♢

Due to Remark 3.1, we instead choose to model the TRS as in (8), which allows for positive objective values. To this end, we define the matrices

$$A_0 = \begin{bmatrix} I_n & 0 & 0 \\ 0^\top & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} Q & g & 0 \\ g^\top & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}, \tag{15}$$

and the corresponding sets

$$\begin{aligned}
\mathcal{F}_0^+ &= \{x : \|y\|^2 \le x_{n+1}^2,\ x_{n+1} \ge 0\} = \left\{x : x^\top A_0 x \le 0,\ x_{n+1} \ge 0\right\}, \\
\mathcal{F}_1 &= \left\{x : y^\top Q y + 2g^\top y x_{n+1} \le x_{n+1}x_{n+2}\right\} = \left\{x : x^\top A_1 x \le 0\right\}, \\
\mathcal{K}^+ &= \{x : Ay - bx_{n+1} \in \mathcal{K}\}, \\
H^1 &= \{x : x_{n+1} = 1\}.
\end{aligned} \tag{16}$$

With these definitions, the epigraph X from (9) can be written as

$$X = \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1.$$

It is mentioned in [13] that the matrices (15) do not satisfy the necessary conditions to apply Theorem 3.1 directly. In particular, Condition 3.3 is violated for the choice of matrices (15); and as a result, the reformulation of the TRS with matrices (13) were opted in [13, Section 7.2] instead. In contrast, we next show that for the special case of the TRS, via a direct analysis, finding the convex hull through linear aggregation of constraints will still carry through for the matrices in (15). This indicates that while Condition 3.3 is sufficient, it is not necessary to obtain the convex hull result. In fact, we show that the value of $s = \frac{1}{1-\lambda_Q}$ that works for the matrices (13) will also work for our matrices (15). More precisely, for $s = \frac{1}{1-\lambda_Q}$, we define

$$\mathcal{F}_s = \left\{x : x^\top A_s x \le 0\right\} = \left\{x : y^\top (Q - \lambda_Q I_n) y + 2g^\top y x_{n+1} + \lambda_Q x_{n+1}^2 \le x_{n+1}x_{n+2}\right\}, \tag{17}$$

and prove that $\operatorname{conv}(X) = \operatorname{conv}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1) = \mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$ directly under the following condition.

Condition 3.6. There exists a vector $d \ne 0$ such that $Qd = \lambda_Q d$ and $\pm Ad \in \mathcal{K}$.

Remark 3.2. Condition 3.6 implies Condition 2.1. To see this, suppose $d \ne 0$ satisfies Condition 3.6. Then if $g^\top d \le 0$, d satisfies Condition 2.1 also. Otherwise, $-d$ will satisfy Condition 2.1. We demonstrate that Condition 2.1 does not imply Condition 3.6 in Example 3.5.

Furthermore, Condition 3.6 holds whenever Condition 2.2 of [25] is satisfied because Condition 2.2 implies that there exists d such that $Qd = \lambda_Q d$ and $Ad = 0$ and since $\mathcal{K}$ is a closed convex cone, $\pm Ad = 0 \in \mathcal{K}$ as well. $\diamondsuit$

One of the ingredients of our convex hull result is given in the next lemma.

Lemma 3.2. *Let $\mathcal{F}_s$ be defined as in (17). Then the cone $\mathcal{F}_s \cap \{x : x_{n+1} > 0\}$ is convex; and the set $\mathcal{F}_s \cap H^1$ where H^1 is as defined in (16) is SOC representable.*

Proof. Let $x = [y; x_{n+1}; x_{n+2}] \in \mathbb{R}^{n+2}$. Note that by definition, we have

$$\begin{aligned}
&\mathcal{F}_s \cap \{x : x_{n+1} > 0\} \\
&= \left\{x :\ y^\top (Q - \lambda_Q I_n) y + 2g^\top y x_{n+1} + \lambda_Q x_{n+1}^2 \le x_{n+1}x_{n+2},\ x_{n+1} > 0\right\} \\
&= \left\{x :\ y^\top (Q - \lambda_Q I_n) y \le x_{n+1}(x_{n+2} - 2g^\top y - \lambda_Q x_{n+1}),\ x_{n+1} > 0\right\} \\
&= \left\{x :\ \begin{array}{l} y^\top (Q - \lambda_Q I_n) y \le x_{n+1}(x_{n+2} - 2g^\top y - \lambda_Q x_{n+1}), \\ x_{n+1} > 0,\quad x_{n+2} - 2g^\top y - \lambda_Q x_{n+1} \ge 0 \end{array}\right\},
\end{aligned}$$

where the last equation follows because $Q - \lambda_Q I_n \succeq 0$, we have $y^\top (Q - \lambda_Q I_n) y \geq 0$ for all y and then $x_{n+1} > 0$ implies $x_{n+2} - 2g^\top y - \lambda_Q x_{n+1} \geq 0$. As a result, $x_{n+1} + x_{n+2} - 2g^\top y - \lambda_Q x_{n+1} \geq 0$ holds for all $x \in \mathcal{F}_s \cap \{x : x_{n+1} > 0\}$. In addition, from these derivations, we immediately conclude that the set $\mathcal{F}_s \cap \{x : x_{n+1} = 1\}$ is an SOC representable set. □

Theorem 3.3. *Let $\mathcal{F}_0^+, \mathcal{F}_1, H^1, \mathcal{K}^+, \mathcal{F}_s$ be defined as in (16) and (17). Assume that Condition 3.6 holds. Then*

$$\operatorname{conv}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1) = \mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1.$$

Proof. Let $x = [y; x_{n+1}; x_{n+2}]$ be a vector in $\mathcal{F}_0^+ \cap \mathcal{K}^+ \cap H^1 \cap \mathcal{F}_s$. Then x satisfies

$$\begin{aligned} x^\top A_0 x &\leq 0, \\ Ay - b x_{n+1} &\in \mathcal{K}, \\ x_{n+1} &= 1, \\ x^\top A_s x &\leq 0. \end{aligned}$$

We will show that $x \in \operatorname{conv}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1)$. If $x \in \mathcal{F}_1$, then we are done. Suppose $x \notin \mathcal{F}_1$, that is, $x^\top A_1 x > 0$. Then from the definition of s, $0 < x^\top A_1 x$, and $x^\top A_s x \leq 0$, we have

$$0 < s(x^\top A_1 x) - x^\top A_s x = -(1-s) x^\top A_0 x = \frac{\lambda_Q}{1 - \lambda_Q}(x_{n+1}^2 - \|y\|^2).$$

Because $\lambda_Q < 0$, this implies $\|y\|^2 < x_{n+1}^2$. Let d be the vector given by Condition 3.6 such that $Qd = \lambda_Q d$, $\pm Ad \in \mathcal{K}$ and $\|d\|^2 = 1$. We now consider the points $x^\eta := [y + \eta d;\ x_{n+1};\ x_{n+2} + 2g^\top d\eta]$ for $\eta \in \mathbb{R}$. We first argue that $x^\eta \in \mathcal{F}_s$ holds for all $\eta \in \mathbb{R}$. To see this, note that

$$\begin{aligned} &(y+\eta d)^\top (Q - \lambda_Q I_n)(y + \eta d) + 2g^\top (y + \eta d) x_{n+1} + \lambda_Q x_{n+1}^2 \\ &= (y+\eta d)^\top Q (y+\eta d) + 2g^\top (y+\eta d) x_{n+1} + \lambda_Q (x_{n+1}^2 - \|y + \eta d\|^2) \\ &= y^\top Q y + 2\underbrace{y^\top Q d}_{=\lambda_Q y^\top d}\eta + \underbrace{d^\top Q d}_{=\lambda_Q}\eta^2 + 2g^\top y + 2g^\top d x_{n+1}\eta + \lambda_Q(x_{n+1}^2 - \|y\|^2 - 2y^\top d\eta - \eta^2) \\ &= y^\top Q y + 2g^\top y x_{n+1} + \lambda_Q (x_{n+1}^2 - \|y\|^2) + 2g^\top d x_{n+1}\eta \\ &= y^\top (Q - \lambda_Q I_n) y + 2g^\top y x_{n+1} + \lambda_Q x_{n+1}^2 + 2g^\top d x_{n+1}\eta \\ &= x_{n+1}x_{n+2} + (1 - \lambda_Q)(x^\top A_s x) + 2g^\top d x_{n+1}\eta \\ &\leq x_{n+1}x_{n+2} + 2g^\top d x_{n+1}\eta \\ &= x_{n+1}(x_{n+2} + 2g^\top d\eta), \end{aligned} \tag{18}$$

where the third equation follows from $Qd = \lambda_Q d$ and $\|d\|^2 = 1$, and the inequality holds because $x^\top A_s x \leq 0$ and $\lambda_Q < 0$. Then from the inequality (18) and the definition of $\mathcal{F}_s$ in (17), we conclude $x^\eta \in \mathcal{F}_s$ for all $\eta \in \mathbb{R}$. Moreover, because $\|y\|^2 < x_{n+1}^2$ and $d \neq 0$, there must exist $\delta, \epsilon > 0$ such that $\|y - \delta d\|^2 = \|y + \epsilon d\|^2 = x_{n+1}^2$. We define

$$\begin{aligned} x^\delta &:= [y - \delta d;\ x_{n+1};\ x_{n+2} - 2g^\top d\delta] \\ x^\epsilon &:= [y + \epsilon d;\ x_{n+1};\ x_{n+2} + 2g^\top d\epsilon]. \end{aligned}$$

Then by our choice of δ, ϵ, we have $x^\delta, x^\epsilon \in \mathrm{bd}(\mathcal{F}_0^+)$. From $s \in (0,1)$, $x^\eta \in \mathcal{F}_s$ for all $\eta \in \mathbb{R}$, and the relation

$$(x^\eta)^\top A_s x^\eta = (1-s)[(x^\eta)^\top A_1 x^\eta] + s[(x^\eta)^\top A_0 x^\eta],$$

we conclude that $x^\eta \in \mathcal{F}_1$ for all η such that $x^\eta \in \mathrm{bd}(\mathcal{F}_0^+)$. In particular, $x^\delta, x^\epsilon \in \mathcal{F}_1$. Furthermore, by Condition 3.6, $\pm Ad \in \mathcal{K}$, and since $\mathcal{K}$ is a cone, $-Ad\delta, Ad\epsilon \in \mathcal{K}$; thus $x^\delta, x^\epsilon \in \mathcal{K}^+$. Finally, $x_{n+1} = 1$ in both x^δ, x^ϵ; so we have $x^\delta, x^\epsilon \in \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1$. Now it is easy to see that

$$x = \frac{\epsilon}{\delta+\epsilon} x^\delta + \frac{\delta}{\delta+\epsilon} x^\epsilon \in \mathrm{conv}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1).$$

As a consequence, we have the relation

$$\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1 \subseteq \mathrm{conv}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1).$$

By Lemma 3.2, the set $\mathcal{F}_s \cap H^1$ is SOC representable and hence convex; this implies that $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$ is convex also. Then taking the convex hull of all terms in the above inequality gives us the result. □

Note that the set $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$ is closed and our explicit convex hull result becomes a simple corollary of Theorem 3.3.

Corollary 3.4. *Let X be the set defined in (9). Then under Condition 3.6*

$$\mathrm{conv}(X) = \left\{ x = [y; 1; x_{n+2}] : \begin{array}{rcl} \|y\| & \leq & 1 \\ y^\top (Q - \lambda_Q I_n) y + 2g^\top y + \lambda_Q & \leq & x_{n+2} \\ Ay - b & \in & \mathcal{K} \end{array} \right\}.$$

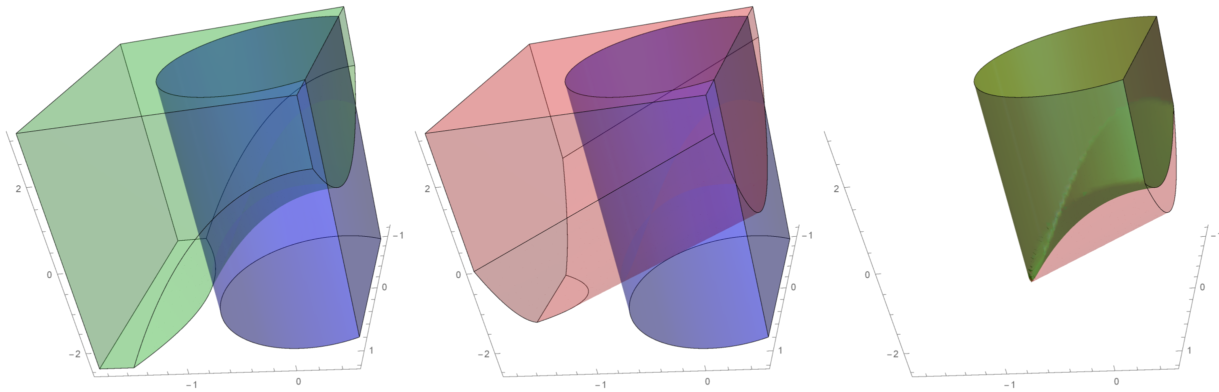
As a result,

$$\begin{aligned} \mathrm{Opt}_h &= \min_y \left\{ h(y) = y^\top Q y + 2g^\top y : \begin{array}{rcl} \|y\| & \leq & 1 \\ Ay-b & \in & \mathcal{K} \end{array} \right\} \\ &= \min_y \left\{ f(y) = y^\top (Q - \lambda_Q I_n) y + 2g^\top y + \lambda_Q : \begin{array}{rcl} \|y\| & \leq & 1 \\ Ay-b & \in & \mathcal{K} \end{array} \right\}. \end{aligned}$$

Remark 3.3. In general, a tight convex relaxation for a nonconvex optimization problem does not necessarily imply that the epigraph of the convex relaxation is giving the exact convex hull of the epigraph of the nonconvex optimization problem. However, in the particular case of the TRS with additional conic constraints, i.e., problem (2), under Condition 3.6, Corollary 3.4 shows that not only our convex relaxation given by (4) is tight but also we can characterize the convex hull of its epigraph exactly. ♢

As a consequence of Corollary 3.4 and Remark 3.2, we note that in all of the cases where Jeyakumar and Li [25] show the tightness of their convex reformulation, i.e. robust least squares and robust SOC programming, we can further give the exact convex hull characterizations.

We next discuss the necessity of Condition 3.6 for giving $\mathrm{conv}(X)$ via the following example.



(a) $X = \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1$ (b) $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$ (c) X vs $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$

Figure 2: Plots of the epigraph of Example 3.5.

Example 3.5. *Consider the following problem with the data given by*

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad b = \frac{1}{2}, \quad \mathcal{K} = \mathbb{R}_+.$$

Note that Condition 3.6 is violated. To see this, any vector d such that $Qd = \lambda_Q d$ is of the form $d = [0; d_2]$. But then $Ad = -d_2$. Hence, if $d_2 > 0$ then $Ad \notin \mathcal{K}$; and similarly, if $d_2 < 0$ then $-Ad \notin \mathcal{K}$.

Figure 2(c) shows that the convex relaxation for the epigraph $X = \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1$ given by $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$ does not give the convex hull of X. Note also that Condition 2.1 is satisfied for this example by taking $d = [0; 1]$; so by Theorem 2.4, the SOC optimization problem (4) is a tight relaxation for (2). Despite this, we cannot give the exact convex hull characterization because Condition 3.6 is violated.

Note that in this example, our convex relaxation simply gives the convex hull of the set obtained by eliminating the additional linear inequality constraint because Condition 3.6 is satisfied for that set. This is illustrated in Figure 3 below. ◇

3.1 Additional Convex and Nonconvex Constraints

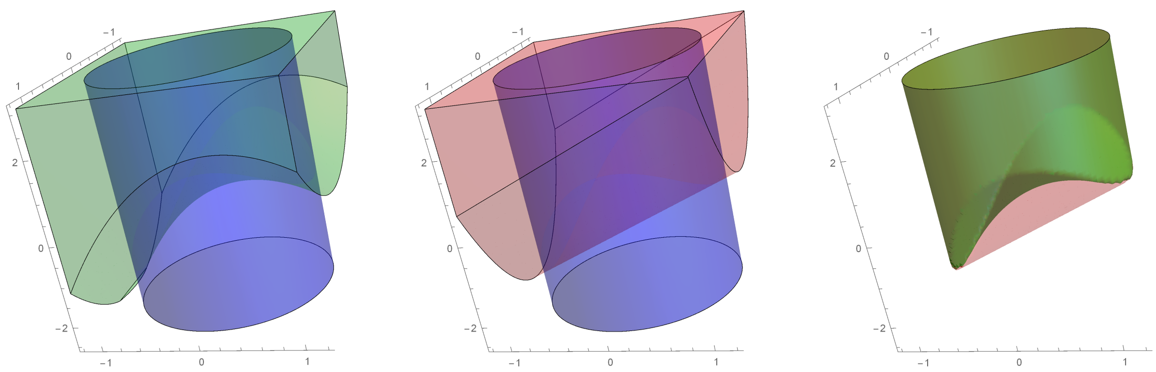
In this section we explore additional constraints $y \in \mathcal{H}$ included in the domain of the TRS, where $\mathcal{H}$ is a given, convex or nonconvex set. More precisely, we derive the convex hull of the set $X \cap \mathcal{H}^+ = \mathcal{F}_0 \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1 \cap \mathcal{H}^+$ where $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{K}^+$, and H^1 are as defined in (16), and $\mathcal{H}^+ := \{[y; x_{n+1}; x_{n+2}] : y \in \mathcal{H}\}$.

Our analysis relies on the following technical lemma which is a generalization of [13, Lemma 5].

Lemma 3.6. *Let F and $\mathcal{H}^+$ be sets such that* $\mathrm{conv}(F) \cap \mathrm{bd}(\mathcal{H}^+) \subseteq F \cap \mathcal{H}^+$. *Then*

$$\mathrm{conv}(F) \cap \mathcal{H}^+ \subseteq \mathrm{conv}(F \cap \mathcal{H}^+).$$

Proof. Consider a point x from $\mathrm{conv}(F) \cap \mathcal{H}^+$. If $x \in \mathrm{bd}(\mathcal{H}^+)$, then by the hypothesis of the lemma, $x \in F \cap \mathcal{H}^+ \subseteq \mathrm{conv}(F \cap \mathcal{H}^+)$.



(a) $X = \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1$ (b) $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$ (c) X vs $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$

Figure 3: Plots of the epigraph of Example 3.5 without the linear inequality.

Now suppose $x \in \operatorname{relint}(\mathcal{H}^+)$. Since $x \in \operatorname{conv}(F)$, we can write x as a finite convex combination $x = \sum_k \lambda_k x^k$ where $\lambda_k > 0$, $\sum_k \lambda_k = 1$ and $x^k \in F$. Let $I = \{k : x^k \in \mathcal{H}^+\}$ and $J = \{k : x^k \notin \mathcal{H}^+\}$. If $J = \emptyset$, then we are done since $x \in \operatorname{conv}(F \cap \mathcal{H}^+)$. If not, for each $j \in J$, the line segment between x and x^j must intersect $\operatorname{bd}(\mathcal{H}^+)$ because $x \in \operatorname{relint}(\mathcal{H}^+)$ and $x^j \notin \mathcal{H}^+$. Thus, there exists a point $z^j = \alpha_j x + (1-\alpha_j)x^j \in \operatorname{bd}(\mathcal{H}^+)$ for some $\alpha_j \in (0,1)$. Note that $z^j \in \operatorname{conv}(F)$ because it is a convex combination of $x, x^j \in \operatorname{conv}(F)$. By a reasoning similar to the above, we deduce that $z^j \in F \cap \operatorname{bd}(\mathcal{H}^+)$. We now rewrite

$$x = \sum_{i \in I} \lambda_i x^i + \sum_{j \in J} \frac{\lambda_j}{1-\alpha_j}(z^j - \alpha_j x) \iff \left(1 + \sum_{j \in J} \frac{\alpha_j \lambda_j}{1-\alpha_j}\right) x = \sum_{i \in I} \lambda_i x^i + \sum_{j \in J} \frac{\lambda_j}{1-\alpha_j} z^j.$$

Notice also that

$$\begin{aligned} 1 + \sum_{j \in J} \frac{\alpha_j \lambda_j}{1-\alpha_j} &= \sum_{i \in I} \lambda_i + \sum_{j \in J} \lambda_j + \sum_{j \in J} \frac{\alpha_j \lambda_j}{1-\alpha_j} \\ &= \sum_{i \in I} \lambda_i + \sum_{j \in J} \frac{(1-\alpha_j)\lambda_j}{1-\alpha_j} + \sum_{j \in J} \frac{\alpha_j \lambda_j}{1-\alpha_j} \\ &= \sum_{i \in I} \lambda_i + \sum_{j \in J} \frac{\lambda_j}{1-\alpha_j}. \end{aligned}$$

Then we conclude that x is a convex combination of points $x^i, z^j \in F \cap \mathcal{H}^+$ as desired. □

3.1.1 Additional Hollow Constraints

Here we consider a hollow constraint $y \in \mathcal{H}$ given by $\mathcal{H} = \mathbb{R}^n \setminus \mathcal{P}$ where $\mathcal{P}$ is a given possibly nonconvex set. We impose the following condition on $\mathcal{H} = \mathbb{R}^n \setminus \mathcal{P}$.

Condition 3.7. The set $\mathcal{P} \subseteq \mathbb{R}^n$ satisfies $\mathcal{P} \subseteq \left\{y : \|y\|^2 < 1,\ Ay - b \in \mathcal{K}\right\}$.

If $\mathcal{P} = \bigcup_{i=1}^m E_i$ is a union of open ellipsoids $E_i = \left\{y : y^\top A_i y + 2b_i^\top y + c_i \le 0\right\}$, then Condition 3.7 can be checked by solving

$$v_i = \min_y \left\{1 - \|y\|^2 :\ y^\top A_i y + 2b_i^\top y + c_i \le 0\right\}.$$

In particular, E_i satisfies Condition 3.7 if and only if $v_i > 0$. While this is a nonconvex quadratic program, our developments from the previous section give a tight SOC reformulation for it. In addition, the S-lemma ensures that the associated semidefinite relaxation is tight. Thus, Condition 3.7 can be verified efficiently when $\mathcal{P}$ is a union of ellipsoids.

A number of papers have studied conditions similar to Condition 3.7. Most notably, the *generalized trust region subproblem* corresponds to the case when $\mathcal{H}$ is a single lower bounded, nonconvex quadratic constraint $y^\top Dy \geq l$; this is studied in [5, 8, 32, 36, 39] for example. More recently, Yang et al. [38] study the case when the hollow set $\mathcal{P}$ is the disjoint union of ellipsoids which do not intersect the boundary of the unit ball $\{y : \|y\| \leq 1\}$. A number of these papers [32, 38, 39] show that the natural SDP relaxation is tight. As opposed to these results on tight SDP relaxations, it is shown in [7] that the general quadratic programming problem

$$\min_y \left\{ y^\top Q_0 y + 2g_0^\top y : y^\top Q_i y + 2g_i^\top y + c_i \leq 0,\ i = 1, \ldots, m \right\}$$

is polynomially solvable using a weak feasibility oracle, under the assumption that at least one quadratic constraint $y^\top Q_i y + 2g_i^\top y + c_i \leq 0$ is strictly convex. In a similar vein, [9] also study the TRS with additional ellipsoidal hollow constraints. In [9], instead of giving the convex hull, they explore conditions that allow for polynomial solvability using a combinatorial enumeration technique and thus are able to cover cases where the hollow set $\mathcal{P}$ may not be contained in the unit ball. On a related domain, [8] studies the characterization and separation of valid linear inequalities that convexify the epigraph of a convex, differentiable function whose domain is restricted to the complement of a convex set defined by linear or convex quadratic inequalities.

We note that these papers [5, 7, 8, 9, 32, 36, 38, 39] consider the more general case of minimizing an arbitrary quadratic objective, which can be convex, over a domain given by possibly nonconvex quadratic constraints. On the other hand, our result applies to the special case of minimizing a nonconvex quadratic over the unit ball, a convex quadratic constraint. As a result, we are able to relax the assumptions that the hollow set $\mathcal{P}$ is generated by quadratics and the ellipsoidal hollows are disjoint. Specifically, we show that under Condition 3.7, the original SOC based reformulation of the TRS with additional conic constraints obtained by ignoring $\mathcal{H}$ is tight.

We start with the following lemma on the structure of extreme points of $\operatorname{conv}(X)$. Given a set S, we let $\operatorname{Ext}(S)$ denote the set of extreme points of S and $\operatorname{Rec}(S)$ denote its recession cone.

Lemma 3.7. *Let X be defined as in (9); and suppose that Condition 3.6 holds. Let $x = [y; 1; x_{n+2}]$ be an extreme point of $\operatorname{conv}(X)$. Then x must satisfy $\|y\| = 1$.*

Proof. Suppose for contradiction that $\|y\| < 1$. By Corollary 3.4, $x \in \operatorname{conv}(X)$ must satisfy

$$\begin{aligned} \|y\| &\leq 1 \\ x_{n+2} &\geq y^\top (Q - \lambda_Q I_n) y + 2g^\top y + \lambda_Q \\ Ay - b &\in \mathcal{K}. \end{aligned}$$

Let $d \neq 0$ be the vector given by Condition 3.6. Since d satisfies $Qd = \lambda_Q d$ and $\pm Ad \in \mathcal{K}$, for any $\epsilon \in \mathbb{R}$,

$$\begin{aligned} &(y + \epsilon d)^\top (Q - \lambda_Q I_n)(y + \epsilon d) + 2g^\top (y + \epsilon d) + \lambda_Q \\ &\quad = [y^\top (Q - \lambda_Q I_n) y + 2g^\top y + \lambda_Q] + 2g^\top d\epsilon \\ &\quad \leq x_{n+2} + 2g^\top d\epsilon. \end{aligned}$$

Since $\|y\|^2 < 1$, there exists $\epsilon > 0$ sufficiently small such that $\|y \pm \epsilon d\|^2 \le 1$. Note that $A(y \pm \epsilon d) - b \in \mathcal{K}$ since $\pm Ad \in \mathcal{K}$. Now define the points

$$x^+ := [y + \epsilon d; 1; x_{n+2} + 2g^\top d\epsilon] \quad \text{and} \quad x^- := [y - \epsilon d; 1; x_{n+2} - 2g^\top d\epsilon].$$

Both x^+, x^- satisfy all constraints of $\operatorname{conv}(X)$, and hence $x^+, x^- \in \operatorname{conv}(X)$. In particular, $x^+ \neq x \neq x^-$, but $x = \frac{1}{2}(x^+ + x^-)$; and thus x cannot be an extreme point of $\operatorname{conv}(X)$. $\square$

Theorem 3.8. *Let X be defined in (9) and $\mathcal{H} = \mathbb{R}^n \setminus \mathcal{P}$ be a set satisfying Condition 3.7. Assume Condition 3.6 also holds. Then*

$$\operatorname{conv}\left(\left\{ [y; 1; x_{n+2}] : \begin{array}{rcl} \|y\| & \le & 1 \\ y & \in & \mathcal{H} \\ Ay - b & \in & \mathcal{K} \\ y^\top Q y + 2g^\top y & \le & x_{n+2} \end{array} \right\}\right) = \operatorname{conv}(X).$$

Proof. Define $\mathcal{P}^+ = \left\{[y; 1; x_{n+2}] : y \in \mathcal{P},\ y^\top Q y + 2g^\top y \le x_{n+2}\right\}$. Then

$$\mathcal{H}^+ = \mathbb{R}^{n+2} \setminus \mathcal{P}^+ = \{[y; 1; x_{n+2}] : y \in \mathcal{H}\} \cup \left\{[y; 1; x_{n+2}] : y^\top Q y + 2g^\top y > x_{n+2}\right\},$$

and hence

$$X \cap \mathcal{H}^+ = \left\{ [y; 1; x_{n+2}] : \begin{array}{rcl} \|y\| & \le & 1 \\ y & \in & \mathcal{H} \\ Ay - b & \in & \mathcal{K} \\ y^\top Q y + 2g^\top y & \le & x_{n+2} \end{array} \right\}.$$

Now by our definition of $\mathcal{P}^+$ and the fact that $\mathcal{P} \subseteq \{y : \|y\| < 1,\ Ay - b \in \mathcal{K}\}$, we have $\operatorname{bd}(\mathcal{P}^+) = \operatorname{bd}(\mathcal{H}^+)$ and $\operatorname{conv}(X) \cap \operatorname{bd}(\mathcal{H}^+) \subseteq X$ because $\operatorname{bd}(\mathcal{P}^+) \subseteq X$. By Lemma 3.6, we have

$$X \cap \mathcal{H}^+ \subseteq \operatorname{conv}(X) \cap \mathcal{H}^+ \subseteq \operatorname{conv}(X \cap \mathcal{H}^+),$$

and taking the convex hull of all sides gives us

$$\operatorname{conv}(\operatorname{conv}(X) \cap \mathcal{H}^+) = \operatorname{conv}(X \cap \mathcal{H}^+).$$

It remains to show that $\operatorname{conv}(\operatorname{conv}(X) \cap \mathcal{H}^+) = \operatorname{conv}(X)$. We can immediately deduce that $\operatorname{conv}(X) \supseteq \operatorname{conv}(\operatorname{conv}(X) \cap \mathcal{H}^+)$; we next prove the reverse direction.

By Condition 3.7, $\mathcal{P} \subseteq \left\{y : \|y\|^2 < 1\right\}$, so by Lemma 3.7, we have $\operatorname{Ext}(\operatorname{conv}(X)) \cap \mathcal{P}^+ = \emptyset$, or equivalently

$$\operatorname{Ext}(\operatorname{conv}(X)) \subseteq \mathcal{H}^+.$$

Intersecting both sides with $\operatorname{conv}(X)$, we obtain $\operatorname{Ext}(\operatorname{conv}(X)) \subseteq \operatorname{conv}(X) \cap \mathcal{H}^+$. Note that $\operatorname{conv}(X) = \operatorname{conv}(\operatorname{Ext}(\operatorname{conv}(X))) + \operatorname{Rec}(\operatorname{conv}(X))$; so

$$\operatorname{conv}(X) = \operatorname{conv}(\operatorname{Ext}(\operatorname{conv}(X))) + \operatorname{Rec}(\operatorname{conv}(X)) \subseteq \operatorname{conv}(\operatorname{conv}(X) \cap \mathcal{H}^+) + \operatorname{Rec}(\operatorname{conv}(X)).$$

Furthermore, from Corollary 3.4, it is easy to see that

$$\operatorname{Rec}(\operatorname{conv}(X) \cap \mathcal{H}^+) = \{[0; 0; x_{n+2}] : x_{n+2} \ge 0\} = \operatorname{Rec}(\operatorname{conv}(X)).$$

Hence, we deduce that

$$\begin{aligned}\operatorname{conv}(X) &\subseteq \operatorname{conv}(\operatorname{conv}(X)\cap\mathcal{H}^+)+\operatorname{Rec}(\operatorname{conv}(X))\\ &= \operatorname{conv}(\operatorname{conv}(X)\cap\mathcal{H}^+)+\operatorname{Rec}(\operatorname{conv}(X)\cap\mathcal{H}^+)\\ &= \operatorname{conv}(\operatorname{conv}(X)\cap\mathcal{H}^+).\end{aligned}$$

The last equality holds because $\operatorname{conv}(S)=\operatorname{conv}(S)+\operatorname{Rec}(S)$ for any set S. This gives us the result. □

3.1.2 Additional Convex Constraints

Lemma 3.6 allows us to add further convex constraints $x\in\mathcal{H}$ as long as these new constraints satisfy the following boundary condition.

Condition 3.8. The new set $\mathcal{H}$ satisfies $\mathcal{F}_0^+\cap\mathcal{F}_s\cap\mathcal{K}^+\cap H^1\cap\operatorname{bd}(\mathcal{H})\subseteq\mathcal{F}_1$.

Corollary 3.9. *Suppose $\mathcal{H}$ is convex and Conditions 3.6 and 3.8 are satisfied. Then we have*

$$\operatorname{conv}(\mathcal{F}_0^+\cap\mathcal{F}_1\cap\mathcal{K}^+\cap H^1\cap\mathcal{H})=\mathcal{F}_0^+\cap\mathcal{F}_s\cap\mathcal{K}^+\cap H^1\cap\mathcal{H}.$$

Proof. From Theorem 3.3, we have $\operatorname{conv}(\mathcal{F}_0^+\cap\mathcal{F}_1\cap\mathcal{K}^+\cap H^1)=\mathcal{F}_0^+\cap\mathcal{F}_s\cap\mathcal{K}^+\cap H^1$. When $\mathcal{H}$ is convex, this immediately implies $\operatorname{conv}(\mathcal{F}_0^+\cap\mathcal{F}_1\cap\mathcal{K}^+\cap H^1\cap\mathcal{H})\subseteq\mathcal{F}_0^+\cap\mathcal{F}_1\cap\mathcal{K}^+\cap H^1\cap\mathcal{H}$. Condition 3.8 gives us $\mathcal{F}_0^+\cap\mathcal{F}_1\cap\mathcal{K}^+\cap H^1\cap\operatorname{bd}(\mathcal{H})\subseteq\mathcal{F}_1$. Then by selecting $F=\mathcal{F}_0^+\cap\mathcal{F}_1\cap\mathcal{K}^+\cap H^1$, we satisfy the premise $\operatorname{conv}(F)\cap\operatorname{bd}(\mathcal{H})\subseteq F\cap\mathcal{H}$ of Lemma 3.6. From Lemma 3.6, we deduce

$$\operatorname{conv}\left(\mathcal{F}_0^+\cap\mathcal{F}_1\cap\mathcal{K}^+\cap H^1\cap\mathcal{H}\right)\supseteq\mathcal{F}_0^+\cap\mathcal{F}_s\cap\mathcal{K}^+\cap H^1\cap\mathcal{H}.$$

□

Acknowledgments

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