Low-Complexity Relaxations and Convex Hulls of Disjunctions on the Positive Semidefinite Cone and General Regular Cones

Sercan Yıldız[1] and Fatma Kılınç-Karzan*[1]

[1] Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA, 15213, USA.
{syildiz,fkilinc}@andrew.cmu.edu

07 April 2016

Abstract

In this paper we analyze general two-term disjunctions on a regular cone $\mathbb{K}$ and derive a general form for a family of convex inequalities which are valid for the resulting nonconvex sets. Under mild technical assumptions, these inequalities collectively describe the closed convex hulls of these disjunctions, and if additional conditions are satisfied, a single inequality from this family is sufficient. In the cases where $\mathbb{K}$ is the positive semidefinite cone or a direct product of second-order cones and a nonnegative orthant, we show that these convex inequalities admit equivalent conic forms for certain choices of disjunctions. Our approach relies on and generalizes the work of Kılınç-Karzan and Yıldız which considers general two-term disjunctions on the second-order cone. Along the way, we establish a connection between two-term disjunctions and nonconvex sets defined by rank-two quadratics, through which we extend our convex hull results to intersections of a regular cone with such quadratic sets.

1 Introduction

Let $\mathbb{E}$ be a finite-dimensional Euclidean space equipped with the inner product $\langle \cdot, \cdot \rangle$. In this paper, we consider nonconvex sets which result from the application of a linear two-term disjunction on a regular (full-dimensional, closed, convex, and pointed) cone. Specifically, we consider a two-term disjunction $\langle c_1, x\rangle \geq c_{1,0} \vee \langle c_2, x\rangle \geq c_{2,0}$ on a regular cone $\mathbb{K} \subset \mathbb{E}$. In reference to the disjunction, we define the sets

$$\mathbb{C}_i := \{x \in \mathbb{K} : \langle c_i, x\rangle \geq c_{i,0}\} \quad \text{for } i \in \{1, 2\}. \tag{1}$$

The purpose of this paper is to study the structure of the closed convex hull of the disjunctive conic set $\mathbb{C}_1 \cup \mathbb{C}_2$ and describe it explicitly with convex inequalities in the space of the original variables. We also develop various techniques for constructing low-complexity convex relaxations of $\mathbb{C}_1 \cup \mathbb{C}_2$ in the same space.

Disjunctive conic sets of the form $\mathbb{C}_1 \cup \mathbb{C}_2$ are at the core of convex optimization based solution methods to conic programs with integrality requirements on the variables and other types of non-convex constraints. In the context of mixed-integer conic programs (MICPs), integrality conditions are naturally relaxed into disjunctions satisfied by all feasible solutions; convex inequalities that

*Corresponding author. The research of this author is supported in part by the NSF grant CMMI 1454548.

are valid for the resulting nonconvex sets can then be added to the problem formulation to obtain a tighter description of the integer hull. Such inequalities are known as *disjunctive inequalities* [4]. We comment further on the use of disjunctive inequalities in the solution of MICPs in the following paragraphs. In addition, two-term disjunctions are closely related to nonconvex sets defined by rank-two quadratics of the form

$$\mathbb{X} := \{x \in \mathbb{E} : \left(c_{1,0} - \langle c_1, x\rangle\right)\left(c_{2,0} - \langle c_2, x\rangle\right) \leq 0\}.$$

For instance, given that there does not exist any point $x \in \mathbb{K}$ which satisfies both $\langle c_1, x\rangle \geq c_{1,0}$ and $\langle c_2, x\rangle \geq c_{2,0}$ strictly, a two-term disjunction on $\mathbb{K}$ can be represented using the set $\mathbb{X}$: $\mathbb{C}_1 \cup \mathbb{C}_2 = \mathbb{K} \cap \mathbb{X}$. We explore this relationship further in Section 2.3.

A conic program is the problem of optimizing a linear function over the intersection of a regular cone with an affine subspace. An MICP is a conic program where some decision variables are constrained to take integer values. In the special case where the regular cone which underlies the problem is a nonnegative orthant, MICPs reduce to mixed-integer linear programs (MILPs). The combined representation power of integer variables and conic constraints makes MICPs an attractive framework for modeling optimization problems which require discrete decisions. Following the development of stable and efficient algorithms for solving second-order cone programs and semidefinite programs, MICPs with second-order cone and positive semidefinite cone constraints have received significant attention in the recent years. These problems find applications in optimization under uncertainty as well as in engineering design and statistical learning. We refer the reader to [9, 10] for recent surveys on applications of MICPs. Motivated by these applications, in this paper we place special emphasis on the cases where $\mathbb{K}$ is the nonnegative orthant $\mathbb{R}^n_+ := \left\{x \in \mathbb{R}^n : x_j \geq 0 \ \forall j \in \{1, \ldots, n\}\right\}$, the second-order (Lorentz) cone $\mathbb{L}^n := \left\{x \in \mathbb{R}^n : \sqrt{\sum_{j=1}^{n-1} x_j^2} \leq x_n\right\}$, the positive semidefinite cone $\mathbb{S}^n_+ := \left\{x \in \mathbb{R}^{n\times n} : x^\top = x, \ a^\top x\, a \geq 0 \ \forall a \in \mathbb{R}^n\right\}$, or one of their direct products.

Disjunctive inequalities, introduced in the early 1970s in the context of MILPs [4], are a main ingredient of today's successful integer programming technology. In their most general form, disjunctive inequalities are inequalities which are valid for nonconvex sets obtained from disjunctions on a convex relaxation of an integer program. Despite their simplicity, the most powerful disjunctions in integer programming are *split disjunctions*, which have the form $\langle c_1, x\rangle \geq c_{1,0} \vee \langle c_2, x\rangle \geq c_{2,0}$ for $c_1, c_2 \in \mathbb{E}$ selected as negative multiples of each other. Disjunctive inequalities derived using split disjunctions are called *split inequalities* [17]. Some of the most well-known families of cutting-planes for MILPs are split inequalities: Chvátal-Gomory inequalities [21, 16], Gomory mixed-integer inequalities [22], mixed-integer rounding inequalities [34], lift-and-project inequalities [5]... More general two-term disjunctions are used for complementarity problems [26, 37] and integer programs with nonconvex quadratic constraints [6, 14]. There has been a lot of recent interest in extending the theory of disjunctive inequalities from the setting of MILPs to that of MICPs. Stubbs and Mehrotra [35, 36] generalized lift-and-project inequalities to mixed-integer convex programs with 0-1 variables. Çezik and Iyengar [15] investigated Chvátal-Gomory inequalities for pure-integer conic programs and lift-and-project inequalities for mixed-integer conic programs with 0-1 variables. Kılınç, Linderoth, and Luedtke [27] and Bonami [12] suggested improved methods for generating lift-and-project inequalities for mixed-integer convex programs. Atamtürk and Narayanan [3] presented a method to lift a valid conic inequality in a low-dimensional restriction to a valid conic inequality for the original mixed-integer conic set. As a special class of MICPs, mixed-integer second-order cone programs have received particular attention [19, 2, 32]. Several authors have studied the problem of representing the closed convex hull of a two-term disjunction on the second-order cone or one of its affine cross-sections in the space of the original variables with closed-form convex inequalities [18, 1, 7, 8, 31, 29, 30, 38]. These results have been generalized to intersections of the second-order cone or one of its affine cross-sections with a general quadratic [13, 33]. In a different vein, Bienstock and Michalka [11] studied the characterization and separation of linear inequalities which are valid for the epigraph of a convex, differentiable function restricted to a nonconvex domain.

The set $\mathbb{C}_1 \cup \mathbb{C}_2$ exemplifies the simplest form of a disjunctive conic set as defined by Kılınç-Karzan [28]. Kılınç-Karzan studied more general disjunctive conic sets in [28] and established that the *minimality* of a valid linear inequality defined with respect to the underlying cone $\mathbb{K}$ of the disjunctive conic set determines a hierarchy for valid linear inequalities in terms of their dominance relations. Based on this, she introduced and examined $\mathbb{K}$-minimal valid linear inequalities for general disjunctive conic sets and showed that these inequalities generate the associated closed convex hulls under a mild technical condition which is also satisfied in our setup. In [29, 30], we established necessary conditions for $\mathbb{K}$-minimal and tight valid linear inequalities for sets of the form $\mathbb{C}_1 \cup \mathbb{C}_2$. In the case where $\mathbb{K}$ is the second-order cone, we showed that families of these linear inequalities can be grouped into convex inequalities with a second-order cone structure. We also gave general sufficient conditions which guarantee that a single convex inequality from this family yields the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. In this paper, we continue our work in [29, 30] and complement the literature on closed convex hull descriptions of two-term disjunctions on regular cones. First, for a two-term disjunction on a general regular cone $\mathbb{K}$, we introduce a family of structured convex inequalities which together characterize the closed convex hull $\mathbb{C}_1 \cup \mathbb{C}_2$. As in [29, 30], under certain conditions, a single inequality from this family produces the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. Along the way, we establish a connection between two-term disjunctions and nonconvex sets defined by simple quadratics. Through this connection, our results also yield valid convex inequalities and closed convex hull descriptions for sets of the form $\mathbb{K} \cap \mathbb{X}$. We note that our results on disjunctions on regular cones easily extend to disjunctions on homogeneous cross-sections of regular cones if we work in the linear subspace which defines the cross-section. We then specialize these results to the case where $\mathbb{K}$ is the positive semidefinite cone. For $\mathbb{K} = \mathbb{S}^n_+$, we identify elementary disjunctions where these inequalities can be expressed in a simple second-order conic form. For more general disjunctions on $\mathbb{S}^n_+$, we suggest low-complexity conic inequalities which provide relaxations for the closed convex hull. To the best of our knowledge, none of the papers from the previous literature provide closed convex hull characterizations of two-term disjunctions on the positive semidefinite cone in the space of the original variables.

The remainder of the paper is organized as follows: In Section 2 we introduce the basic elements of our study. In Section 2.1 we describe our notation and terminology. In Section 2.2 we define the sets $\mathbb{C}_1$ and $\mathbb{C}_2$, identify the basic setup for our analysis with Conditions 2.1 and 2.2, and characterize the set of valid linear inequalities which are of interest to us in this paper. In Section 2.3, we establish a connection between two-term disjunctions and sets of the form $\mathbb{K} \cap \mathbb{X}$; this connection carries over to closed convex hulls of these sets as well. In Section 3 we consider two-term disjunctions on a general regular cone $\mathbb{K}$. In Section 3.1 we formulate the general form of a class of convex inequalities. These inequalities are our main object of study in this paper; we explore their structure in Section 3.1. In the case where $\mathbb{K}$ is a direct product of second-order cones and a nonnegative orthant, we give structured closed-form equivalents of these convex inequalities, recovering the earlier results of [2, 1, 31, 29, 30] on disjunctions on a *single* second-order cone and extending them to direct products multiple cones. In Section 3.2, using the connection established in Section 2.3, we utilize these results to develop convex inequalities for sets of the form $\mathbb{K} \cap \mathbb{X}$. In Section 3.3 we show how the results of Section 3.1 can be strengthened when $\mathbb{C}_1, \mathbb{C}_2$ satisfy a certain disjointness condition. In Section 4 we specialize the results of Section 3 to the case where $\mathbb{K}$ is the positive semidefinite cone. In particular, our results demonstrate that the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ can be described with a single second-order cone inequality for certain choices of disjunctions on the positive semidefinite cone. For more general disjunctions, we present several techniques to generate low-complexity conic inequalities valid for $\mathbb{C}_1 \cup \mathbb{C}_2$. Although, we do not explicitly focus on affine cross-sections of regular cones, our approach immediately leads to valid convex (or conic) inequalities for two-term disjunctions applied to those sets. We comment on such extensions in Section 5.

2 Preliminaries

2.1 Notation and Terminology

Let $\mathbb{E}$ be a finite-dimensional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$. The (standard) Euclidean norm on $\mathbb{E}$ is defined as $\|x\| := \sqrt{\langle x, x \rangle}$ for any $x \in \mathbb{E}$. If $\mathbb{E}$ is a direct product $\mathbb{E} = \prod_{j=1}^{p} \mathbb{E}^j$ of lower-dimensional Euclidean spaces $\mathbb{E}^j$, we define $\langle \cdot, \cdot \rangle$ as the sum of individual inner products $\langle \cdot, \cdot \rangle_j$ on $\mathbb{E}^j$. We assume that $\mathbb{R}^n$ is equipped with the inner product $\langle \alpha, x \rangle = \alpha^\top x$. For $i \in \{1, \ldots, n\}$, we let e^i be the i-th unit vector in $\mathbb{R}^n$, and for a vector $x \in \mathbb{R}^n$, we use $\tilde{x}$ to denote the subvector $\tilde{x} := (x_1; \ldots; x_{n-1})$. We assume that $\mathbb{S}^n := \{x \in \mathbb{R}^{n \times n} : \ x^\top = x\}$ has the (Frobenius) inner product $\langle \alpha, x \rangle = \mathrm{Tr}(\alpha x)$.

Throughout the paper, we consider a regular cone $\mathbb{K} \subset \mathbb{E}$. In the case where $\mathbb{E} = \prod_{j=1}^{p} \mathbb{E}^j$, if $\mathbb{K}^j \subset \mathbb{E}^j$ is a regular cone for each $j \in \{1, \ldots, p\}$, then the direct product $\mathbb{K} = \prod_{j=1}^{p} \mathbb{K}^j$ is also a regular cone in $\mathbb{E}$. The *dual cone* of a cone $\mathbb{K}$ is $\mathbb{K}_* := \{\alpha \in \mathbb{E} : \ \langle x, \alpha \rangle \geq 0 \ \forall x \in \mathbb{K}\}$. The dual cone $\mathbb{K}_*$ of a regular cone $\mathbb{K}$ is also regular, and the dual of $\mathbb{K}_*$ is $\mathbb{K}$ itself. Moreover, when $\mathbb{K}$ is the nonnegative orthant, the second-order cone, the positive semidefinite cone, or any of their direct products, $\mathbb{K}$ is self-dual, that is, $\mathbb{K}_* = \mathbb{K}$.

Given a set $\mathbb{A} \subset \mathbb{E}$, we let $\mathrm{conv}(\mathbb{A})$, $\overline{\mathrm{conv}}(\mathbb{A})$, $\mathrm{int}(\mathbb{A})$, and $\mathrm{bd}(\mathbb{A})$ denote the convex hull, closed convex hull, topological interior, and boundary of $\mathbb{A}$, respectively. For any positive integer k, we let $[k] := \{1, \ldots, k\}$.

2.2 Two-Term Disjunctions on a Regular Cone

Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. In this section we consider $\mathbb{C}_1 \cup \mathbb{C}_2$ where

$$\mathbb{C}_i := \{x \in \mathbb{K} : \ \langle c_i, x \rangle \geq c_{i,0}\} \quad \text{for } i \in \{1, 2\}.$$

2.2.1 The Basic Setup

In this section we describe conditions which simplify our analysis of the set $\mathbb{C}_1 \cup \mathbb{C}_2$ and its closed convex hull.

The inequalities $\langle c_1, x \rangle \geq c_{1,0}$ and $\langle c_2, x \rangle \geq c_{2,0}$ can always be scaled so that their right-hand sides are 0 or ± 1. Therefore, we assume $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ for convenience. Furthermore, when $\mathbb{C}_1 \subset \mathbb{C}_2$, we have $\overline{\mathrm{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \mathbb{C}_2$. Similarly, when $\mathbb{C}_1 \supset \mathbb{C}_2$, we have $\overline{\mathrm{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \mathbb{C}_1$. In the remainder we assume $\mathbb{C}_1 \not\subset \mathbb{C}_2$ and $\mathbb{C}_1 \not\supset \mathbb{C}_2$.

Condition 2.1. $\mathbb{C}_1 \not\subset \mathbb{C}_2$ and $\mathbb{C}_1 \not\supset \mathbb{C}_2$.

In particular, Condition 2.1 implies $\mathbb{C}_1, \mathbb{C}_2 \neq \emptyset$ and $\mathbb{C}_1, \mathbb{C}_2 \neq \mathbb{K}$. Hence, $c_i \notin -\mathbb{K}_*$ when $c_{i,0} = +1$ and $c_i \notin \mathbb{K}_*$ when $c_{i,0} \in \{0, -1\}$. We also use the following technical condition in our analysis.

Condition 2.2. $\mathbb{C}_1$ and $\mathbb{C}_2$ are strictly feasible. That is, $\mathbb{C}_1 \cap \operatorname{int} \mathbb{K} \neq \emptyset$ and $\mathbb{C}_2 \cap \operatorname{int} \mathbb{K} \neq \emptyset$.

Throughout the paper, we are mainly interested in sets $\mathbb{C}_1$ and $\mathbb{C}_2$ which are defined as in (1), $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$, and satisfy Conditions 2.1 and 2.2. We say that such sets $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the *basic disjunctive setup.*

Condition 2.1 has a simple implication which we state next. We refer to [30, Lemma 2] for its proof.

Lemma 2.1 ([30])**.** *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in (1). Suppose Condition 2.1 holds. Then there does not exist any $\beta_1 \geq 0$ such that $\beta_1 c_{1,0} \geq c_{2,0}$ and $c_2 - \beta_1 c_1 \in \mathbb{K}_*$. Similarly, there does not exist any $\beta_2 \geq 0$ such that $\beta_2 c_{2,0} \geq c_{1,0}$ and $c_1 - \beta_2 c_2 \in \mathbb{K}_*$.*

2.2.2 Properties of Valid Linear Inequalities

In this section we study the structure of valid linear inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$.

Because $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy Condition 2.2, strong conic programming duality implies that a linear inequality $\langle \mu, x \rangle \geq \mu_0$ is valid for the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ if and only if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies

$$\begin{aligned} \mu = \alpha_1 + \beta_1 c_1, &\quad \mu = \alpha_2 + \beta_2 c_2, \\ \beta_1 c_{1,0} \geq \mu_0, &\quad \beta_2 c_{2,0} \geq \mu_0, \\ \alpha_1 \in \mathbb{K}_*,\ \beta_1 \in \mathbb{R}_+, &\quad \alpha_2 \in \mathbb{K}_*,\ \beta_2 \in \mathbb{R}_+. \end{aligned} \tag{2}$$

Consider $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ which satisfies (2). If $\mu_0 < \beta_1 c_{1,0}$ and $\mu_0 < \beta_2 c_{2,0}$, then $\langle \mu, x \rangle \geq \mu_0$ is not tight on $\mathbb{C}_1 \cup \mathbb{C}_2$. Any such inequality is dominated by $\langle \mu, x \rangle \geq \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}$ which has a larger right-hand side. Furthermore, if $\beta_1 = 0$ or $\beta_2 = 0$, then $\langle \mu, x \rangle \geq \mu_0$ is implied by the cone constraint $x \in \mathbb{K}$. Therefore, for a complete outer description of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$, one only needs to consider inequalities $\langle \mu, x \rangle \geq \mu_0$ where $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies

$$\begin{aligned} \mu = \alpha_1 + \beta_1 c_1, &\quad \mu = \alpha_2 + \beta_2 c_2, \\ \min\{\beta_1 c_{1,0}, &\beta_2 c_{2,0}\} = \mu_0, \\ \alpha_1 \in \mathbb{K}_*,\ \beta_1 \in \mathbb{R}_+ \setminus \{0\}, &\quad \alpha_2 \in \mathbb{K}_*,\ \beta_2 \in \mathbb{R}_+ \setminus \{0\}. \end{aligned} \tag{3}$$

The system (3) characterizes necessary conditions for linear inequalities which are valid for and tight on $\mathbb{C}_1 \cup \mathbb{C}_2$ and which are not implied by the cone constraint $x \in \mathbb{K}$. Our approach is based on grouping these linear inequalities from (3). Earlier research [29, 30] focused on identifying and grouping tight and $\mathbb{K}$-minimal valid linear inequalities. A valid inequality $\langle \mu, x \rangle \geq \mu_0$ is said to be $\mathbb{K}$-*minimal* if there does not exist another valid inequality of the form $\langle \mu - \delta, x \rangle \geq \mu_0$ with $\delta \in \mathbb{K}_* \setminus \{0\}$ (see [28] for further details). In our setup, tight and $\mathbb{K}$-minimal linear inequalities are known [28] to produce a complete outer description of $\overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$. In [29, 30], the more refined characterization of the system (3) implied by the $\mathbb{K}$-minimality concept was used to derive finer structural results for $\overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$ in the form of conic inequalities; we comment more on this in Remarks 3.5 and 3.12. Nevertheless, as opposed to the previous approach, in this paper, we focus on grouping the larger class of valid linear inequalities in (3) which are tight and not implied by

$x \in \mathbb{K}$ but which are not necessarily $\mathbb{K}$-minimal because we could not find a characterization of $\mathbb{K}$-minimal inequalities for general regular cones $\mathbb{K}$ that can be exploited in a structurally useful form in our developments.

For given $\beta_1, \beta_2 > 0$, we develop structured valid nonlinear inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$ by grouping the linear inequalities associated with the pair (β_1, β_2) in the system (3). For ease of notation, let us define the scalar $\mu_0(\beta_1, \beta_2) := \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}$ and the set

$$\mathbb{M}(\beta_1, \beta_2) := \{\mu \in \mathbb{E} : \quad \exists \alpha_1, \alpha_2 \in \mathbb{K}_*, \quad \mu = \alpha_1 + \beta_1 c_1 = \alpha_2 + \beta_2 c_2\}.$$

Then a point $x \in \mathbb{E}$ satisfies $\langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2)$ for all $\mathbb{M}(\beta_1, \beta_2)$ if and only if it satisfies

$$\inf_{\mu \in \mathbb{M}(\beta_1, \beta_2)} \langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2). \tag{4}$$

Then the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is

$$\begin{aligned} \overline{\mathrm{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) &= \{x \in \mathbb{K} : \quad \langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2) \quad \forall \mu \in \mathbb{M}(\beta_1, \beta_2), \quad \beta_1, \beta_2 > 0\} \\ &= \left\{ x \in \mathbb{K} : \inf_{\mu \in \mathbb{M}(\beta_1, \beta_2)} \langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2) \quad \forall \beta_1, \beta_2 > 0 \right\}. \end{aligned}$$

We note that (4) is valid for $\mathbb{C}_1 \cup \mathbb{C}_2$, regardless of whether or not $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Nevertheless, there are important known cases where (4) associated with a single choice of (β_1, β_2) yields the complete closed convex hull description of $\mathbb{C}_1 \cup \mathbb{C}_2$, together with the cone constraint $x \in \mathbb{K}$. We summarize them in the next lemma.

Lemma 2.2. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ which satisfy the basic disjunctive setup.*

i. [30, Lemma 3] If $c_1 \in \mathbb{K}_*$ or $c_2 \in \mathbb{K}_*$, then

$$\overline{\mathrm{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \left\{ x \in \mathbb{K} : \inf_{\mu \in \mathbb{M}(1,1)} \langle \mu, x \rangle \geq \min\{c_{1,0}, c_{2,0}\} \right\}.$$

ii. [30, Lemma 4 and Proposition 5] If the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed and $c_{1,0} = c_{2,0} \in \{\pm 1\}$, then

$$\mathrm{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) = \left\{ x \in \mathbb{K} : \inf_{\mu \in \mathbb{M}(1,1)} \langle \mu, x \rangle \geq \min\{c_{1,0}, c_{2,0}\} \right\}.$$

In the remainder, we consider the inequality (4) associated with a fixed pair (β_1, β_2) where $\beta_1, \beta_2 > 0$. As Lemma 2.2 demonstrates, an inequality of this form can describe the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ in various cases of interest. In general, however, this inequality can be considered a valid inequality derived for a relaxation $\langle \beta_1 c_1, x \rangle \geq \mu_0(\beta_1, \beta_2) \vee \langle \beta_2 c_2, x \rangle \geq \mu_0(\beta_1, \beta_2)$ of the original disjunction on the cone $\mathbb{K}$. From now on, we let $d_i := \beta_i c_i$, suppress the indices on $\mathbb{M}(\beta_1, \beta_2)$ and $\mu_0(\beta_1, \beta_2)$, and concentrate on the closed convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$ where

$$\mathbb{D}_i := \{x \in \mathbb{K} : \langle d_i, x \rangle \geq \mu_0\} \quad \text{for } i \in \{1, 2\}. \tag{5}$$

Given $\mathbb{C}_1$ and $\mathbb{C}_2$ which satisfy the basic disjunctive setup, the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ always satisfy Condition 2.2 because $\mathbb{D}_1 \supset \mathbb{C}_1$ and $\mathbb{D}_2 \supset \mathbb{C}_2$. However, they may violate Condition 2.1. When this is the case, the convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$ is equal to one of $\mathbb{D}_1$ or $\mathbb{D}_2$. Therefore, we are primarily interested in cases where $\mathbb{D}_1$ and $\mathbb{D}_2$ also satisfy Condition 2.1. By Lemma 2.1, this can happen only if $r := d_2 - d_1 \notin \pm\mathbb{K}_*$. Therefore, while studying convex relaxations for $\mathbb{D}_1 \cup \mathbb{D}_2$ in subsequent sections, we sometimes state our results under the assumption that $r \notin \pm\mathbb{K}_*$.

In Sections 3.1 and 3.3, we study the general form of (4) under various assumptions on the structure of $\mathbb{D}_1$ and $\mathbb{D}_2$.

2.3 Intersection of a Regular Cone with Nonconvex Rank-Two Quadratics

Let $\mathbb{V} \subset \mathbb{E}$ be any convex set. In this section we consider the set $\mathbb{V} \cap \mathbb{X}$ where

$$\mathbb{X} := \{x \in \mathbb{E} : \ (c_{1,0} - \langle c_1, x\rangle)(c_{2,0} - \langle c_2, x\rangle) \leq 0\} \tag{6}$$

is a nonconvex rank-two quadratic. Under a disjointness assumption, the two-term disjunction $\langle c_1, x\rangle \geq c_{1,0} \vee \langle c_2, x\rangle \geq c_{2,0}$ on $\mathbb{V}$ can be written as the intersection of $\mathbb{V}$ with the nonconvex set $\mathbb{X}$. We discuss this connection further in Section 3.3. Note that $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$ where

$$\begin{aligned}\mathbb{X}_1 &:= \{x \in \mathbb{E} : \ \langle c_1, x\rangle \geq c_{1,0}, \ \langle c_2, x\rangle \leq c_{2,0}\},\\ \mathbb{X}_2 &:= \{x \in \mathbb{E} : \ \langle c_1, x\rangle \leq c_{1,0}, \ \langle c_2, x\rangle \geq c_{2,0}\}.\end{aligned}$$

Associated with $\mathbb{X}, \mathbb{V} \subset \mathbb{E}$, we define the sets $\mathbb{C}_i^+, \mathbb{C}_i^- \subset \mathbb{E}$ where

$$\mathbb{C}_i^+ := \{x \in \mathbb{V} : \langle c_i, x\rangle \geq c_{i,0}\}, \quad \mathbb{C}_i^- := \{x \in \mathbb{V} : \langle c_i, x\rangle \leq c_{i,0}\} \quad \text{for } i \in \{1, 2\}. \tag{7}$$

Then $\mathbb{V} \cap \mathbb{X}_1 = \mathbb{C}_1^+ \cap \mathbb{C}_2^-$ and $\mathbb{V} \cap \mathbb{X}_2 = \mathbb{C}_1^- \cap \mathbb{C}_2^+$. Furthermore,

$$\mathbb{V} \cap \mathbb{X} = \left(\mathbb{C}_1^+ \cup \mathbb{C}_2^+\right) \cap \left(\mathbb{C}_1^- \cup \mathbb{C}_2^-\right).$$

In Proposition 2.3 below, we show that the *convex hull* of $\mathbb{V} \cap \mathbb{X}$ equals the intersection of the convex hulls of $\mathbb{C}_1^+ \cup \mathbb{C}_2^+$ and $\mathbb{C}_1^- \cup \mathbb{C}_2^-$.

Proposition 2.3. *Let $\mathbb{V} \subset \mathbb{E}$ be a convex set. Let $\mathbb{X} \subset \mathbb{E}$ and $\mathbb{C}_i^+, \mathbb{C}_i^- \subset \mathbb{E}$ be defined as in (6) and (7), respectively.*

i. $\operatorname{conv}(\mathbb{V} \cap \mathbb{X}) = \operatorname{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \operatorname{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$.

ii. Suppose $\mathbb{V}$ is closed. Then $\overline{\operatorname{conv}}(\mathbb{V} \cap \mathbb{X}) = \overline{\operatorname{conv}}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \overline{\operatorname{conv}}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$.

Proof. First we prove (i). Because $\mathbb{V} \cap \mathbb{X} = (\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap (\mathbb{C}_1^- \cup \mathbb{C}_2^-)$, we immediately have $\operatorname{conv}(\mathbb{V} \cap \mathbb{X}) \subset \operatorname{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \operatorname{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$. If $\operatorname{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \operatorname{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-) = \emptyset$, then we have equality throughout. Let $x \in \operatorname{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \operatorname{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$. We will show $x \in \operatorname{conv}(\mathbb{V} \cap \mathbb{X})$. If $x \in \mathbb{X}$, then we are done, because $\operatorname{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \operatorname{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-) \subset \mathbb{V}$. Hence, we assume $x \notin \mathbb{X}$. Then $x \in \mathbb{T}^+ \cup \mathbb{T}^-$ where $\mathbb{T}^+ := \{x \in \mathbb{E} : \ \langle c_1, x\rangle > c_{1,0}, \ \langle c_2, x\rangle > c_{2,0}\}$ and $\mathbb{T}^- := \{x \in \mathbb{E} : \ \langle c_1, x\rangle < c_{2,0}, \ \langle c_2, x\rangle < c_{2,0}\}$.

Consider the case where $x \in \mathbb{T}^+$. The case for $x \in \mathbb{T}^-$ is similar. Because $x \in \mathbb{T}^+$, we have $\langle c_1, x\rangle > c_{1,0}$ and $\langle c_2, x\rangle > c_{2,0}$. Because $x \in \operatorname{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$, there exists $x_1, x_2 \in \mathbb{C}_1^- \cup \mathbb{C}_2^-$ such that $x \in \operatorname{conv}\{x_1, x_2\}$. We claim $x_1, x_2 \in \mathbb{X}$. Suppose not. Then $x_1 \in \mathbb{T}^-$ or $x_2 \in \mathbb{T}^-$. In the first case, x_1 satisfies $\langle c_1, x_1\rangle < c_{1,0}$ and $\langle c_2, x_1\rangle < c_{2,0}$, whereas $x_2 \in \mathbb{C}_1^- \cup \mathbb{C}_2^-$ implies that x_2 satisfies at least one of $\langle c_1, x_2\rangle \leq c_{1,0}$ or $\langle c_2, x_2\rangle \leq c_{2,0}$. This contradicts $x \in \mathbb{T}^+$. The case where $x_2 \in \mathbb{T}^-$ is analogous and leads to the same conclusion.

Now we prove (ii). The inclusion $\overline{\text{conv}}(\mathbb{V}\cap\mathbb{X})\subset\overline{\text{conv}}(\mathbb{C}_1^+\cup\mathbb{C}_2^+)\cap\overline{\text{conv}}(\mathbb{C}_1^-\cup\mathbb{C}_2^-)$ follows from statement (i). As in the proof of statement (i), we can assume $\overline{\text{conv}}(\mathbb{C}_1^+\cup\mathbb{C}_2^+)\cap\overline{\text{conv}}(\mathbb{C}_1^-\cup\mathbb{C}_2^-)\neq\emptyset$. Let $x\in\overline{\text{conv}}(\mathbb{C}_1^+\cup\mathbb{C}_2^+)\cap\overline{\text{conv}}(\mathbb{C}_1^-\cup\mathbb{C}_2^-)$. We will show $x\in\overline{\text{conv}}(\mathbb{V}\cap\mathbb{X})$. Because $x\in\mathbb{V}$, it is enough to consider $x\notin\mathbb{X}$. Suppose $x\in\mathbb{T}^+$. Because $x\in\overline{\text{conv}}(\mathbb{C}_1^-\cup\mathbb{C}_2^-)$, there exists a sequence $\{u^i\}_{i=1}^\infty\subset\text{conv}(\mathbb{C}_1^-\cup\mathbb{C}_2^-)$ which converges to x. The subsequence $\{u^i\}_{i=1}^\infty\cap\mathbb{T}^+$ is infinite, contained in $\text{conv}(\mathbb{C}_1^+\cup\mathbb{C}_2^+)\cap\text{conv}(\mathbb{C}_1^-\cup\mathbb{C}_2^-)$, and also converges to x. By statement (i), this subsequence is also contained in $\text{conv}(\mathbb{V}\cap\mathbb{X})$. Therefore, $x\in\overline{\text{conv}}(\mathbb{V}\cap\mathbb{X})$. $\square$

3 Nonlinear Inequalities with Special Structure

3.1 Inequalities for Two-Term Disjunctions

Let $\mathbb{K}$ be a regular cone. In this section we consider sets $\mathbb{D}_1$ and $\mathbb{D}_2$ which are defined as in (5). Let $\mathbb{M}:=\{\mu\in\mathbb{E}:\ \exists\alpha_1,\alpha_2\in\mathbb{K}_*,\ \mu=\alpha_1+d_1=\alpha_2+d_2\}$. As discussed in Section 2.2, any point $x\in\mathbb{D}_1\cup\mathbb{D}_2$ satisfies

$$\inf_{\mu\in\mathbb{M}}\langle\mu,x\rangle\geq\mu_0,\tag{8}$$

regardless of whether or not $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the basic disjunctive setup. Furthermore, whenever $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the conditions of Lemma 2.2, the inequality (8) describes the closed convex hull of $\mathbb{D}_1\cup\mathbb{D}_2$. Our main purpose is to investigate the general form of this inequality under minimal assumptions on the structure of $\mathbb{K}$. Through this framework, we also recover previous results about two-term disjunctions on the nonnegative orthant and the second-order cone and extend them to direct products of these cones.

Throughout this section, we denote $r=d_2-d_1\in\mathbb{E}$. We start with a simple observation which yields an alternate representation of the disjunction $\langle d_1,x\rangle\geq\mu_0\vee\langle d_2,x\rangle\geq\mu_0$.

Remark 3.1. A point $x\in\mathbb{E}$ satisfies the disjunction $\langle d_1,x\rangle\geq\mu_0\vee\langle d_2,x\rangle\geq\mu_0$ if and only if it satisfies

$$|\langle r,x\rangle|\geq 2\mu_0-\langle d_1+d_2,x\rangle.\tag{9}$$

$\diamondsuit$

The following lemma is used in the proof of Proposition 3.2, which states (8) in an alternate form.

Lemma 3.1. *Let $\mathbb{K}\subset\mathbb{E}$ be a regular cone. For any $r\in\mathbb{E}$, there exist $\alpha_1,\alpha_2\in\mathbb{K}_*$ such that $\alpha_1-\alpha_2=r$.*

Proof. The dual cone $\mathbb{K}_*$ is also a regular cone. Let $e\in\text{int}\,\mathbb{K}_*$. Then there exists $\epsilon>0$ such that $e+\mathbb{B}(\epsilon)\subset\mathbb{K}_*$ where $\mathbb{B}(\epsilon):=\{x\in\mathbb{E}:\|x\|\leq\epsilon\}$. Let $r\in\mathbb{E}$. Then $\frac{\epsilon}{\|r\|}r\in\mathbb{B}(\epsilon)$. Hence, $e+\frac{\epsilon}{\|r\|}r\in\mathbb{K}_*$. After scaling, we obtain $\frac{\|r\|}{\epsilon}e+r\in\mathbb{K}_*$, which implies that r can be written as the difference of some point in $\mathbb{K}_*$ and $\frac{\|r\|}{\epsilon}e$. $\square$

Proposition 3.2. *Let $\mathbb{K}\subset\mathbb{E}$ be a regular cone. A point $x\in\mathbb{E}$ satisfies (8) if and only if it satisfies*

$$f_{\mathbb{K},r}(x)\geq 2\mu_0-\langle d_1+d_2,x\rangle\tag{10}$$

where $f_{\mathbb{K},r}:\mathbb{E}\to\mathbb{R}\cup\{-\infty\}$ is defined as

$$f_{\mathbb{K},r}(x):=\inf_{\alpha_1,\alpha_2}\{\langle\alpha_1+\alpha_2,x\rangle:\quad\alpha_1-\alpha_2=r,\quad\alpha_1,\alpha_2\in\mathbb{K}_*\}\tag{11}$$

$$=\max_{\rho}\{\langle r,\rho\rangle:\quad x-\rho\in\mathbb{K},\quad x+\rho\in\mathbb{K}\}.\tag{12}$$

Proof. Consider (8). Note that

$$\begin{aligned}\inf_{\mu}\{\langle\mu,x\rangle:\mu\in\mathbb{M}\} &= \inf_{\mu,\alpha_1,\alpha_2}\{\langle\mu,x\rangle:\quad \mu=\alpha_1+d_1,\quad \mu=\alpha_2+d_2,\quad \alpha_1,\alpha_2\in\mathbb{K}_*\}\\ &= \frac{1}{2}\langle d_1+d_2,x\rangle+\frac{1}{2}\inf_{\alpha_1,\alpha_2}\left\{\langle\alpha_1+\alpha_2,x\rangle:\quad \begin{array}{c}\alpha_1-\alpha_2=r,\\ \alpha_1,\alpha_2\in\mathbb{K}_*\end{array}\right\}\\ &= \frac{1}{2}\langle d_1+d_2,x\rangle+\frac{1}{2}f_{\mathbb{K},r}(x).\end{aligned}$$

Therefore, (8) is equivalent to (10). Lemma 3.1(i) shows that there always exist $\widehat{\alpha}_1,\widehat{\alpha}_2\in\mathbb{K}_*$ such that $\widehat{\alpha}_1-\widehat{\alpha}_2=r$. Hence, (11) is always feasible. Indeed, this minimization problem is strictly feasible because, for any $e\in\operatorname{int}\mathbb{K}_*$, we have $\widehat{\alpha}_1+e,\widehat{\alpha}_2+e\in\operatorname{int}\mathbb{K}_*$ and $(\widehat{\alpha}_1+e)-(\widehat{\alpha}_2+e)=r$. Therefore, strong conic programming duality applies, and the dual problem (12) is solvable whenever the optimal value of (11) is bounded from below. □

Next, we make a series of immediate observations on the function $f_{\mathbb{K},r}(x)$.

Remark 3.2. Let $\mathbb{K}\subset\mathbb{E}$ be a regular cone. Fix $r\in\mathbb{E}$.

i. As a function of x, $-f_{\mathbb{K},r}(x)$ is the support function of a nonempty set (see (11)). Therefore, it is closed and sublinear. Furthermore, the value of $-f_{\mathbb{K},r}(x)$ is finite if and only if $x\in\mathbb{K}$.

ii. The function $f_{\mathbb{K},r}(x)$ satisfies $f_{\mathbb{K},r}(x)\geq|\langle r,x\rangle|$ for any $x\in\mathbb{K}$. If x is an extreme ray of $\mathbb{K}$, then $f_{\mathbb{K},r}(x)=|\langle r,x\rangle|$.

Proof. We only prove statement (ii). Let $x\in\mathbb{K}$. Both x and $-x$ are feasible solutions to (12). Therefore, $f_{\mathbb{K},r}(x)\geq|\langle r,x\rangle|$. Now suppose x is an extreme ray of $\mathbb{K}$. Let $\rho\in\mathbb{E}$ be any feasible solution to (12). We show $\rho\in\operatorname{conv}\{x,-x\}$. First, note that $\frac{1}{2}(x-\rho)+\frac{1}{2}(x+\rho)=x$. Because x is an extreme ray of K, there must exist $\lambda_1,\lambda_2\geq 0$ such that $x-\rho=\lambda_1 x$ and $x+\rho=\lambda_2 x$. It follows that $\rho=(1-\lambda_1)x=(\lambda_2-1)x$ and $\lambda_1+\lambda_2=2$, which completes the proof of the claim. □

Recall from Remark 3.1 that (9) provides an exact representation of the disjunction $\langle d_1,x\rangle\geq\mu_0\ \vee\ \langle d_2,x\rangle\geq\mu_0$. Remark 3.2 shows that $f_{\mathbb{K},r}(x)$ is a concave function of x which satisfies $f_{\mathbb{K},r}(x)\geq|\langle r,x\rangle|$ for any $x\in\mathbb{K}$. Replacing the term $|\langle r,x\rangle|$ on the left-hand side of (9) with any such function would define a convex relaxation of $\mathbb{D}_1\cup\mathbb{D}_2$ inside the cone $\mathbb{K}$. On the other hand, $f_{\mathbb{K},r}(x)$ is a "tight" concave overestimator of the function $x\mapsto|\langle r,x\rangle|:\mathbb{E}\to\mathbb{R}$ over $\mathbb{K}$: It satisfies $f_{\mathbb{K},r}(x)=|\langle r,x\rangle|$ whenever x is an extreme ray of $\mathbb{K}$. This implies that an extreme ray $x\in\mathbb{K}$ satisfies (10) if and only if $x\in\mathbb{D}_1\cup\mathbb{D}_2$. Furthermore, if the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the conditions of Lemma 2.2, the inequality (10) defines the closed convex hull of $\mathbb{D}_1\cup\mathbb{D}_2$.

Remark 3.3. Let $\mathbb{K}\subset\mathbb{E}$ be a regular cone. Fix $x\in\mathbb{K}$.

i. As a function of r, $f_{\mathbb{K},r}(x)$ is the support function of a bounded set which contains the origin (see (12)). Therefore, it is nonnegative, finite-valued, and sublinear.

ii. As a function of r, $f_{\mathbb{K},r}(x)$ is symmetric with respect to the origin, that is, $f_{\mathbb{K},r}(x)=f_{\mathbb{K},-r}(x)$ for any $r\in\mathbb{E}$.

◇

Remark 3.4. Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Fix $x \in \mathbb{K}$. If $r \in \mathbb{K}_*$, then $f_{\mathbb{K},r}(x) = \langle r, x\rangle$; if $-r \in \mathbb{K}_*$. then $f_{\mathbb{K},r}(x) = \langle -r, x\rangle$. Thus, $f_{\mathbb{K},r}(x) = |\langle r, x\rangle|$ if $r \in \pm\mathbb{K}_*$. $\diamondsuit$

Remark 3.5. Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone; and consider $r \notin \pm\mathbb{K}_*$. Then from the definitions of r and the set $\mathbb{M}$, we immediately deduce that $\alpha_1, \alpha_2 \in \mathbb{K}_* \setminus \{0\}$ in (3). Moreover, using the necessary conditions for tight and $\mathbb{K}$-minimal inequalities $\langle \mu, x\rangle \geq \mu_0$, it was shown in [30, Proposition 1] that the conditions $\alpha_1, \alpha_2 \in \mathbb{K}_*$ of (3) can be strengthened into $\alpha_1, \alpha_2 \in \mathrm{bd}(\mathbb{K}_*)$. Then the sufficiency of tight, $\mathbb{K}$-minimal inequalities immediately implies $f_{\mathbb{K},r}(x) := f'_{\mathbb{K},r}(x)$ where $f'_{\mathbb{K},r} : \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$f'_{\mathbb{K},r}(x) = \inf_{\alpha_1,\alpha_2} \{\langle \alpha_1 + \alpha_2, x\rangle :\ \alpha_1 - \alpha_2 = r,\ \alpha_1, \alpha_2 \in \mathrm{bd}(\mathbb{K}_*)\} \tag{13}$$

whenever $r \notin \pm\mathbb{K}_*$. $\diamondsuit$

We can use Proposition 3.2 together with Remarks 3.2(i) and 3.3(i) to build simple convex inequalities for $\mathbb{D}_1 \cup \mathbb{D}_2$.

Remark 3.6. Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Fix $r \in \mathbb{E}$. For any $r_1, \ldots, r_\ell \in \mathbb{E}$ such that $r = \sum_{i=1}^{\ell} r_i$, we have $\sum_{i=1}^{\ell} f_{\mathbb{K},r_i}(x) \geq f_{\mathbb{K},r}(x)$. Therefore, the inequality $\sum_{i=1}^{\ell} f_{\mathbb{K},r_i}(x) \geq 2\mu_0 - \langle d_1 + d_2, x\rangle$ is a relaxation of (10). Furthermore, note from Remark 3.2(i) that each function $f_{\mathbb{K},r_i}(x)$ is a concave function of x; hence, the resulting inequality is convex. $\diamondsuit$

Remark 3.6 suggests a general procedure for developing convex inequalities for $\mathbb{D}_1 \cup \mathbb{D}_2$ which might have nicer structural properties than (10). Furthermore, it allows great flexibility in the choice of the decomposition $r = \sum_{i=1}^{\ell} r_i$. For certain choices of $r_1, \ldots, r_\ell \in \mathbb{E}$, the relaxation suggested in Remark 3.6 has the interpretation of relaxing the underlying disjunction. We comment more on this interpretation in Section 4.4.2. Next we consider an immediate application of the procedure outlined in Remark 3.6 which gives valid linear inequalities for $\mathbb{D}_1 \cup \mathbb{D}_2$ as a consequence of Remark 3.4(i).

Remark 3.7. Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Fix $r \in \mathbb{E}$. By Lemma 3.1, there exists $r_+, r_- \in \mathbb{K}_*$ such that $r = r_+ - r_-$. Remark 3.3(i) shows that $f_{\mathbb{K},r}(x) \leq f_{\mathbb{K},r_+}(x) + f_{\mathbb{K},-r_-}(x) = f_{\mathbb{K},r_+}(x) + f_{\mathbb{K},r_-}(x)$. Moreover, because $r_+, r_- \in \mathbb{K}_*$, Remark 3.4(i) implies $f_{\mathbb{K},r_+}(x) = \langle r_+, x\rangle$ and $f_{\mathbb{K},r_-}(x) = \langle r_-, x\rangle$. Finally, using Proposition 3.2, we conclude that any $x \in \mathbb{D}_1 \cup \mathbb{D}_2$ satisfies the linear inequality

$$\langle r_+ + r_-, x\rangle \geq 2\mu_0 - \langle d_1 + d_2, x\rangle. \tag{14}$$

Note that any possible choice of $r_+, r_- \in \mathbb{K}_*$ satisfying $r = r_+ - r_-$ leads to a different inequality of the form (14). Given a two-term disjunction and a point $x \in \mathbb{K}$ that is desired to be cut off, we can select the best possible inequality of the form (14) via a conic optimization problem. $\diamondsuit$

Remark 3.8. Let $\mathbb{K} \subset \mathbb{E}$ and $\mathbf{K} \subset \mathbb{E}$ be regular cones such that $\mathbf{K} \supset \mathbb{K}$. Then $\mathbf{K}_* \subset \mathbb{K}_*$, and for any $x, r \in \mathbb{E}$, we have $f_{\mathbf{K},r}(x) \geq f_{\mathbb{K},r}(x)$. $\diamondsuit$

The monotonicity result from Remark 3.8 can be useful when one would like to develop structured convex relaxations of $\mathbb{D}_1 \cup \mathbb{D}_2$ by replacing $\mathbb{K}$ with a regular cone $\mathbf{K} \supset \mathbb{K}$ such that an expression for $f_{\mathbf{K},r}(x)$ is readily available.

Remark 3.9. Let $\mathbb{E} = \prod_{j=1}^{p} \mathbb{E}^j$ be a direct product of finite-dimensional Euclidean spaces. Suppose $\mathbb{K} = \prod_{j=1}^{p} \mathbb{K}^j$ and each $\mathbb{K}^j \subset \mathbb{E}^j$ is a regular cone. Then

$$f_{\mathbb{K},r}(x) = \sum_{j=1}^{p} \inf_{\alpha_1^j,\alpha_2^j} \left\{\langle \alpha_1^j + \alpha_2^j, x^j\rangle_j :\ \alpha_1^j - \alpha_2^j = r^j,\ \alpha_1^j, \alpha_2^j \in \mathbb{K}_*^j\right\} = \sum_{j=1}^{p} f_{\mathbb{K}^j,r^j}(x^j).$$

$\diamondsuit$

Under the hypotheses of Remark 3.9, let us define the following sets with respect to $r = (r^1, \dots, r^p) \in \mathbb{E}$:

$$\mathbb{P}^{+} := \left\{ j \in [p] \colon -r^j \in \mathbb{K}_*^j \right\}, \ \mathbb{P}^{-} := \left\{ j \in [p] \colon r^j \in \mathbb{K}_*^j \right\}, \ \mathbb{P}^{\circ} := \left\{ j \in [p] \colon r^j \notin \pm\mathbb{K}_*^j \right\}. \tag{15}$$

Next we state a consequence of Proposition 3.2 and Remarks 3.4(i) and 3.9.

Proposition 3.3. *Let $\mathbb{E} = \prod_{j=1}^p \mathbb{E}^j$ be a direct product of finite-dimensional Euclidean spaces. Suppose $\mathbb{K} = \prod_{j=1}^p \mathbb{K}^j$ and each $\mathbb{K}^j \subset \mathbb{E}^j$ is a regular cone. Define the sets $\mathbb{P}^+$, $\mathbb{P}^-$, and $\mathbb{P}^\circ$ as in (15).*

i. A point $x \in \mathbb{K}$ satisfies (10) if and only if it satisfies

$$\sum_{j \in \mathbb{P}^\circ} f_{\mathbb{K}^j, r^j}(x^j) + \sum_{j \in \mathbb{P}^\circ} \langle d_1^j + d_2^j, x^j \rangle_j + 2 \sum_{j \in \mathbb{P}^+} \langle d_1^j, x^j \rangle_j + 2 \sum_{j \in \mathbb{P}^-} \langle d_2^j, x^j \rangle_j \geq 2\mu_0. \tag{16}$$

ii. A point $x \in \mathbb{K}$ satisfies (16) if and only if there exist $z^j \in \mathbb{R}$, $j \in [p]$, such that

$$f_{\mathbb{K}^j, r^j}(x^j) \geq |2z^j - \langle d_1^j + d_2^j, x^j \rangle| \quad \forall j \in [p], \tag{17a}$$

$$\sum_{j=1}^p z^j \geq \mu_0. \tag{17b}$$

Furthermore, for each $j \in [p]$, (17a) is equivalent to

$$\left[f_{\mathbb{K}^j, r^j}(x^j) \right]^2 - \langle r^j, x^j \rangle_j^2 \geq 4(z^j - \langle d_1^j, x^j \rangle_j)(z^j - \langle d_2^j, x^j \rangle_j). \tag{18}$$

Proof. Statement (i) follows directly from Proposition 3.2 and Remarks 3.4(i) and 3.9. Fix $x \in \mathbb{K}$. The "if" part of statement (ii) is clear. To show the "only if" part, let $\bar{z}^j := \frac{1}{2}(f_{\mathbb{K}^j, r^j}(x^j) + \langle d_1^j + d_2^j, x^j \rangle_j)$ for each $j \in [p]$. Recall from Remark 3.3(i) that each $f_{\mathbb{K}^j, r^j}(x^j)$ is finite and nonnegative. Then $2\bar{z}^j - \langle d_1^j + d_2^j, x^j \rangle_j = f_{\mathbb{K}^j, r^j}(x^j) \geq 0$. Hence, $(\bar{z}^1, \dots, \bar{z}^p)$ satisfies (17).

To finish the proof, we show that (17a) is equivalent to $[f_{\mathbb{K}^j, r^j}(x^j)]^2 - \langle r^j, x^j \rangle_j^2 \geq 4(z^j - \langle d_1^j, x^j \rangle_j)(z^j - \langle d_2^j, x^j \rangle_j)$ for any $z^j \in \mathbb{R}$. The nonnegativity of $f_{\mathbb{K}^j, r^j}(x^j)$ implies

$$\begin{aligned} f_{\mathbb{K}^j, r^j}(x^j) \geq |2z^j - \langle d_1^j + d_2^j, x^j \rangle_j| \quad &\Leftrightarrow \quad \left[f_{\mathbb{K}^j, r^j}(x^j) \right]^2 \geq (2z^j - \langle d_1^j + d_2^j, x^j \rangle_j)^2 \\ &\Leftrightarrow \quad \left[f_{\mathbb{K}^j, r^j}(x^j) \right]^2 - \langle r^j, x^j \rangle_j^2 \geq 4(z^j - \langle d_1^j, x^j \rangle_j)(z^j - \langle d_2^j, x^j \rangle_j). \end{aligned}$$

$\square$

Remark 3.10. Under the hypotheses of Proposition 3.3, Remark 3.4(i) shows that $f_{\mathbb{K}^j, r^j}(x^j) = |\langle r^j, x^j \rangle|$ for $j \in \mathbb{P}^+ \cup \mathbb{P}^-$. Therefore, (17a) simplifies to $\langle d_1^j, x^j \rangle \geq z^j \geq \langle d_2^j, x^j \rangle$ for $j \in \mathbb{P}^+$ and to $\langle d_2^j, x^j \rangle \geq z^j \geq \langle d_1^j, x^j \rangle$ for $j \in \mathbb{P}^-$. Hence, the auxiliary variables z^j, $j \in \mathbb{P}^+ \cup \mathbb{P}^-$, can be eliminated from (17) after setting them equal to their corresponding upper bounds. $\diamondsuit$

Remark 3.11. Let $\mathbb{E} = \mathbb{R}^p$ and $\mathbb{K} = \mathbb{R}^p_+$. Note that $\mathbb{R}^p_+$ is a decomposable cone: It can be seen as a direct product $\prod_{j=1}^{p} \mathbb{K}^j$ where $\mathbb{K}^j = \mathbb{R}_+$ for all $j \in [p]$. Then Remark 3.4(i), together with the fact that $r^j \in \pm\mathbb{R}_+$ for all $j \in [p]$, implies $f_{\mathbb{R}^p_+,r}(x) = \sum_{j=1}^{p} |r^j x^j| = \sum_{j=1}^{p} |r^j| x^j$ for all $x \in \mathbb{R}^p_+$. Proposition 3.2 shows that the inequality $\sum_{j=1}^{p} |r^j| x^j \geq 2\mu_0 - \langle d_1 + d_2, x\rangle$ is valid for $\mathbb{D}_1 \cup \mathbb{D}_2$. This inequality can be further simplified into

$$\sum_{j=1}^{p} \max\left\{d_1^j, d_2^j\right\} x^j \geq \mu_0.$$

$\diamondsuit$

In the case of $\mathbb{K} = \mathbb{L}^n$, it was identified in [29, 30] that the structural properties of tight, $\mathbb{K}$-minimal inequalities lead to a nice characterization of the function $f_{\mathbb{L}^n,r}(x)$. These functions, $f_{\mathbb{L}^n,r}(x)$, then induce second-order conic inequalities in a lifted space with one additional variable. Next we summarize these results from [29, 30].

Remark 3.12. Let $\mathbb{E} = \mathbb{R}^n$ and $\mathbb{K} = \mathbb{L}^n$. Suppose $r \notin \pm\mathbb{L}^n$.

i. Let $x \in \mathbb{L}^n$. Recall from Remark 3.5 that $f_{\mathbb{L}^n,r}(x) = f'_{\mathbb{L}^n,r}(x)$. It was shown in [30, Theorem 3] that

$$f'_{\mathbb{L}^n,r}(x) = \sqrt{\langle r, x\rangle^2 + (\|\tilde{r}\|^2 - r_n^2)(x_n^2 - \|\tilde{x}\|^2)}.$$

Then (10) reduces to

$$\sqrt{\langle r, x\rangle^2 + (\|\tilde{r}\|^2 - r_n^2)(x_n^2 - \|\tilde{x}\|^2)} \geq 2\mu_0 - \langle d_1 + d_2, x\rangle. \tag{19}$$

ii. It was further shown in [30, Proposition 3] that, for any $x \in \mathbb{L}^n$ and $z \in \mathbb{R}$, the inequality $[f_{\mathbb{L}^n,r}(x)]^2 - \langle r, x\rangle^2 \geq 4(z - \langle d_1, x\rangle)(z - \langle d_2, x\rangle)$ can be represented in second-order conic form as

$$\left(\|\tilde{r}\|^2 - r_n^2\right) x - 2(\langle d_1, x\rangle - z) \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n. \tag{20}$$

Therefore, according to Proposition 3.3(ii), a point $x \in \mathbb{L}^n$ satisfies (19) if and only if there exists $z \geq \mu_0$ such that (20) holds.

$\diamondsuit$

In parallel to Remark 3.12, the necessary conditions for tight, $\mathbb{K}$-minimal inequalities also lead to such refined characterizations of the functions $f_{\mathbb{K},r}(x)$ and a corresponding family of conic inequalities for general p-order cones with $p \in (1, \infty)$ for certain elementary disjunctions.

The following corollary presents an extension of Remark 3.12 to the case where $\mathbb{K}$ is a direct product of multiple second-order cones and nonnegative rays.

Corollary 3.4. *Suppose* $\mathbb{E} = \prod_{j=1}^{p_1+p_2} \mathbb{E}^j$ *and* $\mathbb{K} = \prod_{j=1}^{p_1+p_2} \mathbb{K}^j$ *where* $\mathbb{E}^j = \mathbb{R}^{n^j}$ *and* $\mathbb{K}^j = \mathbb{L}^{n^j}$ *for* $j \in [p_1]$ *and* $\mathbb{E}^{p_1+j} = \mathbb{R}$ *and* $\mathbb{K}^{p_1+j} = \mathbb{R}_+$ *for* $j \in [p_2]$. *Let*

$$\mathbb{P}_1^+ := \left\{j \in [p_1] : -r^j \in \mathbb{L}^{n^j}\right\}, \ \mathbb{P}_1^- := \left\{j \in [p_1] : r^j \in \mathbb{L}^{n^j}\right\}, \ \mathbb{P}_1^\circ := \left\{j \in [p_1] : r^j \notin \pm\mathbb{L}^{n^j}\right\}.$$

i. A point $x \in \mathbb{K}$ satisfies (10) if and only if it satisfies

$$\sum_{j\in\mathbb{P}_1^\circ} f_{\mathbb{L}^{n^j},r^j}(x^j) + \sum_{j\in\mathbb{P}_1^\circ} \langle d_1^j + d_2^j, x^j\rangle_j + 2\sum_{j\in\mathbb{P}_1^+}\langle d_1^j, x^j\rangle_j + 2\sum_{j\in\mathbb{P}_1^-}\langle d_2^j, x^j\rangle_j + 2\sum_{j=p_1+1}^{p_1+p_2} \max\left\{d_1^j, d_2^j\right\}x^j \geq 2\mu_0 \tag{21}$$

where $f_{\mathbb{L}^{n^j},r^j}(x^j) = \sqrt{\langle r^j, x^j\rangle_j^2 + (\|\tilde{r}^j\|^2 - (r^j_{n^j})^2)((x^j_{n^j})^2 - \|\tilde{x}^j\|^2)}$ *for* $j \in \mathbb{P}_1^\circ$.

ii. A point $x \in \mathbb{K}$ satisfies (21) if and only if there exist $z^j \in \mathbb{R}$, $j \in \mathbb{P}_1^\circ$, such that

$$\Big(\|\tilde{r}^j\|^2 - (r^j_{n^j})^2\Big)x^j - 2\Big(\langle d_1^j, x^j\rangle_j - z^j\Big)\begin{pmatrix} -\tilde{r}^j \\ r^j_{n^j}\end{pmatrix} \in \mathbb{L}^{n^j} \qquad \forall j \in \mathbb{P}_1^\circ, \tag{22a}$$

$$\sum_{j\in\mathbb{P}_1^\circ} z^j + \sum_{j\in\mathbb{P}_1^+}\langle d_1^j, x^j\rangle_j + \sum_{j\in\mathbb{P}_1^-}\langle d_2^j, x^j\rangle_j + \sum_{j=p_1+1}^{p_1+p_2}\max\{d_1^j, d_2^j\}x^j \geq \mu_0. \tag{22b}$$

Proof. Fix $x \in \mathbb{K}$. By Proposition 3.3 and Remarks 3.11 and 3.12(i), the inequality (10) reduces to (21). To show statement (*ii*), consider Proposition 3.3(ii). Remark 3.10 shows that the auxiliary variables z^j can be eliminated from (17) for $j \in \mathbb{P}_1^+ \cup \mathbb{P}_1^-$. Furthermore, as discussed in Remark 3.12(ii), the inequalities $\left[f_{\mathbb{L}^{n^j},r^j}(x^j)\right]^2 - \langle r^j, x^j\rangle_j^2 \geq 4(z^j - \langle d_1^j, x^j\rangle_j)(z^j - \langle d_2^j, x^j\rangle_j)$ can be represented in second-order conic form as (22a) for $j \in \mathbb{P}_1^\circ$. Hence, (17) reduces to (22). $\square$

3.2 Inequalities for Intersections with Rank-Two Nonconvex Quadratics

In this section, we consider sets of the form $\mathbb{K} \cap \mathbb{F}$ where $\mathbb{K} \subset \mathbb{E}$ is a regular cone and $\mathbb{F} \subset \mathbb{E}$ is a nonconvex set defined by a rank-two quadratic inequality:

$$\mathbb{F} := \{x \in \mathbb{E} : \ (\mu_0 - \langle d_1, x\rangle)(\mu_0 - \langle d_2, x\rangle) \leq 0\}. \tag{23}$$

We will show how the results of Sections 2.3 and 3.1 can be combined to build convex relaxations and convex hull descriptions for $\mathbb{K} \cap \mathbb{F}$.

As in the previous section, we denote $r = d_2 - d_1 \in \mathbb{E}$. We start with a simple observation on an alternate representation of $\mathbb{F}$, which parallels Remark 3.1.

Remark 3.13. A point $x \in \mathbb{E}$ satisfies $(\mu_0 - \langle d_1, x\rangle)(\mu_0 - \langle d_2, x\rangle) \leq 0$ if and only if it satisfies

$$|\langle r, x\rangle| \geq |2\mu_0 - \langle d_1 + d_2, x\rangle| \tag{24}$$

$\diamondsuit$

The following result is a consequence of Remark 3.3(ii) and Propositions 2.3 and 3.2.

Proposition 3.5. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{F} \subset \mathbb{E}$ defined as in (23). Let $\mathbb{D}_i^+ := \{x \in \mathbb{K} : \langle d_i, x\rangle \geq \mu_0\}$ and $\mathbb{D}_i^- := \{x \in \mathbb{K} : \langle d_i, x\rangle \leq \mu_0\}$ for $i \in \{1, 2\}$.*

i. Any point $x \in \mathbb{K} \cap \mathbb{F}$ satisfies

$$f_{\mathbb{K},r}(x) \geq |2\mu_0 - \langle d_1 + d_2, x\rangle|. \tag{25}$$

ii. Suppose $\overline{\text{conv}}(\mathbb{D}_1^+ \cup \mathbb{D}_2^+) = \mathbb{K}$ *or the sets* $\mathbb{D}_1^+$ *and* $\mathbb{D}_2^+$ *satisfy the conditions of Lemma 2.2. Suppose also that* $\overline{\text{conv}}(\mathbb{D}_1^- \cup \mathbb{D}_2^-) = \mathbb{K}$ *or the sets* $\mathbb{D}_1^-$ *and* $\mathbb{D}_2^-$ *satisfy the conditions of Lemma 2.2. Then*

$$\overline{\text{conv}}(\mathbb{K} \cap \mathbb{F}) = \{x \in \mathbb{K} : \ f_{\mathbb{K},r}(x) \geq |2\mu_0 - \langle d_1 + d_2, x\rangle|\} \,. \tag{26}$$

Proof. Note that $\mathbb{K} \cap \mathbb{F} = (\mathbb{D}_1^+ \cup \mathbb{D}_2^+) \cap (\mathbb{D}_1^- \cup \mathbb{D}_2^-)$. Using Proposition 3.2 for $\mathbb{D}_1^+ \cup \mathbb{D}_2^+$ and $\mathbb{D}_1^- \cup \mathbb{D}_2^-$ shows that the inequalities $f_{\mathbb{K},r}(x) \geq 2\mu_0 - \langle d_1 + d_2, x\rangle$ and $f_{\mathbb{K},-r}(x) \geq -2\mu_0 + \langle d_1 + d_2, x\rangle$ are both valid for $\mathbb{K} \cap \mathbb{F}$. By Remark 3.3(ii), $f_{\mathbb{K},-r}(x) = f_{\mathbb{K},r}(x)$ for any $r \in \mathbb{E}$ and $x \in \mathbb{K}$. Therefore, the two inequalities together are equivalent to (25). Under the hypotheses of statement (*ii*), we have

$$\begin{aligned}
\overline{\text{conv}}(\mathbb{D}_1^+ \cup \mathbb{D}_2^+) &= \{x \in \mathbb{K} : \ f_{\mathbb{K},r}(x) \geq 2\mu_0 - \langle d_1 + d_2, x\rangle\} \ \text{ and} \\
\overline{\text{conv}}(\mathbb{D}_1^- \cup \mathbb{D}_2^-) &= \{x \in \mathbb{K} : \ f_{\mathbb{K},-r}(x) \geq -2\mu_0 + \langle d_1 + d_2, x\rangle\} \,.
\end{aligned}$$

Then Proposition 2.3 shows (26). □

The next proposition shows that the linear inequality in (17) can be replaced with a linear equality when we consider the intersection of $\mathbb{K}$ with a rank-two nonconvex quadratic instead of a two-term disjunction.

Proposition 3.6. *Let* $\mathbb{E} = \prod_{j=1}^p \mathbb{E}^j$ *be a direct product of finite-dimensional Euclidean spaces. Suppose* $\mathbb{K} = \prod_{j=1}^p \mathbb{K}^j$ *and each* $\mathbb{K}^j \subset \mathbb{E}^j$ *is a regular cone. A point* $x \in \mathbb{K}$ *satisfies (25) if and only if there exist* $z^j \in \mathbb{R}$, $j \in [p]$, *such that (17a) (or, equivalently (18)) holds together with* $\sum_{j=1}^p z^j = \mu_0$.

Proof. Fix $x \in \mathbb{K}$. The "if" part follows from the triangle inequality. To show the "only if" part, recall from Proposition 3.3(ii) that x satisfies $f_{\mathbb{K}^j,r^j}(x^j) \geq 2\mu_0 - \langle d_1^j + d_2^j, x^j\rangle$ if and only if there exist $t_1^j \in \mathbb{R}$, $j \in [p]$, such that

$$f_{\mathbb{K}^j,r^j}(x^j) \geq |2t_1^j - \langle d_1^j + d_2^j, x^j\rangle| \quad \forall j \in [p], \tag{27a}$$

$$\sum_{j=1}^p t_1^j \geq \mu_0. \tag{27b}$$

Furthermore, x satisfies $f_{\mathbb{K}^j,r^j}(x^j) \geq -2\mu_0 + \langle d_1^j + d_2^j, x^j\rangle$ if and only if there exist $t_2^j \in \mathbb{R}$, $j \in [p]$, such that

$$f_{\mathbb{K}^j,r^j}(x^j) \geq |-2t_2^j + \langle d_1^j + d_2^j, x^j\rangle| \quad \forall j \in [p], \tag{28a}$$

$$-\sum_{j=1}^p t_2^j \geq -\mu_0. \tag{28b}$$

Let $0 \leq \delta \leq 1$ such that $\delta \sum_{j=1}^p t_1^j - (1-\delta)\sum_{j=1}^p t_2^j = \mu_0$. For all $j \in [p]$, we also define $z^j := \delta t_1^j - (1-\delta)t_2^j$. Then $\sum_{j=1}^p z^j = \mu_0$. For any $j \in [p]$, combining (27a) and (28a) with weights δ and $1-\delta$, we have

$$\begin{aligned}
f_{\mathbb{K}^j,r^j}(x^j) &\geq \delta|2t_1^j - \langle d_1^j + d_2^j, x^j\rangle| + (1-\delta)|-2t_2^j + \langle d_1^j + d_2^j, x^j\rangle| \\
&= \delta|2t_1^j - \langle d_1^j + d_2^j, x^j\rangle| + (1-\delta)|2t_2^j - \langle d_1^j + d_2^j, x^j\rangle| \\
&\geq |2z^j - \langle d_1^j + d_2^j, x^j\rangle|,
\end{aligned}$$

where the second inequality holds because the function $z \mapsto |2z - \langle d_1^j + d_2^j, x^j \rangle| : \mathbb{R} \to \mathbb{R}$ is convex. This completes the proof of the first part. Finally, we note that the equivalence of (17a) to $[f_{\mathbb{K}^j, r^j}(x^j)]^2 - \langle r^j, x^j \rangle_j^2 \geq 4(z^j - \langle d_1^j, x^j \rangle_j)(z^j - \langle d_2^j, x^j \rangle_j)$ can be shown as in the proof of Proposition 3.3. □

We close this section by presenting a result which complements the relation between convex hulls of nonconvex quadratic sets of form $\mathbb{K} \cap \mathbb{F}$ and the associated disjunctions given in Proposition 3.5. In particular, we show that given a structured and explicit characterization of the closed convex hull of $\mathbb{F} \cap \mathbb{K}$), we can obtain a convex hull characterization of $\mathbb{D}_1 \cup \mathbb{D}_2$ even when the disjointness assumption is violated.

Proposition 3.7. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{D}_1, \mathbb{D}_2 \subset \mathbb{E}$ defined as in (5) and $\mathbb{F} \subset \mathbb{E}$ defined as in (23). Let $g(x) : \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous, concave function such that $g(x) \geq 0$ for any $x \in \mathbb{K}$ and $\mathbb{K} \cap \mathbb{F} \subseteq \{x \in \mathbb{K} : \ g(x) \geq |2\mu_0 - \langle d_1 + d_2, x \rangle|\}$.*

i. Any point $x \in \mathbb{D}_1 \cup \mathbb{D}_2$ satisfies the convex inequality

$$g(x) \geq 2\mu_0 - \langle d_1 + d_2, x \rangle. \tag{29}$$

ii. If $\overline{\text{conv}}(\mathbb{K} \cap \mathbb{F}) = \{x \in \mathbb{K} : \ g(x) \geq |2\mu_0 - \langle d_1 + d_2, x \rangle|\}$, then

$$\overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) = \{x \in \mathbb{K} : \ g(x) \geq 2\mu_0 - \langle d_1 + d_2, x \rangle\}. \tag{30}$$

Proof. Note that $\mathbb{D}_1 \cup \mathbb{D}_2 = (\mathbb{K} \cap \mathbb{F}) \cup (\mathbb{D}_1 \cap \mathbb{D}_2)$. Our hypotheses ensure that any $x \in \mathbb{K} \cap \mathbb{F}$ satisfies (29). Moreover, for any $x \in \mathbb{D}_1 \cap \mathbb{D}_2$, we have $0 \geq 2\mu_0 - \langle d_1 + d_2, x \rangle$. Then (29) is valid for $\mathbb{D}_1 \cap \mathbb{D}_2$ because $g(\cdot)$ is nonnegative at $\mathbb{K}$.

Statement (i), together with the concavity of $g(\cdot)$, shows that (29) is valid for $\text{conv}(\mathbb{D}_1 \cup \mathbb{D}_2)$. The continuity of $g(\cdot)$ implies the validity of (29) for $\overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2)$. If $\overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) = \mathbb{K}$, then (29) is redundant. Suppose $\overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) \neq \mathbb{K}$. Assume for contradiction that there exists $\bar{x} \in \mathbb{K}$ satisfying (29) but $\bar{x} \notin \overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2)$. Then $\bar{x} \notin \overline{\text{conv}}(\mathbb{K} \cap \mathbb{F})$ as well; thus $g(\bar{x}) < |2\mu_0 - \langle d_1 + d_2, \bar{x} \rangle|$. Combining this with (29), we arrive at

$$|2\mu_0 - \langle d_1 + d_2, \bar{x} \rangle| > g(\bar{x}) \geq 2\mu_0 - \langle d_1 + d_2, \bar{x} \rangle,$$

which implies $0 > 2\mu_0 - \langle d_1 + d_2, \bar{x} \rangle$. Then at least one of $0 > \mu_0 - \langle d_1, \bar{x} \rangle$ or $0 > \mu_0 - \langle d_2, \bar{x} \rangle$ must hold. Hence, $\bar{x} \in \mathbb{D}_1 \cup \mathbb{D}_2$, contradicting the assumption $\bar{x} \notin \overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2)$. This proves the relation stated in (30). □

3.3 Inequalities for Disjoint Two-Term Disjunctions

As in Section 3.1, we consider sets $\mathbb{D}_1$ and $\mathbb{D}_2$ defined as in (5). In the case of $\mathbb{K} = \mathbb{L}^n$, it was identified in [30] that when the interiors of the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ do not intersect, the convex inequality (10) can be expressed in an equivalent second-order conic form without recourse to auxiliary variables. Such a condition turns out to be relevant in our current setup for general $\mathbb{K}$ as well. In this section, we assume $\{x \in \mathbb{K} : \ \langle d_1, x \rangle > \mu_0, \ \langle d_2, x \rangle > \mu_0\} = \emptyset$. Whenever this is the case, we say that $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the *disjointness condition.* Such sets $\mathbb{D}_1$ and $\mathbb{D}_2$ are naturally associated with rank-two quadratic constraints: In particular, under the disjointness condition, $\mathbb{D}_1 \cup \mathbb{D}_2 = \mathbb{K} \cap \mathbb{F}$ where $\mathbb{F}$ is defined as in (23). Therefore, we can immediately use the results of Section 3.2 in this case. Specifically, we have the following result.

Corollary 3.8. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Suppose $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the disjointness condition. Then a point $x \in \mathbb{K}$ satisfies (10) if and only if it satisfies (25).*

Proof. Under the disjointness condition, any point $x \in \mathbb{K}$ satisfies the disjunction $\langle d_1, x\rangle \le \mu_0 \vee \langle d_2, x\rangle \le \mu_0$. Therefore, the inequality $f_{\mathbb{K},-r}(x) \ge -2\mu_0 + \langle d_1 + d_2, x\rangle$ is valid for the whole of $\mathbb{K}$. Recall from Remark 3.3(ii) that $f_{\mathbb{K},-r}(x) = f_{\mathbb{K},r}(x)$ for any $r \in \mathbb{E}$ and $x \in \mathbb{K}$. Hence, (10) and (25) are equivalent for any $x \in \mathbb{K}$. ☐

For instance, when $\mathbb{K} = \mathbb{L}^n$, $r \notin \pm\mathbb{L}^n$, and $\{x \in \mathbb{L}^n : \langle d_1, x\rangle > \mu_0, \langle d_2, x\rangle > \mu_0\} = \emptyset$, Corollary 3.8 and Proposition 3.6 show that any point $x \in \mathbb{K}$ satisfies (19) if and only if it satisfies $[f_{\mathbb{L}^n,r}(x)]^2 - \langle r, x\rangle^2 \ge 4(\mu_0 - \langle d_1, x\rangle)(\mu_0 - \langle d_2, x\rangle)$. Using Remark 3.12(ii), this inequality can be represented in second-order conic form as

$$\left(\|\tilde{r}\|^2 - r_n^2\right) x - 2(\langle d_1, x\rangle - \mu_0) \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n,$$

recovering [30, Proposition 3].

4 Two-Term Disjunctions on the Positive Semidefinite Cone

In this section, we concentrate our study on the case $\mathbb{E} = \mathbb{S}^n$ where $\mathbb{S}^n$ is the space of symmetric $n \times n$ matrices with real entries. We assume that $\mathbb{S}^n$ is equipped with the (Frobenius) inner product $\langle A, X\rangle = \mathrm{Tr}(AX)$. From now on, we distinguish between the elements of $\mathbb{R}^n$ and $\mathbb{S}^n$: We denote the elements of $\mathbb{R}^n$ with lowercase letters and the elements of $\mathbb{S}^n$ with uppercase letters. With this notation, we have $\mathbb{S}^n = \{X \in \mathbb{R}^{n\times n} : \ X^\top = X\}$. We study sets arising from two-term disjunctions $\langle D_1, X\rangle \ge \mu_0 \vee \langle D_2, X\rangle \ge \mu_0$ on the positive semidefinite cone $\mathbb{K} = \mathbb{S}^n_+ = \{X \in \mathbb{S}^n : a^\top X a \ge 0 \ \forall a \in \mathbb{R}^n\}$. We rewrite the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ as

$$\mathbb{D}_i = \left\{X \in \mathbb{S}^n_+ : \quad \langle D_i, X\rangle \ge \mu_0\right\} \quad \text{for } i \in \{1, 2\}. \tag{31}$$

In addition, we consider the intersection $\mathbb{F} \cap \mathbb{S}^n_+$ where $\mathbb{F} \subset \mathbb{S}^n$ is the nonconvex set

$$\mathbb{F} = \{X \in \mathbb{S}^n : \quad (\mu_0 - \langle D_1, X\rangle)(\mu_0 - \langle D_2, X\rangle) \ge 0\}. \tag{32}$$

As in Section 3, we would like to develop structured nonlinear valid inequalities for $\mathbb{D}_1 \cup \mathbb{D}_2$ and $\mathbb{F} \cap \mathbb{S}^n_+$. Whenever we consider the sets $\mathbb{D}_1$ and $\mathbb{D}_2$, we are primarily interested in the cases where $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy Condition 2.1. Hence, we assume $R = D_2 - D_1 \notin \pm\mathbb{S}^n_+$ when necessary.

Given a matrix $A \in \mathbb{R}^{n\times n}$, we let $\lambda(A)$ denote the vector of the eigenvalues of A arranged in nonincreasing order and $\lambda_i(A)$ denote its i-th eigenvalue. If $A \in \mathbb{S}^n$, then the eigenvalues of A are real. Furthermore, $A \in \mathbb{S}^n$ is positive semidefinite (resp. positive definite) if and only if $\lambda_i(A) \ge 0$ (resp. $\lambda_i(A) > 0$) for all $i \in [n]$. As a reminder, the dual cone of $\mathbb{S}^n_+$ is again $\mathbb{S}^n_+$. Given a matrix $A \in \mathbb{R}^{n\times n}$ and $\mathbb{J} \subset [n]$, we let $A[\mathbb{J}]$ denote the principal submatrix of A whose rows and columns are indexed by the elements of $\mathbb{J}$. We let $I_n \in \mathbb{S}^n$ represent the $n \times n$ identity matrix.

4.1 A Transformation to Simplify Disjunctions on the Positive Semidefinite Cone

In this section, we establish a linear correspondence which reduces the closed convex hull description of any two-term disjunction on $\mathbb{S}^n_+$ to the closed convex hull description of an associated disjunction for which the matrix $R = D_2 - D_1$ is diagonal. We first prove the following more general result.

Proposition 4.1. *Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be a linear map. Consider $\mathbb{C}_1, \mathbb{C}_2 \subset \mathbb{S}^n$ defined as $\mathbb{C}_i := \{X \in \mathbb{S}^n_+ : \mathcal{A}X = b, \langle C_i, X\rangle \geq c_{i,0}\}$. Let $Q \in \operatorname{int}(\mathbb{S}^n_+)$ and $U \in \mathbb{R}^{n\times n}$ be a diagonal matrix and an orthogonal matrix, respectively. Define the linear map $\mathcal{A}' : \mathbb{S}^n \to \mathbb{R}^m$ as $\mathcal{A}'X := \mathcal{A}UQXQU^\top$, the matrices $C'_i := QU^\top C_i UQ$, and the sets $\mathbb{C}'_i := \{X \in \mathbb{S}^n_+ : \mathcal{A}'X = b, \langle C'_i, X\rangle \geq c_{i,0}\}$ for $i \in \{1, 2\}$. Then*

i. $\mathbb{C}_i = UQ\mathbb{C}'_i QU^\top$ *for* $i \in \{1, 2\}$,

ii. $\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) = UQ \operatorname{conv}(\mathbb{C}'_1 \cup \mathbb{C}'_2)QU^\top$.

iii. $\overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = UQ\, \overline{\operatorname{conv}}(\mathbb{C}'_1 \cup \mathbb{C}'_2)QU^\top$.

Proof. First we prove (i). Note that $C_i = UQ^{-1}C'_iQ^{-1}U^\top$ for $i \in \{1, 2\}$. We can write

$$\begin{aligned}
\mathbb{C}_i &= \big\{X \in \mathbb{S}^n_+ : \quad \mathcal{A}X = b, \quad \langle C_i, X\rangle \geq c_{i,0}\big\} \\
&= \Big\{UQYQU^\top \in \mathbb{S}^n_+ : \mathcal{A}UQYQU^\top = b, \big\langle UQ^{-1}C'_iQ^{-1}U^\top, UQYQU^\top\big\rangle \geq c_{i,0}\Big\} \\
&= \Big\{UQYQU^\top : \quad \mathcal{A}'Y = b, \quad \langle C'_i, Y\rangle \geq c_{i,0}, \quad Y \in \mathbb{S}^n_+\Big\} \\
&= UQ\mathbb{C}'_iQU^\top.
\end{aligned}$$

The third equality above uses the observation that $UQYQU^\top \in \mathbb{S}^n_+$ if and only $Y \in \mathbb{S}^n_+$, which is true because $QU^\top$ is a nonsingular matrix.

Statement (ii) follows from (i) and the observation that convex combinations are invariant under the linear transformations $X \mapsto UQXQU^\top : \mathbb{S}^n \to \mathbb{S}^n$ and $X \mapsto Q^{-1}U^\top XUQ^{-1} : \mathbb{S}^n \to \mathbb{S}^n$. Statement (iii) follows from (ii) and the observation that the linear transformations $X \mapsto UQXQU^\top : \mathbb{S}^n \to \mathbb{S}^n$ and $X \mapsto Q^{-1}U^\top XUQ^{-1} : \mathbb{S}^n \to \mathbb{S}^n$ are continuous. □

Corollary 4.2. *Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be a linear map. Consider $\mathbb{V}, \mathbb{X} \subset \mathbb{S}^n$ defined as $\mathbb{V} := \{X \in \mathbb{S}^n_+ : \mathcal{A}X = b\}$ and $\mathbb{X} := \{X \in \mathbb{S}^n : (c_{1,0} - \langle C_1, X\rangle)(c_{2,0} - \langle C_2, X\rangle) \geq 0\}$. Let $Q \in \operatorname{int}(\mathbb{S}^n_+)$ and $U \in \mathbb{R}^{n\times n}$ be a diagonal matrix and an orthogonal matrix, respectively. Define the linear map $\mathcal{A}' : \mathbb{S}^n \to \mathbb{R}^m$ as $\mathcal{A}'X := \mathcal{A}UQXQU^\top$, the matrices $C'_i := QU^\top C_iUQ$, and the sets $\mathbb{V}' := \{X \in \mathbb{S}^n_+ : \mathcal{A}'X = b\}$ and $\mathbb{X}' := \{X \in \mathbb{E} : (c_{1,0} - \langle C'_1, X\rangle)(c_{2,0} - \langle C'_2, X\rangle) \geq 0\}$. Then*

i. $\operatorname{conv}(\mathbb{V} \cap \mathbb{X}) = UQ \operatorname{conv}(\mathbb{V}' \cap \mathbb{X}')QU^\top$.

ii. $\overline{\operatorname{conv}}(\mathbb{V} \cap \mathbb{X}) = UQ\, \overline{\operatorname{conv}}(\mathbb{V}' \cap \mathbb{X}')QU^\top$.

Proof. For $i \in \{1, 2\}$, let $\mathbb{C}^+_i := \{X \in \mathbb{V} : \langle C_i, X\rangle \geq c_{i,0}\}$ and $\mathbb{C}^-_i := \{X \in \mathbb{V} : \langle C_i, X\rangle \leq c_{i,0}\}$. Similarly, define $(\mathbb{C}^+_i)' := \{X \in \mathbb{V}' : \langle C'_i, X\rangle \geq c_{i,0}\}$ and $(\mathbb{C}^-_i)' := \{X \in \mathbb{V}' : \langle C'_i, X\rangle \leq c_{i,0}\}$. Then $\mathbb{V}\cap\mathbb{X} = (\mathbb{C}^+_1 \cup \mathbb{C}^+_2) \cap (\mathbb{C}^-_1 \cup \mathbb{C}^-_2)$ and $\mathbb{V}'\cap\mathbb{X}' = ((\mathbb{C}^+_1)' \cup (\mathbb{C}^+_2)') \cap ((\mathbb{C}^-_1)' \cup (\mathbb{C}^-_2)')$. To prove statement (i), note that

$$\begin{aligned}
\operatorname{conv}(\mathbb{V} \cap \mathbb{X}) &= \operatorname{conv}(\mathbb{C}^+_1 \cup \mathbb{C}^+_2) \cap \operatorname{conv}(\mathbb{C}^-_1 \cup \mathbb{C}^-_2) \\
&= UQ\big[\operatorname{conv}((\mathbb{C}^+_1)' \cup (\mathbb{C}^+_2)') \cap \operatorname{conv}((\mathbb{C}^-_1)' \cup (\mathbb{C}^-_2)')\big]QU^\top \\
&= UQ \operatorname{conv}(\mathbb{V}' \cap \mathbb{X}')QU^\top.
\end{aligned}$$

The first and third equalities above hold as a result of Proposition 2.3; and the second equality follows from Proposition 4.1(ii). Statement (ii) follows similarly from the same results. □

Remark 4.1. Based on Proposition 4.1, we can assume without any loss of generality that the matrices $D_1, D_2 \in \mathbb{S}^n$ which define the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ are such that the matrix $R = D_2 - D_1$ is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. To see this, consider the eigenvalue decomposition of $R = U\Lambda U^\top$ where $U \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and $\Lambda \in \mathbb{S}^n$ is a diagonal matrix whose entries are the eigenvalues of R sorted in nonincreasing order. Let $Q \in \operatorname{int}(\mathbb{S}^n_+)$ be the diagonal matrix with diagonal entries $Q_{ii} = \frac{1}{\sqrt{|\Lambda_{ii}|}}$ if Λ_{ii} is nonzero and $Q_{ii} = 1$ otherwise. By Proposition 4.1(iii), we have $\overline{\operatorname{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) = UQ\,\overline{\operatorname{conv}}(\mathbb{D}'_1 \cup \mathbb{D}'_2)QU^\top$ where $\mathbb{D}'_i := \left\{X \in \mathbb{S}^n_+ : \langle D'_i, X\rangle \ge \mu_0\right\}$ and $D'_i := QU^\top D_i UQ$ for $i \in \{1, 2\}$. Furthermore, $R' := D'_2 - D'_1 = QU^\top RUQ = Q\Lambda Q$ is a diagonal matrix with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. When $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy Condition 2.1, Lemma 2.1 implies $R \notin \pm\mathbb{S}^n_+$, in which case R' has at least one diagonal entry equal to 1 and one diagonal entry equal to -1. Analogously, based on Corollary 4.2, we can assume that the matrices $D_1, D_2 \in \mathbb{S}^n$ which define $\mathbb{F}$ are such that the matrix $R = D_2 - D_1$ is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. $\diamondsuit$

In order to simplify the presentation of certain results in the rest of the paper, we sometimes make the assumption that R is a diagonal matrix whose diagonal elements are from $\{0, \pm 1\}$ and sorted in nonincreasing order. Proposition 4.1, Corollary 4.2, and Remark 4.1 show that this assumption is without any loss of generality.

4.2 General Two-Term Disjunctions on the Positive Semidefinite Cone

We specialize Proposition 3.2 to the case of $\mathbb{E} = \mathbb{S}^n$ and $\mathbb{K} = \mathbb{S}^n_+$ in Theorem 4.4. This result is based on the following lemma.

Lemma 4.3. *For any $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$, we have $f_{\mathbb{S}^n_+,R}(X) = \left\|\lambda\left(X^{1/2}RX^{1/2}\right)\right\|_1$.*

Proof. The dual cone of $\mathbb{S}^n_+$ is again $\mathbb{S}^n_+$. Hence, by Proposition 3.2, we have

$$f_{\mathbb{S}^n_+,R}(X) = \max_P \left\{\langle R, P\rangle :\ X - P \in \mathbb{S}^n_+,\ X + P \in \mathbb{S}^n_+\right\}.$$

First consider the case $X \in \operatorname{int}(\mathbb{S}^n_+)$. Then there exists a matrix $X^{1/2} \in \operatorname{int}(\mathbb{S}^n_+)$ such that $X = X^{1/2}X^{1/2}$. A matrix $P \in \mathbb{S}^n$ satisfies $X - P \in \mathbb{S}^n_+$ and $X + P \in \mathbb{S}^n_+$ if and only if it satisfies $I_n - X^{-1/2}PX^{-1/2} \in \mathbb{S}^n_+$ and $I_n + X^{-1/2}PX^{-1/2} \in \mathbb{S}^n_+$. Therefore, after introducing a new variable $Q := X^{-1/2}PX^{-1/2}$, we can write

$$\begin{aligned}
f_{\mathbb{S}^n_+,R}(X) &= \max_Q \left\{\left\langle R, X^{1/2}QX^{1/2}\right\rangle :\ I_n - Q \in \mathbb{S}^n_+,\ I_n + Q \in \mathbb{S}^n_+\right\}\\
&= \max_Q \left\{\left\langle X^{1/2}RX^{1/2}, Q\right\rangle :\ I_n - Q \in \mathbb{S}^n_+,\ I_n + Q \in \mathbb{S}^n_+\right\}\\
&= \max_Q \left\{\left\langle X^{1/2}RX^{1/2}, Q\right\rangle :\ \|\lambda(Q)\|_\infty \le 1\right\} = \left\|\lambda\left(X^{1/2}RX^{1/2}\right)\right\|_1.
\end{aligned}$$

Now consider the more general case $X \in \mathbb{S}^n_+$. For $\epsilon > 0$, let $X^\epsilon := X + \epsilon I_n$. Then $X^\epsilon \in \operatorname{int}(\mathbb{S}^n_+)$ and $\lambda_i\left((X^\epsilon)^{1/2}\right) = \sqrt{\lambda_i(X) + \epsilon}$ for all $i \in [n]$. Furthermore, $\lim_{\epsilon\downarrow 0}\left\|(X^\epsilon)^{1/2}R(X^\epsilon)^{1/2} - X^{1/2}RX^{1/2}\right\| = 0$. The function $A \mapsto \|\lambda(A)\|_1 : \mathbb{S}^n \to \mathbb{R}$ is convex and finite everywhere; therefore, it is continuous. It follows that $\lim_{\epsilon\downarrow 0}\left\|\lambda\left((X^\epsilon)^{1/2}R(X^\epsilon)^{1/2}\right)\right\|_1 = \left\|\lambda\left(X^{1/2}RX^{1/2}\right)\right\|_1$. On the other hand, according to Remark 3.2, the function $-f_{\mathbb{S}^n_+,R}(X)$ is a closed convex function of X; therefore,

$\lim_{\epsilon\downarrow 0} f_{\mathbb{S}^n_+,R}(X^\epsilon) = f_{\mathbb{S}^n_+,R}(X)$ (see, for instance, [23, Proposition B.1.2.5]). Putting these together, we get

$$f_{\mathbb{S}^n_+,R}(X) = \lim_{\epsilon\downarrow 0} f_{\mathbb{S}^n_+,R}(X^\epsilon) = \lim_{\epsilon\downarrow 0} \left\| \lambda\big((X^\epsilon)^{1/2} R (X^\epsilon)^{1/2}\big) \right\|_1 = \left\| \lambda\big(X^{1/2} R X^{1/2}\big) \right\|_1.$$

□

We note that, for any $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$, the eigenvalues of $X^{1/2} R X^{1/2}$ are real because it is real symmetric. Lemma 4.3 implies the following result.

Theorem 4.4. *Let $\mathbb{E} = \mathbb{S}^n$ and $\mathbb{K} = \mathbb{S}^n_+$. Then a point $X \in \mathbb{S}^n_+$ satisfies (8) if and only if it satisfies*

$$\left\| \lambda\big(X^{1/2} R X^{1/2}\big) \right\|_1 \geq 2\mu_0 - \langle D_1 + D_2, X \rangle. \tag{33}$$

Similarly, a point $X \in \mathbb{S}^n_+$ satisfies (25) if and only if it satisfies

$$\left\| \lambda\big(X^{1/2} R X^{1/2}\big) \right\|_1 \geq |2\mu_0 - \langle D_1 + D_2, X \rangle|. \tag{34}$$

Theorem 4.4 and Proposition 3.2 indicate that (33) is a valid convex inequality for $\mathbb{D}_1 \cup \mathbb{D}_2$, where $\mathbb{D}_1, \mathbb{D}_2 \subset \mathbb{S}^n_+$ are defined as in (31). Furthermore, if $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the conditions of Lemma 2.2, the inequality (33) describes the closed convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$, together with the cone constraint $X \in \mathbb{S}^n_+$. If $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the disjointness condition, then Corollary 3.8 shows that a point $X \in \mathbb{S}^n_+$ satisfies (33) if and only if it satisfies (34). On the other hand, Theorem 4.4 and Proposition 3.5(i) indicate that (34) is a valid convex inequality for $\mathbb{F} \cap \mathbb{S}^n_+$, where $\mathbb{F} \subset \mathbb{S}^n$ is defined as in (32). Furthermore, if $\mathbb{F}$ satisfies the conditions of Proposition 3.5(ii), then (34) describes the closed convex hull of $\mathbb{F} \cap \mathbb{S}^n_+$.

The lemma below can be used to simplify the term $\left\| \lambda\big(X^{1/2} R X^{1/2}\big) \right\|_1$ on the left-hand side of (33); we refer to [25, Theorem 1.3.22] for a proof of this result.

Lemma 4.5. *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $m \leq n$. Then the n eigenvalues of BA are the m eigenvalues of AB together with $n - m$ zeroes.*

Corollary 4.6. *For any $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$, we have $\lambda\big(X^{1/2} R X^{1/2}\big) = \lambda(RX)$. In particular:*

i. The eigenvalues of RX are real.

ii. $f_{\mathbb{S}^n_+,R}(X) = \|\lambda(RX)\|_1$.

Corollary 4.7. *Let $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$. Suppose R is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $\operatorname{supp}(R) \subset [n]$ be the set of indices of the nonzero elements of the diagonal of R. Then*

i. The eigenvalues of $R[\operatorname{supp}(R)]X[\operatorname{supp}(R)]$ are real,

ii.

$$\begin{aligned} f_{\mathbb{S}^n_+,R}(X) &= \left\| \lambda(X[\operatorname{supp}(R)]^{1/2} R[\operatorname{supp}(R)] X[\operatorname{supp}(R)]^{1/2}) \right\|_1 \\ &= \left\| \lambda(R[\operatorname{supp}(R)] X[\operatorname{supp}(R)]) \right\|_1. \end{aligned}$$

Proof. Let t^+, t^-, and t^0 be the number of diagonal elements of R which are equal to $+1$, -1, and 0, respectively. Then $t^+ + t^- = |\operatorname{supp}(R)|$. Let $P \in \mathbb{R}^{n\times(t^+ + t^-)}$ be the matrix whose i-th row is e^i if $i \in [t^+]$, e^{i-t^0} if $i \in [n] \setminus [t^+ + t^0]$, and the zero vector otherwise. Then $R = PR[\operatorname{supp}(R)]P^\top$ and

$$X^{1/2}RX^{1/2} = X^{1/2}PR[\operatorname{supp}(R)]P^\top X^{1/2}.$$

Note that the eigenvalues of $X^{1/2}PR[\operatorname{supp}(R)]P^\top X^{1/2}$ are real because it is real symmetric. By Lemma 4.5, the n eigenvalues of $X^{1/2}PR[\operatorname{supp}(R)]P^\top X^{1/2}$ are the $t^+ + t^-$ eigenvalues of $R[\operatorname{supp}(R)]P^\top XP = R[\operatorname{supp}(R)]X[\operatorname{supp}(R)]$ together with t^0 zeroes. Noting $X[\operatorname{supp}(R)] \in \mathbb{S}_+^{t^+ + t^-}$ and applying Lemma 4.5 again, we see that the eigenvalues of $R[\operatorname{supp}(R)]X[\operatorname{supp}(R)]$ are the same as the eigenvalues of $X[\operatorname{supp}(R)]^{1/2}R[\operatorname{supp}(R)]X[\operatorname{supp}(R)]^{1/2}$. $\square$

We use the next result in the proof of Lemma 4.9, which provides an alternate representation of $\|\lambda(X^{1/2}RX^{1/2})\|_1$.

Lemma 4.8. *Let $R \in \mathbb{S}^n$ and $X \in \mathbb{S}_+^n$. The number of positive (resp. negative) eigenvalues of $X^{1/2}RX^{1/2}$ is less than or equal to the number of positive (resp. negative) eigenvalues of R.*

Proof. Consider the eigenvalue decomposition of $X = U_x D_x U_x^\top$ with an orthogonal matrix U_x and a diagonal matrix D_x. Note $\lambda(X^{1/2}RX^{1/2}) = \lambda(D_x^{1/2}U_x R U_x^\top D_x^{1/2})$. Let I_x be a diagonal matrix which has $(I_x)_{ii} = (D_x)_{ii}$ if $(D_x)_{ii} > 0$ and $(I_x)_{ii} = 1$ if $(D_x)_{ii} = 0$. Let P_x be a diagonal matrix which has $(P_x)_{ii} = 1$ if $(D_x)_{ii} > 0$ and $(I_x)_{ii} = 0$ if $(D_x)_{ii} = 0$. Then $D_x^{1/2}U_x R U_x^\top D_x^{1/2} = P_x(I_x^{1/2}U_x R U_x^\top I_x^{1/2})P_x$. The matrix $I_x^{1/2}U_x R U_x^\top I_x^{1/2}$ has the same inertia as R because $I_x^{1/2}U_x$ is nonsingular. Because $P_x(I_x^{1/2}U_x R U_x^\top I_x^{1/2})P_x$ is a principal submatrix of $I_x^{1/2}U_x R U_x^\top I_x^{1/2}$, we deduce the result from Cauchy's interlacing eigenvalue theorem. $\square$

Lemma 4.9. *Let $R \in \mathbb{S}^n$ and $X \in \mathbb{S}_+^n$. Suppose $R \notin \pm\mathbb{S}_+^n$ and it is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $n^+ := \max\{k : R_{kk} = 1\}$, $n^- := \min\{k : R_{kk} = -1\}$, and $\mathbb{J} := \{(i,j) : 1 \le i \le n^+,\ n^- \le j \le n\}$. Then*

$$\left\|\lambda\left(X^{1/2}RX^{1/2}\right)\right\|_1 = \sqrt{\langle R, X\rangle^2 - 4\sum_{(i,j)\in\mathbb{J}} \lambda_i\left(X^{1/2}RX^{1/2}\right)\lambda_j\left(X^{1/2}RX^{1/2}\right)}.$$

Proof. Note that $\langle R, X\rangle = \operatorname{Tr}(RX) = \sum_{i=1}^n \lambda_i(RX) = \sum_{i=1}^n \lambda_i\left(X^{1/2}RX^{1/2}\right)$ where the last equality follows from Corollary 4.6. Furthermore, $X^{1/2}RX^{1/2}$ has at most n^+ positive and at most

$n - n^- + 1$ negative eigenvalues because of Lemma 4.8. Hence, we can write

$$
\begin{aligned}
\big\|\lambda\big(X^{1/2}RX^{1/2}\big)\big\|_1^2 - \langle R, X\rangle^2 &= \big\|\lambda\big(X^{1/2}RX^{1/2}\big)\big\|_1^2 - \Big(\sum_{i=1}^{n} \lambda_i\big(X^{1/2}RX^{1/2}\big)\Big)^2 \\
&= \Big[\sum_{i=1}^{n^+} \lambda_i\big(X^{1/2}RX^{1/2}\big) - \sum_{i=n^-}^{n} \lambda_i\big(X^{1/2}RX^{1/2}\big)\Big]^2 \\
&\qquad - \Big[\sum_{i=1}^{n^+} \lambda_i\big(X^{1/2}RX^{1/2}\big) + \sum_{i=n^-}^{n} \lambda_i\big(X^{1/2}RX^{1/2}\big)\Big]^2 \\
&= -4\Big[\sum_{i=1}^{n^+} \lambda_i\big(X^{1/2}RX^{1/2}\big)\Big]\Big[\sum_{i=n^-}^{n} \lambda_i\big(X^{1/2}RX^{1/2}\big)\Big] \\
&= -4 \sum_{(i,j)\in\mathbb{J}} \lambda_i\big(X^{1/2}RX^{1/2}\big)\lambda_j\big(X^{1/2}RX^{1/2}\big).
\end{aligned}
$$

The result follows from the nonnegativity of $\big\|\lambda\big(X^{1/2}RX^{1/2}\big)\big\|_1$. □

Lemmas 4.3 and 4.9, along with Propositions 3.3(ii) and 3.6(ii), have the following consequence.

Corollary 4.10. *Let $\mathbb{E} = \mathbb{S}^n$ and $\mathbb{K} = \mathbb{S}^n_+$. Suppose $R \notin \pm\mathbb{S}^n_+$ and it is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $n^+ := \max\{k : R_{kk} = 1\}$, $n^- := \min\{k : R_{kk} = -1\}$, and $\mathbb{J} := \{(i,j) : 1 \le i \le n^+,\ n^- \le j \le n\}$. Then a point $X \in \mathbb{S}^n_+$ satisfies (33) if and only if there exists $z \ge \mu_0$ such that*

$$
- \sum_{(i,j)\in\mathbb{J}} \lambda_i\big(X^{1/2}RX^{1/2}\big)\lambda_j\big(X^{1/2}RX^{1/2}\big) \ge (z - \langle D_1, X\rangle)(z - \langle D_2, X\rangle). \tag{35}
$$

Similarly, a point $X \in \mathbb{S}^n_+$ satisfies (34) if and only if it satisfies (35) together with $z = \mu_0$.

Proof. Lemmas 4.3 and 4.9 show

$$
\begin{aligned}
[f_{\mathbb{S}^n_+, R}(X)]^2 - \langle R, X\rangle^2 &= \|\lambda(X^{1/2}RX^{1/2})\|_1^2 - \langle R, X\rangle^2 \\
&= - \sum_{(i,j)\in\mathbb{J}} \lambda_i\big(X^{1/2}RX^{1/2}\big)\lambda_j\big(X^{1/2}RX^{1/2}\big).
\end{aligned}
$$

Then the two claims follow from Propositions 3.3(ii) and 3.6(ii), respectively. □

4.3 Elementary Disjunctions on the Positive Semidefinite Cone

Although it provides a closed-form equivalent for (8) in the case of disjunctions on the positive semidefinite cone, (33) can pose challenges from a computational perspective. In this section, we identify a class of two-term disjunctions for which (33) can be exactly represented in a tractable form.

We say that the disjunction $\langle D_1, X\rangle \ge \mu_0 \vee \langle D_2, X\rangle \ge \mu_0$ is *elementary* when the matrix $R = D_2 - D_1 \in \mathbb{S}^n$ has exactly one positive and one negative eigenvalue. In this section we consider sets $\mathbb{D}_1, \mathbb{D}_2 \subset \mathbb{S}^n_+$ which are defined by an elementary disjunction $\langle D_1, X\rangle \ge \mu_0 \vee \langle D_2, X\rangle \ge \mu_0$. By Remark 4.1, we assume without any loss of generality that R is diagonal and has exactly one positive entry $R_{11} = 1$ and one negative entry $R_{nn} = -1$. In this case, using Lemma 4.8, the matrix $X^{1/2}RX^{1/2}$ has at most one positive and at most one negative eigenvalue for any $X \in \mathbb{S}^n_+$. The largest and smallest eigenvalues of $X^{1/2}RX^{1/2}$ are

$$\lambda_1\big(X^{1/2}RX^{1/2}\big) = \frac{1}{2}\left(X_{11} - X_{nn} + \sqrt{(X_{11} - X_{nn})^2 + 4(X_{11}X_{nn} - X_{1n}^2)}\right), \tag{36a}$$

$$\lambda_n\big(X^{1/2}RX^{1/2}\big) = \frac{1}{2}\left(X_{11} - X_{nn} - \sqrt{(X_{11} - X_{nn})^2 + 4(X_{11}X_{nn} - X_{1n}^2)}\right). \tag{36b}$$

Hence, Lemma 4.3 and Theorem 4.4 reduce to the statement below for elementary disjunctions on the positive semidefinite cone.

Corollary 4.11. *Let $\mathbb{E} = \mathbb{S}^n$ and $\mathbb{K} = \mathbb{S}^n_+$. Suppose $R = D_2 - D_1$ is a diagonal matrix with exactly one positive entry $R_{11} = 1$ and one negative entry $R_{nn} = -1$. Then $f_{\mathbb{S}^n_+,R}(X) = \sqrt{(X_{11} - X_{nn})^2 + 4(X_{11}X_{nn} - X_{1n}^2)}$ for any $X \in \mathbb{S}^n_+$. Furthermore, a point $X \in \mathbb{S}^n_+$ satisfies (33) if and only if it satisfies*

$$\sqrt{(X_{11} - X_{nn})^2 + 4(X_{11}X_{nn} - X_{1n}^2)} \geq 2\mu_0 - \langle D_1 + D_2, X\rangle. \tag{37}$$

Proof. The proof follows from noting that $\|\lambda(X^{1/2}RX^{1/2})\|_1 = \lambda_1(X^{1/2}RX^{1/2}) - \lambda_n(X^{1/2}RX^{1/2})$ where $\lambda_1(X^{1/2}RX^{1/2})$ and $\lambda_n(X^{1/2}RX^{1/2})$ are as in (36). $\square$

Corollary 4.10 leads to equivalent second-order cone representations for (37) in the case of both disjoint and non-disjoint disjunctions.

Theorem 4.12. *Let $\mathbb{E} = \mathbb{S}^n$ and $\mathbb{K} = \mathbb{S}^n_+$. Suppose $R = D_2 - D_1$ is a diagonal matrix with exactly one positive entry $R_{11} = 1$ and one negative entry $R_{nn} = -1$. Then a point $X \in \mathbb{S}^n_+$ satisfies (33) if and only if there exists $z \geq \mu_0$ such that*

$$X[\{1,n\}] - (z - \langle D_1, X\rangle)R[\{1,n\}] \in \mathbb{S}^2_+. \tag{38}$$

Similarly, a point $X \in \mathbb{S}^n_+$ satisfies (34) if and only if it satisfies (38) together with $z = \mu_0$. Furthermore, the inequality (38) can be represented as a second-order cone constraint.

Proof. Fix $X \in \mathbb{S}^n_+$. The first part of Corollary 4.10 shows that X satisfies (33) if and only if there exists $z \geq \mu_0$ such that

$$(X_{11}X_{nn} - X_{1n}^2) \geq (z - \langle D_1, X\rangle)(z - \langle D_2, X\rangle).$$

This inequality can be rewritten as

$$\begin{aligned}
&[X_{11}X_{nn} - X_{1n}^2] \geq (z - \langle D_1, X\rangle)(z - \langle D_1, X\rangle - \langle R, X\rangle) \\
&\Leftrightarrow [X_{11}X_{nn} - X_{1n}^2] \geq (z - \langle D_1, X\rangle)^2 - (z - \langle D_1, X\rangle)[X_{11} - X_{nn}] \\
&\Leftrightarrow X_{11}X_{nn} + (z - \langle D_1, X\rangle)[X_{11} - X_{nn}] - (z - \langle D_1, X\rangle)^2 - X_{1n}^2 \geq 0 \\
&\Leftrightarrow [X_{11} - (z - \langle D_1, X\rangle)]\,[X_{nn} + (z - \langle D_1, X\rangle)] - X_{1n}^2 \geq 0.
\end{aligned} \tag{39}$$

The left-hand side of (39) is equal to the determinant of the matrix

$$\begin{pmatrix} X_{11} - (z - \langle D_1, X\rangle) & X_{1n} \\ X_{1n} & X_{nn} + (z - \langle D_1, X\rangle) \end{pmatrix}.$$

This matrix equals $X[\{1,n\}] - (z - \langle D_1, X\rangle)R[\{1,n\}]$ which also appears in (38).

To finish the proof, we show that the diagonal elements of the matrix on the left-hand side of (38) are nonnegative for any $X \in \mathbb{S}^n_+$ and $z \in \mathbb{R}$ which satisfy (39). That is, we show $X_{11} - (z - \langle D_1, X\rangle) \geq 0$ and $X_{nn} + (z - \langle D_1, X\rangle) \geq 0$. When X and z satisfy $\langle D_1, X\rangle = z$, the hypothesis that $X \in \mathbb{S}^n_+$ implies this immediately. Therefore, we can assume $\langle D_1, X\rangle \neq z$. Note that (39) implies

$$[X_{11} - (z - \langle D_1, X\rangle)]\,[X_{nn} + (z - \langle D_1, X\rangle)] \geq 0.$$

Because $\langle D_1, X\rangle \neq z$ and $X_{11}, X_{nn} \geq 0$ for $X \in \mathbb{S}^n_+$, at least one of the terms in the product above is positive; this also implies the nonnegativity of the other term. Hence, (39) is equivalent to (38) for any $X \in \mathbb{S}^n_+$ and $z \in \mathbb{R}$.

The second part of Corollary 4.10 shows that X satisfies (34) if and only if it satisfies (38) together with $z = \mu_0$. $\square$

Remark 4.2. Suppose the hypotheses of Theorem 4.12 are satisfied. Reversing the roles of D_1 and D_2 in the proof of Theorem 4.12, the inequality (38) can be equivalently represented as

$$X[\{1,n\}] + (z - \langle D_2, X\rangle)R[\{1,n\}] \in \mathbb{S}^2_+.$$

$\diamondsuit$

4.4 Low-Complexity Inequalities for General Disjunctions

In this section, in a spirit similar to Remark 3.7, we study structured conic inequalities valid for two-term disjunctions on $\mathbb{S}^n_+$. Section 4.3 showed that (33) admits an exact second-order cone representation when we consider elementary disjunctions on the positive semidefinite cone. However, the structure of (33) can be more complicated in the case of general two-term disjunctions. In this section, we introduce and discuss simpler conic inequalities which provide good relaxations to (33) at a significantly lower cost of computational complexity.

4.4.1 Relaxing the Inequality

We are going to use a classical result from matrix analysis to arrive at the results of this section. We state this result as Lemma 4.13 below; see [25, Theorem 1.2.16] for a proof.

Lemma 4.13. *Let $A \in \mathbb{R}^{n\times n}$. Then*

$$\sum_{1\leq i<j\leq n} \det(A[\{i,j\}]) = \sum_{1\leq i<j\leq n} \lambda_i(A)\lambda_j(A).$$

Using Lemma 4.13, we prove the following result.

Lemma 4.14. *Let $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$. Suppose $R \notin \pm\mathbb{S}^n_+$ and R is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $n^+ := \max\{k : R_{kk} = 1\}$, $n^- := \min\{k : R_{kk} = -1\}$, and $\mathbb{J} := \{(i,j) : 1 \leq i \leq n^+,\ n^- \leq j \leq n\}$. Then*

$$\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}]) \geq - \sum_{(i,j)\in\mathbb{J}} \lambda_i\big(X^{1/2}RX^{1/2}\big)\lambda_j\big(X^{1/2}RX^{1/2}\big). \qquad (40)$$

Proof. Let $Y := RX$. From Corollary 4.6, $\lambda(Y) = \lambda(X^{1/2}RX^{1/2})$; therefore, the right-hand side of (40) is exactly equal to $-\sum_{(i,j)\in\mathbb{J}} \lambda_i(Y)\lambda_j(Y)$. Define the sets $\mathbb{J}^+ := \{(i,j) : 1 \le i < j \le n^+\}$ and $\mathbb{J}^- := \{(i,j) : n^- \le i < j \le n\}$. Note that $\det(Y[\{i,j\}]) = \det(X[\{i,j\}])$ if $(i,j) \in \mathbb{J}^+ \cup \mathbb{J}^-$, $\det(Y[\{i,j\}]) = -\det(X[\{i,j\}])$ if $(i,j) \in \mathbb{J}$, and $\det(Y[\{i,j\}]) = 0$ otherwise. Furthermore, Y has at most n^+ positive and at most $n - n^- + 1$ negative eigenvalues. Then

$$\begin{aligned}
\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}]) &= -\sum_{(i,j)\in\mathbb{J}} \det(Y[\{i,j\}]) \\
&= -\sum_{1\le i<j\le n} \det(Y[\{i,j\}]) + \sum_{(i,j)\in\mathbb{J}^+} \det(Y[\{i,j\}]) + \sum_{(i,j)\in\mathbb{J}^-} \det(Y[\{i,j\}]) \\
&= -\sum_{1\le i<j\le n} \lambda_i(Y)\lambda_j(Y) + \sum_{(i,j)\in\mathbb{J}^+} \det(X[\{i,j\}]) + \sum_{(i,j)\in\mathbb{J}^-} \det(X[\{i,j\}]) \\
&= -\sum_{(i,j)\in\mathbb{J}} \lambda_i(Y)\lambda_j(Y) + \left[\sum_{(i,j)\in\mathbb{J}^+} \det(X[\{i,j\}]) - \sum_{(i,j)\in\mathbb{J}^+} \lambda_i(Y)\lambda_j(Y)\right] \\
&\qquad + \left[\sum_{(i,j)\in\mathbb{J}^-} \det(X[\{i,j\}]) - \sum_{(i,j)\in\mathbb{J}^-} \lambda_i(Y)\lambda_j(Y)\right].
\end{aligned}$$

In order to reach (40), we show

$$\sum_{(i,j)\in\mathbb{J}^+} \det(X[\{i,j\}]) \ge \sum_{(i,j)\in\mathbb{J}^+} \lambda_i(Y)\lambda_j(Y), \tag{41a}$$

$$\sum_{(i,j)\in\mathbb{J}^-} \det(X[\{i,j\}]) \ge \sum_{(i,j)\in\mathbb{J}^-} \lambda_i(Y)\lambda_j(Y). \tag{41b}$$

Let $P^+ \in \mathbb{S}^n_+$ be the diagonal matrix with diagonal entries $P^+_{ii} = 1$ if $i \in [n^+]$ and zero otherwise. Let $P^- \in \mathbb{S}^n_+$ be the matrix $P^- := P^+ - R$. Define $X^+ := P^+XP^+$ and $X^- := P^-XP^-$. Then $X^+, X^- \in \mathbb{S}^n_+$. Furthermore, X^+ (resp. X^-) has at most n^+ (resp. $n - n^- + 1$) nonzero (positive) eigenvalues. We first prove (41a). Note that

$$\begin{aligned}
\sum_{(i,j)\in\mathbb{J}^+} \det(X[\{i,j\}]) &= \sum_{1\le i<j\le n} \det(X^+[\{i,j\}]) = \sum_{1\le i<j\le n} \lambda_i(X^+)\lambda_j(X^+) \\
&= \sum_{(i,j)\in\mathbb{J}^+} \lambda_i(X^+)\lambda_j(X^+),
\end{aligned}$$

where the second equation follows from Lemma 4.13 and the last one from the fact that X^+ has at most n^+ positive eigenvalues. From $(P^+)^2 = P^+$ and Lemma 4.5, we have $\lambda(X^+) = \lambda(P^+XP^+) = \lambda(P^+X) = \lambda(X^{1/2}P^+X^{1/2})$. From Corollary 4.6, we have $\lambda(Y) = \lambda(X^{1/2}RX^{1/2})$. Note $X^{1/2}P^+X^{1/2} - X^{1/2}RX^{1/2} = X^{1/2}P^-X^{1/2} \in \mathbb{S}^n_+$; hence, $\lambda(X^{1/2}P^+X^{1/2}) \ge \lambda(X^{1/2}RX^{1/2})$. Note from Lemma 4.8 that $X^{1/2}RX^{1/2}$ has at most $n - n^- + 1$ negative eigenvalues; hence, the largest n^+ eigenvalues of $X^{1/2}RX^{1/2}$ are all nonnegative. Then we have $\sum_{(i,j)\in\mathbb{J}^+} \lambda_i(X^+)\lambda_j(X^+) \ge \sum_{(i,j)\in\mathbb{J}^+} \lambda_i(Y)\lambda_j(Y)$ because the first n^+ coordinates of both $\lambda(X^+)$ and $\lambda(Y)$ are nonnegative and $\lambda(X^+) \ge \lambda(Y)$. This proves (41a). The proof of (41b) follows in a similar manner. □

Remark 4.3. Suppose the hypotheses of Lemma 4.14 are satisfied. Then Remark 3.2(ii) and Lemmas 4.3, 4.9, and 4.14 imply that, for any $X \in \mathbb{S}^n_+$, we have

$$\sqrt{\langle R, X\rangle^2 + 4 \sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}])} \geq \|\lambda(X^{1/2}RX^{1/2})\|_1 \geq |\langle R, X\rangle|.$$

If the rank of $X \in \mathbb{S}^n_+$ is one, then $\det(X[\{i,j\}) = 0$ for all $(i,j) \in \mathbb{J}$; therefore, both inequalities above hold at equality. ♢

Remark 3.2 entails an attractive feature of the inequality (33): any rank-one matrix $X \in \mathbb{S}^n_+$ satisfies (33) if and only if $X \in \mathbb{D}_1 \cup \mathbb{D}_2$. Next we use Remark 4.3 to construct a relaxation of (33) which shares the same feature.

Proposition 4.15. *Let $\mathbb{E} = \mathbb{S}^n$ and $\mathbb{K} = \mathbb{S}^n_+$. Suppose $R \notin \pm\mathbb{S}^n_+$ and it is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $n^+ := \max\{k : R_{kk} = 1\}$, $n^- := \min\{k : R_{kk} = -1\}$, and $\mathbb{J} := \{(i,j) : 1 \leq i \leq n^+,\ n^- \leq j \leq n\}$. Let $g_{\mathbb{S}^n_+,R} : \mathbb{S}^n \to \mathbb{R} \cup \{-\infty\}$ be defined as*

$$g_{\mathbb{S}^n_+,R}(X) = \begin{cases} \sqrt{\langle R, X\rangle^2 + 4\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}])} & \text{if } X \in \mathbb{S}^n_+, \\ -\infty & \text{otherwise.} \end{cases}$$

i. Any point $X \in \mathbb{S}^n_+$ which satisfies (33) also satisfies

$$g_{\mathbb{S}^n_+,R}(X) \geq 2\mu_0 - \langle D_1 + D_2, X\rangle. \tag{42}$$

Similarly, any point $X \in \mathbb{S}^n_+$ which satisfies (34) also satisfies

$$g_{\mathbb{S}^n_+,R}(X) \geq |2\mu_0 - \langle D_1 + D_2, X\rangle|. \tag{43}$$

ii. Any point $X \in \mathbb{S}^n_+$ satisfies (42) if and only if there exists $z \geq \mu_0$ such that

$$\left[\sum_{i=1}^{n^+} X_{ii} - (z - \langle D_1, X\rangle)\right] \left[\sum_{j=n^-}^{n} X_{jj} + (z - \langle D_1, X\rangle)\right] \geq \sum_{(i,j)\in\mathbb{J}} X_{ij}^2, \tag{44a}$$

$$\sum_{i=1}^{n^+} X_{ii} - (z - \langle D_1, X\rangle) \geq 0, \qquad \sum_{j=n^-}^{n} X_{jj} + (z - \langle D_1, X\rangle) \geq 0. \tag{44b}$$

Similarly, any point $X \in \mathbb{S}^n_+$ satisfies (43) if and only if it satisfies (44) together with $z = \mu_0$. Furthermore, (44) can be represented as a single second-order cone constraint.

Proof. By Remark 4.3, $g_{\mathbb{S}^n_+,R}(X) \geq f_{\mathbb{S}^n_+,R}(X)$ for all $X \in \mathbb{S}^n_+$. Then statement (i) follows from Theorem 4.4. As in Proposition 3.3(ii), we can show that a point $X \in \mathbb{S}^n_+$ satisfies (42) if and only if there exists $z \geq \mu_0$ such that

$$\left[g_{\mathbb{S}^n_+,R}(X)\right]^2 - \langle R, X\rangle^2 \geq 4(z - \langle D_1, X\rangle)(z - \langle D_2, X\rangle). \tag{45}$$

Similarly, as in Proposition 3.6(ii), we can show that a point $X \in \mathbb{S}^n_+$ satisfies (43) if and only if it satisfies (45) together with $z = \mu_0$. We show that (45) can be represented as (44). The inequality

(45) is identical to $\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}]) \geq (z - \langle D_1, X\rangle)(z - \langle D_2, X\rangle)$. Following steps similar to those in the proof of Theorem 4.12, we rewrite it as

$$
\begin{aligned}
&\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}]) \geq (z - \langle D_1, X\rangle)(z - \langle D_1, X\rangle - \langle R, X\rangle) \\
\Leftrightarrow &\sum_{(i,j)\in\mathbb{J}} [X_{ii}X_{jj} - X_{ij}^2] \geq (z - \langle D_1, X\rangle)^2 - (z - \langle D_1, X\rangle)\left[\sum_{i=1}^{n^+} X_{ii} - \sum_{j=n^-}^{n} X_{jj}\right] \\
\Leftrightarrow &\left[\sum_{i=1}^{n^+} X_{ii} - (z - \langle D_1, X\rangle)\right]\left[\sum_{j=1}^{n^-} X_{jj} + (z - \langle D_1, X\rangle)\right] - \sum_{(i,j)\in\mathbb{J}} X_{ij}^2 \geq 0.
\end{aligned}
$$

The final form is the same as (44a). Furthermore, as in the proof of Theorem 4.12, we can show $\sum_{i=1}^{n^+} X_{ii} - (z - \langle D_1, X\rangle) \geq 0$ and $\sum_{j=n^-}^{n} X_{jj} + (z - \langle D_1, X\rangle) \geq 0$ for any $X \in \mathbb{S}_+^n$ and $z \in \mathbb{R}$ satisfying (44a). Observing that the inequalities (44) can be written as a rotated second-order cone constraint completes the proof. □

Remark 4.4. Under the hypotheses of Proposition 4.15, the inequality (42) defines a convex region in $\mathbb{S}_+^n$. To see this, note that the set of points satisfying (42) and $X \in \mathbb{S}_+^n$ is precisely the projection of the set of points satisfying (44) and $X \in \mathbb{S}_+^n$ onto the space of X variables. Because projection of a convex set is convex, this immediately proves the convexity of the region defined by (42) inside $\mathbb{S}_+^n$. ♢

Remark 4.5. Proposition 4.15 immediately implies the results presented in Section 4.3 because in the particular case of elementary disjunctions, (40) holds at equality. This can be seen by noting that $\mathbb{J}^+ = \mathbb{J}^- = \emptyset$ in the proof of Lemma 4.14. Therefore, in the case of elementary disjunctions, (42) does not only define a relaxation of (33); it is also equivalent to (33). Despite this connection, we have opted to keep Section 4.3 due to its more transparent derivation. ♢

Example 4.16. *Consider the split disjunction* $-\frac{1}{2}(X_{11}+X_{22}-X_{33}) \geq 1 \vee \frac{1}{2}(X_{11}+X_{22}-X_{33}) \geq 1$ *on* $\mathbb{S}_+^3$. *The sets* $\mathbb{D}_1$ *and* $\mathbb{D}_2$ *are defined as in* (31) *with* $D_1 := -\frac{1}{2}\left((e^1)(e^1)^\top + (e^2)(e^2)^\top - (e^3)(e^3)^\top\right)$, $D_2 := -D_1$, *and* $\mu_0 = 1$. *Proposition 4.15(ii) shows that the inequalities*

$$
\begin{gathered}
\left[\frac{1}{2}(X_{11}+X_{22}+X_{33}) - 1\right]\left[\frac{1}{2}(X_{11}+X_{22}+X_{33}) + 1\right] \geq X_{13}^2 + X_{23}^2, \\
\frac{1}{2}(X_{11}+X_{22}+X_{33}) - 1 \geq 0 \qquad \frac{1}{2}(X_{11}+X_{22}+X_{33}) + 1 \geq 0
\end{gathered}
$$

are valid for $\mathbb{D}_1 \cup \mathbb{D}_2$. *Furthermore, these inequalities can be represented as the second-order cone constraint*

$$
\begin{pmatrix} 2X_{13} \\ 2X_{23} \\ 2 \\ X_{11}+X_{22}+X_{33} \end{pmatrix} \in \mathbb{L}^4. \tag{46}
$$

Let $\mathbb{G}$ *denote the region defined by* (46). *Figure 1 shows the intersection of various two-dimensional linear spaces with* $\mathbb{D}_1 \cup \mathbb{D}_2$, $\mathbb{S}_+^3$, *and* $\mathbb{G}$. *Each two-dimensional linear space has the form* $\mathbb{W} :=$

$\{x\pi\pi^\top + y\psi\psi^\top : (x, y) \in \mathbb{R}^2\}$ *where* $\pi, \psi \in \mathbb{R}^3$ *are chosen such that* $\pi_1 = \frac{\sqrt{5}}{2}$, $\psi_3 = \sqrt{2}$, *and the remaining components of* π *and* ψ *are random numbers from the interval* $[-1, 1]$. *The intersection of* $\mathbb{W}$ *with* $\mathbb{S}^3_+$ *corresponds to the nonnegative orthant in the* (x, y) *space. Each image depicts the intersection of* $\mathbb{W}$ *with* $\mathbb{D}_1 \cup \mathbb{D}_2$ *(blue meshed area) and* $\mathbb{G}$ *(red unmeshed area) in the* (x, y) *space.*

We remind the reader that (46) *is valid for all of* $\mathbb{D}_1 \cup \mathbb{D}_2$ *and not just* $\mathbb{D}_1 \cup \mathbb{D}_2 \cap \mathbb{W}$. *Hence, even in the cases where* $\overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) = \mathbb{S}^3_+ \cap \mathbb{G}$, *we cannot in general expect to have* $\overline{\text{conv}}((\mathbb{D}_1 \cup \mathbb{D}_2) \cap \mathbb{W}) = \mathbb{S}^3_+ \cap \mathbb{G} \cap \mathbb{W}$. □

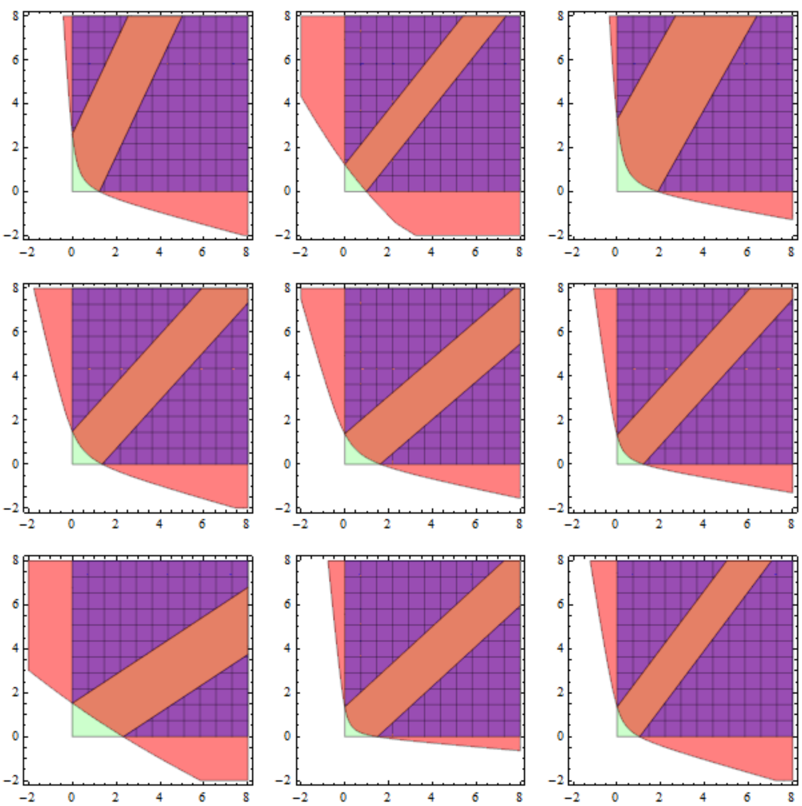

Figure 1: Sets associated with the disjunction $-\frac{1}{2}(X_{11} + X_{22} - X_{33}) \geq 1 \vee \frac{1}{2}(X_{11} + X_{22} - X_{33}) \geq 1$ on $\mathbb{S}^3_+$.

In the next remark, we discuss how we can utilize our results for elementary disjunctions in the light of Remark 3.7 to build structured relaxations of (33).

Remark 4.6. Suppose $R \notin \pm\mathbb{S}^n_+$ is a diagonal matrix with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $R_+, R_- \in \mathbb{S}^n_+$ and $R_1, \ldots, R_\ell \notin \pm\mathbb{S}^n_+$ be such that $R = R_+ - R_- + \sum_{k=1}^{\ell} R_k$ and $\text{rank}(R_k) = 2$. Remark 3.6 indicates that any $X \in \mathbb{D}_1 \cup \mathbb{D}_2$ satisfies the convex inequality

$$f_{\mathbb{S}^n_+, R_+}(X) + f_{\mathbb{S}^n_+, -R_-}(X) + \sum_{k=1}^{\ell} f_{\mathbb{S}^n_+, R_k}(X) \geq 2\mu_0 - \langle D_1 + D_2, X \rangle.$$

Note that, for any $X \in \mathbb{S}_+^n$, $f_{\mathbb{S}_+^n,R_+}(X) = \langle R_+, X\rangle$ and $f_{\mathbb{S}_+^n,-R_-}(X) = \langle R_-, X\rangle$. Now, for each $k \in [\ell]$, consider the eigenvalue decomposition of $R_k = U_k D_k U_k^\top$, and define $Q_k \in \operatorname{int}(\mathbb{S}_+^n)$ as in Remark 4.1. Then $J := Q_k U_k^\top R_k U_k Q_k$ is a diagonal matrix with exactly one positive entry $J_{11} = 1$ and exactly one negative entry $J_{nn} = -1$. Furthermore, Lemmas 4.3 and 4.5 show

$$\begin{aligned} f_{\mathbb{S}_+^n,R_k}(X) &= \big\|\lambda\big(R_k X\big)\big\|_1 = \big\|\lambda\big(J(Q_k^{-1}U_k^\top X U_k Q_k^{-1})\big)\big\|_1 \\ &= f_{\mathbb{S}_+^n,J}\big(Q_k^{-1}U_k^\top X U_k Q_k^{-1}\big). \end{aligned}$$

The function $f_{\mathbb{S}_+^n,J}(\cdot)$ has the form given in Corollary 4.11. It follows that any inequality constructed through this approach admits a second-order conic representation in a lifted space. We note that there is a lot of flexibility in the choice of the matrices R_+, R_-, and R_k and each selection will lead to a different valid inequality. $\diamondsuit$

Our simple numerical experimentation has demonstrated that neither the relaxed conic inequality from Proposition 4.15 nor the family of inequalities derived from the procedure described in Remark 4.6 dominates each other.

4.4.2 Relaxing the Disjunction

Another approach to using our results on elementary disjunctions for arbitrary two-term disjunctions might be through relaxing the underlying disjunction. To illustrate this point, consider a disjunction $\langle D_1, X\rangle \geq \mu_0 \vee \langle D_2, X\rangle \geq \mu_0$. Let $R_+, R_- \in \mathbb{S}_+^n$ be such that $R' := R - R_+ + R_- \notin \pm\mathbb{S}_+^n$ and has rank two. Define $D_1' := D_1 + R_-$ and $D_2' := D_2 + R_+$. The matrices D_1' and D_2' define a relaxation $\langle D_1', X\rangle \geq \mu_0 \vee \langle D_2', X\rangle \geq \mu_0$ of the original disjunction because any $X \in \mathbb{S}_+^n$ satisfying $\langle D_i, X\rangle \geq \mu_0$ also satisfies $\langle D_i', X\rangle \geq \mu_0$ for $i \in \{1,2\}$. Therefore, any inequality valid for the relaxed disjunction is also valid for the original. Because $R' \notin \pm\mathbb{S}_+^n$ and has rank two, it has exactly one positive and one negative eigenvalue. The relaxed disjunction is elementary, and the results of Section 4.3 can be used to derive structured nonlinear valid inequalities for $\mathbb{D}_1 \cup \mathbb{D}_2$. In particular, this approach leads to the inequality

$$\begin{aligned} & f_{\mathbb{S}_+^n,R'}(X) \geq 2\mu_0 - \langle D_1' + D_2', X\rangle = 2\mu_0 - \langle D_1 + D_2, X\rangle - \langle R_+ + R_-, X\rangle \\ \iff & \langle R_+ + R_-, X\rangle + f_{\mathbb{S}_+^n,R'}(X) \geq 2\mu_0 - \langle D_1 + D_2, X\rangle \\ \iff & f_{\mathbb{S}_+^n,R_+}(X) + f_{\mathbb{S}_+^n,-R_-}(X) + f_{\mathbb{S}_+^n,R'}(X) \geq 2\mu_0 - \langle D_1 + D_2, X\rangle. \end{aligned}$$

We note, however, that the inequality above can also be obtained through the approach outlined in Remark 4.6. Therefore, the approach of Remark 4.6 is a more powerful method to build structured relaxations of (33).

5 Conclusion

In this paper we have considered two-term disjunctions on a regular cone $\mathbb{K}$ and intersections of a regular cone $\mathbb{K}$ with rank-two nonconvex quadratics. These sets provide fundamental nonconvex relaxations for conic programs with integrality requirements and other types of nonconvex constraints. We have developed a general theory for constructing closed convex hull descriptions and low-complexity relaxations of such sets in the space of the original variables or using a small number of auxiliary variables. These relaxations can be used to strengthen convex relaxations of conic programs with nonconvex constraints. In the second part of the paper, we have specialized these results to the case where $\mathbb{K}$ is the positive semidefinite cone.

We note that our results immediately extend to cases where the base convex set is the intersection of a regular cone $\mathbb{K}$ with homogeneous half-spaces through [13, Lemma 5] (or its generalization given in [24, Lemma 3.6]) and to cases where it corresponds to certain cross-sections of $\mathbb{K}$ through [13, Lemma 7]. Nonetheless, studying closed convex hulls of disjunctions on general cross-sections of $\mathbb{K}$ is a topic of future research. Particular cross-sections of the positive semidefinite cone deserve specific interest from the point of view of combinatorial optimization. For instance, in the case of the maximum cut problem, it is well-known that the elliptope $\{X \in \mathbb{S}^n_+ : X_{ii} = 1 \ \forall i \in [n]\}$ provides a good outer approximation to the cut polytope, which is the convex hull of (± 1) characteristic vectors of all cuts in a complete graph on n vertices. Goemans and Williamson [20] used this observation to develop the approximation algorithm with the best known approximation guarantee for the maximum cut problem. Furthermore, the elliptope provides a valid integer programming formulation for the maximum cut problem in the sense that any $X \in \{\pm 1\}^{n\times n}$ in the elliptope corresponds to the characteristic vector of a cut. On this cross-section of the positive semidefinite cone, we can easily transform any two-term disjunction into an elementary disjunction. Thus, the results of Section 4.3 can be relevant. We hope that these results will be instrumental to the development of more practical algorithms for maximum cut and other hard combinatorial problems.

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