Reminder

- Quiz for Lecture 1 (9/1, 10pm)
- Quiz for Lecture 2 (9/3, 10pm)

- Paper Bidding (9/6, 10pm)
- Paper Reading Assignment 1 (9/13, 10pm)
  - Peer reviewed (Due 1 week after assignment due)
- Confirm group members for course project (9/13, 10pm)

Revisit Lec 1, Quiz 3

- Consider the following two LPs (LP-L and LP-R) where $b \geq 0$

LP-L

$$\min_{x,z} 1^T z$$

$$\text{s.t. } Ax + z = b$$

$$x, z \geq 0$$

LP-R

$$\min_{x} c^T x$$

$$\text{s.t. } Ax = b$$

$$x \geq 0$$

- Applying simplex algorithm to LP-L with the initial vertex $x_0 = 0$, $z_0 = b$. Denote the optimal solution as (x^*, z^*). If $z^* = 0$, then which of the following claims are true about x^*?
  - A: x^* is not in the feasible region of LP-R
  - B: x^* is in the feasible region of LP-R
  - C: x^* is a vertex of the feasible region of LP-R
  - D: x^* is an optimal solution of LP-R

Revisit Lec 1, Quiz 3

- Consider the following two LPs (LP-L and LP-R) where $b \geq 0$

LP-L

$$\begin{aligned} \min_{x,z} \; & 1^T z \\ \text{s.t. } & Ax + z = b \\ & x, z \geq 0 \end{aligned}$$

LP-R

$$\begin{aligned} \min_{x} \; & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}$$

- Denote the optimal solution of LP-L as (x^*, z^*). If $z^* = 0$, then x^* is a vertex of the feasible region of LP-R
- (x^*, z^*) is a vertex of feasible region of LP-L. LP-L has $m + n$ variables, $m (< n)$ equality constraints, $m + n$ inequality constraints (non-negative constraints). So the vertex is defined by the m equality constraints $Ax + z = b$ and make n non-negative constraints equality constraints, i.e., n of x and z variables have to be 0. Since $z^* = 0$ (m variables), we know that at least $m - n$ variables in x^* are 0. So x^* only has m non-zero values and satisfies $Ax^* = b$, which means x^* is a vertex of LP-R

Advanced Topics in
Machine Learning and Game Theory
Lecture 2: Introduction to Game Theory

17599/17759

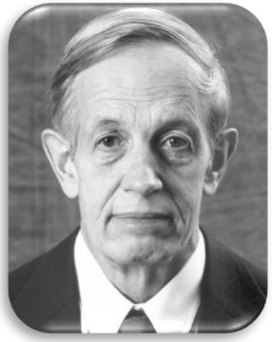
Fei Fang
feifang@cmu.edu

From Games to Game Theory

- The study of mathematical models of conflict and cooperation between intelligent decision makers
- Used in economics, political science etc

John von Neumann

John Nash



Heinrich Freiherr von Stackelberg

Winners of Nobel Memorial Prize in Economic Sciences

Outline

- Normal-Form Games
- Solution Concepts
- Linear Programming-based Equilibrium Computation
- Extensive-Form Games

Some Classical Games

- Rock-Paper-Scissors (RPS)
- Prisoner's Dilemma (PD)
  - If both Cooperate: 1 year in jail each
  - If one Defect, one Cooperate: 0 year for (D), 3 years for (C)
  - If both Defect: 2 years in jail each
- Football vs Concert (FvsC)
  - Historically known as Battle of Sexes
  - If football together: Alex ☺☺, Berry ☺
  - If concert together: Alex ☺, Berry ☺☺
  - If not together: Alex ☹, Berry ☹

Normal-Form Games

- A finite, n-player normal-form game is described by a tuple (N, A, u)
  - Set of players $N = \{1..n\}$
  - Set of joint actions $A = \prod_i A_i$
    - $\mathbf{a} = (a_1, \dots, a_n) \in A$ is an action profile
  - Payoffs / Utility functions $u_i: A \rightarrow \mathbb{R}$
    - $u_i(a_1, \dots, a_n)$ or $u_i(\mathbf{a})$
- Players move simultaneously and then game ends immediately
- Zero-Sum Game: $\sum_i u_i(\mathbf{a}) = 0, \forall \mathbf{a}$

May also be called matrix form, strategic form, or standard form

Payoff Matrix

- A two-player normal-form game with finite actions can be represented by a (bi)matrix
  - Player 1: Row player, Player 2: Column player
  - First number is the utility for Player 1, second for Player 2

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Q: What if we have more than 2 players?

Pure Strategy, Mixed Strategy, Support

- Pure strategy: choose an action deterministically
- Mixed strategy: choose action randomly
- Given action set A_i, player i's strategy set is $S_i = \Delta^{|A_i|}$

- Support: set of actions chosen with non-zero probability

- Let $s_i = \left(x_1, \ldots, x_{|A_i|}\right)^T$ where x_j is the probability of choosing the j^{th} action of player i, then
  - Pure strategy:
  - Mixed strategy:
  - Support$\triangleq$

Pure Strategy, Mixed Strategy, Support

- Pure strategy: choose an action deterministically
- Mixed strategy: choose action randomly
- Given action set A_i, player i's strategy set is $S_i = \Delta^{|A_i|}$

- Support: set of actions chosen with non-zero probability

- Let $s_i = \left(x_1, \ldots, x_{|A_i|}\right)^T$ where x_j is the probability of choosing the j^{th} action of player i, then
  - Pure strategy: $\exists j^*, x_{j^*} = 1$
  - Mixed strategy: $\exists j_1, j_2$ where $j_1 \neq j_2, x_{j_1} > 0, x_{j_2} > 0$
  - Support$\triangleq \{j: x_j > 0\}$

Expected Utility

- Given players' strategy profile $\mathbf{s} = (s_1, \ldots, s_n)$, what is the expected utility for each player?
- Let $s_i(a)$ be the probability of choosing action $a \in A_i$, then
  - $u_i(s_1, \ldots, s_n) =$

Expected Utility

- Given players' strategy profile $\mathbf{s} = (s_1, \dots, s_n)$, what is the expected utility for each player?
- Let $s_i(a)$ be the probability of choosing action $a \in A_i$, then
  - $u_i(s_1, \dots, s_n) = \sum_{\mathbf{a} \in \mathrm{A}} P(\mathbf{a}) u_i(\mathbf{a}) = \sum_{\mathbf{a} \in \mathrm{A}} u_i(\mathbf{a}) \prod_{i'} s_{i'}(a_{i'})$

Outline

- Normal-Form Games
- Solution Concepts
- Linear Programming-based Equilibrium Computation
- Extensive-Form Games

Best Response

- Let $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$.
- An action profile can be denoted as $\mathbf{a} = (a_i, a_{-i})$
- Similarly, define u_{-i} and s_{-i}

- Best Response: Set of actions or strategies leading to highest expected utility given the strategies or actions of other players
  - $a_i^* \in BR(a_{-i})$ iff
  - $s_i^* \in BR(s_{-i})$ iff

- Theorem (Nash 1951): A mixed strategy is BR iff all actions in the support are BR
  - $s_i \in BR(s_{-i})$ iff

Best Response

- Let $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots a_n)$.
- An action profile can be denoted as $\mathbf{a} = (a_i, a_{-i})$
- Similarly, define u_{-i} and s_{-i}

- Best Response: Set of actions or strategies leading to highest expected utility given the strategies or actions of other players
  - $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$
  - $s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$

- Theorem (Nash 1951): A mixed strategy is BR iff all actions in the support are BR
  - $s_i \in BR(s_{-i})$ iff $\forall a_i: s_i(a_i) > 0, a_i \in BR(s_{-i})$

Dominant Strategy

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

- Dominant Strategy
  - One strategy is always better/never worse/never worse and sometimes better than any other strategy
  - Focus on single player's strategy
  - Not always exist

s_i strictly dominates s_i' if

s_i very weakly dominates s_i' if

s_i weakly dominates s_i' if

s_i is a (strictly/very weakly/weakly) dominant strategy if it dominates $s_i', \forall s_i' \in S_i$

Dominant Strategy

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

- Dominant Strategy
  - One strategy is always better/never worse/never worse and sometimes better than any other strategy
  - Focus on single player's strategy
  - Not always exist

s_i strictly dominates s_i' if $\forall s_{-i}, u_i\ (s_i, s_{-i}) > u_i(s_i', s_{-i})$

s_i very weakly dominates s_i' if $\forall s_{-i}, u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$

s_i weakly dominates s_i' if $\forall s_{-i}, u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$
and $\exists s_{-i}, u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

s_i is a (strictly/very weakly/weakly) dominant strategy if it dominates $s_i', \forall s_i' \in S_i$

Dominant Strategy Equilibrium or Dominant Strategy Solution

- Dominant strategy equilibrium/solution
  - Every player plays a dominant strategy
  - Focus on strategy profile for all players
  - Not always exist
  - Can be found through enumerating pure strategies for each player

Q: Is there a dominant strategy equilibrium in the following game?

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

	c	d
a	2,1	4,0
b	1,0	3,2

Nash Equilibrium

- Nash Equilibrium (NE)
  - $\mathbf{s} = \langle s_1, \ldots, s_n \rangle$ is NE if $\forall i, s_i \in BR(s_{-i})$
  - Everyone's strategy is a BR to others' strategy profile
  - Focus on strategy profile for all players
  - One cannot gain by unilateral deviation
  - Pure Strategy Nash Equilibrium (PSNE)
    - $\mathbf{a} = \langle a_1, \ldots, a_n \rangle$ is PSNE if $\forall i, a_i \in BR(a_{-i})$
  - Mixed Strategy NE: at least one player use a mixed strategy

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Q: What are the PSNEs in this game?

Quiz 1

$\mathbf{s} = \langle s_1, \dots, s_n \rangle$ is NE if $\forall i, s_i \in BR(s_{-i})$

Is the following strategy profile an NE?
Alex: (2/3,1/3), Berry: (1/3,2/3)

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Quiz 1

$\mathbf{s} = \langle s_1, \ldots, s_n \rangle$ is NE if $\forall i, s_i \in BR(s_{-i})$

Is the following strategy profile an NE?
Alex: (2/3,1/3), Berry: (1/3,2/3)

$$u_A(s_A, s_B) = \frac{2}{3} * \frac{1}{3} * 2 + \frac{1}{3} * \frac{2}{3} * 1 = 2/3$$

$$u_A(F, s_B) = 2 * \frac{1}{3} = \frac{2}{3}$$

$$u_A(C, s_B) = 1 * \frac{2}{3} = \frac{2}{3}$$

So $u_A(s_A', s_B) = \epsilon u_A(F, s_B) + (1-\epsilon) u_A(C, s_B) = 2/3$

So Alex has no incentive to deviate (u_A cannot increase)

Similar reasoning goes for u_B

Berry

Alex		Football	Concert
	Football	2,1	0,0
	Concert	0,0	1,2

Nash Equilibrium

- Theorem (Nash 1951): NE always exists in finite games
  - Finite game: $n < \infty, |A| < \infty$
  - NE: pure or mixed

Maximin Strategy

- Maximin Strategy (applicable to multiplayer games)
  - Maximize worst case expected utility
  - Maximin strategy for player i is $\underset{s_i}{\operatorname{argmax}} \min_{s_{-i}} u_i(s_i, s_{-i})$
  - Maximin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$

    (Also called safety level)

  - Focus on single player's strategy
  - Can be computed through linear programming

Minimax Strategy

- Minimax Strategy in two-player games:
  - Minimize best case expected utility for the other player (just want to harm your opponent)
  - Minimax strategy for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$
  - Minimax value for player $-i$ is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$
  - Focus on single player's strategy
  - Can be computed through linear programming

Minimax Strategy

- Minimax Strategy in n-player games:
  - Coordinate with other players to minimize best case expected utility for a particular player (just want to harm that player)
  - Minimax strategy for player i against player j is i's component of s_{-j} in $\arg\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$
  - Minimax value for player j is $\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$
  - Focus on single player's strategy
  - Can be computed through linear programming (treating all players other than j as a meta-player)

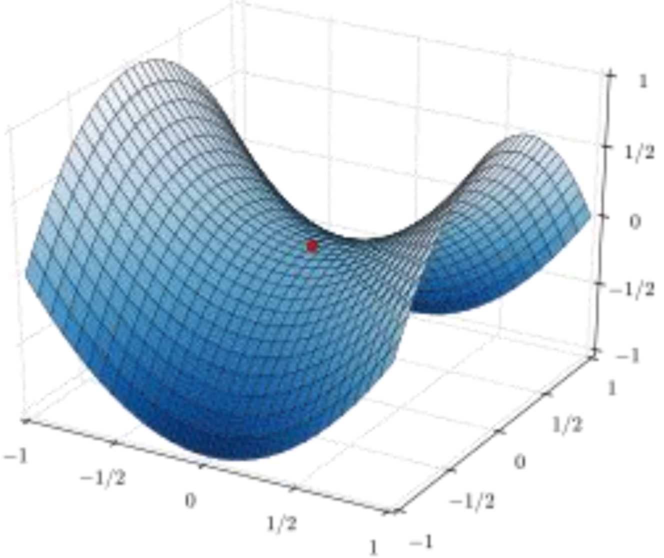
Minimax Theorem

- Theorem (von Neumann 1928, Nash 1951):
  - Informal: Minimax value=Maximin value=NE value in finite 2-player zero-sum games
  - Formally
    - $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$
    - $\exists v \in \mathbb{R}$ such that Player 1 can guarantee value at least v and Player 2 can guarantee loss at most v (v is called value of the game)
  - Indication: All NEs leads to the same utility profile in a finite two-player zero-sum game

Minimax Theorem

- Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ be compact convex sets
- If $f: X \times Y \rightarrow R$ is a continuous concave-convex function, i.e., $f(\cdot, y)$ is a concave function of x for any fixed y, $f(x,\cdot)$ is a convex function of y for any fixed x
- Then

$$\max_x \min_y f(x,y) = \min_y \max_x f(x,y)$$



$f(x,y) = x^2 - y^2$

https://en.wikipedia.org/wiki/Saddle_point

Power of Commitment

- NE utility=$(2,1)$
- If leader (player 1) commits to playing b, then player has to play d, leading to a utility of 3 for leader
- If leader (player 1) commits to playing a and b uniformly randomly, then player still has to play d, leading to a utility of 3.5 for leader

Player 2

Player 1

	c	d
a	2,1	4,0
b	1,0	3,2

Best Response Function

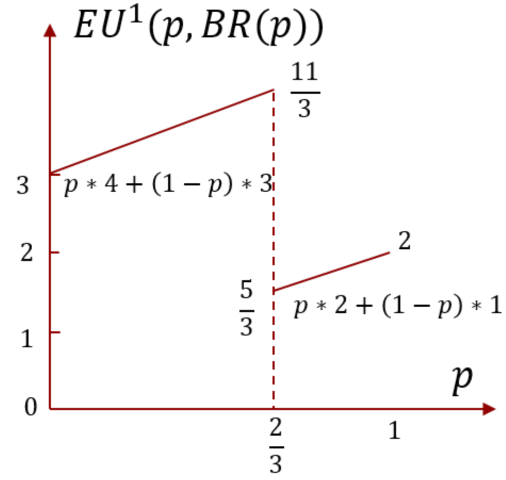
- Recall: Best response: Set of actions or strategies leading to highest expected utility given the strategies or actions of other players
  - $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$
  - $s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$

- Best Response Function
  - A mapping from a strategy of one player to a strategy of another player in the best response set
  - $f: S_1 \rightarrow S_2$ is a best response function iff $u_2\left(s_1, f(s_1)\right) \geq u_2(s_1, s_2), \forall s_1 \in S_1, s_2 \in S_2$. Or equivalently, $u_2\left(s_1, f(s_1)\right) \geq u_2(s_1, a_2), \forall s_1 \in S_1, a_2 \in A_2$

Stackelberg Equilibrium

		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2

- Stackelberg Equilibrium
  - Focus on strategy profile for all players
  - Follower responds according a best response function
  - $(s_1, f(s_1))$ is a Stackelberg Equilibrium iff
    - 1) f is a best response function
    - 2) $u_1\left(s_1, f(s_1)\right) \geq u_1\left(s_1', f(s_1')\right), \forall s_1' \in S_1$
  - There may exist many Stackelberg Equilibria due to different best response functions. For some best response functions, the Stackelberg Equilibrium may not exist



If $f\left(p = \frac{2}{3}\right) = d$, then SE is $s_1 = \left(\frac{2}{3}, \frac{1}{3}\right), s_2 = (0,1)$

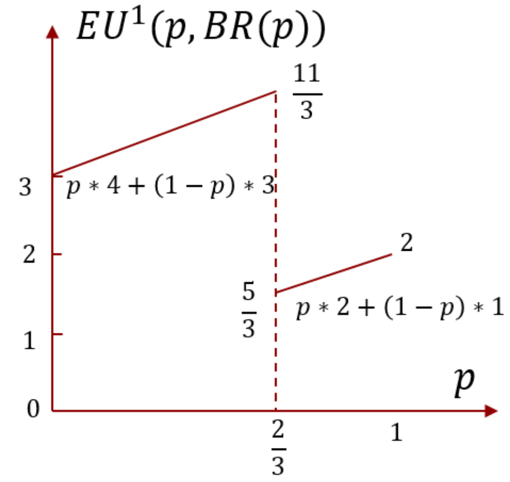
If $f\left(p = \frac{2}{3}\right) = c$, then SE does not exist

Quiz 2

Player 1

	c	d
a	2,1	4,0
b	1,0	3,2

- If the best response function breaks tie uniform randomly, does Stackelberg Equilibrium exist in this game?
- Yes
- No



Strong Stackelberg Equilibrium

- Strong Stackelberg Equilibrium (SSE)
  - Follower breaks tie in favor of the leader
  - $(s_1, f(s_1))$ is a Strong Stackelberg Equilibrium iff
    - 1) f is a best response function
    - 2) $f(s) \in \underset{s_2 \in BR(s)}{\operatorname{argmax}} u_1(s, s_2)$
    - 3) $u_1\big(s_1, f(s_1)\big) \geq u_1\big(s_1', f(s_1')\big), \forall s_1' \in S_1$
  - There may exist many SSEs but the leader's utility is the same in all these equilibria
  - Leader can induce the follower to breaks tie in favor of the leader by perturbing the strategy in the right direction
  - SSE always exist in two-player finite games

Outline

- Normal-Form Games
- Solution Concepts
- Linear Programming-based Equilibrium Computation
- Extensive-Form Games

Find All NEs (PSNE and Mixed Strategy NE)

- Special case: Two player, finite, zero-sum game
  - NE=Minimax=Maximin (Minimax theorem)
  - Solved by LP

- General case: PPAD-Complete (Chen & Deng, 2006)
  - Unlikely to have polynomial time algorithm
  - Conjecture: slightly easier than NP-Complete problems

- Two-player, general-sum bimatrix game: Support Enumeration Method

Compute Maximin Strategy

- For bimatrix games, maximin strategy can be computed through linear programming
- Let U_{ij}^1 be player 1's payoff value when player 1 choose action i and player 2 choose action j

Denote $s_1 = \langle x_1, \ldots, x_{|A_1|} \rangle$ where x_i is the probability of choosing the i^{th} action of player 1

Compute Maximin Strategy

- For bimatrix games, maximin strategy can be computed through linear programming
- Let U_{ij}^1 be player 1's payoff value when player 1 choose action i and player 2 choose action j

To get $\underset{s_1}{\operatorname{argmax}}\, \underset{s_2}{\min}\, u_1(s_1, s_2)$, we denote $s_1 = \langle x_1, \dots, x_{|A_1|} \rangle$ where x_i is the probability of choosing the i^{th} action of player 1. Now we need to find the value of x_i

$$\max_{x_1,\dots,x_{|A_1|}} \min_j \sum_i x_i U_{ij}^1$$

$$\text{s.t.} \sum_i x_i = 1$$

$$x_i \geq 0$$

Only need to check pure strategies. Recall the theorem of BR: A mixed strategy is BR iff all actions in the support are BR

Compute Maximin Strategy

- Convert to LP

$\mathcal{P}_1$

$$\max_x \min_j \sum_i x_i U_{ij}^1$$

$$\text{s.t. } \textstyle\sum_i x_i = 1$$

$$x_i \geq 0$$

$\mathcal{P}_2$ -- LP

$$\max_{x,v} v$$

$$\text{s.t. } v \leq \textstyle\sum_i x_i U_{ij}^1, \forall j$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

- Claim: x^* is optimal solution for $\mathcal{P}_1$ iff it is optimal solution for $\mathcal{P}_2$

Compute Maximin Strategy

- Convert to LP

$\mathcal{P}_1$

$$\max_x \min_j \sum_i x_i U^1_{ij}$$

$$\text{s.t. } \sum_i x_i = 1$$

$$x_i \geq 0$$

→

$\mathcal{P}_2$ -- LP

$$\max_{x,v} v$$

$$\text{s.t. } v \leq \sum_i x_i U^1_{ij}, \forall j$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

- Claim: x^* is optimal solution for $\mathcal{P}_1$ iff it is optimal solution for $\mathcal{P}_2$

Let U^1 be the payoff matrix for player 1 (row player). Then $\mathcal{P}_2$ can be rewritten in matrix form

$$\max_{\mathbf{x},v} v$$

$$\text{s.t. } v \leq \left(\mathbf{x}^{\mathrm{T}} U^1\right)_j, \forall j$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{1} = 1$$

$$\mathbf{x} \geq \mathbf{0}$$

Compute Maximin Strategy

$$\max_{x,v} v$$

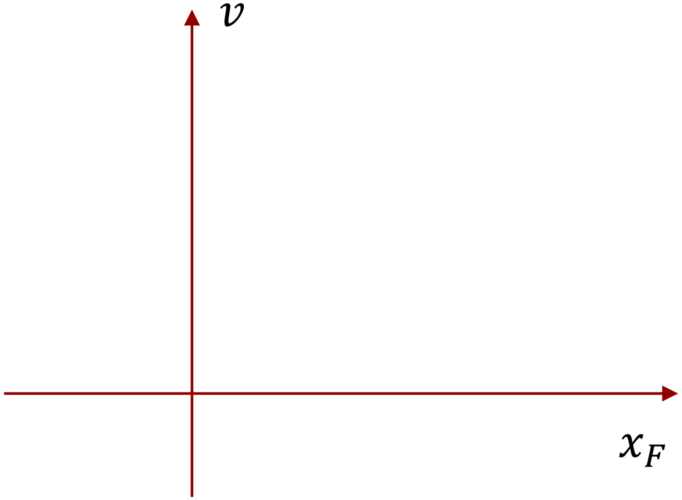
s.t. $v \leq \sum_i x_i U^1_{ij}, \forall j$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

Berry

Alex		Football	Concert
	Football	2,1	0,0
	Concert	0,0	1,2



Compute Maximin Strategy

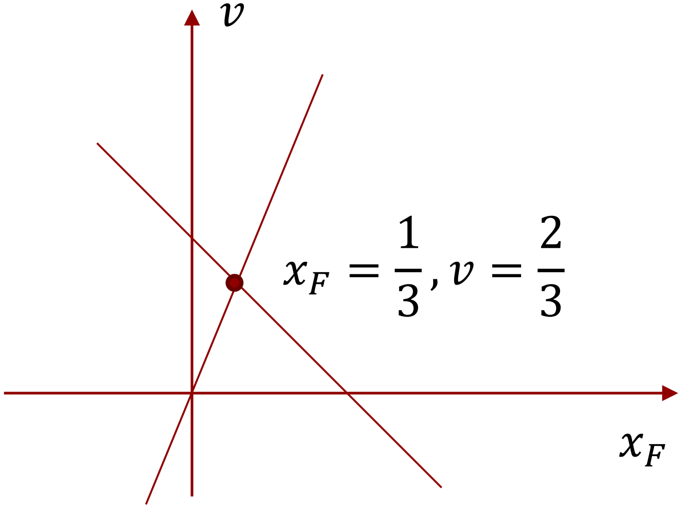
$$\max_{x,v} v$$
$$\text{s.t. } v \le \sum_i x_i U_{ij}^1, \forall j$$
$$\sum_i x_i = 1$$
$$x_i \ge 0$$

$$\max_{x_F, x_C, v} \ v$$
$$\text{s.t. } v \le x_F * 2 + x_C * 0$$
$$v \le x_F * 0 + x_C * 1$$
$$x_F + x_C = 1$$
$$x_F \ge 0, x_C \ge 0$$

$$\max_{x_F, v} v$$
$$\text{s.t. } v \le 2x_F$$
$$v \le 1 - x_F$$
$$0 \le x_F \le 1$$

Berry

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2



$$x_F = \frac{1}{3}, v = \frac{2}{3}$$

Compute Minimax Strategy

$\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

- For bimatrix games, minimax strategy can be computed through linear programming
- Let U_{ij}^2 be player 2's payoff value when player 1 choose action i and player 2 choose action j. Denote $s_1 = \langle x_1, \dots, x_{|A_1|} \rangle$ where x_i is the probability of choosing the i^{th} action of player 1. Then the minimax strategy can be found through solving the following LP

$$\min_{x,v} v$$

$$\text{s.t. } v \geq \sum_i x_i U_{ij}^2, \forall j$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

Quiz 3

- What is the minimax value for player 2 in the following game?
  - A: 1/3
  - B: 2/3
  - C: 0
  - D: 1

Berry

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

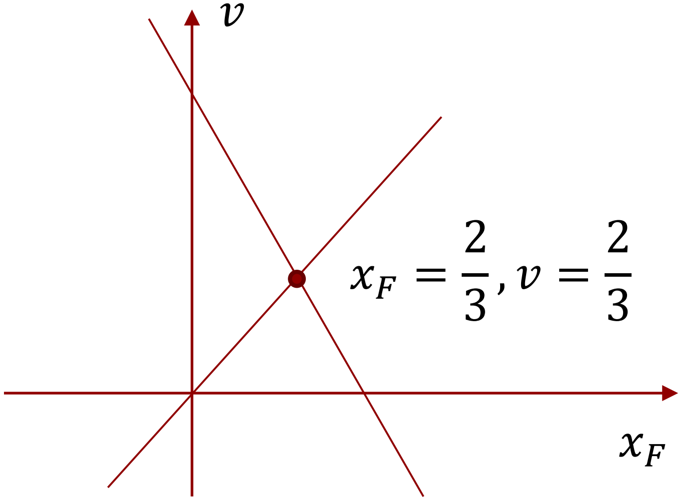
Quiz 3

$$\min_{x,v} v$$
$$\text{s.t. } v \geq \sum_i x_i U_{ij}^2, \forall j$$
$$\sum_i x_i = 1$$
$$x_i \geq 0$$

$$\min_{x_F, x_C, v} \; v$$
$$\text{s.t. } v \geq x_F * 1 + x_C * 0$$
$$v \geq x_F * 0 + x_C * 2$$
$$x_F + x_C = 1$$
$$x_F \geq 0, x_C \geq 0$$

$$\min_{x_F, v} v$$
$$\text{s.t. } v \geq x_F$$
$$v \geq 2(1 - x_F)$$
$$0 \leq x_F \leq 1$$

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2



Find All NEs

- Recall: A mixed strategy is BR iff all actions in the support are BR
- To find all NEs, think from the inverse direction: enumerate support
  - If we know in a NE, for player i, action $1, 2$, and 3 are in the support of s_i, action $4, 5$ are not what does it mean?
    - (1)
    - (2)
    - (3)
    - (4)

Find All NEs

- Recall: A mixed strategy is BR iff all actions in the support are BR
- To find all NEs, think from the inverse direction: enumerate support
  - If we know in a NE, for player i, action $1, 2,$ and 3 are in the support of s_i, action $4, 5$ are not what does it mean?
    - (1) Action $1, 2,$ and 3 are chosen with non-zero probability, action 4,5 are chosen with zero probability
    - (2) The probability of choosing action $1, 2, 3$ sum up to 1
    - (3) Action $1, 2,$ and 3 lead to the exactly same expected utility
    - (4) The expected utility of taking action $1, 2,$ and 3 is not lower than action $4, 5$

Find All NEs

- If support for both Alex and Berry is (F, C), then action F and C should lead to same expected utility for Alex when fixing Berry's strategy and vice versa
- Assume Alex's strategy is $s_A = (x_1, x_2)$ and Berry's strategy is $s_B = (y_1, y_2)$ then similar to (1)-(4) in the previous slide, we know

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Find All NEs

- If support for both Alex and Berry is (F, C), then action F and C should lead to same expected utility for Alex when fixing Berry's strategy and vice versa
- Assume Alex's strategy is $s_A = (x_1, x_2)$ and Berry's strategy is $s_B = (y_1, y_2)$ then similar to (1)-(4) in the previous slide, we know

(1): $x_1 > 0, x_2 > 0, y_1 > 0, y_2 > 0$
(2): $x_1 + x_2 = 1, y_1 + y_2 = 1$
(3): $u_A(F, s_B) = u_A(C, s_B), u_B(s_A, F) = u_B(s_A, C)$

$$u_A(F, s_B) = 2 \times y_1 + 0 \times y_2$$
$$u_A(C, s_B) = 0 \times y_1 + 1 \times y_2$$

So $2y_1 = y_2$

$$u_B(s_A, F) = 1 \times x_1 + 0 \times x_2$$
$$u_B(s_A, C) = 0 \times x_1 + 2 \times x_2$$

So $x_1 = 2x_2$

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Solve the equations in (2)(3) and get $s_A = \left(\frac{2}{3}, \frac{1}{3}\right), s_B = \left(\frac{1}{3}, \frac{2}{3}\right)$ which satisfy (1). It is indeed a NE with specified support.

Find All NEs

- Support Enumeration Method (for bimatrix games)
  - Enumerate all support pairs with the same size for size=1 to $\min_{\mathrm{i}} |A_i|$
  - For each possible support pair J_1, J_2, build and solve a LP
    - An NE is found if the LP has a feasible solution

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  - Enumerate all support pairs with the same size for size=1 to $\min_{\mathrm{i}} |A_i|$
  - For each possible support pair J_1, J_2, build and solve a LP

$$\max_{x,y,v} 1$$

$$x_i \geq 0, \forall i; y_j \geq 0, \forall j$$

$$x_i = 0, \forall i \notin J_1; y_j = 0, \forall j \notin J_2$$

$$\sum_{i \in J_1} x_i = 1$$

$$\sum_{j \in J_2} y_j = 1$$

$$\sum_{j \in J_2} y_j u_1(i,j) = v_1, \forall i \in J_1$$

$$\sum_{i \in J_1} x_i u_2(i,j) = v_2, \forall j \in J_2$$

$$\sum_{j \in J_2} y_j u_1(i,j) \leq v_1, \forall i \notin J_1$$

$$\sum_{i \in J_1} x_i u_2(i,j) \leq v_2, \forall j \notin J_2$$

  - An NE is found if the LP has a feasible solution

Find All NEs

- Support Enumeration Method (for bimatrix games)
  - Enumerate all support pairs with the same size for size=1 to $\min_{\mathrm{i}} |A_i|$
  - For each possible support pair J_1, J_2, build and solve a LP
    - Variables: $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, v_1, v_2$
    - Objective: a dummy one $\max_{x,y,v} 1$
    - Constraints (1b,1c): Probabilities are nonnegative, probability of actions not in the support is zero
      - ☐ $x_i \geq 0, \forall i; y_j \geq 0, \forall j; x_i = 0, \forall i \notin J_1; y_j = 0, \forall j \notin J_2$
    - Constraints (2): Probability of taking actions in the support sum up to 1
      - ☐ $\sum_{i \in J_1} x_i = 1; \sum_{j \in J_2} y_j = 1$
    - Constraints (3): Expected utility (EU) of choosing any action is the support is the same when fixing the other player's strategy
      - ☐ $\sum_{j \in J_2} y_j u_1(i,j) = v_1, \forall i \in J_1; \sum_{i \in J_1} x_i u_2(i,j) = v_2, \forall j \in J_2$
    - Constraints (4): Actions not in support does not lead to higher expected utility
      - ☐ $\sum_{j \in J_2} y_j u_1(i,j) \leq v_1, \forall i \notin J_1; \sum_{i \in J_1} x_i u_2(i,j) \leq v_2, \forall j \notin J_2$
    - An NE is found if the LP has a feasible solution

Compute Nash Equilibrium

- Find all Nash Equilibrium (two-player)
  - Support Enumeration Method
  - Lemke-Howson Algorithm
    - Linear Complementarity (LCP) formulation (another special class of optimization problem)
    - Solve by pivoting on support (similar to Simplex algorithm)
  - In practice, available solvers/packages: nashpy (python), gambit project (http://www.gambit-project.org/)

Compute Strong Stackelberg Equilibrium

- Find Strong Stackelberg Equilibrium (not restricted to pure strategy)
  - Finite zero-sum games: SSE=NE=Minimax=Maximin
  - General case: solve multiple linear programs or a mixed integer linear program
  - For some security games: greedy algorithm

Compute Strong Stackelberg Equilibrium

- Find Strong Stackelberg Equilibrium (not restricted to pure strategy)
  - Special case (zero-sum): SSE=NE=Minimax=Maximin
  - General case: Solve Multiple Linear Programs
    - Key idea: Enumerate the follower's best response (similar to support enumeration method for finding NE)
    - If the leader (player 1) plays a mixed strategy $s_1 = \langle x_1, \ldots, x_{|A_1|} \rangle$, and follower's (player 2) best response is action j, then
      - ☐ 1) $x_1, \ldots, x_{|A_1|}$ sum up to 1
      - ☐ 2) All actions other than j lead to no higher expected utility for player 2
    - No matter what the leader plays, one of the actions in A_2 is a best response for player 2

Compute Strong Stackelberg Equilibrium

- Solve Multiple Linear Programs

Let U_{ij}^1 be player 1's payoff value when player 1 choose action i and player 2 choose action j

For each $j = 1..|A_2|$, solve the following LP

Then pick the solution with the highest optimal objective value among all j's

Compute Strong Stackelberg Equilibrium

- Solve Multiple Linear Programs

Let U_{ij}^1 be player 1's payoff value when player 1 choose action i and player 2 choose action j

For each $j = 1..|A_2|$, solve the following LP

$$\max_x \sum_i x_i U_{ij}^1$$

s.t.

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

$$\sum_i x_i U_{ij}^2 \geq \sum_i x_i U_{ij'}^2, \forall j' \in A_2$$

Then pick the solution with the highest optimal objective value among all j's

Multiple LP

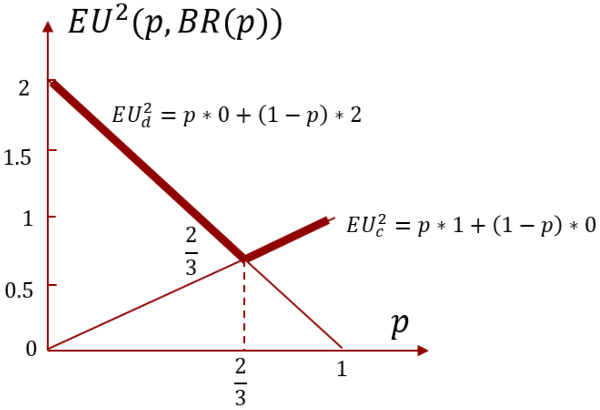
Player 1

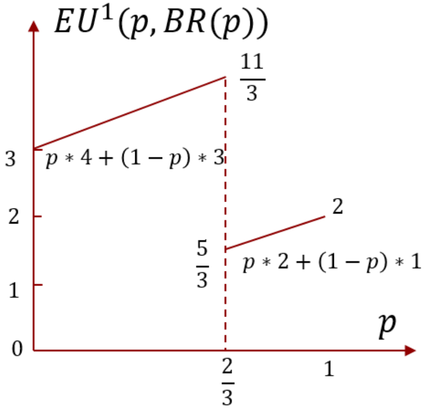
	c	d
a	2,1	4,0
b	1,0	3,2

Let $s_1 = \langle p, 1-p \rangle$

If BR is c, solve

If BR is d, solve





Multiple LP

		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2

Let $s_1 = \langle p, 1-p \rangle$

If BR is c, solve

$$\max_p EU^1(p,c) = p*2+(1-p)*1$$
$$\text{s.t.} 0 \le p \le 1$$
$$EU_c^2 = p*1+(1-p)*0 \ge EU_d^2$$
$$= p*0+(1-p)*2$$

Get $p = 1, EU^1(p,c) = 2$

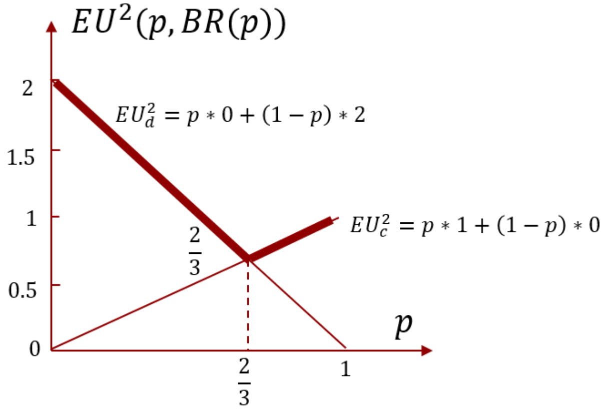
If BR is d, solve

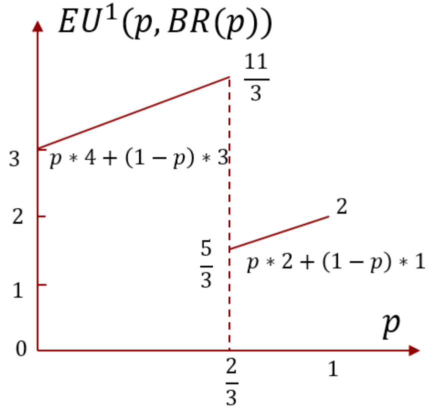
$$\max_p EU^1(p,d) = p*4+(1-p)*3$$
$$\text{s.t.} 0 \le p \le 1$$
$$EU_d^2 = p*0+(1-p)*2$$
$$\ge EU_c^2 = p*1+(1-p)*0$$

Get $p = \frac{2}{3}, \max_p EU^1(p,d) = \frac{11}{3}$

Compare the optimal objective value, pick the second LP.

So $p = \frac{2}{3}$, SSE is $s_1 = \left(\frac{2}{3}, \frac{1}{3}\right), s_2 = (0,1)$



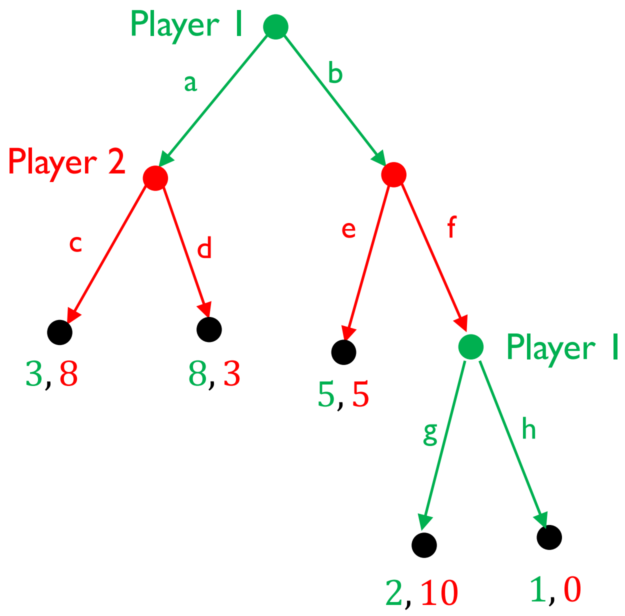


Outline

- Normal-Form Games
- Solution Concepts
- Linear Programming-based Equilibrium Computation
- Extensive-Form Games

Extensive-Form Games

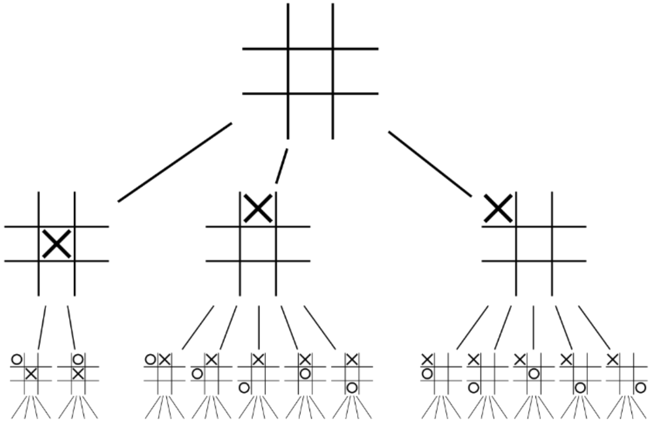
- A game in extensive-form
  - Timing, sequence of move
  - Can be represented by a game tree with information sets



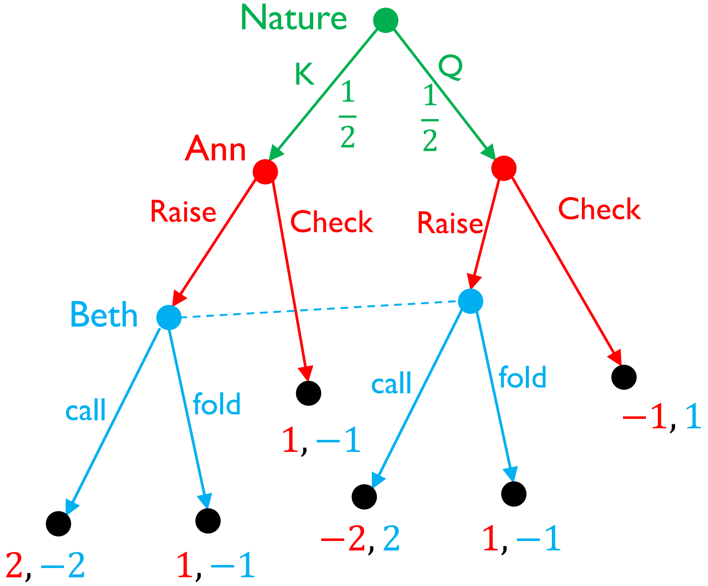
Extensive-Form Games

- Perfect information vs Imperfect information
- Special fictitious player: Nature or Chance

Tic-Tac-Toe



Myerson Poker



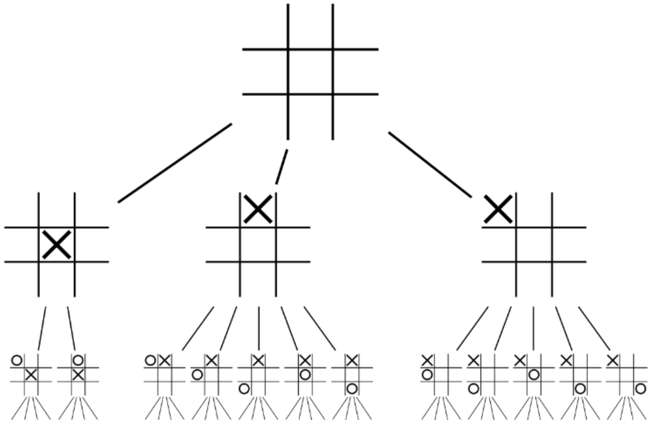
Perfect-Information Extensive-Form Game

- A finite, n-player perfect-information extensive-form game is described by a tuple $(N, A, H, Z, \chi, \rho, \sigma, u)$
  - $N = \{1..n\}$: Set of players
  - A: Set of actions Note: not joint actions
  - H: Set of non-terminal nodes in the game tree
  - Z: Set of terminal nodes
  - $\chi: H \mapsto \{0,1\}^{|A|}$ specifies actions available at each node
  - $\rho: H \mapsto N$ specifies the acting player at each node
  - $\sigma: H \times A \mapsto H \cup Z$ is the successor function, specifies the successor node after an action is taken at a node
  - Payoffs / Utility functions $u_i: Z \mapsto \mathbb{R}$

Perfect-Information Extensive-Form Game

- $(N, A, H, Z, \chi, \rho, \sigma, u)$

Tic-Tac-Toe



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Payoffs / Utility functions $u_i: Z \mapsto \mathbb{R}$

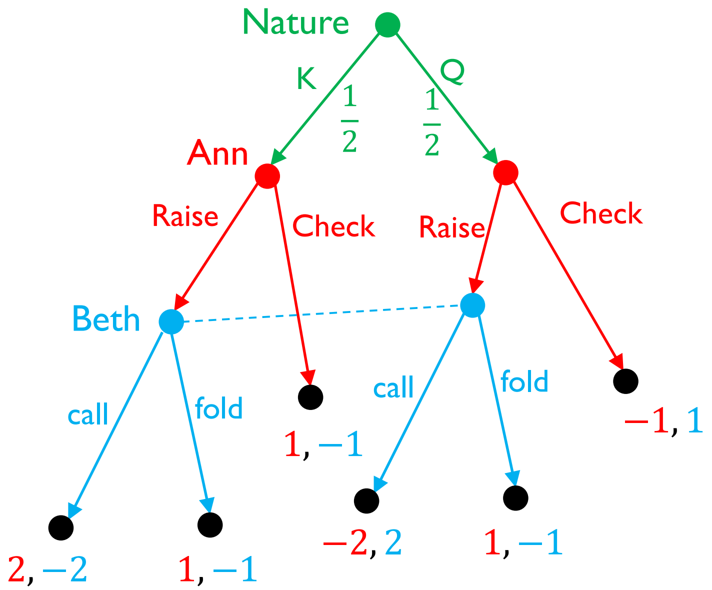
Imperfect-Information Extensive-Form Game

- A finite, n-player imperfect-information extensive-form game is described by a tuple $(N, A, H, Z, \chi, \rho, \sigma, u, I)$
  - I specifies the information sets (infosets in short)
  - $I = (I_1, \dots, I_n)$
  - $I_i = (I_{i1}, \dots, I_{ik_i})$
  - I_i is a partition of the set of nodes belonging to player i
    - $\cap I_{ij} = \emptyset, \forall i, j$
    - $\cup_j I_{ij} = \{h: \rho(h) = i\}$
  - Nodes in the same information set should have the same acting player and the same available actions
    - $\rho(h) = \rho(h'), \chi(h) = \chi(h')$

Imperfect-Information Extensive-Form Game

- $(N, A, H, Z, \chi, \rho, \sigma, u, I)$

Myerson Poker



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$$\cap \, I_{ij} = \emptyset, \forall i, j$$

$$\cup_j \, I_{ij} = \{h: \rho(h) = i\}$$

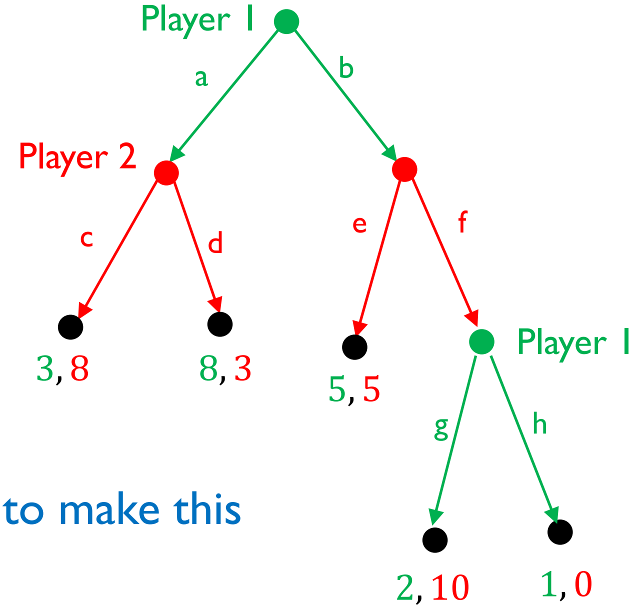
Nodes in the same information set should have the same acting player and the same available actions

$$\rho(h) = \rho(h'), \chi(h) = \chi(h')$$

Q: How many infosets are there in this game (exclude Nature)?

Extensive-Form Games

- An EFG has perfect recall if all players remember their own past actions
  - Nodes in the same infoset has the same "path" if we only consider the actions and decision points of the acting player
- We focus on games with perfect recall



Specify an information set to make this game have imperfect recall

Extensive-Form Games

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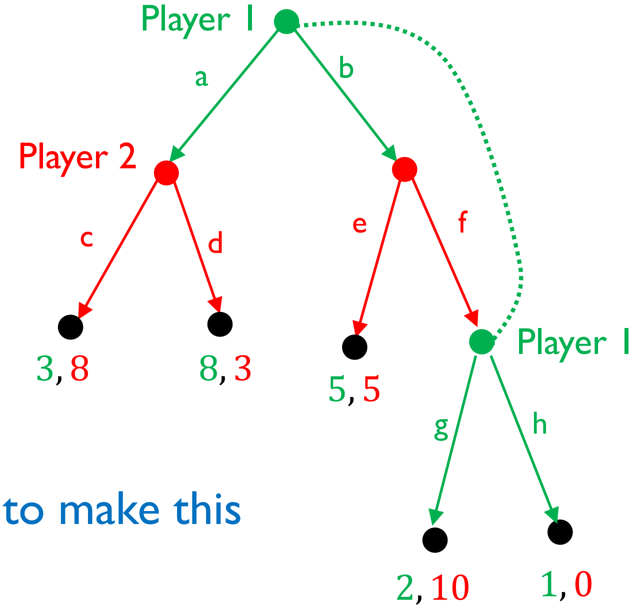
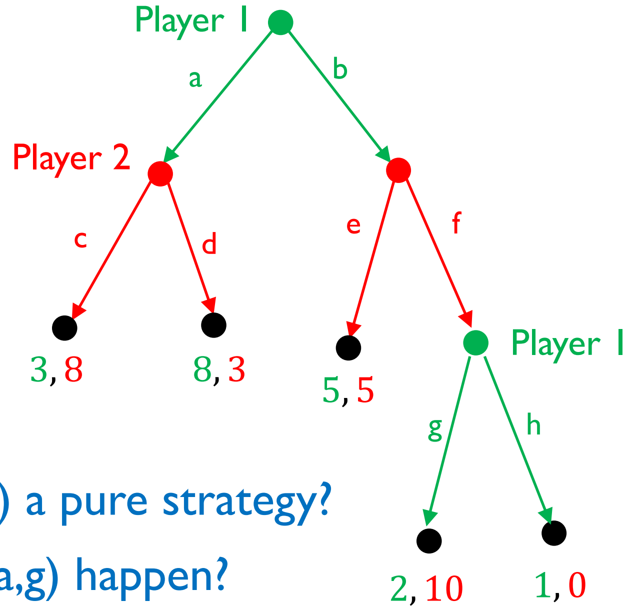


Specify an information set to make this game have imperfect recall

Pure Strategy

- A pure strategy of a player is a complete contingent-plan determining the action to take at each infoset he is to move
  - A mapping from an infoset belonging to that player to an available action at that infoset
- Reduced-form strategy: only specify actions at infosets that are not precluded by the plan

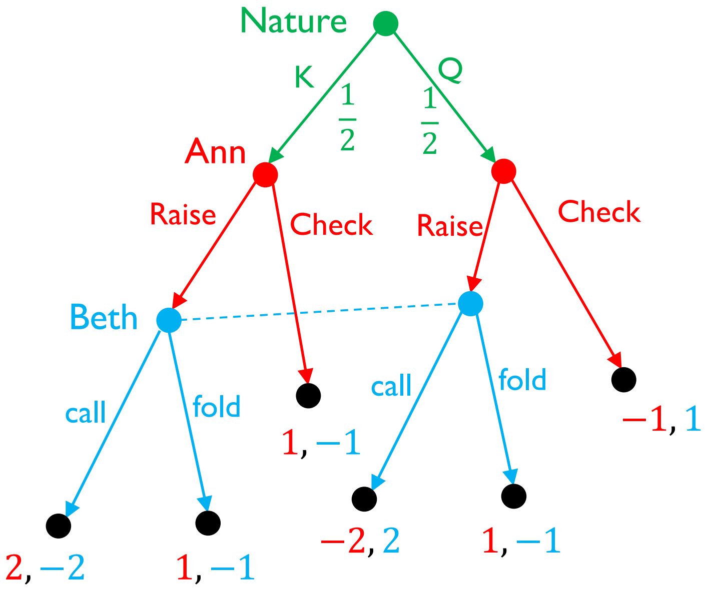


Is (a,g) a pure strategy?

Can (a,g) happen?

Extensive-Form Games

- A game in extensive form can be converted into a game in normal form

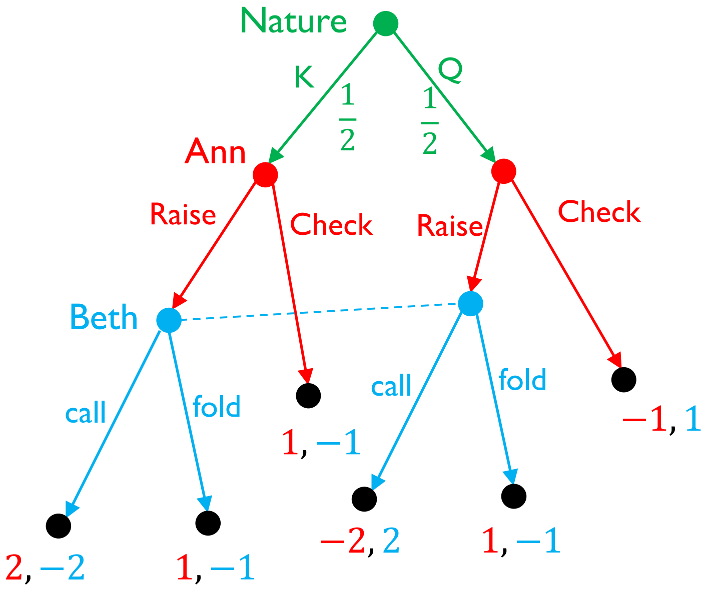


Beth

Ann

Extensive-Form Games

- A game in extensive form can be converted into a game in normal form



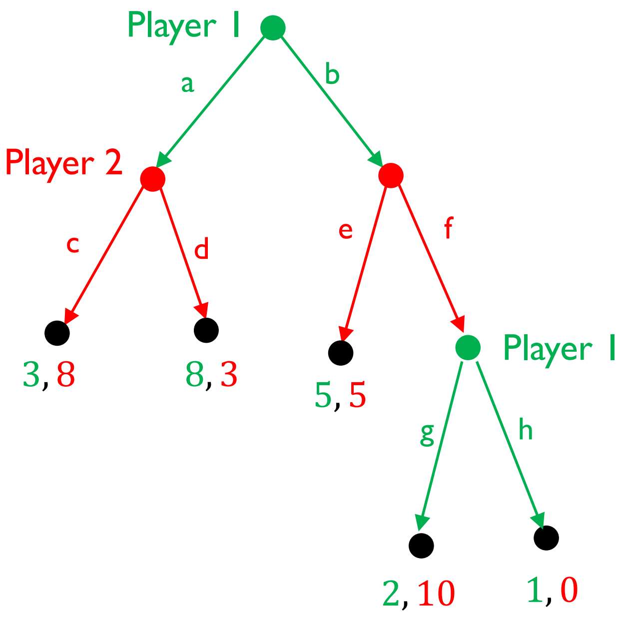
Beth

Ann

	Call	Fold
Raise if K, Raise if Q	EU^{Ann}=0	0
Raise if K, Check if Q	0.5	0
Check if K, Raise if Q	-0.5	1
Check if K, Check if Q	0	0

Extensive-Form Games

- Sometimes we only use reduced-form strategies in the converted normal-form game



Player 2

		c	d	e	f
Player 1	a				
	b, g				
	b, h				

Extensive-Form Games

- Can we represent a normal-form game (e.g., Rock-Paper-Scissors) as an extensive form game?

Extensive-Form Games

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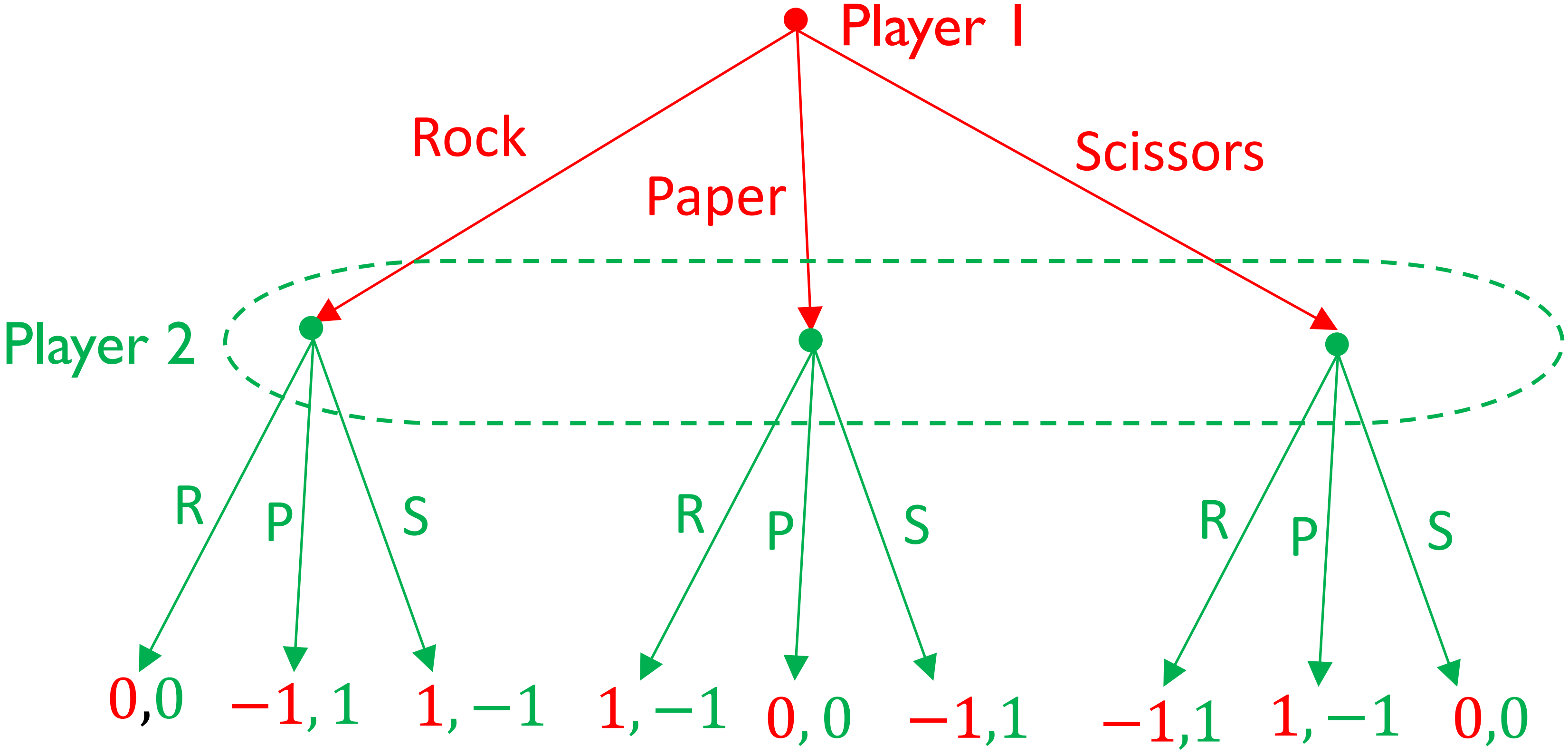
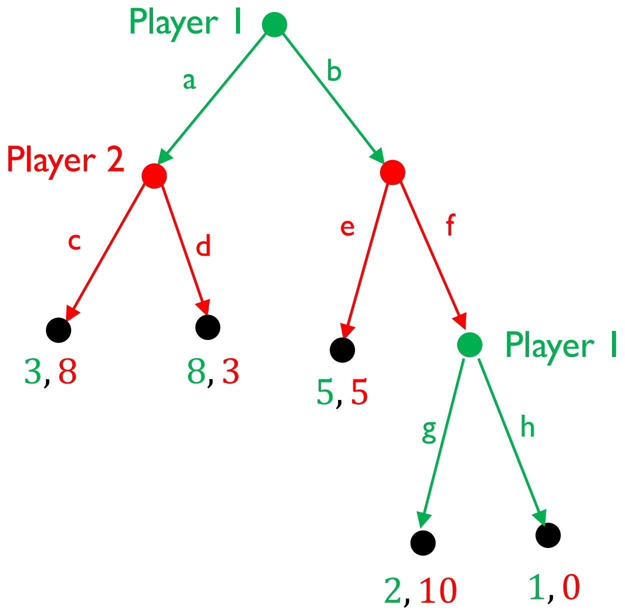


Randomized Strategy

- How to represent randomness in the strategy?
  - Option 1 (mixed strategy): Prob. distribution over pure strategies
  - Option 2 (behavioral strategy): Prob. distribution over actions at each infoset



Mixed: $\mathbb{P}(a, g) = 0.3, \mathbb{P}(b, h) = 0.7$

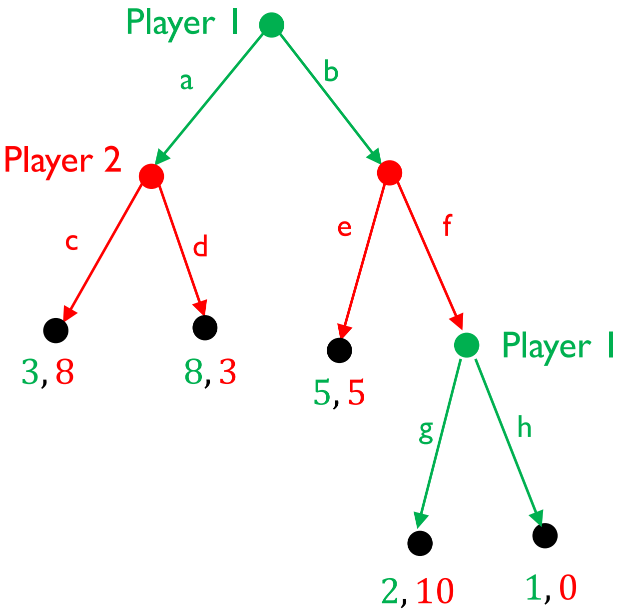
Behavioral: $\mathbb{P}(a) = 0.5, \mathbb{P}(g) = 0.6$

Which one is better?

Randomized Strategy

- In games with perfect recall:
  - Any mixed (behavioral) strategy of an agent can be replaced by an equivalent behavioral (mixed) strategy
  - Two strategies of a player are equivalent if they induce the same probabilities on outcomes given any fixed strategy profile of the other players

Provide an example mixed / behavioral strategy of the game



Mixed: $\mathbb{P}(a, g) = 0.3, \mathbb{P}(b, h) = 0.7$

Behavioral: $\mathbb{P}(a) = 0.5, \mathbb{P}(g) = 0.6$

Summary

Solution Concepts	Key Algorithm In Class
Minimax/Maximin	LP
Nash Equilibrium	LP for zero-sum, Support enumeration for general-sum
Strong Stackelberg Equilibrium	LP for zero-sum, multiple LP or MILP for general-sum

Game Theory: Additional Resources

- *Algorithmic Game Theory 1st Edition, Chapters 1-3*

  Noam Nisan (Editor), Tim Roughgarden (Editor), Eva Tardos (Editor), Vijay V. Vazirani (Editor)
  - http://www.cs.cmu.edu/~sandholm/cs15-892F13/algorithmic-game-theory.pdf
- Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations, Chp 3,4
- Online course
  - https://www.youtube.com/user/gametheoryonline