Artificial Intelligence Methods for Social Good

M2-1 [Game Theory]:

Basics of Game Theory

08-537 (9-unit) and 08-737 (12-unit)

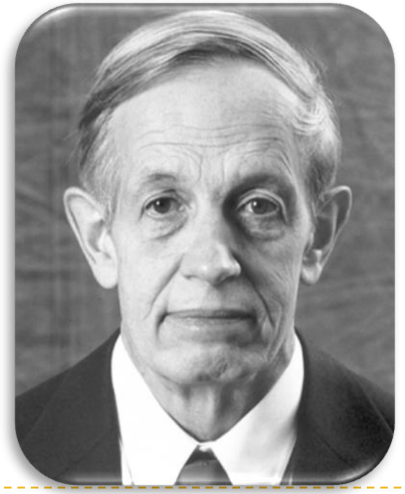
Instructor: Fei Fang
feifang@cmu.edu
Wean Hall 4126

Quiz 1: Recap: Optimization Problem

- Given coordinates of n residential areas in a city (assuming 2-D plane), denoted as $x^1, \dots, x^n$, the government wants to find a location that minimizes the sum of (Euclidean) distances to all residential areas to build a hospital. The optimization problem can be written as
  - A: $\min_x \sum_i |x^i - x|$
  - B: $\min_x \sum_i \|x^i - x\|_2$
  - C: $\min_x \sum_i (x^i - x)^2$
  - D: none of above

From Games to Game Theory

- The study of mathematical models of conflict and cooperation between intelligent decision makers
- Used in economics, political science etc



Outline

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Strong Stackelberg Equilibrium

Learning Objectives

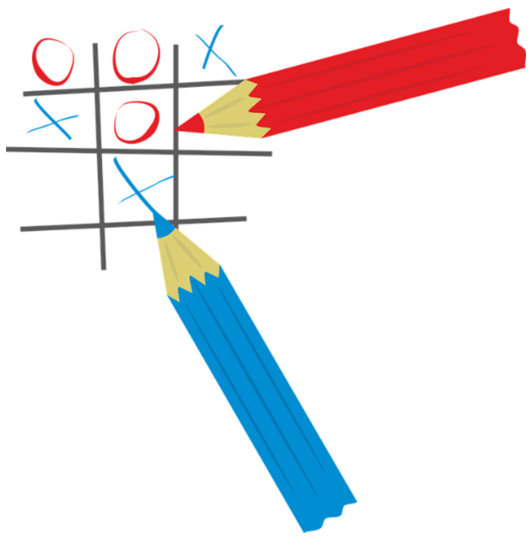
- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium
- Describe Minimax Theory
- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (multiple LP and MILP)
- Know how to find the method/algorithm/solver/package you can use for solving the games
- Compute NE/SSE by hand or by calling a solver for small games

Let's Play! Classical Games

- Exp 1: Rock-Paper-Scissors (RPS)
  - Rock beats Scissors
  - Scissors beats Paper
  - Paper beats Rock
- Exp 2: Prisoner's Dilemma (PD)
  - If both Cooperate: 1 year in jail each
  - If one Defect, one Cooperate: 0 year for (D), 3 years for (C)
  - If both Defect: 2 years in jail each

Let's Play! Classical Games

- Exp 3: Battle of Sexes (BoS)
  - If football together: Alex ☺☺, Berry ☺
  - If concert together: Alex ☺, Berry ☺☺
  - If not together: Alex ☹, Berry ☹
- Tic-Tac-Toe (TTT)



Basic Concepts in Games

- Def 1: Game
  - Players
  - Actions
  - Payoffs

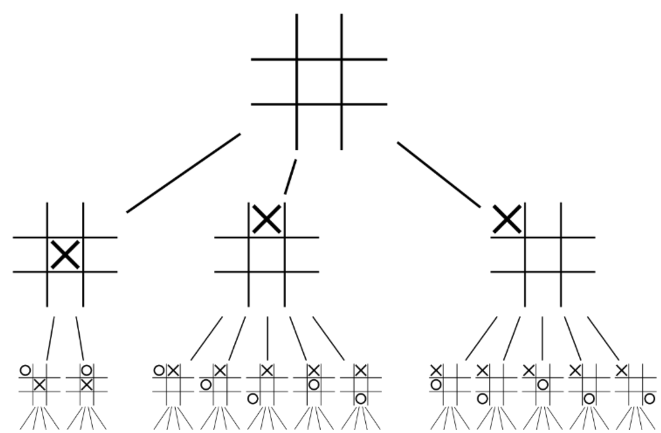
Basic Concepts in Games

- Representation
  - Normal form (Matrix form, Strategic form, Standard form)
    - Move simultaneously
    - Bimatrix game (Two-player)
      - Exp 2: PD

Player 2

Player 1	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

  - Extensive form
    - Timing, Sequence of move
    - Game tree
    - Information



  - Natural description

Basic Concepts in Games

- Pure Strategy
  - Choose one action deterministically
- Def 2: Mixed Strategy
  - Play randomly
  - Support: chosen with non-zero probability
- Def 3: Expected utility
  - Average utility weighted by probability

Quiz 2: Basic Concepts in Games

- In Exp 1 (Rock-Paper-Scissors), if $s_1 = (\frac{1}{3}, \frac{2}{3}, 0)$, $s_2 = (0, \frac{1}{2}, \frac{1}{2})$, what is u?
  - $u = (0,0)$
  - $u = (-\frac{1}{3}, \frac{1}{3})$
  - $u = (-\frac{1}{2}, \frac{1}{3})$
  - $u = (-\frac{1}{2}, \frac{1}{2})$

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Basic Concepts in Games

- Def 4: Best Response
  - Set of actions or strategies leading to highest expected utility
- Thm 1 (Nash 1951): A mixed strategy is BR iff all actions in the support are BR

Learning Objectives

- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium
- Describe Minimax Theory
- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (multiple LP and MILP)
- Know how to find the method/algorithm/solver/package you can use for solving the games
- Compute NE/SSE by hand or by calling a solver for small games

Outline

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Strong Stackelberg Equilibrium

Basic Solution Concepts

- How should a player play?
- Def 5: Dominant Strategy
  - One strategy is always better/never worse/never worse and sometimes better than any other strategy
  - Focus on single player's strategy
  - Exp 2: PD

Player 2

Player 1	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

- Dominant strategy equilibrium/solution
  - Every player plays a dominant strategy
  - Focus on strategy profile for all players

Basic Solution Concepts

- Def 6: Nash Equilibrium
  - Every player's strategy is a best response to others' strategy profile
  - Focus on strategy profile for all players
  - One cannot gain by unilateral deviation
  - Pure Strategy Nash Equilibrium (PSNE)
  - Mixed Strategy Nash Equilibrium (MSNE)
  - Exp 2: PD

Player 2

Player 1	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Basic Solution Concepts

- Thm 2 (Nash 1951): NE always exists in finite games
  - Brouwer's fixed point theorem

Basic Solution Concepts

- Def 7: Maxmin Strategy (applicable to multiplayer games)
  - Maximize worst case expected utility
  - Focus on single player's strategy
- Def 8: Minmax Strategy (two-player games only)
  - Minimize best case expected utility for the other player (just want to harm your opponent)
  - Focus on single player's strategy

Basic Solution Concepts

- Thm 3: (Minimax Theorem, von Neumann 1928) Minmax=Maxmin in 2-player zero-sum games
- Further, Minmax=Maxmin=NE (Nash 1951)

Basic Solution Concepts

- Exp 4: Power of Commitment
  - NE utility=$(2,1)$
  - If leader (player 1) commits to playing b, then player has to play d, leading to a utility of 3 for leader
  - If leader (player 1) commits to playing a and b uniformly randomly, then player still has to play d, leading to a utility of 3.5 for leader

Player 2

Player 1

	c	d
a	2,1	4,0
b	1,0	3,2

Basic Solution Concepts

- Def 9: Best Response Function
  - A mapping from a strategy of one player to a strategy of another player in the best response set
- Def 10: Stackelberg Equilibrium
  - Leader vs follower game
  - Leader commits to a strategy
  - Follower responds according a best response function
  - Focus on strategy profile for all players

Basic Solution Concepts

- Def 11: Strong Stackelberg Equilibrium (SSE)
  - Follower breaks tie in favor of the leader

Basic Solution Concepts

- Def 11: Strong Stackelberg Equilibrium (SSE)
  - Follower breaks tie in favor of the leader
  - Leader can induce the follower to do so by perturbing the strategy in the right direction
  - SSE always exist in two-player finite games

Quiz 3: Basic Solution Concepts

- What is the relationship of leader's expected utility in SSE and NE in two-player games?
  - $u^{SSE} \geq u^{NE}$
  - $u^{SSE} = u^{NE}$
  - $u^{SSE} \leq u^{NE}$
  - none of the above

Learning Objectives

- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium
- Describe Minimax Theory
- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (multiple LP and MILP)
- Know how to find the method/algorithm/solver/package you can use for solving the games
- Compute NE/SSE by hand or by calling a solver for small games

Outline

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Stackelberg Equilibrium

Compute Nash Equilibrium

- Find pure strategy Nash Equilibrium (PSNE)
  - Enumerate all action profile
  - Check if no incentive to deviate

Compute Nash Equilibrium

- Find all Nash Equilibrium (two-player)
  - Special case: Zero-sum game
    - Polynomial time solvable (minmax or maxmin LP)
  - General case
    - PPAD-Complete (Chen & Deng, 2006)
      - Unlikely to have polynomial time algorithm
      - Conjecture: slightly easier than NP-Complete problems

Compute Nash Equilibrium

- Find all Nash Equilibrium (two-player)
  - Support Enumeration Method
    - Enumerate support pair
    - For each possible support pair
      - Compute the probability so as to keep the other player indifferent among actions in the support
      - Check if no incentive to deviate
      - Or combine the two steps: solve an LP

Compute Nash Equilibrium

- Exp 3 (Battle of Sexes)
  - Enumerate support pair
    - Support size=1: PSNE!
    - Support size=2: Alex: (Football, Concert), Berry: (Football, Concert)
    - Different support size: no NE exist (#constraints>#variables)
  - Compute the probability so as to keep the other player indifferent among actions in the support
  - Why? (check Thm 1)

Berry

Alex		Football	Concert
	Football	2,1	0,0
	Concert	0,0	1,2

Quiz 4: Compute Nash Equilibrium

- What is the probability of Berry choosing Football in NE of Exp 3 (Battle of Sexes) with support size=2?
  - 0
  - 1
  - $\frac{1}{3}$
  - $\frac{2}{3}$

Berry (columns) / Alex (rows)

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Compute Nash Equilibrium

- Find all Nash Equilibrium (two-player)
  - Support Enumeration Method
    - Enumerate support pair
    - For each possible support pair
      - Compute the probability so as to keep the other player indifferent among actions in the support
      - Check if no incentive to deviate
      - Or combine the two steps: solve an LP
  - Lemke-Howson Algorithm
    - Linear Complementarity (LCP) formulation (another special class of optimization problem)
    - Solve by pivoting on support (similar to Simplex algorithm)
  - In practice, available solvers/packages: nashpy (python), gambit project (http://www.gambit-project.org/)

Learning Objectives

- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium
- Describe Minimax Theory
- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (MILP)
- Know how to find the method/algorithm/solver/package you can use for solving the games
- Compute NE/SSE by hand or by calling a solver for small games

Outline

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Strong Stackelberg Equilibrium

Compute Strong Stackelberg Equilibrium

- Find Strong Stackelberg Equilibrium (not restricted to pure strategy)
  - Special case (zero-sum): SSE=NE=Minmax=Maxmin
  - General case: Polynomial time solvable
    - Multiple Linear Programming
    - Mixed Integer Linear Programming
      - Exponential in theory
      - Efficient in practice in some cases
  - Solvers available? Course project idea (9 units only): develop a SSE solver package in Python

Learning Objectives

- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium
- Describe Minimax Theory
- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (MILP)
- Know how to find the method/algorithm/solver/package you can use for solving the games
- Compute NE/SSE by hand or by calling a solver for small games

Summary

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Strong Stackelberg Equilibrium

- Key take-away:
  - There are various solution concepts in games
  - In two-player zero-sum games, many solution concepts lead to same strategy and utility
  - Finding NE or SSE can be formulated as one or more optimization problem

Summary

Solution Concepts	Key Algorithm In Class
Minmax/Maxmin	LP
Nash Equilibrium	LP for zero-sum, Support enumeration for general-sum
Strong Stackelberg Equilibrium	LP for zero-sum, multiple LP or MILP for general-sum

Game Theory: Additional Resources

- Text book
  - *Algorithmic Game Theory 1st Edition, Chapters 1-3*
  - *Noam Nisan (Editor), Tim Roughgarden (Editor), Eva Tardos (Editor), Vijay V. Vazirani (Editor)*
  - http://www.cs.cmu.edu/~sandholm/cs15-892F13/algorithmic-game-theory.pdf
- Online course
  - https://www.youtube.com/user/gametheoryonline