# Infinitary proof theory of first order linear logic with fixed points

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# Induction and coinduction

| Induction | Coinduction - Bisimulation |
|---|---|

# Induction and coinduction

| | |
|---|---|
| Termination, Progress;<br>A property eventually holds | Productivity, Equality of streams;<br>A property holds infinitely often |
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# Induction and coinduction

| Finite data types | Infinite data types |
| --- | --- |
| Natural numbers, Lists, etc. | Streams, Infinite trees, etc. |
| Termination, Progress;<br>A property eventually holds | Productivity, Equality of streams;<br>A property holds infinitely often |
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# Induction and coinduction

| Least fixed points | Greatest fixed points |
| --- | --- |
| Finite data types | Infinite data types |
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| Termination, Progress;<br>A property eventually holds | Productivity,Equality of streams;<br>A property holds infinitely often |
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# Mutual least and greatest fixed points

1. Examples?
2. Induction/Coinduction?
3. Termination/productivity?

# Prove theorems using induction and coinduction - Previous works

- Induction principle
- Bisimulation
- Coinduction principle [Kozen and Silva]
- An infinitary calculus for first-order logic with inductive definitions [Brotherston]
- A finitary calculus for least and greatest fixed points in linear logic [Baelde]
- Well founded recursion with copatterns and sized types [Abel and Pientka]

# Our contribution

**A first order calculus for proving properties about mutual least and greatest fixed points, in particular Session-typed processes**

1. Add fixed points and assign priorities to them,
2. Use circular edges in the proof for inductive and coinductive steps,
3. Impose a validity condition to ensure soundness of this proof system.

We use priorities in the validity condition to ensure valid simultaneous induction and conduction.

# Finite lists: Example of least fixed points

## Natural numbers

$\mathtt{nat} =^1_\mu \oplus\{\mathsf{zero} : 1, \mathsf{succ} : \mathtt{nat}\}$

$\overline{3} = \mathsf{succ}\ \mathsf{succ}\ \mathsf{succ}\ \mathsf{zero}$

## Lists of natural numbers

$\mathtt{list}_{\mathsf{nat}} =^1_\mu \oplus\{\mathsf{nil} : 1, \mathsf{cons} : \mathsf{nat} \otimes \mathtt{list}_{\mathsf{nat}}\}$

$\overline{[3, 3]} = \mathsf{cons}(\overline{3}, \mathsf{cons}(\overline{3}, \mathsf{nil}))$

# Programming with finite lists

Terminating

## Append two lists

$$l \leftarrow \texttt{Append} \leftarrow l_1, l_2 =$$
$$\textbf{case}\, l_1\, (\mu_{list} \Rightarrow$$
$$\textbf{case}\, l_1\, (nil \Rightarrow \textbf{wait}\, l_1; l \leftarrow l_2$$
$$cons \Rightarrow l.\mu_{list};$$
$$x \leftarrow \textbf{recv}\, l;$$
$$l.cons; \textbf{send}\, l\, x; l \leftarrow \texttt{Append} \leftarrow l_1\, l_2))$$

If l1 is an empty list (nil): forward l2 to l.

If l1=cons(x, --): send x to l and call Append on l2 and the remaining of l1.

**I use linear binary session typed processes for programming examples. See [1,2] for more info.**

# Termination and List as first order predicates

$$List(l_1) \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow l_1 \; \_)$$

$$Terminate(\_ \leftarrow \text{Append} \leftarrow \texttt{nil} \; \_) =^1_\mu 1$$
$$Terminate(\_ \leftarrow \text{Append} \leftarrow (\text{cons}(x) :: l'_1) \; \_) =^1_\mu Terminate(\_ \leftarrow \text{Append} \leftarrow l'_1 \; \_)$$

$$List(\texttt{nil}) =^1_\mu 1$$
$$List(\text{cons}(x) :: l'_1) =^1_\mu List(l'_1)$$

# Append terminates - proof

$List(l_1) \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow l_1 \ \_)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot \vdash 1}\ 1R}{1 \vdash 1}\ 1L}{1 \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow \texttt{nil}\ \_)}\ \mu R}{\dagger\ \ List(\texttt{nil}) \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow \texttt{nil}\ \_)}\ \mu L$$

$$\dfrac{\dfrac{\overset{\dagger}{List(l_1') \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow l_1'\ \_)}}{List(l_1') \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow (\text{cons}(x) :: l_1')\ \_)}\ \mu R}{\dagger\ \ List(\text{cons}(x) :: l_1') \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow (\text{cons}(x) :: l_1')\ \_)}\ \mu L$$

# Programming with streams: example of greatest fixed points

Productive

$\mathsf{merge}$

Merge two streams into a single stream by alternatively outputting an element of each.

$\mathsf{split}_1$

Return the odd elements of a stream.

$\mathsf{split}_2$

Return the even elements of a stream.

$$\mathsf{merge}(\mathsf{split}_1(t), \mathsf{split}_2(t)) = t$$

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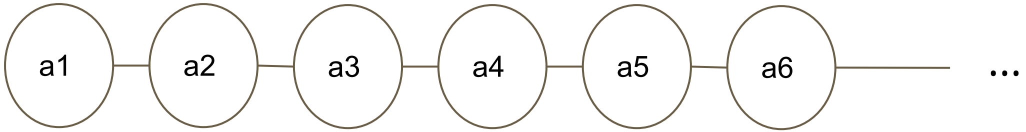

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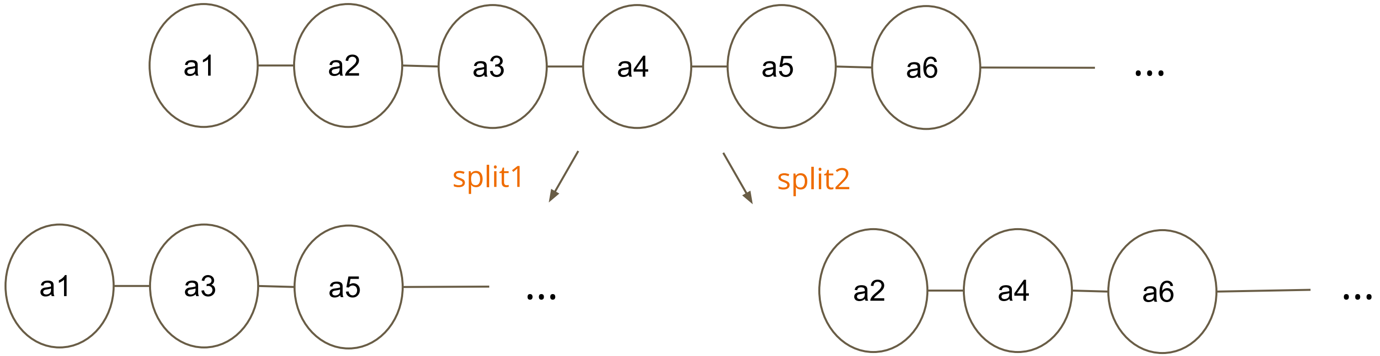

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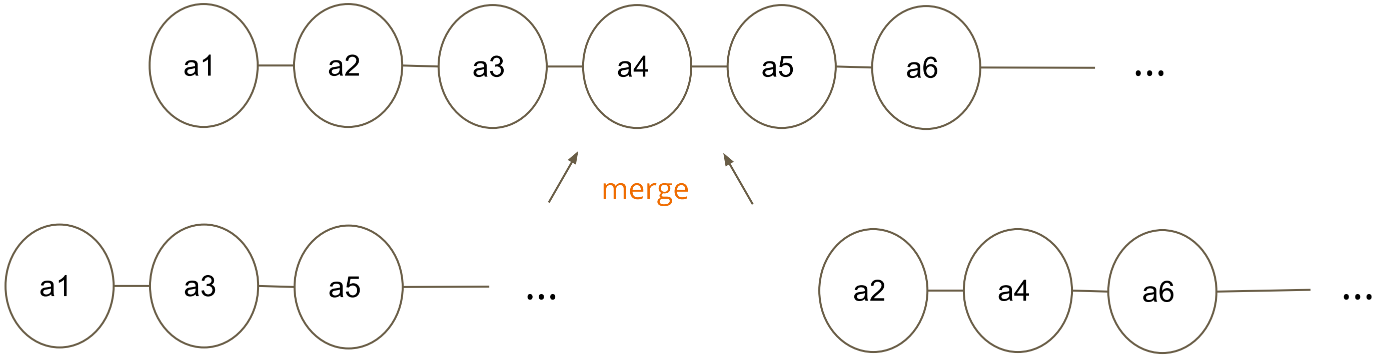

# Programming with streams

**Define properties of merge and splits as:**

$$
\begin{aligned}
\mathtt{Merge}(x, y, z) &\ =^1_\nu\ (\mathsf{hd}\, z = \mathsf{hd}\, x \,\&\, \mathtt{Merge}\,(y, \mathsf{tl}\, x, \mathsf{tl}\, z)) \\
\mathtt{Split}_1(x, y) &\ =^1_\nu\ (\mathsf{hd}\, y = \mathsf{hd}\, x \,\&\, \mathtt{Split}_2(\mathsf{tl}\, x, \mathsf{tl}\, y)) \\
\mathtt{Split}_2(x, y) &\ =^1_\nu\ (1 \,\&\, \mathtt{Split}_1(\mathsf{tl}\, x, y))
\end{aligned}
$$

# Operations merge and split□ are inverses

$$\dfrac{}{\cdot \vdash \mathsf{hd}\, y_1 = \mathsf{hd}\, y_1}\ {=}R$$

$$\dfrac{\cdot \vdash \mathsf{hd}\, y_1 = \mathsf{hd}\, y_1}{\mathsf{hd}\, y_1 = \mathsf{hd}\, x \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1}\ {=}L$$

$$\dfrac{\mathsf{hd}\, y_1 = \mathsf{hd}\, x \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1}{\mathsf{hd}\, y_1 = \mathsf{hd}\, x, 1 \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1}\ 1L$$

$$\dfrac{\mathsf{hd}\, y_1 = \mathsf{hd}\, x, 1 \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1}{\mathsf{hd}\, y_1 = \mathsf{hd}\, x, 1\&\mathtt{S}_1(\mathtt{tl}\, x, y_2) \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1}\ \&L$$

$$\dfrac{\mathsf{hd}\, y_1 = \mathsf{hd}\, x, 1\&\mathtt{S}_1(\mathtt{tl}\, x, y_2) \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1}{\mathsf{hd} y_1 = \mathsf{hd} x \& \mathtt{S}_2(\mathsf{tl} x, \mathsf{tl} y_1), 1\&\mathtt{S}_1(\mathtt{tl} x, y_2) \vdash \mathsf{hd} x = \mathsf{hd} y_1}\ \&L$$

$$\dfrac{\mathtt{S}_2(x, y_2), \mathtt{S}_1(x, y_1) \vdash \mathtt{M}(y_1, y_2, x)}{\mathtt{S}_2(\mathsf{tl} x, \mathsf{tl} y_1), \mathtt{S}_1(\mathtt{tl} x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl} y_1, \mathsf{tl} x)}\ \mathsf{Sub}_{[\mathsf{tl} x, \mathsf{tl} y_1, y_2 / x, y_2, y_1]}$$

$$\dfrac{\mathtt{S}_2(\mathsf{tl} x, \mathsf{tl} y_1), \mathtt{S}_1(\mathtt{tl} x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl} y_1, \mathsf{tl} x)}{\mathtt{S}_2(\mathsf{tl} x, \mathsf{tl} y_1), 1\&\mathtt{S}_1(\mathtt{tl} x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl} y_1, \mathsf{tl} x)}\ \&L$$

$$\dfrac{\mathtt{S}_2(\mathsf{tl} x, \mathsf{tl} y_1), 1\&\mathtt{S}_1(\mathtt{tl} x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl} y_1, \mathsf{tl} x)}{\mathsf{hd} y_1 = \mathsf{hd} x \& \mathtt{S}_2(\mathsf{tl} x, \mathsf{tl} y_1), 1\&\mathtt{S}_1(\mathtt{tl} x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl} y_1, \mathsf{tl} x)}\ \&L$$

$$\dfrac{\mathsf{hd} y_1 = \mathsf{hd} x \& \mathtt{S}_2(\mathsf{tl} x, \mathsf{tl} y_1), 1\&\mathtt{S}_1(\mathtt{tl} x, y_2) \vdash \mathsf{hd} x = \mathsf{hd} y_1 \qquad \mathsf{hd} y_1 = \mathsf{hd} x \& \mathtt{S}_2(\mathsf{tl} x, \mathsf{tl} y_1), 1\&\mathtt{S}_1(\mathtt{tl} x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl} y_1, \mathsf{tl} x)}{\mathsf{hd}\, y_1 = \mathsf{hd}\, x \,\&\, \mathtt{S}_2(\mathsf{tl}\, x, \mathsf{tl}\, y_1), 1\&\mathtt{S}_1(\mathtt{tl}\, x, y_2) \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1 \,\&\, \mathtt{M}(y_2, \mathsf{tl}\, y_1, \mathsf{tl}\, x)}\ \&R$$

$$\dfrac{\mathsf{hd}\, y_1 = \mathsf{hd}\, x \,\&\, \mathtt{S}_2(\mathsf{tl}\, x, \mathsf{tl}\, y_1), 1\&\mathtt{S}_1(\mathtt{tl}\, x, y_2) \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1 \,\&\, \mathtt{M}(y_2, \mathsf{tl}\, y_1, \mathsf{tl}\, x)}{\mathsf{hd}\, y_1 = \mathsf{hd}\, x \,\&\, \mathtt{S}_2(\mathsf{tl}\, x, \mathsf{tl}\, y_1), \mathtt{S}_2(x, y_2) \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1 \,\&\, \mathtt{M}(y_2, \mathsf{tl}\, y_1, \mathsf{tl}\, x)}\ \nu L$$

$$\dfrac{\mathsf{hd}\, y_1 = \mathsf{hd}\, x \,\&\, \mathtt{S}_2(\mathsf{tl}\, x, \mathsf{tl}\, y_1), \mathtt{S}_2(x, y_2) \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1 \,\&\, \mathtt{M}(y_2, \mathsf{tl}\, y_1, \mathsf{tl}\, x)}{\mathtt{S}_1(x, y_1), \mathtt{S}_2(x, y_2) \vdash \mathsf{hd}\, x = \mathsf{hd}\, y_1 \,\&\, \mathtt{M}(y_2, \mathsf{tl}\, y_1, \mathsf{tl}\, x)}\ \nu L$$

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# Programming with mutual least and greatest fixed points

*run(x,t)*: A stream producer where x is the list of operations, and t is the output stream.

Skip one step and do nothing

Put z as the head of output stream and inserts the new list of operations x to the original one.

$$
\begin{aligned}
\mathtt{run}(\cdot\,, t) &=^1_\mu \quad 1 \\
\mathtt{run}(skip; x, t) &=^1_\mu \quad \mathtt{run}(x, t) \\
\mathtt{run}(put\langle x\rangle; y, t) &=^1_\mu \quad \mathtt{nrun}\,(x, y, t) \\
\mathtt{nrun}(x, y, t) &=^2_\nu \quad \mathtt{hd}\, t = \mathtt{o} \,\&\, \mathtt{run}(x; y, \mathtt{tl}\; t)
\end{aligned}
$$

# Run on any list of operations produces a (possibly infinite) list of elements "o"

$$\begin{aligned}
\mathtt{run}(\cdot\,, t) &=^1_\mu 1 \\
\mathtt{run}(skip; x, t) &=^1_\mu \mathtt{run}(x, t) \\
\mathtt{run}(put\langle x\rangle; y, t) &=^1_\mu \mathtt{nrun}\,(x, y, t) \\
\mathtt{nrun}(x, y, t) &=^2_\nu \mathtt{hd}\, t = \mathtt{o} \mathbin{\&} \mathtt{run}(x; y, \mathtt{tl}\, t)
\end{aligned}$$

$$\begin{aligned}
\mathtt{list}_\mathtt{o}(t) &=^1_\mu \oplus\{\mathsf{nil} : 1, \mathsf{next} : \mathtt{stream}_\mathtt{o}(t)\} \\
\mathtt{stream}_\mathtt{o}(t) &=^2_\nu \&\{\mathsf{hd} : \mathtt{hd}\, t = \mathtt{o}, \mathsf{tl} : \mathtt{list}_\mathtt{o}\,(\mathtt{tl}\, t)\}
\end{aligned}$$

$$(\dagger)\ \mathtt{run}(x, t) \vdash \mathtt{list}_\mathtt{o}(t)$$

$$(\star)\ \mathtt{nrun}(x, y, t) \vdash \mathtt{stream}_\mathtt{o}(t)$$

# Run produces a $\text{list}_{\circ}$ - proof

$$
\begin{array}{lll}
\mathtt{run}(\cdot\,, t) & =^{1}_{\mu} & 1 \\
\mathtt{run}(skip; x, t) & =^{1}_{\mu} & \mathtt{run}(x, t) \\
\mathtt{run}(put\langle x\rangle; y, t) & =^{1}_{\mu} & \mathtt{nrun}\,(x, y, t) \\
\mathtt{nrun}(x, y, t) & =^{2}_{\nu} & \mathsf{hd}\,t = \mathsf{o} \,\&\, \mathtt{run}(x; y, \mathtt{tl}\,t)
\end{array}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot \vdash 1}\;1R}{\cdot \vdash \oplus\{\mathsf{nil} : 1, \mathsf{next} : \mathtt{stream}_{\circ}(t)\}}\;\oplus R}{\cdot \vdash \mathtt{list}_{\circ}(t)}\;\mu_{\mathtt{list}_{\circ}}R}{\dfrac{1 \vdash \mathtt{list}_{\circ}(t)}{\dagger\,\mathtt{run}(\cdot\,, t) \vdash \mathtt{list}_{\circ}(t)}\;\mu_{\mathtt{run}}L}\;1L
$$

$$
\dfrac{\dfrac{\dagger}{\mathtt{run}(x, t) \vdash \mathtt{list}_{\circ}(t)}}{\dagger\,\mathtt{run}((skip; x), t) \vdash \mathtt{list}_{\circ}(t)}\;\mu_{\mathtt{run}}L
$$

$$
\dfrac{\dfrac{\dfrac{\star}{\mathtt{nrun}\,(x, y, t) \vdash \mathtt{stream}_{\circ}(t)}}{\mathtt{nrun}\,(x, y, t) \vdash \oplus\{\mathsf{nil} : 1, \mathsf{next} : \mathtt{stream}_{\circ}(t)\}}\;\oplus R}{\dfrac{\mathtt{nrun}\,(x, y, t) \vdash \mathtt{list}_{\circ}(t)}{\dagger\,\mathtt{run}(put\langle x\rangle; y, t) \vdash \mathtt{list}_{\circ}(t)}\;\mu_{\mathtt{run}}L}\;\mu_{\mathtt{list}_{\circ}}R
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{hd}\,t = \mathsf{o} \vdash \mathsf{hd}\,t = \mathsf{o}}\;\mathtt{ID}}{\&\{\mathsf{hd} : \mathsf{hd}\,t = \mathsf{o}, \mathsf{tl} : \mathtt{run}(x; y, \mathtt{tl}\,t)\} \vdash \mathsf{hd}\,t = \mathsf{o}}\;\&L}{\mathtt{nrun}\,(x, y, t) \vdash \mathsf{hd}\,t = \mathsf{o}}\;\nu_{\mathtt{nrun}}L \qquad \dfrac{\dfrac{\dfrac{\dagger}{\mathtt{run}(x; y, \mathtt{tl}\,t) \vdash \mathtt{list}_{\circ}(\mathsf{tl}\,t)}}{\&\{\mathsf{hd} : \mathsf{hd}\,t = \mathsf{o}, \mathsf{tl} : \mathtt{run}(x; y, \mathtt{tl}\,t)\} \vdash \mathtt{list}_{\circ}(\mathsf{tl}\,t)}\;\&L}{\mathtt{nrun}\,(x, y, t) \vdash \mathtt{list}_{\circ}(\mathsf{tl}\,t)}\;\nu_{\mathtt{nrun}}L}{\dfrac{\mathtt{nrun}\,(x, y, t) \vdash \&\{\mathsf{hd} : \mathsf{hd}\,t = \mathsf{o},\ \mathsf{tl} : \mathtt{list}_{\circ}(\mathsf{tl}\,t)\}}{\star\,\mathtt{nrun}\,(x, y, t) \vdash \mathtt{stream}_{\circ}(t)}\;\nu_{\mathtt{stream}_{\circ}}R}\;\&R
$$

# Strong progress and Validity condition

A process satisfies *strong progress*, if after *finite number of steps*, it either becomes *empty* or attempts to *communicate to the left or right* [2].

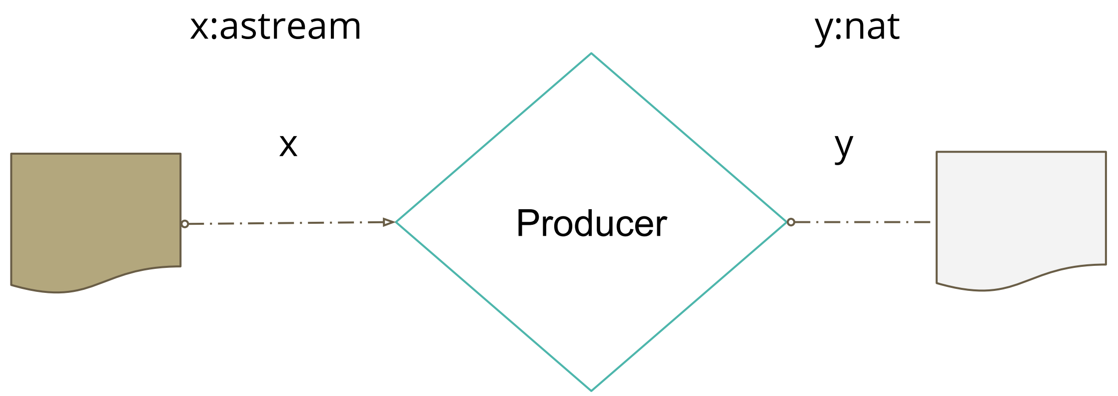
***Theorem.*** Our *validity condition* on session-typed processes ensures *strong progress* [2].

**We want to prove this directly using our calculus.**

# Producer/Idle: a locally valid program

$$\Sigma := \mathsf{ack} =^1_\mu \oplus\{ack : \mathsf{astream}\},$$
$$\mathsf{astream} =^2_\nu \&\{head : \mathsf{ack},\quad tail : \mathsf{astream}\},$$
$$\mathsf{nat} =^3_\mu \oplus\{z : 1,\quad s : nat\}$$

$$z : \mathsf{ack} \vdash w \leftarrow \mathtt{Idle} \leftarrow z :: (w : \mathsf{nat})$$
$$x : \mathsf{astream} \vdash y \leftarrow \mathtt{Producer} \leftarrow x :: (y : \mathsf{nat}),$$



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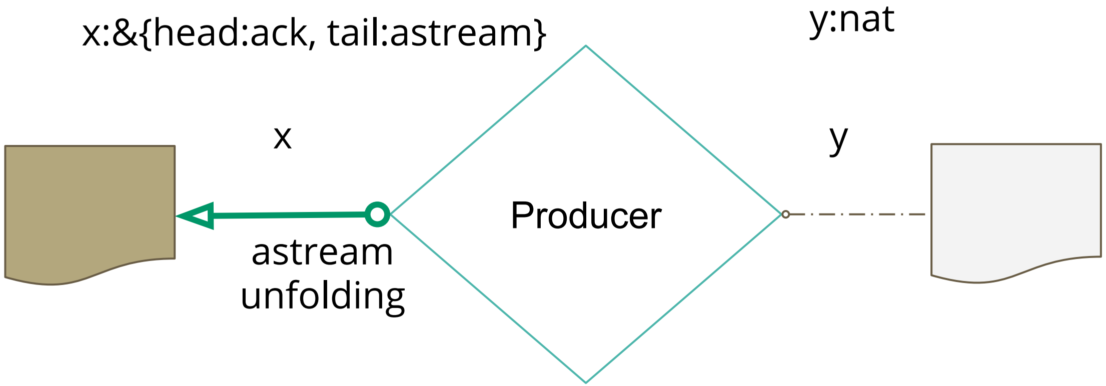

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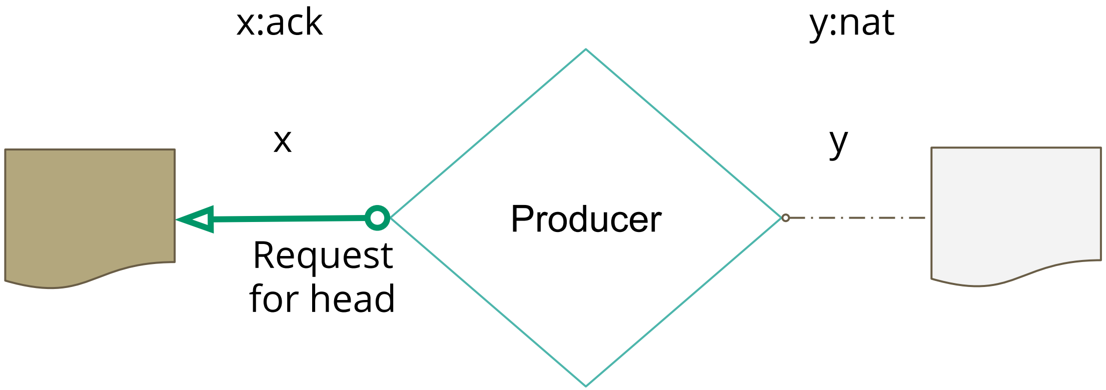

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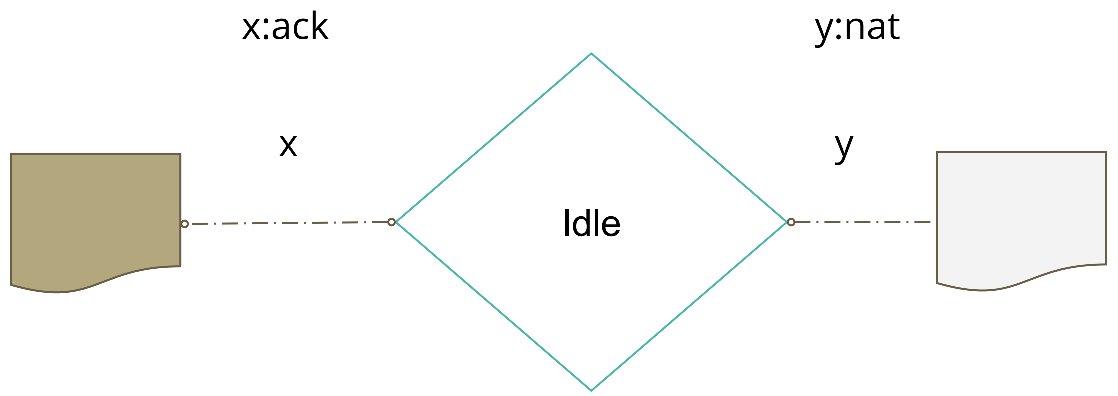

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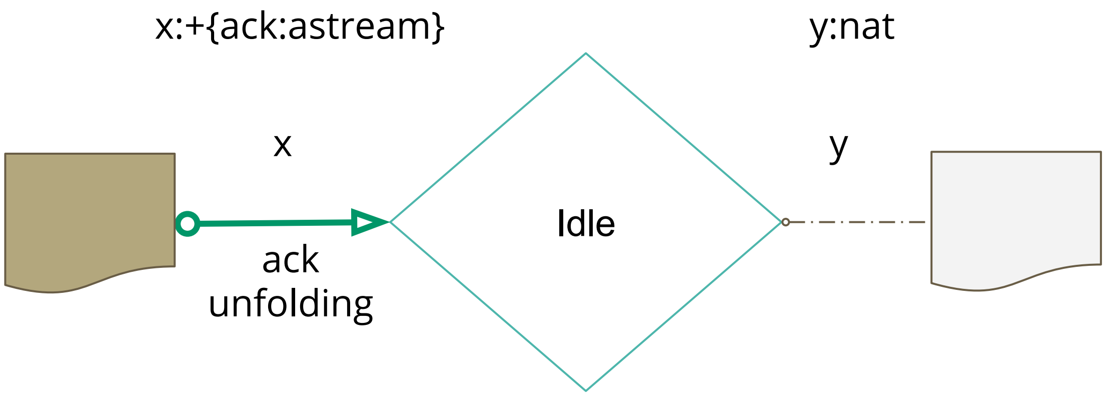

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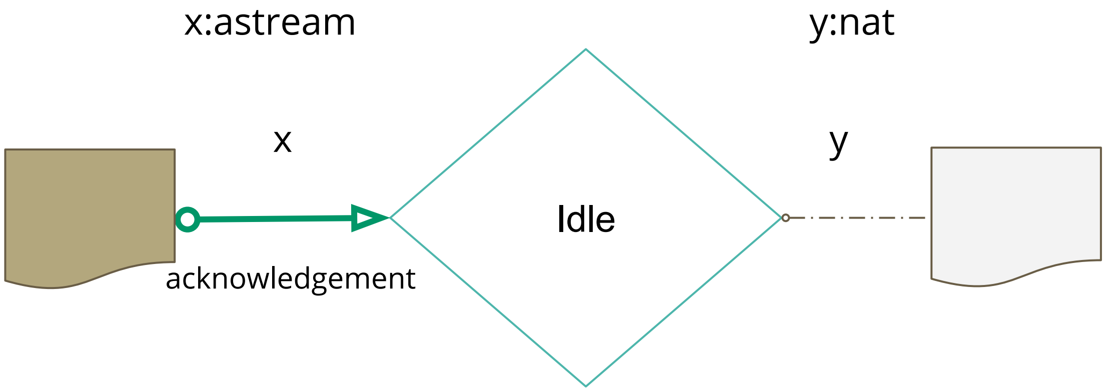

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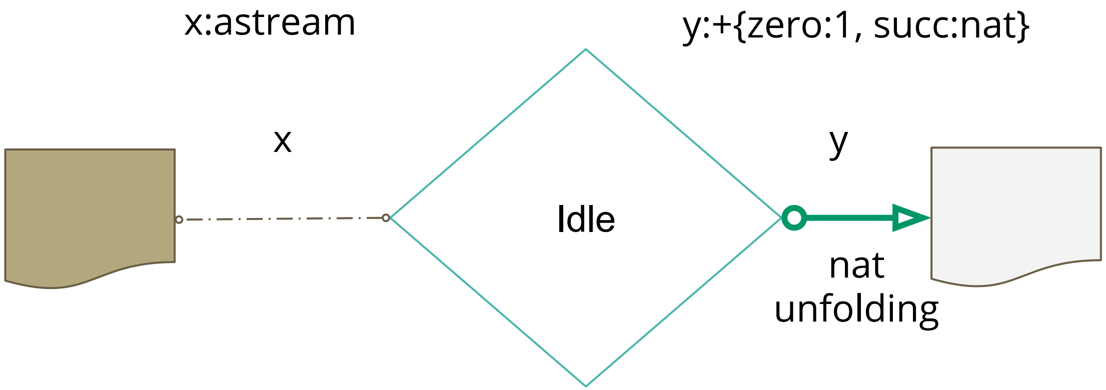

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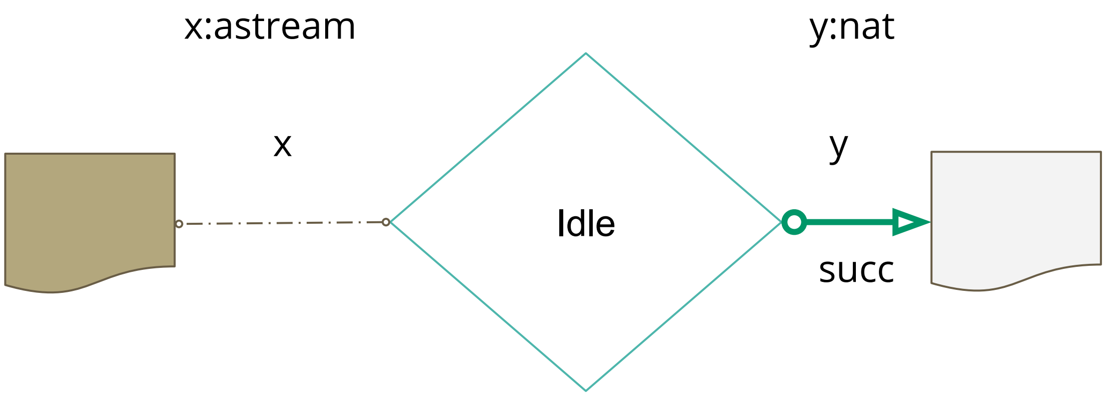

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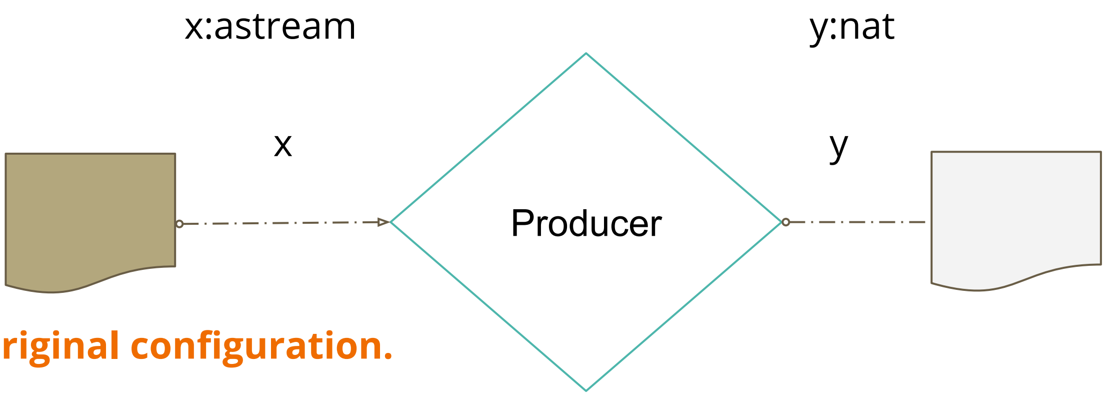



**Back to the original configuration.**

# Producer/Idle: a locally valid program - code

$$\Sigma := \mathsf{ack} =^1_\mu \oplus\{ack : \mathsf{astream}\},$$
$$\mathsf{astream} =^2_\nu \&\{head : \mathsf{ack}, \quad tail : \mathsf{astream}\},$$
$$\mathsf{nat} =^3_\mu \oplus\{z : 1, \quad s : nat\}$$

Eventually communicate with its external channels

$$z : \mathsf{ack} \vdash w \leftarrow \texttt{Idle} \leftarrow z :: (w : \mathsf{nat})$$
$$x : \mathsf{astream} \vdash y \leftarrow \texttt{Producer} \leftarrow x :: (y : \mathsf{nat}),$$

$$\begin{aligned} &y^0 \leftarrow \texttt{Producer} \leftarrow x^0 = \\ &\quad Lx^0.\nu_{astream}; \\ &\qquad Lx^1.head; y^0 \leftarrow \texttt{Idle} \leftarrow x^1 \end{aligned}$$

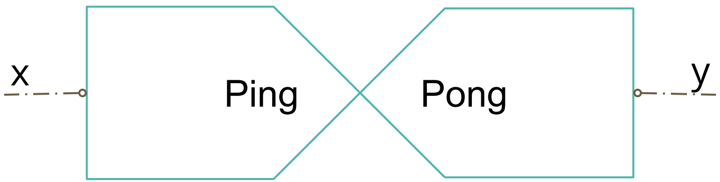
$$\begin{aligned} &y^0 \leftarrow \texttt{Idle} \leftarrow x^1 = \\ &\quad \textbf{case } Lx^1\ (\mu_{ack} \Rightarrow \\ &\qquad \textbf{case } Lx^2\ (ack \Rightarrow Ry^0.\mu_{nat}; \\ &\qquad\quad Ry^1.s; y^1 \leftarrow \texttt{Producer} \leftarrow x^2)) \end{aligned}$$

**This example is adapted from [2].**

# Ping-Pong: an invalid program

$$\Sigma := \mathsf{ack} =^1_\mu \oplus\{ack : \mathsf{astream}\},$$
$$\mathsf{astream} =^2_\nu \&\{head : \mathsf{ack}, \quad tail : \mathsf{astream}\},$$
$$\mathsf{nat} =^3_\mu \oplus\{z : 1, \quad s : nat\}$$

$$x : \mathsf{nat} \vdash \mathtt{Ping} :: (w : \mathsf{astream})$$
$$w : \mathsf{astream} \vdash \mathtt{Pong} :: (y : \mathsf{nat})$$
$$x : \mathsf{nat} \vdash \mathtt{PingPong} :: (y : \mathsf{nat})$$



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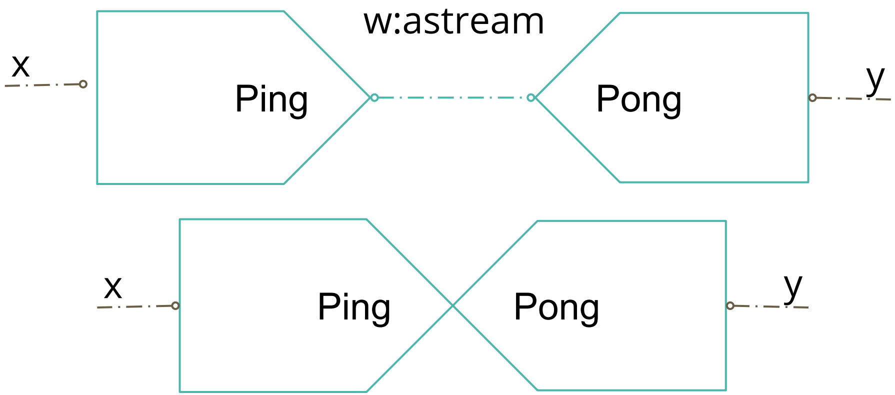

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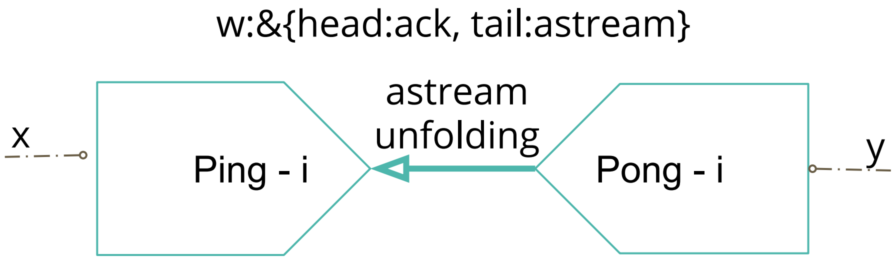
$$w : \mathsf{astream} \vdash \texttt{Pong} :: (y : \mathsf{nat})$$
$$x : \mathsf{nat} \vdash \texttt{PingPong} :: (y : \mathsf{nat})$$



# Ping-Pong: an invalid program

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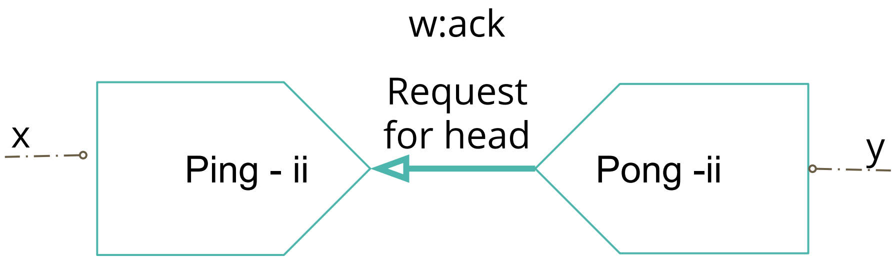

# Ping-Pong: an invalid program

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$$x : \mathsf{nat} \vdash \mathtt{Ping} :: (w : \mathsf{astream})$$
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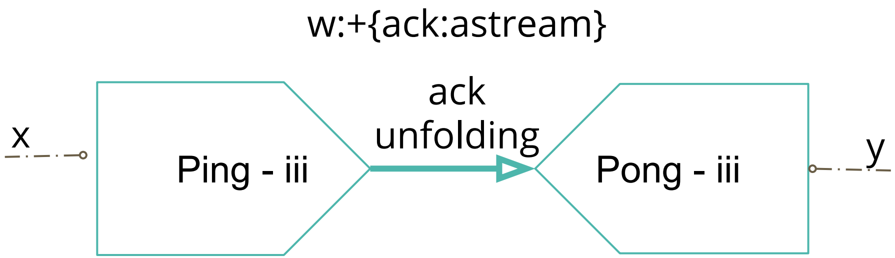

# Ping-Pong: an invalid program

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$$x : \mathsf{nat} \vdash \mathtt{Ping} :: (w : \mathsf{astream})$$
$$w : \mathsf{astream} \vdash \mathtt{Pong} :: (y : \mathsf{nat})$$
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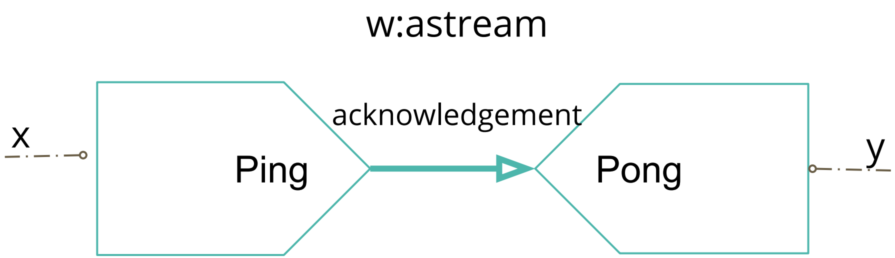


**Back to the original configuration.**

# Ping-Pong: an invalid program - code

$y \leftarrow \texttt{PingPong} \leftarrow x =$

$\quad w \leftarrow \texttt{Ping} \leftarrow x;$ $\qquad$ % *spawn process* $\texttt{Ping}_w$

$\qquad y \leftarrow \texttt{Pong} \leftarrow w$ $\qquad$ % *continue with a tail call*

Keep calling itself without communicating with its external channels

| | |
|---|---|
| $w \leftarrow \texttt{Ping} \leftarrow x =$ | $[0, 0\ , 0\ , 0, 0, 0]$ |
| $\quad \textbf{case}\ Rw\ (\nu_{astream} \Rightarrow$ | $[0, 0, -1, 0, 0, 0]$ |
| $\qquad \textbf{case}\ Rw\ (head \Rightarrow Rw.\mu_{ack};$ | $[0, 1, -1, 0, 0, 0]$ |
| $\qquad\qquad Rw.ack; w \leftarrow \texttt{Ping} \leftarrow x$ | $[0, 1, -1, 0, 0, 0]$ |
| $\qquad\quad \mid tail \Rightarrow w \leftarrow \texttt{Ping} \leftarrow x))$ | $[0, 0, -1, 0, 0, 0]$ |

| | |
|---|---|
| $y \leftarrow \texttt{Pong} \leftarrow w =$ | $[0, 0, 0, 0, 0, 0]$ |
| $\quad Lw.\nu_{astream};$ | $[0, 0, 0, 1, 0, 0]$ |
| $\quad Lw.head;$ | $[0, 0, 0, 1, 0, 0]$ |
| $\quad \textbf{case}\ Lw\ (\mu_{ack} \Rightarrow$ | $[-1, 0, 0, 1, 0, 0]$ |
| $\qquad \textbf{case}\ Lw\ ($ | $[-1, 0, 0, 1, 0, 0]$ |
| $\qquad\quad ack \Rightarrow Ry.\mu_{nat};$ | $[-1, 0, 0, 1, 0, 1]$ |
| $\qquad\qquad Ry.s;$ | $[-1, 0, 0, 1, 0, 1]$ |
| $\qquad\qquad y \leftarrow \texttt{Pong} \leftarrow w))$ | $[-1, 0, 0, 1, 0, 1]$ |

# A valid configuration of processes satisfies strong progress

**We define strong progress as a predicate**

$$\mathcal{C} \in [\![x : A]\!]$$

$$\cdot \vdash \mathcal{C} :: (x{:}A) \quad \xLeftrightarrow{\text{Bisimulation}} \quad \cdot \vdash \mathcal{C} \in [\![x : A]\!]$$

***Theorem.*** *If configuration C is well-typed then there is an infinite derivation for its strong progress property. Moreover, if it C is valid, the infinite derivation is a proof.*

# Conclusion

**We introduced an infinitary sequent calculus for first order intuitionistic multiplicative additive linear logic with fixed points [2].**

**Our main motivation for introducing this calculus is to reason about programs behaviour. In particular we use this calculus to give a direct proof for the strong progress property of locally valid binary session typed processes [2]. The importance of a direct proof other than its elegance is that it can be adapted for a more general validity condition on processes without the need to prove cut elimination productivity for their underlying derivations.**

# Send me an Email!

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