# Lecture 13

# Homotopy pushouts

We can use higher inductive types to attach cells to types. For example, when we are given a type $A$, and we have a map $f : \mathbb{S}^1 \to A$ describing a circle in $A$. Then we can form a new type $A'$ in which we attach a disc by 'gluing' the boundary of the disc to the circle in $A$. Using higher inductive types, this process of attaching a disc works as follows:

(i) First we add all the points of $A$ to $A'$, i.e. $A'$ comes equipped with a map

$$i : A \to A'$$

(ii) Next, we add a new point, which is to be thought of as the center of the disc that we're attaching. In other words, $A'$ comes equipped with

$$\mathrm{pt} : A'$$

(iii) Finally, for each point $x$ on the circle we add a path from the center of the disc to $i(f(x))$. In other words, $A'$ comes equipped with a path constructor

$$r : \prod_{(x:\mathbb{S}^1)} \mathrm{pt} = i(f(x)).$$

Moreover, since we're only attaching a disc to $A$ along $f$, we suppose that $A'$ satisfies an induction principle with respect to the constructors $i$, pt, and $r$.

The process of attaching a disc to a type $A$ along a map $f : \mathbb{S}^1 \to A$ can be generalized, so that we will also be able to attach cells of different shapes to a type. This generalization is called homotopy pushouts. Homotopy pushouts are dual to homotopy pullbacks. However, unlike pullbacks we will *assume* that pushouts exist by postulating rules for higher inductive types. For the purpose of this course, the only higher inductive types that we add to our type theory are the pushouts. Some of the more exotic higher inductive types, including the Cauchy real numbers, are described in [2].

## 13.1 Pushouts as higher inductive types

The idea of pushouts is to glue two types $A$ and $B$ together using a mediating type $S$ and maps $f : S \to A$ and $g : S \to B$. In other words, we start with a diagram of the form

$$A \xleftarrow{f} S \xrightarrow{g} B.$$

We call such a triple $\mathcal{S} \equiv (S, f, g)$ a **span** from $A$ to $B$. A span from $A$ to $B$ can be thought of as a relation from $A$ to $B$, relating $f(x)$ to $g(x)$ for any $x : S$. Indeed, an equivalence between the type of all spans and the type of relations from $A$ to $B$ is established in Exercise 13.1.

Given a span $\mathcal{S}$ from $A$ to $B$, we form the higher inductive type $A \sqcup^{\mathcal{S}} B$. It comes equipped with the following constructors

$$\begin{aligned} \mathsf{inl} &: A \to A \sqcup^{\mathcal{S}} B \\ \mathsf{inr} &: B \to A \sqcup^{\mathcal{S}} B \\ \mathsf{glue} &: \textstyle\prod_{(x:S)} \mathsf{inl}(f(x)) = \mathsf{inr}(g(x)) \end{aligned}$$

and we require that it satisfies an induction principle and computation rules.

To see what the induction principle has to be, consider first a dependent function $s : \prod_{(x:A \sqcup^{\mathcal{S}} B)} P(x)$. When we evaluate this function at the constructors, we obtain

$$\begin{aligned} s \circ \mathsf{inl} &: \textstyle\prod_{(a:A)} P(\mathsf{inl}(a)) \\ s \circ \mathsf{inr} &: \textstyle\prod_{(b:B)} P(\mathsf{inr}(b)) \\ \mathsf{apd}_s \circ \mathsf{glue} &: \textstyle\prod_{(x:S)} \mathsf{tr}_P(\mathsf{glue}(x), s(f(x))) = s(g(x)). \end{aligned}$$

**Definition 13.1.1.** Consider a span $\mathcal{S} \equiv (S, f, g)$ from $A$ to $B$, and let $P$ be a family over $A \sqcup^{\mathcal{S}} B$. The **dependent action on generators** is defined to be the map

$$\mathsf{dgen}_{\mathcal{S}}^P : \Big(\textstyle\prod_{(x:A \sqcup^{\mathcal{S}} B)} P(x)\Big) \to \Big(\textstyle\sum_{(f':\prod_{(a:A)} P(\mathsf{inl}(a)))} \textstyle\sum_{(g':\prod_{(b:B)} P(\mathsf{inr}(b)))} \textstyle\prod_{(x:S)} \mathsf{tr}_P(\mathsf{glue}(x), f'(f(x))) = g'(g(x))\Big).$$

given by $s \mapsto (s \circ \mathsf{inl}, s \circ \mathsf{inr}, \mathsf{apd}_s \circ \mathsf{glue})$.

We can now fully specify homotopy pushouts.

**Definition 13.1.2.** Given a span $\mathcal{S} \equiv (S, f, g)$, the **(homotopy) pushout** $A \sqcup^{\mathcal{S}} B$ of $\mathcal{S}$ is defined to be the higher inductive type equipped with

$$\begin{aligned}\mathsf{inl} &: A \to A \sqcup^{\mathcal{S}} B \\ \mathsf{inr} &: B \to A \sqcup^{\mathcal{S}} B \\ \mathsf{glue} &: \textstyle\prod_{(x:S)} \mathsf{inl}(f(x)) = \mathsf{inr}(g(x)),\end{aligned}$$

satisfying the **induction principle** for pushouts, which asserts that for each type family $P$ over $A \sqcup^{\mathcal{S}} B$ the map $\mathsf{dgen}_{\mathcal{S}}^{P}$ has a section.

*Remark* 13.1.3. The induction principle of the pushout $A \sqcup^{\mathcal{S}} B$ provides us with a dependent function

$$\mathsf{ind}_{\mathcal{S}}(f', g', G) : \textstyle\prod_{(x:A \sqcup^{\mathcal{S}} B)} P(x),$$

for every

$$\begin{aligned}f' &: \textstyle\prod_{(a:A)} P(\mathsf{inl}(a)) \\ g' &: \textstyle\prod_{(b:B)} P(\mathsf{inr}(b)) \\ G &: \textstyle\prod_{(x:S)} \mathsf{tr}_P(\mathsf{glue}(x), f'(f(x))) = g'(g(x))\end{aligned}$$

Moreover, the function $\mathsf{ind}_{\mathcal{S}}(f', g', G)$ comes equipped with an identification

$$\mathsf{dgen}_{\mathcal{S}}(\mathsf{ind}_{\mathcal{S}}(f', g', G)) = (f', g', G).$$

Writing $s :\equiv \mathsf{ind}_{\mathcal{S}}(f', g', G)$, we see that such an identification between triples is equivalently described by a triple $(H, K, L)$ consisting of

$$\begin{aligned}H &: s \circ \mathsf{inl} \sim f' \\ K &: s \circ \mathsf{inr} \sim g'\end{aligned}$$

and a homotopy $L$ witnessing that the square

$$\begin{array}{ccc}\mathsf{tr}_P(\mathsf{glue}(x), s(\mathsf{inl}(f(x)))) & \xlongequal{\mathsf{ap}_{\mathsf{tr}_P(\mathsf{glue}(x))}(H(x))} & \mathsf{tr}_P(\mathsf{glue}(x), f'(f(x))) \\ {\scriptstyle \mathsf{apd}_s(\mathsf{glue}(x))}\Big\| & & \Big\|{\scriptstyle G(x)} \\ s(\mathsf{inr}(g(x))) & \xlongequal[K(x)]{} & g'(g(x))\end{array}$$

commutes, for every $x : S$. These are the **computation rules** for pushouts.

## 13.2 Examples of pushouts

Many interesting types can be defined as homotopy pushouts.

**Definition 13.2.1.** Let $X$ be a type. We define the **suspension** $\Sigma X$ of $X$ to be the pushout of the span

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathsf{inr}} \\ \mathbf{1} & \xrightarrow[\mathsf{inl}]{} & \Sigma X \end{array}$$

**Definition 13.2.2.** We define the $n$**-sphere** $\mathbb{S}^n$ for any $n : \mathbb{N}$ by induction on $n$, by taking

$$\begin{aligned} \mathbb{S}^0 &:\equiv \mathbf{2} \\ \mathbb{S}^{n+1} &:\equiv \Sigma \mathbb{S}^n. \end{aligned}$$

**Definition 13.2.3.** Given a map $f : A \to B$, we define the **cofiber** $\mathsf{cofib}_f$ of $f$ as the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathsf{inr}} \\ \mathbf{1} & \xrightarrow[\mathsf{inl}]{} & \mathsf{cofib}_f. \end{array}$$

The cofiber of a map is sometimes also called the **mapping cone**.

*Example* 13.2.4. The suspension $\Sigma X$ of $X$ is the cofiber of the map $X \to \mathbf{1}$.

**Definition 13.2.5.** We define the **join** $X * Y$ of $X$ and $Y$ to be the pushout

$$\begin{array}{ccc} X \times Y & \xrightarrow{\mathsf{pr}_2} & Y \\ {\scriptstyle \mathsf{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{inr}} \\ X & \xrightarrow[\mathsf{inl}]{} & X * Y. \end{array}$$

**Definition 13.2.6.** Suppose $A$ and $B$ are pointed types, with base points $a_0$ and $b_0$, respectively. The **(binary) wedge** $A \vee B$ of $A$ and $B$ is defined as the pushout

$$\begin{array}{ccc} \mathbf{2} & \longrightarrow & A + B \\ \big\downarrow & & \big\downarrow \\ \mathbf{1} & \longrightarrow & A \vee B. \end{array}$$

**Definition 13.2.7.** Given a type $I$, and a family of pointed types $A$ over $i$, with base points $a_0(i)$. We define the **(indexed) wedge** $\bigvee_{(i:I)} A_i$ as the pushout

$$\begin{array}{ccc} I & \xrightarrow{\lambda i.\,(i,a_0(i))} & \sum_{(i:I)} A_i \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \bigvee_{(i:I)} A_i. \end{array}$$

## 13.3 The universal property of pushouts

**Definition 13.3.1.** Consider a span $\mathcal{S} \equiv (S, f, g)$ from $A$ to $B$, and let $X$ be a type. A **cocone** with vertex $X$ on $\mathcal{S}$ is a triple $(i, j, H)$ consisting of maps $i : A \to X$ and $j : B \to X$, and a homotopy $H : i \circ f \sim j \circ g$ witnessing that the square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & X \end{array}$$

commutes. We write $\mathsf{cocone}_{\mathcal{S}}(X)$ for the type of cocones on $\mathcal{S}$ with vertex $X$.

**Definition 13.3.2.** Consider a cocone $(i, j, H)$ with vertex $X$ on the span $\mathcal{S} \equiv (S, f, g)$, as indicated in the following commuting square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & X. \end{array}$$

For every type $Y$, we define the map

$$\mathsf{cocone\_map}(i, j, H) : (X \to Y) \to \mathsf{cocone}(Y)$$

by $f \mapsto (f \circ i, f \circ j, f \cdot H)$.

**Definition 13.3.3.** A commuting square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & X. \end{array}$$

with $H : i \circ f \sim j \circ g$ is said to be a **(homotopy) pushout square** if the cocone $(i, j, H)$ with vertex $X$ on the span $\mathcal{S} \equiv (S, f, g)$ satisfies the **universal property of pushouts**, which asserts that the map

$$\mathsf{cocone\_map}(i, j, H) : (X \to Y) \to \mathsf{cocone}(Y)$$

is an equivalence for any type $Y$. Sometimes pushout squares are also called **cocartesian squares**.

**Lemma 13.3.4.** *For any span $\mathcal{S} \equiv (S, f, g)$ from $A$ to $B$, and any type $X$ the square*

$$\begin{array}{ccc} \mathsf{cocone}_{\mathcal{S}}(X) & \xrightarrow{\pi_2} & X^B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle -\circ g} \\ X^A & \xrightarrow[-\circ f]{} & X^S, \end{array}$$

*which commutes by the homotopy $\pi_3' :\equiv \lambda(i, j, H).\,\mathsf{eq\_htpy}(H)$, is a pullback square.*

*Proof.* The gap map $\mathsf{cocone}_{\mathcal{S}}(X) \to X^A \times_{X^S} X^B$ is the function

$$\lambda(i, j, H).\,(i, j, \mathsf{eq\_htpy}(H)).$$

This is an equivalence by Theorem 7.1.3, since it is the induced map on total spaces of the fiberwise equivalence $\mathsf{eq\_htpy}$. Therefore, the square is a pullback square by Theorem 10.2.6. □

In the following theorem we establish an alternative characterization of the universal property of pushouts.

**Theorem 13.3.5.** *Consider a commuting square*

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & X, \end{array}$$

*with $H : i \circ f \sim j \circ g$. The following are equivalent:*

(i) *The square is a pushout square.*

*(ii) The square*

$$\begin{array}{ccc} T^X & \xrightarrow{-\circ j} & T^B \\ {\scriptstyle -\circ i}\downarrow & & \downarrow{\scriptstyle -\circ g} \\ T^A & \xrightarrow[-\circ f]{} & T^S \end{array}$$

*which commutes by the homotopy*

$$\lambda h.\,\mathsf{eq\_htpy}(h \cdot H)$$

*is a pullback square, for every type $T$.*

*Proof.* It is straightforward to verify that the triangle

$$\begin{array}{ccc} & T^X & \\ {\scriptstyle \mathsf{cocone\_map}(i,j,H)}\swarrow & & \searrow{\scriptstyle \mathsf{gap}(-\circ i,-\circ j,\mathsf{eq\_htpy}(-\cdot H))} \\ \mathsf{cocone}(T) & \xrightarrow[\mathsf{gap}(i,j,\mathsf{eq\_htpy}(H))]{} & T^A \times_{T^S} T^B \end{array}$$

commutes. Since the bottom map is an equivalence by Lemma 13.3.4, it follows that if either one of the remaining maps is an equivalence, so is the other. The claim now follows by Theorem 10.2.6. □

*Example* 13.3.6. By Exercise 12.1 and the second characterization of pushouts in Theorem 13.3.5 it follows that the circle is a pushout

$$\begin{array}{ccc} \mathbf{2} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbb{S}^1. \end{array}$$

In other words, $\mathbb{S}^1 \simeq \mathbf{\Sigma 2}$.

**Theorem 13.3.7.** *Consider a span $\mathcal{S} \equiv (S, f, g)$ from $A$ to $B$. Then the square*

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{inr}} \\ A & \xrightarrow[\mathsf{inl}]{} & A \sqcup^{\mathcal{S}} B \end{array}$$

*is a pushout square.*

*Proof.* Let $X$ be a type. Our goal is to show that the map

$$\mathsf{cocone\_map}(\mathsf{inl}, \mathsf{inr}, \mathsf{glue}) : (A \sqcup^{\mathcal{S}} B \to X) \to \mathsf{cocone}_{\mathcal{S}}(X)$$

is an equivalence. For notational breveity we will just write $\mathsf{gen}_{\mathcal{S}}$ for $\mathsf{cocone\_map}_{\mathcal{S}}(\mathsf{inl}, \mathsf{inr}, \mathsf{glue})$, because $\mathsf{cocone\_map}_{\mathcal{S}}(\mathsf{inl}, \mathsf{inr}, \mathsf{glue})$ is just the action on generators.

We first note that by Exercise 4.3 there is a commuting triangle

$$\begin{array}{ccc} & X^{A\sqcup^{\mathcal{S}}B} & \\ {\scriptstyle \mathsf{gen}_{\mathcal{S}}}\swarrow & & \searrow{\scriptstyle \mathsf{dgen}_{\mathcal{S}}} \\ \mathsf{cocone}_{\mathcal{S}}(X) & \xrightarrow{\simeq} & \mathsf{cocone}'_{\mathcal{S}}(X) \end{array}$$

where we write

$$\mathsf{cocone}'_{\mathcal{S}}(X) : \Big(\sum_{(f':A\to X)}\sum_{(g':A\to X)} \prod_{(x:S)} \mathsf{tr}_{W_{A\sqcup^{\mathcal{S}}B}(X)}(\mathsf{glue}(x), f'(f(x))) = g'(g(x))\Big).$$

By the induction principle for $A \sqcup^{\mathcal{S}} B$ we have a section $\mathsf{ind}_{\mathcal{S}}$ of $\mathsf{dgen}_{\mathcal{S}}$. Thus we obtain a section $\mathsf{rec}_{\mathcal{S}}$ of $\mathsf{gen}_{\mathcal{S}}$. Our goal is now to show that $\mathsf{rec}_{\mathcal{S}}$ is also a retraction of $\mathsf{gen}_{\mathcal{S}}$. We establish in Lemma 13.3.8 that

$$(\mathsf{gen}_{\mathcal{S}}(\mathsf{rec}_{\mathcal{S}}(\mathsf{gen}_{\mathcal{S}}(h))) = \mathsf{gen}_{\mathcal{S}}(h)) \to (\mathsf{rec}_{\mathcal{S}}(\mathsf{gen}_{\mathcal{S}}(h)) = h)$$

Then we obtain that $\mathsf{rec}_{\mathcal{S}}$ is a retraction of $\mathsf{gen}_{\mathcal{S}}$ by using this implication and the fact that $\mathsf{rec}_{\mathcal{S}}$ is a section of $\mathsf{gen}_{\mathsf{S}}$. □

**Lemma 13.3.8.** *Let $h, h' : A \sqcup^{\mathcal{S}} B \to X$ be two functions. Then we have*

$$(\mathsf{gen}_{\mathcal{S}}(h) = \mathsf{gen}_{\mathcal{S}}(h')) \to (h = h').$$

*Proof.* Suppose we have $\mathsf{gen}_{\mathcal{S}}(h) = \mathsf{gen}_{\mathcal{S}}(h')$. This type of equalities between triples is equivalent to the type of triples $(K, L, M)$ consisting of

$$K : h \circ \mathsf{inl} \sim h' \circ \mathsf{inl}$$
$$L : h \circ \mathsf{inr} \sim h' \circ \mathsf{inr},$$

and a homotopy $M$ witnessing that the square

$$\begin{array}{ccc} h \circ \mathsf{inl} \circ f & \xrightarrow{K \cdot f} & h' \circ \mathsf{inl} \circ f \\ {\scriptstyle h \cdot \mathsf{glue}}\downarrow & & \downarrow{\scriptstyle h' \cdot \mathsf{glue}} \\ h \circ \mathsf{inr} \circ f & \xrightarrow[L \cdot g]{} & h' \circ \mathsf{inr} \circ g \end{array}$$

of homotopies commutes. By function extensionality, our goal is equivalent to constructing a homotopy (i.e. a dependent function) of type

$$\prod_{(t:A \sqcup^{\mathcal{S}} B)} f(t) = g(t).$$

We will construct such a function by the induction principle for $A \sqcup^{\mathcal{S}} B$. Therefore it suffices to construct

$$\begin{aligned} K &: h \circ \mathsf{inl} \sim h' \circ \mathsf{inl} \\ L &: h \circ \mathsf{inr} \sim h' \circ \mathsf{inr} \\ M' &: \mathsf{tr}_{E_{h,h'}}(\mathsf{glue}, K) = L \end{aligned}$$

The type of $M'$ is equivalent to the type of $M$, so we obtain the requested structure from our assumptions. □

As a basic application we establish the universal property of suspensions.

**Corollary 13.3.9.** *Let $X$ and $Y$ be types. Then the map*

$$(\Sigma X \to Y) \to \sum_{(y,y':Y)} X \to (y = y')$$

*given by $f \mapsto (f(\mathsf{inl}(\star)), f(\mathsf{inr}(\star)), \mathsf{ap}_f(\mathsf{glue}(-)))$ is an equivalence.*

*Proof.* We have equivalences

$$\begin{aligned} (\Sigma X \to Y) &\simeq \sum_{(y,y':\mathbf{1} \to Y)} X \to (y(\star) = y'(\star)) \\ &\simeq \sum_{(y,y':Y)} X \to (y = y'). \end{aligned}$$ □

## 13.4 The pasting property for pushouts

**Theorem 13.4.1.** *Consider the following configuration of commuting squares:*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{k} & C \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[j]{} & Y & \xrightarrow[l]{} & Z \end{array}$$

*with homotopies $H : j \circ f \sim g \circ i$ and $K : l \circ g \sim h \circ k$, and suppose that the square on the left is a pushout square. Then the square on the right is a pushout square if and only if the outer rectangle is a pushout square.*

*Proof.* Let $T$ be a type. Taking the exponent $T^{(-)}$ of the entire diagram of the statement of the theorem, we obtain the following commuting diagram

$$\begin{array}{ccccc} T^Z & \xrightarrow{-\circ l} & T^Y & \xrightarrow{-\circ j} & T^X \\ {\scriptstyle -\circ h}\downarrow & & {\scriptstyle -\circ g}\downarrow & & \downarrow{\scriptstyle -\circ f} \\ T^C & \xrightarrow[-\circ k]{} & T^B & \xrightarrow[-\circ i]{} & T^A. \end{array}$$

By the assumption that $Y$ is the pushout of $B \leftarrow A \rightarrow X$, it follows that the square on the right is a pullback square. It follows by Theorem 10.6.1 that the rectangle on the left is a pullback if and only if the outer rectangle is a pullback. Thus the statement follows by the second characterization in Theorem 13.3.5. □

**Lemma 13.4.2.** *Consider a map $f : A \to B$. Then the cofiber of the map $\mathsf{inr} : B \to \mathsf{cofib}_f$ is equivalent to the suspension $\Sigma A$ of $A$.*

## Exercises

13.1 Use Theorems 9.1.4 and 11.4.4 and Corollary 9.2.2 to show that the type

$$\mathsf{span}(A, B) :\equiv \textstyle\sum_{(S:\mathcal{U})}(S \to A) \times (S \to B)$$

of small spans from $A$ to $B$ is equivalent to the type $A \to (B \to \mathcal{U})$ of small relations from $A$ to $B$.

13.2 Use Theorems 9.3.3 and 13.3.5 and Corollary 10.5.6 to show that for any commuting square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & C \end{array}$$

where $f$ is an equivalence, the square is a pushout square if and only if $j : B \to C$ is an equivalence. Use this observation to conclude the following:

(i) If $X$ is contractible, then $\Sigma X$ is contractible.

(ii) The cofiber of any equivalence is contractible.

(iii) The cofiber of a point in $B$ (i.e. of a map of the type $\mathbf{1} \to B$) is equivalent to $B$.

(iv) There is an equivalence $X \simeq \mathbf{0} * X$.

(v) If $X$ is contractible, then $X * Y$ is contractible.

(vi) If $A$ is contractible, then there is an equivalence $A \vee B \simeq B$ for any pointed type $B$.

13.3 Let $P$ and $Q$ be propositions.

(a) Show that $P * Q$ satisfies the *universal property of disjunction*, i.e. that for any proposition $R$, the map

$$(P * Q \to R) \to (P \to R) \times (Q \to R)$$

given by $f \mapsto (f \circ \mathsf{inl}, f \circ \mathsf{inr})$, is an equivalence.

(b) Use the proposition $R :\equiv \mathsf{is\_contr}(P * Q)$ to show that $P * Q$ is again a proposition.

13.4 Let $Q$ be a proposition, and let $A$ be a type. Show that the following are equivalent:

(a) The map $(Q \to A) \to (\mathbf{0} \to A)$ is an equivalence.
(b) The type $A^Q$ is contractible.
(c) There is a term of type $Q \to \mathsf{is\_contr}(A)$.
(d) The map $\mathsf{inr} : A \to Q * A$ is an equivalence.

13.5 Let $P$ be a proposition. Show that $\Sigma P$ is a set, with an equivalence

$$\big(\mathsf{inl}(\star) = \mathsf{inr}(\star)\big) \simeq P.$$

13.6 Show that $A \sqcup^{\mathcal{S}} B \simeq B \sqcup^{\mathcal{S}^{\mathsf{op}}} A$, where $\mathcal{S}^{\mathsf{op}} :\equiv (S, g, f)$ is the **opposite span** of $\mathcal{S}$.

13.7 Use Exercise 10.6.b to show that if

$$\begin{array}{ccc} S & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is a pushout square, then so is

$$\begin{array}{ccc} A \times S & \longrightarrow & A \times Y \\ \downarrow & & \downarrow \\ A \times X & \longrightarrow & A \times Z \end{array}$$

for any type $A$.

13.8 Use Exercise 10.5 to show that if

$$\begin{array}{ccc} S_1 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & Z_1 \end{array} \qquad \begin{array}{ccc} S_2 & \longrightarrow & Y_2 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & Z_2 \end{array}$$

are pushout squares, then so is

$$\begin{array}{ccc} S_1 + S_2 & \longrightarrow & Y_1 + Y_2 \\ \downarrow & & \downarrow \\ X_1 + X_2 & \longrightarrow & Z_1 + Z_2. \end{array}$$

13.9 (a) Consider a span $(S, f, g)$ from $A$ to $B$. Use Exercise 10.4 to show that the square

$$\begin{array}{ccc} S + S & \xrightarrow{[\mathsf{id},\mathsf{id}]} & S \\ {\scriptstyle f+g}\downarrow & & \downarrow{\scriptstyle \mathsf{inr} \circ g} \\ A + B & \xrightarrow[{[\mathsf{inl},\mathsf{inr}]}]{} & A \sqcup^S B \end{array}$$

is again a pushout square.

(b) Show that $\Sigma X \simeq \mathbf{2} * X$.

13.10 Consider a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & X & \end{array}$$

with $H : f \sim g \circ h$.

(a) Construct a map $\mathsf{cofib}_{(h,H)} : \mathsf{cofib}_g \to \mathsf{cofib}_f$.

(b) Use Exercise 10.10 to show that $\mathsf{cofib}_{\mathsf{cofib}(h,H)} \simeq \mathsf{cofib}_h$.

13.11 Use Exercise 12.9 to show that for $n \geq 0$, $X$ is an $n$-type if and only if the map

$$\lambda x.\,\mathsf{const}_x : X \to (\mathbb{S}^{n+1} \to X)$$

is an equivalence.

13.12 (a) Construct for every $f : X \to Y$ a function

$$\Sigma f : \Sigma X \to \Sigma Y.$$

(b) Show that if $f \sim g$, then $\Sigma f \sim \Sigma g$.

(c) Show that $\Sigma \mathsf{id}_X \sim \mathsf{id}_{\Sigma X}$
(d) Show that
$$\Sigma(g \circ f) \sim (\Sigma g) \circ (\Sigma f).$$
for any $f : X \to Y$ and $g : Y \to Z$.

13.13 Consider a commuting diagram of the form

$$\begin{array}{ccccc} A_0 & \longleftarrow & B_0 & \longrightarrow & C_0 \\ \uparrow & & \uparrow & & \uparrow \\ A_1 & \longleftarrow & B_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_2 & \longleftarrow & B_2 & \longrightarrow & C_2 \end{array}$$

with homotopies filling the (small) squares. Use Exercise 10.11 to construct an equivalence

$$\begin{aligned}(A_0 \sqcup^{B_0} C_0) \sqcup^{(A_1 \sqcup^{B_1} C_1)} (A_2 \sqcup^{B_2} C_2) \\ \simeq (A_0 \sqcup^{A_1} A_2) \sqcup^{(B_0 \sqcup^{B_1} B_2)} (C_0 \sqcup^{C_1} C_2).\end{aligned}$$

This is known as the **3-by-3 lemma** for pushouts.

13.14 (a) Let $I$ be a type, and let $A$ be a family over $I$. Construct an equivalence
$$\Big(\bigvee_{(i:I)} \Sigma A_i\Big) \simeq \Sigma\Big(\bigvee_{(i:I)} A_i\Big).$$
(b) Show that for any type $X$ there is an equivalence
$$\Big(\bigvee_{(x:X)} \mathbf{2}\Big) \simeq X + 1.$$
(c) Construct an equivalence
$$\Sigma(\mathsf{Fin}(n+1)) \simeq \bigvee_{(i:\mathsf{Fin}(n))} \mathbb{S}^1.$$

13.15 Show that $\mathsf{Fin}(n+1) * \mathsf{Fin}(m+1) \simeq \bigvee_{(i:\mathsf{Fin}(n \cdot m))} \mathbb{S}^1$, for any $n, m : \mathbb{N}$.