Lecture 10

Homotopy pullbacks

Suppose we are given a map $f : A \to B$, and type families P over A, and Q over B. Then any fiberwise map

$$g : \prod_{(x:A)} P(x) \to Q(f(x))$$

gives rise to a commuting square

$$\begin{array}{ccc}
\sum_{(x:A)} P(x) & \xrightarrow{\mathsf{total}_f(g)} & \sum_{(y:B)} Q(y) \\
{\scriptstyle \mathsf{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{pr}_1} \\
A & \xrightarrow[f]{} & B
\end{array}$$

where $\mathsf{total}_f(g)$ is defined as $\lambda(x,p).\,(f(x), g(x,y))$. We will show in Theorem 10.5.2 that g is a fiberwise equivalence if and only if this square is a *pullback square*. This generalization of Theorem 7.1.3 is therefore abstracting away from the notion of fiberwise equivalence, and it serves as our motivating theorem to introduce pullbacks. The connection between pullbacks and fiberwise equivalences has an important role in the descent theorem in Lecture 14.

10.1 Cartesian squares

Recall that a square

$$\begin{array}{ccc}
C & \xrightarrow{q} & B \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\
A & \xrightarrow[g]{} & X
\end{array}$$

is said to **commute** if there is a homotopy $H : f \circ p \sim g \circ q$. The pullback property is a *universal property* of the upper left corner of a commuting square (in our case C), characterizing the maps *into* it.

To describe the universal property of pullbacks we first need to have a closer look at the *anatomy* of commuting squares.

Definition 10.1.1. A commuting square

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f \circ p \sim g \circ q$ can be dissected into three parts, consisting of a *cospan*, a type, and a *cone*, where

(i) A **cospan** consists of three types A, X, and B, and maps $f : A \to X$ and $g : B \to X$.

(ii) Given a type C, a **cone** on the cospan $A \xrightarrow{f} X \xleftarrow{g} B$ with **vertex** C consists of maps $p : C \to A$, $q : C \to B$ and a homotopy $H : f \circ p \sim g \circ q$. We write

$$\mathsf{cone}(C) :\equiv \textstyle\sum_{(p:C\to A)}\sum_{(q:C\to B)} f \circ p \sim g \circ q$$

for the type of cones with vertex C.

Given a cone with vertex C on a span $A \xrightarrow{f} X \xleftarrow{g} B$ and a map $h : C' \to C$, we construct a new cone with vertex C' in the following definition.

Definition 10.1.2. For any cone (p, q, H) with vertex C and any type C', we define a map

$$\mathsf{cone\_map}(p, q, H) : (C' \to C) \to \mathsf{cone}(C')$$

by $h \mapsto (p \circ h, q \circ h, H \circ h)$.

Definition 10.1.3. We say that a commuting square

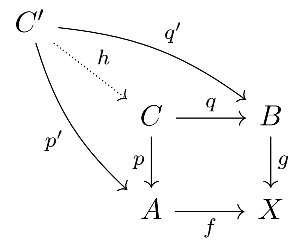
$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f \circ p \sim g \circ q$ is a **pullback square**, or that it is **cartesian**, if it satisfies the **universal property** of pullbacks, which asserts that the map

$$\mathsf{cone\_map}(p, q, H) : (C' \to C) \to \mathsf{cone}(C')$$

is an equivalence for every type C'.

We often indicate the universal property with a diagram as follows:



since the universal property states that for every cone (p', q', H') with vertex C', the type of pairs (h, α) consisting of $h : C' \to C$ equipped with $\alpha : \mathsf{cone\_map}((p, q, H), h) = (p', q', H')$ is contractible by Theorem 6.3.3.

In order to see what goes on in the universal property of pullbacks, we need to first characterize the identity type of $\mathsf{cone}(C)$, for any type C.

Lemma 10.1.4. *Let (p, q, H) and (p', q', H') be cones on a cospan $f : A \to X \leftarrow B : g$, both with vertex C. Then the type $(p, q, H) = (p', q', H')$ is equivalent to the type of triples (K, L, M) consisting of*

$$\begin{aligned} K &: p \sim p' \\ L &: q \sim q' \\ M &: H \cdot (g \cdot L) \sim (f \cdot K) \cdot H' \end{aligned}$$

Remark 10.1.5. The homotopy M witnesses that the square

$$\begin{array}{ccc} f \circ p & \xrightarrow{f \cdot K} & f \circ p' \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H'} \\ g \circ q & \xrightarrow[g \cdot L]{} & g \circ q' \end{array}$$

of homotopies commutes. Therefore M is a homotopy of homotopies, and for each $z : C$ the identification $M(z)$ witnesses that the square of identifications

$$\begin{array}{ccc} f(p(z)) & \overset{\mathsf{ap}_f(K(z))}{=\!=\!=\!=} & f(p'(z)) \\ {\scriptstyle H(z)}\Vert & & \Vert{\scriptstyle H'(z)} \\ g(q(z)) & \underset{\mathsf{ap}_g(L(z))}{=\!=\!=\!=} & g(q'(z)) \end{array}$$

commutes.

Proof of Lemma 10.1.4. By the fundamental theorem of identity types (Theorem 7.2.1) and associativity of Σ-types (Exercise 5.10) it suffices to show that the type

$$\sum_{(p':C\to A)}\sum_{(q':C\to B)}\sum_{(H':f\circ p'\sim g\circ q')}\sum_{(K:p\sim p')}\sum_{(L:q\sim q')} H \cdot (g\cdot L) \sim (f\cdot K)\cdot H'$$

is contractible. Now we apply Exercise 5.11 repeatedly to see that this type is equivalent to the type

$$\sum_{(p':C\to A)}\sum_{(K:p\sim p')}\sum_{(q':C\to B)}\sum_{(L:q\sim q')}\sum_{(H':f\circ p'\sim g\circ q')} H \cdot (g\cdot L) \sim (f\cdot K)\cdot H'.$$

The types $\sum_{(p':C\to A)} p\sim p'$ and $\sum_{(q':C\to B)} q\sim q'$ are contractible by function extensionality, and we have

$$\begin{aligned}(p,\mathsf{htpy.refl}_p) &: \textstyle\sum_{(p':C'\to A)} p\sim p'\\ (q,\mathsf{htpy.refl}_q) &: \textstyle\sum_{(q':C'\to B)} q\sim q'.\end{aligned}$$

Thus we apply Exercise 6.4 to see that the type of tuples (p',K,q',L,H',M) is equivalent to the type

$$\sum_{(H':f\circ p'\sim g\circ q')} H\cdot \mathsf{htpy.refl}_{g\circ q} \sim \mathsf{htpy.refl}_{f\circ p}\cdot H'.$$

Of course, the type $H\cdot\mathsf{htpy.refl}_{g\circ q}\sim\mathsf{htpy.refl}_{f\circ p}\cdot H'$ is equivalent to the type $H\sim H'$, and $\sum_{(H':f\circ p\sim g\circ q)} H\sim H'$ is contractible. □

As a corollary we obtain the following characterization of the universal property of pullbacks.

Theorem 10.1.6. *Consider a commuting square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f\circ p\sim g\circ q$ *Then the following are equivalent:*

(i) The square is a pullback square.

(ii) For every type C' and every cone (p', q', H') with vertex C', the type of quadruples (h, K, L, M) consisting of

$$\begin{aligned} h &: C' \to C \\ K &: p \circ h \sim p' \\ L &: q \circ h \sim q' \\ M &: (H \cdot h) \cdot (g \cdot L) \sim (f \cdot K) \cdot H' \end{aligned}$$

is contractible.

Remark 10.1.7. The homotopy M in Theorem 10.1.6 witnesses that the square

$$\begin{array}{ccc} f \circ p \circ h & \xrightarrow{f \cdot K} & f \circ p' \\ {\scriptstyle H \cdot h}\big\downarrow & & \big\downarrow{\scriptstyle H'} \\ g \circ q \circ h & \xrightarrow[g \cdot L]{} & g \circ q' \end{array}$$

of homotopies commutes.

10.2 The unique existence of pullbacks

Definition 10.2.1. Let $f : A \to X$ and $B \to X$ be maps. Then we define

$$\begin{aligned} A \times_X B &:\equiv \textstyle\sum_{(x:A)} \sum_{(y:B)} f(x) = g(y) \\ \pi_1 &:\equiv \mathsf{pr}_1 && : A \times_X B \to A \\ \pi_2 &:\equiv \mathsf{pr}_1 \circ \mathsf{pr}_2 && : A \times_X B \to B \\ \pi_3 &:\equiv \mathsf{pr}_2 \circ \mathsf{pr}_2 && : f \circ \pi_1 \sim g \circ \pi_2. \end{aligned}$$

The type $A \times_X B$ is called the **canonical pullback** of f and g.

Note that $A \times_X B$ depends on f and g, although this dependency is not visible in the notation.

Theorem 10.2.2. *Given maps $f : A \to X$ and $g : B \to X$, the commuting square*

$$\begin{array}{ccc} A \times_X B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X, \end{array}$$

is a pullback square.

Proof. Let C be a type. Our goal is to show that the map

$$\mathsf{cone\_map}(\pi_1, \pi_2, \pi_3) : (C \to A \times_X B) \to \mathsf{cone}(C)$$

is an equivalence. By double application of Theorem 9.1.4 we obtain equivalences

$$\begin{aligned}(C \to A \times_X B) &\equiv C \to \textstyle\sum_{(x:A)}\sum_{(y:B)} f(x) = g(y) \\ &\simeq \textstyle\sum_{(p:C\to A)}\prod_{(z:C)}\sum_{(y:B)} f(p(z)) = y \\ &\simeq \textstyle\sum_{(p:C\to A)}\sum_{(q:C\to B)}\prod_{(z:C)} f(p(z)) = g(q(z)) \\ &\equiv \mathsf{cone}(C)\end{aligned}$$

The composite of these equivalences is the map

$$\lambda f.\, (\lambda z.\, \mathsf{pr}_1(f(z)), \lambda z.\, \mathsf{pr}_1(\mathsf{pr}_2(f(z))), \lambda z.\, \mathsf{pr}_2(\mathsf{pr}_2(f(z)))),$$

which is *exactly* the map $\mathsf{cone\_map}(\pi_1, \pi_2, \pi_3)$, and since it is a composite of equivalences it follows that it is itself an equivalence. □

In the following lemma we establish the uniqueness of pullbacks up to equivalence via a *3-for-2 property* for pullbacks.

Lemma 10.2.3. *Consider the squares*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array} \qquad\qquad \begin{array}{ccc} C' & \xrightarrow{q'} & B \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with homotopies $H : f \circ p \sim g \circ q$ *and* $H' : f \circ p' \sim g \circ q'$*. Furthermore, suppose we have a map* $h : C' \to C$ *equipped with*

$$\begin{aligned} K &: p \circ h \sim p' \\ L &: q \circ h \sim q' \\ M &: (H \cdot h) \cdot (g \cdot L) \sim (f \cdot K) \cdot H'. \end{aligned}$$

If any two of the following three properties hold, so does the third:

(i) C *is a pullback.*

(ii) C' *is a pullback.*

(iii) h *is an equivalence.*

Proof. By the characterization of the identity type of $\mathsf{cone}(C')$ given in Lemma 10.1.4 we obtain an identification

$$\mathsf{cone\_map}((p,q,H),h) = (p',q',H')$$

from the triple (K, L, M). Let D be a type, and let $k : D \to C'$ be a map. We observe that

$$\begin{aligned}
\mathsf{cone_map}((p,q,H),(h\circ k)) &\equiv (p\circ(h\circ k), q\circ(h\circ k), H\circ(h\circ k)) \\
&\equiv ((p\circ h)\circ k, (q\circ h)\circ k, (H\circ h)\circ k) \\
&\equiv \mathsf{cone_map}(\mathsf{cone_map}((p,q,H),h),k) \\
&= \mathsf{cone_map}((p',q',H'),k).
\end{aligned}$$

Thus we see that the triangle

$$\begin{array}{ccc}
(D \to C') & \xrightarrow{h\circ -} & (D \to C) \\
& {\scriptstyle \mathsf{cone_map}(p',q',H')} \searrow \quad \swarrow {\scriptstyle \mathsf{cone_map}(p,q,H)} & \\
& \mathsf{cone}(D) &
\end{array}$$

commutes. Therefore it follows from the 3-for-2 property of equivalences established in Exercise 5.5, that if any two of the following properties hold, then so does the third:

(i) The map $\mathsf{cone\_map}(p,q,H) : (D \to C) \to \mathsf{cone}(D)$ is an equivalence,

(ii) The map $\mathsf{cone\_map}(p',q',H') : (D \to C') \to \mathsf{cone}(D)$ is an equivalence,

(iii) The map $h \circ - : (D \to C') \to (D \to C)$ is an equivalence.

Thus the 3-for-2 property for pullbacks follows from the fact that h is an equivalence if and only if $h \circ - : (D \to C') \to (D \to C)$ is an equivalence for any type D, which was established in Exercise 9.5. □

Pullbacks are not only unique in the sense that any two pullbacks of the same cospan are equivalent, they are *uniquely unique* in the sense that the type of quadruples (h, K, L, M) as in Lemma 10.2.3 is contractible.

Corollary 10.2.4. *Suppose both commuting squares*

$$\begin{array}{ccc}
C & \xrightarrow{q} & B \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\
A & \xrightarrow[f]{} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
C' & \xrightarrow{q'} & B \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle g} \\
A & \xrightarrow[f]{} & X
\end{array}$$

with homotopies $H : f \circ p \sim g \circ q$ *and* $H' : f \circ p' \sim g \circ q'$ *are pullback squares. Then the type of quadruples* (e, K, L, M) *consisting of an equivalence* $e : C' \simeq C$ *equipped with*

$$\begin{aligned} K &: p \circ e \sim p' \\ L &: q \circ e \sim q' \\ M &: (g \cdot L) \cdot (H \cdot e) \sim (f \cdot K) \cdot H'. \end{aligned}$$

is contractible.

Proof. We have seen that the type of quadruples (h, K, L, M) is equivalent to the fiber of $\mathsf{cone\_map}(p, q, H)$ at (p', q', H'). By Lemma 10.2.3 it follows that h is an equivalence. Since $\mathsf{is\_equiv}(h)$ is a proposition (and hence contractible as soon as it is inhabited) it follows that the type of quadruples (e, K, L, M) is contractible. □

Definition 10.2.5. Given a commuting square

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f \circ p \sim g \circ q$, we define the **gap map**

$$\mathsf{gap}(p, q, H) : C \to A \times_X B$$

by $\lambda z. (p(z), q(z), H(z))$. Furthermore, we will write

$$\mathsf{is\_pullback}(f, g, H) :\equiv \mathsf{is\_equiv}(\mathsf{gap}(p, q, H)).$$

Theorem 10.2.6. *Consider a commuting square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f \circ p \sim g \circ q$*. The following are equivalent:*

(i) *The square is a pullback square*

(ii) *There is a term of type*

$$\mathsf{is\_pullback}(p,q,H) :\equiv \mathsf{is\_equiv}(\mathsf{gap}(p,q,H)).$$

Proof. Note that there are homotopies

$$\begin{aligned} K &: \pi_1 \circ \mathsf{gap}(p,q,H) \sim p \\ L &: \pi_2 \circ \mathsf{gap}(p,q,H) \sim q \\ M &: (\pi_3 \cdot \mathsf{gap}(p,q,H)) \cdot (g \cdot L) \sim (f \cdot K) \cdot H. \end{aligned}$$

given by

$$\begin{aligned} K &:\equiv \lambda z.\, \mathsf{refl}_{p(z)} \\ L &:\equiv \lambda z.\, \mathsf{refl}_{q(z)} \\ M &:\equiv \lambda z.\, \mathsf{right\_unit}(H(z)) \cdot \mathsf{left\_unit}(H(z))^{-1}. \end{aligned}$$

Therefore the claim follows by Lemma 10.2.3. □

10.3 Fiber products

An important special case of pullbacks occurs when the cospan is of the form

$$A \longrightarrow \mathbf{1} \longleftarrow B.$$

In this case, the pullback is just the *cartesian product*.

Lemma 10.3.1. *Let A and B be types. Then the square*

$$\begin{array}{ccc} A \times B & \xrightarrow{\mathsf{pr}_2} & B \\ {\scriptstyle \mathsf{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{const}_\star} \\ A & \xrightarrow[\mathsf{const}_\star]{} & \mathbf{1} \end{array}$$

which commutes by the homotopy $\mathsf{const}_{\mathsf{refl}_\star}$ is a pullback square.

Proof. By Theorem 10.2.6 it suffices to show that

$$\mathsf{gap}(\mathsf{pr}_1, \mathsf{pr}_2, \lambda(a,b).\, \mathsf{refl}_\star)$$

is an equivalence. Its inverse is the map $\lambda(a,b,p).\,(a,b)$. □

The following generalization of Lemma 10.3.1 is the reason why pullbacks are sometimes called **fiber products**.

Theorem 10.3.2. *Let P and Q be families over a type X. Then the square*

$$\begin{array}{ccc} \sum_{(x:X)} P(x) \times Q(x) & \xrightarrow{\lambda(x,(p,q)).\,(x,q)} & \sum_{(x:X)} Q(x) \\ {\scriptstyle \lambda(x,(p,q)).\,(x,p)}\downarrow & & \downarrow{\scriptstyle \mathsf{pr}_1} \\ \sum_{(x:X)} P(x) & \xrightarrow[\mathsf{pr}_1]{} & X, \end{array}$$

which commutes by the homotopy

$$H :\equiv \lambda(x,(p,q)).\,\mathsf{refl}_x,$$

is a pullback square.

Proof. By Theorem 10.2.6 it suffices to show that the gap map is an equivalence. The gap map is homotopic to the function

$$\lambda(x,(p,q)).\,((x,p),(x,q),\mathsf{refl}_x)$$

is an equivalence. The inverse of this function is the map

$$\lambda((x,p),(y,q),\alpha).\,(y,(\mathsf{tr}_P(\alpha,p),q)).$$

□

Corollary 10.3.3. *For any $f : A \to X$ and $g : B \to X$, the square*

$$\begin{array}{ccc} \sum_{(x:X)} \mathsf{fib}_f(x) \times \mathsf{fib}_g(y) & \xrightarrow{\lambda(x,((a,p),(b,q))).\,b} & B \\ {\scriptstyle \lambda(x,((a,p),(b,q))).\,a}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

is a pullback square.

10.4 Fiber sequences

Lemma 10.4.1. *For any function $f : A \to B$, and any $b : B$, consider the square*

$$\begin{array}{ccc} \mathsf{fib}_f(b) & \xrightarrow{\mathsf{const}_\star} & \mathbf{1} \\ {\scriptstyle \mathsf{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{const}_b} \\ A & \xrightarrow[f]{} & B \end{array}$$

which commutes by $\mathsf{pr}_2 : \prod_{(t:\mathsf{fib}_f(b))} f(\mathsf{pr}_1(t)) = b$. This is a pullback square.

Proof. By Theorem 10.2.6 it suffices to show that the gap map is an equivalence. The gap map is homotopic to the function

$$\mathsf{total}(\lambda x.\, \lambda p.\, (\star, p))$$

The map $\lambda x.\, \lambda p.\, (\star, p)$ is a fiberwise equivalence by Exercise 6.4, so it induces an equivalence on total spaces by Theorem 7.1.3. □

Lemma 10.4.1 motivates the following definition of *fiber sequences*, which play an important role in synthetic homotopy theory (and in algebraic topology).

Definition 10.4.2. A **fiber sequence** consists of types F, E, and B with **base points** $x : F$, $y : E$, and $b : B$, and maps

$$F \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B$$

preserving the base points in the sense that $i(x) = y$ and $p(y) = b$, such that the square

$$\begin{array}{ccc} F & \xrightarrow{\ i\ } & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{1} & \xrightarrow[\ b\ ]{} & B \end{array}$$

is a pullback square. We often write $F \hookrightarrow E \twoheadrightarrow B$ to indicate that we have a fiber sequence.

Given a fiber sequence $F \hookrightarrow E \twoheadrightarrow B$, we call B the **base space**, E the **total space**, and F the **fiber**.

Example 10.4.3. For any type family B over A and any $a : A$ the square

$$\begin{array}{ccc} B(a) & \xrightarrow{\ \mathsf{const}_\star\ } & \mathbf{1} \\ {\scriptstyle \lambda y.\,(a,y)}\downarrow & & \downarrow{\scriptstyle \lambda \star.\, a} \\ \sum_{(x:A)} B(x) & \xrightarrow[\ \mathsf{pr}_1\ ]{} & A \end{array}$$

is a pullback square.

To see this, note that the gap map is homotopic to the function

$$e :\equiv \lambda y.\, ((a, y), \mathsf{refl}_a).$$

This function is an equivalence by Exercise 7.5.

Thus we see that if we additionally suppose that there is a term $b : B(a)$, then we obtain a fiber sequence

$$B(a) \hookrightarrow \textstyle\sum_{(x:A)} B(x) \twoheadrightarrow A.$$

10.5 Fiberwise equivalences

Lemma 10.5.1. *Let $f : A \to B$, and let Q be a type family over B. Then the square*

$$\begin{array}{ccc} \sum_{(x:A)} Q(f(x)) & \xrightarrow{\lambda(x,q).\,(f(x),q)} & \sum_{(y:B)} Q(b) \\ {\scriptstyle \mathsf{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{pr}_1} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes by $H :\equiv \lambda(x,q).\,\mathsf{refl}_{f(x)}$. This is a pullback square.

Proof. By Theorem 10.2.6 it suffices to show that the gap map is an equivalence. The gap map is homotopic to the function

$$\lambda(x,q).\,(x,(f(x),q),\mathsf{refl}_{f(x)}).$$

The inverse of this map is given by $\lambda(x,((y,q),p)).\,(x,\mathsf{tr}_Q(p^{-1},q))$, and it is straightforward to see that these maps are indeed mutual inverses. □

Theorem 10.5.2. *Let $f : A \to B$, and let $g : \prod_{(a:A)} P(a) \to Q(f(a))$ be a fiberwise transformation. The following are equivalent:*

(i) The commuting square

$$\begin{array}{ccc} \sum_{(a:A)} P(a) & \xrightarrow{\mathsf{total}_f(g)} & \sum_{(b:B)} Q(b) \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback square.

(ii) g is a fiberwise equivalence.

Proof. The gap map is homotopic to the composite

$$\sum_{(x:A)} P(x) \xrightarrow{\mathsf{total}(g)} \sum_{(x:A)} Q(f(x)) \xrightarrow{\mathsf{gap}'} A \times_B \Big(\sum_{(y:B)} Q(y)\Big)$$

where gap' is the gap map for the square in Lemma 10.5.1. Since gap' is an equivalence, it follows by Exercise 5.5 and Theorem 7.1.3 that the gap map is an equivalence if and only if g is a fiberwise equivalence. □

Lemma 10.5.3. *Consider a commuting square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f \circ p \sim g \circ q$, *and consider the fiberwise transformation*

$$\mathsf{fib}_{(f,q,H)} : \prod_{(a:A)} \mathsf{fib}_p(a) \to \mathsf{fib}_g(f(a))$$

given by $\lambda a.\, \lambda(c,u).\, (q(c), H(c)^{-1} \cdot \mathsf{ap}_f\,(u))$. *Then there is an equivalence*

$$\mathsf{fib}_{\mathsf{gap}(p,q,H)}((a,b,\alpha)) \simeq \mathsf{fib}_{\mathsf{fib}_{(f,q,H)}(a)}((b,\alpha^{-1}))$$

Proof. To obtain an equivalence of the desired type we simply concatenate known equivalences:

$$\begin{aligned} \mathsf{fib}_h((a,b,\alpha)) &\equiv \sum_{(z:C)} (p(z), q(z), H(z)) = (a,b,\alpha) \\ &\simeq \sum_{(z:C)} \sum_{(u:p(z)=a)} \sum_{(v:q(z)=b)} H(z) \cdot \mathsf{ap}_g\,(v) = \mathsf{ap}_f\,(u) \cdot \alpha \\ &\simeq \sum_{((z,u):\mathsf{fib}_p(a))} \sum_{(v:q(z)=b)} H(z)^{-1} \cdot \mathsf{ap}_f\,(u) = \mathsf{ap}_g\,(v) \cdot \alpha^{-1} \\ &\simeq \mathsf{fib}_{\varphi(a)}((b,\alpha^{-1})) \end{aligned}$$

□

Corollary 10.5.4. *Consider a commuting square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f \circ p \sim g \circ q$. *The following are equivalent:*

(i) The square is a pullback square.

(ii) The induced map on fibers

$$\lambda x.\, \lambda(z,\alpha).\, (q(z), H(z)^{-1} \cdot \mathsf{ap}_f\,(\alpha)) : \prod_{(x:A)} \mathsf{fib}_p(x) \to \mathsf{fib}_g(f(x))$$

is a fiberwise equivalence.

Corollary 10.5.5. *Consider a pullback square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X. \end{array}$$

If g is a k-truncated map, then so is p. In particular, if g is an embedding then so is p.

Proof. Since the square is assumed to be a pullback square, it follows from Corollary 10.5.4 that for each $x : A$, the fiber $\mathsf{fib}_p(x)$ is equivalent to the fiber $\mathsf{fib}_g(f(x))$, which is k-truncated. Since k-truncated types are closed under equivalences by Theorem 8.3.3, it follows that p is a k-truncated map. □

Corollary 10.5.6. *Consider a commuting square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X. \end{array}$$

and suppose that g is an equivalence. Then the following are equivalent:

(i) The square is a pullback square.

(ii) The map $p : C \to A$ is an equivalence.

Proof. If the square is a pullback square, then by Theorem 10.5.2 the fibers of p are equivalent to the fibers of g, which are contractible by Theorem 6.3.3. Thus it follows that p is a contractible map, and hence that p is an equivalence.

If p is an equivalence, then by Theorem 6.3.3 both $\mathsf{fib}_p(x)$ and $\mathsf{fib}_g(f(x))$ are contractible for any $x : X$. It follows by Exercise 6.3 that the induced map $\mathsf{fib}_p(x) \to \mathsf{fib}_g(f(x))$ is an equivalence. Thus we apply Corollary 10.5.4 to conclude that the square is a pullback. □

Theorem 10.5.7. *Consider a diagram of the form*

$$\begin{array}{ccc} A & & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[h]{} & Y. \end{array}$$

Then the type of triples (i, H, p) consisting of a map $i : A \to B$, a homotopy $H : h \circ f \sim g \circ i$, and a term p witnessing that the square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[h]{} & Y. \end{array}$$

is a pullback square, is equivalent to the type of fiberwise equivalences

$$\prod_{(x:X)} \mathsf{fib}_f(x) \simeq \mathsf{fib}_g(h(x)).$$

Corollary 10.5.8. *Let $h : X \to Y$ be a map, and let P and Q be families over X and Y, respectively. Then the type of triples (i, H, p) consisting of a map*

$$i : \Big(\sum_{(x:X)} P(x)\Big) \to \Big(\sum_{(y:Y)} Q(y)\Big),$$

a homotopy $H : h \circ \mathsf{pr}_1 \sim \mathsf{pr}_1 \circ i$, and a term p witnessing that the square

$$\begin{array}{ccc} \sum_{(x:X)} P(x) & \xrightarrow{i} & \sum_{(y:Y)} Q(y) \\ {\scriptstyle \mathsf{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{pr}_1} \\ X & \xrightarrow[h]{} & Y. \end{array}$$

is a pullback square, is equivalent to the type of fiberwise equivalences

$$\prod_{(x:X)} P(x) \simeq Q(h(x)).$$

10.6 The pullback pasting property

The following theorem is also called the **pasting property** of pullbacks.

Theorem 10.6.1. *Consider a commuting diagram of the form*

$$\begin{array}{ccccc} A & \xrightarrow{k} & B & \xrightarrow{l} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[i]{} & Y & \xrightarrow[j]{} & Z \end{array}$$

with homotopies $H : i \circ f \sim g \circ k$ and $K : j \circ g \sim h \circ l$, and the homotopy

$$(j \cdot H) \cdot (K \cdot k) : j \circ i \circ f \sim h \circ l \circ k$$

witnessing that the outer rectangle commutes. Furthermore, suppose that the square on the right is a pullback square. Then the following are equivalent:

(i) The square on the left is a pullback square.

(ii) The outer rectangle is a pullback square.

Proof. The commutativity of the two squares induces fiberwise transformations

$$\prod_{(x:X)}\mathsf{fib}_f(x) \to \mathsf{fib}_g(i(x))$$
$$\prod_{(y:Y)}\mathsf{fib}_g(y) \to \mathsf{fib}_h(j(y)).$$

By the assumption that the square on the right is a pullback square, it follows from Corollary 10.5.4 that the fiberwise transformation

$$\prod_{(y:Y)}\mathsf{fib}_g(y) \to \mathsf{fib}_h(j(y))$$

is a fiberwise equivalence. Therefore it follows from 3-for-2 property of equivalences that the fiberwise transformation

$$\prod_{(x:X)}\mathsf{fib}_f(x) \to \mathsf{fib}_g(i(x))$$

is a fiberwise equivalence if and only if the fiberwise transformation

$$\prod_{(x:X)}\mathsf{fib}_f(x) \to \mathsf{fib}_h(j(i(x)))$$

is a fiberwise equivalence. Now the claim follows from one more application of Corollary 10.5.4. □

10.7 The disjointness of coproducts

As an application of the theory of pullbacks, we show that coproducts are disjoint. In this section we will write

$$[f, g] : A + B \to X$$

for the unique map satisfying $[f, g](\mathsf{inl}(x)) \equiv f(x)$ and $[f, g](\mathsf{inr}(y)) \equiv g(y)$, where $f : A \to X$ and $g : B \to X$. Furthermore, we will write

$$f + g :\equiv [\mathsf{inl} \circ f, \mathsf{inr} \circ g] : A + B \to X + B$$

for any $f : A \to X$ and $g : B \to Y$.

Lemma 10.7.1. *Let X be a type. Then we have the pullback squares*

$$\begin{array}{ccc} X & \xrightarrow{\mathsf{const}_\star} & \mathbf{1} \\ {\scriptstyle \mathsf{id}}\downarrow & & \downarrow{\scriptstyle \mathsf{const}_{0_{\mathbf{2}}}} \\ X & \xrightarrow[\mathsf{const}_{0_{\mathbf{2}}}]{} & \mathbf{2} \end{array} \qquad \begin{array}{ccc} \mathbf{0} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow{\scriptstyle \mathsf{const}_{1_{\mathbf{2}}}} \\ X & \xrightarrow[\mathsf{const}_{0_{\mathbf{2}}}]{} & \mathbf{2}, \end{array}$$

and we have similar pullback squares with the roles of $0_{\mathbf{2}}$ and $1_{\mathbf{2}}$ reversed.

Proof. For the first square we observe that both squares and the outer rectangle in the diagram

$$\begin{array}{ccccc} X & \longrightarrow & \mathbf{1} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathsf{const}_{0_{\mathbf{2}}}} \\ X & \xrightarrow[\mathsf{const}_\star]{} & \mathbf{1} & \xrightarrow[\mathsf{const}_{0_{\mathbf{2}}}]{} & \mathbf{2}. \end{array}$$

are pullback squares. To see this, recall that the identity type $0_{\mathbf{2}} = 0_{\mathbf{2}}$ is contractible by Exercise 8.3. Therefore it follows that the square on the right is a pullback square by Exercise 10.1. The square on the left is a pullback square by Corollary 10.5.6. Therefore the outer rectangle is a pullback square by Theorem 10.6.1.

For the second square we observe that both squares end the outer rectangle in the diagram

$$\begin{array}{ccccc} \mathbf{0} & \longrightarrow & \mathbf{0} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathsf{const}_{1_{\mathbf{2}}}} \\ X & \xrightarrow[\mathsf{const}_\star]{} & \mathbf{1} & \xrightarrow[\mathsf{const}_{0_{\mathbf{2}}}]{} & \mathbf{2}. \end{array}$$

are pullback squares. To see this, recall that the identity type $0_{\mathbf{2}} = 1_{\mathbf{2}}$ is equivalent to the empty type by Exercise 8.3. Therefore it follows that the square on the right is a pullback. It is also straightforward to verify that the square on the left is a pullback. Therefore it follows from Theorem 10.6.1 that the outer rectangle is a pullback. □

Lemma 10.7.2. *For any two types A and B, the squares*

$$\begin{array}{ccc} A & \xrightarrow{\mathsf{const}_\star} & \mathbf{1} \\ {\scriptstyle \mathsf{inl}}\downarrow & & \downarrow{\scriptstyle \mathsf{const}_{0_{\mathbf{2}}}} \\ A+B & \xrightarrow[[\mathsf{const}_{0_{\mathbf{2}}},\mathsf{const}_{1_{\mathbf{2}}}]]{} & \mathbf{2} \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\mathsf{const}_\star} & \mathbf{1} \\ {\scriptstyle \mathsf{inr}}\downarrow & & \downarrow{\scriptstyle \mathsf{const}_{1_{\mathbf{2}}}} \\ A+B & \xrightarrow[[\mathsf{const}_{0_{\mathbf{2}}},\mathsf{const}_{1_{\mathbf{2}}}]]{} & \mathbf{2} \end{array}$$

are pullback squares.

Proof. The two cases are similar, so we only give the proof that the left square is a pullback. The left square commutes by the homotopy

$$H :\equiv \mathsf{htpy.refl}_{\mathsf{const}_{0_{\mathbf{2}}}}.$$

To see that the asserted square is a pullback square we use Theorem 10.2.6 and show that the gap map is an equivalence. First we note that the gap map is homotopic to the function $e : A \to (A+B) \times_{\mathbf{2}} \mathbf{1}$ is defined by

$$\lambda x.\, (\mathsf{inl}(x), \star, \mathsf{refl}_{0_{\mathbf{2}}}).$$

The inverse is defined by the induction principle of coproducts by

$$\begin{aligned} e^{-1}(\mathsf{inl}(x), t, \alpha) &:\equiv x \\ e^{-1}(\mathsf{inr}(y), t, \alpha) &:\equiv \mathsf{ind}_{\mathbf{0}}(\zeta(\alpha)), \end{aligned}$$

where $\zeta : \prod_{(x,y:\mathbf{2})}(x = y) \to \mathsf{Eq}_{\mathbf{2}}(x,y)$ is the canonical map of the identity type of $\mathbf{2}$ into the observational equality on $\mathbf{2}$. In the case of $\alpha : 0_{\mathbf{2}} = 1_{\mathbf{2}}$ we obtain a term of $\mathsf{Eq}_{\mathbf{2}}(0_{\mathbf{2}}, 1_{\mathbf{2}}) \equiv \mathbf{0}$. It is immediate from the computation rules that $e^{-1} \circ e \equiv \mathsf{id}$.

The homotopy $e \circ e^{-1} \sim \mathsf{id}$ is again constructed by the induction principle of coproducts. In the inl-case we have $e(e^{-1}(\mathsf{inl}(x), t, \alpha)) \equiv (\mathsf{inl}(x), \star, \mathsf{refl}_{0_{\mathbf{2}}})$. We define the identification

$$(\mathsf{inl}(x), \star, \mathsf{refl}_{0_{\mathbf{2}}}) = (\mathsf{inl}(x), t, \alpha)$$

by singleton induction on $t : \mathbf{1}$ and $\alpha : 0_{\mathbf{2}} = 0_{\mathbf{2}}$ (both of which are terms of contractible types). Thus, it suffices to provide an identification

$$(\mathsf{inl}(x), \star, \mathsf{refl}_{0_{\mathbf{2}}}) = (\mathsf{inl}(x), \star, \mathsf{refl}_{0_{\mathbf{2}}}),$$

which we have by reflexivity. The inr-case is again automatic, since we obtain a term of the empty type from $\alpha : 0_{\mathbf{2}} = 1_{\mathbf{2}}$. This completes the proof that e is an equivalence. □

Corollary 10.7.3. *The maps* $\mathsf{inl} : A \to A + B$ *and* $\mathsf{inr} : B \to A + B$ *are embeddings.*

Proof. By the pullback squares of Lemma 10.7.2 and Corollary 10.5.5 it suffices to show that $\mathbf{1} \to \mathbf{2}$ is an embedding. This is Exercise 8.9. □

Theorem 10.7.4. *Coproducts are* ***disjoint*** *in the sense that for any two types A and B, the commuting square*

$$\begin{array}{ccc} \mathbf{0} & \longrightarrow & B \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathsf{inr}} \\ A & \xrightarrow[\mathsf{inl}]{} & A+B \end{array}$$

is a pullback square.

Proof. Now consider the commuting diagram

$$\begin{array}{ccccc} \mathbf{0} & \longrightarrow & B & \longrightarrow & \mathbf{1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathsf{inr}} & & \big\downarrow{\scriptstyle \mathsf{const}_{1_{\mathbf{2}}}} \\ A & \xrightarrow[\mathsf{inl}]{} & A+B & \xrightarrow[{[\mathsf{const}_{0_{\mathbf{2}}},\mathsf{const}_{1_{\mathbf{2}}}]}]{} & \mathbf{2}. \end{array}$$

By Lemma 10.7.1 it follows that the outer rectangle is a pullback square. The square on the right is a pullback square by Lemma 10.7.2. Therefore the square on the left is a pullback square by Theorem 10.6.1. □

Corollary 10.7.5. *Let A and B be types. There are equivalences*

$$\begin{aligned} (\mathsf{inl}(x) = \mathsf{inl}(x')) &\simeq (x =_A x') \\ (\mathsf{inl}(x) = \mathsf{inr}(y')) &\simeq \mathbf{0} \\ (\mathsf{inr}(y) = \mathsf{inl}(x')) &\simeq \mathbf{0} \\ (\mathsf{inr}(y) = \mathsf{inr}(y')) &\simeq (y =_B y'). \end{aligned}$$

Proof. The cases

$$\begin{aligned} (\mathsf{inl}(x) = \mathsf{inl}(x')) &\simeq (x =_A x') \\ (\mathsf{inr}(y) = \mathsf{inr}(y')) &\simeq (y =_B y'). \end{aligned}$$

follow from Corollary 10.7.3 since both inl and inr are embeddings. The remaining cases follow from the disjointness of coproducts, proven in Theorem 10.7.4. □

Exercises

10.1 (a) Show that the square

$$\begin{array}{ccc} (x=y) & \longrightarrow & \mathbf{1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathsf{const}_y} \\ \mathbf{1} & \xrightarrow[\mathsf{const}_x]{} & A \end{array}$$

is a pullback square.

(b) Show that the square

$$\begin{array}{ccc} (x = y) & \xrightarrow{\mathsf{const}_x} & A \\ {\scriptstyle \mathsf{const}_\star}\downarrow & & \downarrow{\scriptstyle \delta_A} \\ \mathbf{1} & \xrightarrow[\mathsf{const}_{(x,y)}]{} & A \times A \end{array}$$

is a pullback square, where $\delta_A : A \to A \times A$ is the diagonal of A, defined in Exercise 8.1.

10.2 In this exercise we give an alternative characterization of the notion of k-truncated map, compared to Theorem 8.3.7. Given a map $f : A \to X$ define the **diagonal** of f to be the map $\delta_f : A \to A \times_X A$ given by $x \mapsto (x, x, \mathsf{refl}_{f(x)})$.

(a) Construct an equivalence

$$\mathsf{fib}_{\delta_f}((x, y, p)) \simeq \mathsf{fib}_{\mathsf{ap}_f}(p)$$

to show that the square

$$\begin{array}{ccc} \mathsf{fib}_{\mathsf{ap}_f}(p) & \xrightarrow{\mathsf{const}_x} & A \\ {\scriptstyle \mathsf{const}_\star}\downarrow & & \downarrow{\scriptstyle \delta_f} \\ \mathbf{1} & \xrightarrow[\mathsf{const}_{(x,y,p)}]{} & A \times_X A \end{array}$$

is a pullback square, for every $x, y : A$ and $p : f(x) = f(y)$.

(b) Show that a map $f : A \to X$ is $(k+1)$-truncated if and only if δ_f is k-truncated.

Conclude that f is an embedding if and only if δ_f is an equivalence.

10.3 Consider a commuting square

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f \circ p \sim g \circ q$. Show that the following are equivalent:

(i) The square is a pullback square.

(ii) For every type T, the commuting square

$$\begin{array}{ccc} C^T & \xrightarrow{q\circ -} & B^T \\ {\scriptstyle p\circ -}\downarrow & & \downarrow{\scriptstyle g\circ -} \\ A^T & \xrightarrow[f\circ -]{} & X^T \end{array}$$

is a pullback square.

Note: property (ii) is really just a rephrasing of the universal property of pullbacks.

10.4 Consider a commuting square

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f \circ p \sim g \circ q$. Show that the following are equivalent:

(i) The square is a pullback square.

(ii) The square

$$\begin{array}{ccc} C & \xrightarrow{g\circ q} & X \\ {\scriptstyle \lambda x.\,(p(x),q(x))}\downarrow & & \downarrow{\scriptstyle \delta_X} \\ A \times B & \xrightarrow[f\times g]{} & X \times X \end{array}$$

which commutes by $\lambda z.\,\mathsf{eq\_pair}(H(z), \mathsf{refl}_{g(q(z))})$ is a pullback square.

10.5 Show that if

$$\begin{array}{ccc} C_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & X_1 \end{array} \qquad \begin{array}{ccc} C_2 & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & X_2 \end{array}$$

are pullback squares, then so is

$$\begin{array}{ccc} C_1 \times C_2 & \longrightarrow & B_1 \times B_2 \\ \downarrow & & \downarrow \\ A_1 \times A_2 & \longrightarrow & X_1 \times X_2. \end{array}$$

10.6 Consider for each $i : I$ a pullback square

$$\begin{array}{ccc} C_i & \xrightarrow{q_i} & B_i \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle g_i} \\ A_i & \xrightarrow[f_i]{} & X_i \end{array}$$

with $H_i : f_i \circ p_i \sim g_i \circ q_i$.

(a) Show that the square

$$\begin{array}{ccc} \sum_{(i:I)} C_i & \xrightarrow{\mathsf{total}(q)} & \sum_{(i:I)} B_i \\ {\scriptstyle \mathsf{total}(p)}\downarrow & & \downarrow{\scriptstyle \mathsf{total}(g)} \\ \sum_{(i:I)} A_i & \xrightarrow[\mathsf{total}(f)]{} & \sum_{(i:I)} X_i \end{array}$$

which commutes by the homotopy

$$\mathsf{total}(H) :\equiv \lambda(i,c).\, \mathsf{eq\_pair}(\mathsf{refl}_i, H_i(c))$$

is a pullback square.

(b) Show that the commuting square

$$\begin{array}{ccc} \prod_{(i:I)} C_i & \longrightarrow & \prod_{(i:I)} B_i \\ \downarrow & & \downarrow \\ \prod_{(i:I)} A_i & \longrightarrow & \prod_{(i:I)} X_i \end{array}$$

is a pullback square.

10.7 Let B be a type family over A. Show that the square

$$\begin{array}{ccc} \prod_{(x:A)} B(x) & \xrightarrow{\lambda f.\,\lambda x.\,(x,f(x))} & \Big(\sum_{(x:A)} B(x)\Big)^A \\ \downarrow & & \downarrow{\scriptstyle \mathsf{pr}_1\circ -} \\ \mathbf{1} & \xrightarrow[\mathsf{const}_{\mathsf{id}_A}]{} & A^A \end{array}$$

is a pullback square. Conclude that the type $\prod_{(x:A)} B(x)$ is equivalent to the type $\mathsf{sec}(\mathsf{pr}_1)$ of sections of the projection map.

10.8 Consider a pullback square

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X, \end{array}$$

with $H : f \circ p \sim g \circ q$, and let $c_1, c_2 : C$. Show that the square

$$\begin{array}{ccc} (c_1 = c_2) & \xrightarrow{\mathsf{ap}_q} & (q(c_1) = q(c_2)) \\ {\scriptstyle \mathsf{ap}_p}\big\downarrow & & \big\downarrow{\scriptstyle \lambda\beta.\, H(c_1) \cdot \mathsf{ap}_g(\beta)} \\ (p(c_1) = p(c_2)) & \xrightarrow[\lambda\alpha.\, \mathsf{ap}_f(\alpha) \cdot H(c_2)]{} & f(p(c_1)) = g(q(c_2)), \end{array}$$

which commutes by the naturality of homotopies (Definition 6.3.1), is again a pullback square.

10.9 Suppose that

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

with $H : f \circ p \sim g \circ q$ is a pullback square. Show that the square

$$\begin{array}{ccc} C & \xrightarrow{p} & A \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & X \end{array}$$

with $H^{-1} : g \circ q \sim f \circ p$ is again a pullback square.

10.10 Consider a commuting square

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X. \end{array}$$

with $H : f \circ p \sim g \circ q$, and let $h : C \to A \times_X B$ be the map given by $h(z) :\equiv (p(c), q(c), H(c))$. Show that the square

$$\begin{array}{ccc} \mathsf{fib}_{\mathsf{gap}(p,q,H)}((a,b,\alpha)) & \xrightarrow{\lambda(c,\beta).\,(c,\mathsf{ap}_{\pi_1}(\beta))} & \mathsf{fib}_p(a) \\ {\scriptstyle \mathsf{const}_\star}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{fib}_{(f,g,H)}} \\ \mathbf{1} & \xrightarrow[\mathsf{const}_{(b,\alpha^{-1})}]{} & \mathsf{fib}_g(f(a)) \end{array}$$

10.11 Consider a commuting diagram of the form

$$\begin{array}{ccccc} A_0 & \longrightarrow & B_0 & \longleftarrow & C_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 & \longleftarrow & C_1 \\ \uparrow & & \uparrow & & \uparrow \\ A_2 & \longrightarrow & B_2 & \longleftarrow & C_2 \end{array}$$

with homotopies filling the (small) squares. Construct an equivalence

$$\begin{aligned} &(A_0 \times_{B_0} C_0) \times_{(A_1 \times_{B_1} C_1)} (A_2 \times_{B_2} C_2) \\ &\qquad \simeq (A_0 \times_{A_1} A_2) \times_{(B_0 \times_{B_1} B_2)} (C_0 \times_{C_1} C_2). \end{aligned}$$

This is also known as the **3-by-3 lemma** for pullbacks.