Lecture 1

Dependent type theory

1.1 The primitive judgments of type theory

The theory of type dependency is formulated as a deductive system in which derivations establish that a given construction is well-formed. In any dependent type theory there are four **primitive judgments**:

(i) '*A is a well-formed* ***type*** *in context* Γ.'

(ii) '*A and B are* ***judgmentally equal types*** *in context* Γ.'

(iii) '*a is a well-formed* ***term*** *of type A in context* Γ.'

(iv) '*a and b are* ***judgmentally equal terms*** *of type A in context* Γ.'

The symbolic expressions for these four primitive judgments are as follows:

$$\begin{array}{ll} \Gamma \vdash A \ \mathrm{type} & \qquad \Gamma \vdash A \equiv B \ \mathrm{type} \\ \Gamma \vdash a : A & \qquad \Gamma \vdash a \equiv b : A. \end{array}$$

A **context** is an expression of the form

$$x_1 : A_1,\ x_2 : A_2(x_1),\ \ldots,\ x_n : A_n(x_1, \ldots, x_{n-1}),$$

which we often simply write as $x_1 : A_1,\ x_2 : A_2,\ \ldots,\ x_n : A_n$, satisfying the condition that for each $1 \leq k \leq n$ we have that A_k is a well-formed type in context $x_1 : A_1, x_2 : A_2, \ldots, x_{k-1} : A_{k-1}$, i.e.

$$x_1 : A_1, x_2 : A_2, \ldots, x_{k-1} : A_{k-1} \vdash A_k \ \mathrm{type}.$$

We say that a context $x_1 : A_1,\ \ldots,\ x_n : A_n$ **declares the variables** $x_1, \ldots, x_n$. We may use variable names other than $x_1, \ldots, x_n$, as long as *no variable is declared more than once.*

In the special case where $n = 0$, the list $x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$ is empty, which satisfies the well-formedness condition vacuously. In other words, the **empty context** is well-formed. A well-formed type in the empty context is also called a **closed type**, and a well-formed term of a closed type is called a **closed term**.

When B is a type in context $\Gamma, x : A$, we also say that B is a **family of types** over A (in context Γ).

1.2 Renaming variables

In some situations one might want to change the name of a variable in a context. This is allowed, provided that the new variable does not occur anywhere else in the context, so that also after renaming no variable is declared more than once. The inference rules that rename a variable are sometimes called α-conversion rules.

If we are given a type A in context Γ, then for any type B in context $\Gamma, x : A, \Delta$ we can form the type $B[x'/x]$ in context $\Gamma, x' : A, \Delta[x'/x]$, where $B[x'/x]$ is an abbreviation for

$$B(x_1, \ldots, x_{n-1}, x', x_{n+1}, \ldots, x_{n+m-1})$$

This definition of **renaming** the variable x by x' is understood to be recursive over the length of Δ. The first variable renaming rule postulates that the renaming of a variable preserves well-formedness of types:

$$\frac{\Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, x' : A, \Delta[x'/x] \vdash B[x'/x] \text{ type}}\ x'/x$$

Similarly we obtain for any term $b : B$ in context $\Gamma, x : A, \Delta$ a term $b[x'/x] : B[x'/x]$, and there is a variable renaming rule postulating that the renaming of a variable preserves the well-formedness of terms. In fact, there is variable renaming rule for each of the primitive judgments. To avoid having to state essentially the same rule four times in a row, we postulate the four variable renaming rules all at once using a *generic judgment* $\mathcal{J}$.

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x' : A, \Delta[x'/x] \vdash \mathcal{J}[x'/x]}\ x'/x$$

where $\mathcal{J}$ may be a typing judgment, a judgment of equality of types, a term judgment, or a judgment of equality of terms. We will use generic judgments extensively to postulate the rest of the rules of dependent type theory.

1.3 Inference rules governing judgmental equality

Both on types and on terms, we postulate that judgmental equality is an equivalence relation. That is, we provide inference rules for the reflexivity, symmetry and transitivity of both kinds of judgmental equality:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \equiv A \text{ type}} \qquad \frac{\Gamma \vdash A \equiv A' \text{ type}}{\Gamma \vdash A' \equiv A \text{ type}} \qquad \frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma \vdash A' \equiv A'' \text{ type}}{\Gamma \vdash A \equiv A'' \text{ type}}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv a' : A}{\Gamma \vdash a' \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv a' : A \qquad \Gamma \vdash a' \equiv a'' : A}{\Gamma \vdash a \equiv a'' : A}$$

Apart from the rules postulating that judgmental equality is an equivalence relation, there are also **variable conversion rules**. Informally, these are rules stating that if A and A' are judgmentally equal types in context Γ, then any valid judgment in context $\Gamma, x : A$ is also a valid judgment in context $\Gamma, x : A'$. In other words: we can convert the type of a variable to a judgmentally equal type. We state this with a generic judgment $\mathcal{J}$

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x : A', \Delta \vdash \mathcal{J}} \; A'/A$$

An analogous *term conversion rule* stated in Exercise 1.1, converting the type of a term to a judgmentally equal type, is derivable.

1.4 Structural rules of type theory

We complete the specification of dependent type theory by postulating rules for *weakening* and *substitution*, and the *variable rule*:

(i) If we are given a type A in context Γ, then any judgment made in a longer context Γ, Δ can also be made in the context $\Gamma, x : A, \Delta$, for a fresh variable x. The **weakening rule** asserts that weakening by a type A in context preserves well-formedness and judgmental equality of types and terms.

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, \Delta \vdash \mathcal{J}}{\Gamma, x : A, \Delta \vdash \mathcal{J}} \; W_A$$

This process of expanding the context by a fresh variable of type A is called **weakening (by A)**. The type family $W_A(B)$ over A is also called the **constant family** B, or the **trivial family** B.

(ii) If we are given a type A in context Γ, then x is a well-formed term of type A in context $\Gamma, x : A$.

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash x : A}\ \delta_A$$

This is called the **variable rule**. It provides an *identity function* on the type A in context Γ.

(iii) If we are given a term $a : A$ in context Γ, then for any type B in context $\Gamma, x : A, \Delta$ we can form the type $B[a/x]$ in context $\Gamma, \Delta[a/x]$, where $B[a/x]$ is an abbreviation for

$$B(x_1, \ldots, x_{n-1}, a(x_1, \ldots, x_{n-1}), x_{n+1}, \ldots, x_{n+m-1})$$

This definition of substituting a for x is understood to be recursive over the length of Δ. Similarly we obtain for any term $b : B$ in context $\Gamma, x : A, \Delta$ a term $b[a/x] : B[a/x]$. The **substitution rule** asserts that substitution preserves well-formedness and judgmental equality of types and terms:

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}\ S_a$$

Furthermore, we postulate that substitution by judgmentally equal terms results in judgmentally equal types

$$\frac{\Gamma \vdash a \equiv a' : A \qquad \Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, \Delta[a/x] \vdash B[a/x] \equiv B[a'/x] \text{ type}}$$

and it also results in judgmentally equal terms

$$\frac{\Gamma \vdash a \equiv a' : A \qquad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] \equiv b[a'/x] : B[a/x]}$$

When B is a family of types over A and $a : A$, we also say that $B[a/x]$ is the **fiber** of B at a. Often we write $B(a)$ for $B[a/x]$.

Example 1.4.1. To give an example of how the deductive system works, we give a deduction for the **interchange rule**

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma, x : A, y : B, \Delta \vdash \mathcal{J}}{\Gamma, y : B, x : A, \Delta \vdash \mathcal{J}}$$

In other words, if we have two types A and B in context Γ, and we make a judgment in context $\Gamma, x : A, y : B$, then we can make that same judgment in context $\Gamma, y : B, x : A$. The derivation is as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash B \text{ type}}{\Gamma, y : B \vdash y : B}\,\delta_B}{\Gamma, y : B, x : A \vdash y : B}\,W_{W_B(A)} \qquad \dfrac{\dfrac{\Gamma, x : A, y : B, \Delta \vdash \mathcal{J}}{\Gamma, x : A, y' : B, \Delta[y'/y] \vdash \mathcal{J}[y'/y]}\,y'/y}{\Gamma, y : B, x : A, y' : B, \Delta[y'/y] \vdash \mathcal{J}[y'/y]}\,W_B}{\Gamma, y : B, x : A, \Delta \vdash \mathcal{J}}\,S_{W_A(y)}$$

Exercises

1.1 Give a derivation for the following conversion rule:

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma \vdash a : A}{\Gamma \vdash a : A'}$$