Recursion

Varsity Practice 11/15/20

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1 Background

We'll sketch a proof that the n-th Fibonacci number is

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

First, consider the recurrence $z_n = z_{n-1} + z_{n-2}$, or equivalently $z_n - z_{n-1} - z_{n-2} = 0$. If we guess that $z_n = \alpha^n$ for some α, then we have

$$0 = z_n - z_{n-1} - z_{n-2} = \alpha^n - \alpha^{n-1} - \alpha^{n-2} = \alpha^{n-2}(\alpha^2 - \alpha - 1)$$

The nonzero solutions are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\alpha = \frac{1-\sqrt{5}}{2}$. Now notice that since $z_n = (\frac{1+\sqrt{5}}{2})^n$ and $z_n = (\frac{1-\sqrt{5}}{2})^n$ each satisfy the recurrence relation $z_n = z_{n-1}+z_{n-2}$, then $f_n = \beta(\frac{1+\sqrt{5}}{2})^n + \gamma(\frac{1-\sqrt{5}}{2})^n$ also satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$. It remains only to choose β and γ such that $f_0 = 0$ and $f_1 = 1$.

2 Warm-up

1. (PUMaC 2017) Let the sequence $a_1, a_2, \ldots$ be defined recursively as follows: $a_n = 11a_{n-1} - n$. Find the smallest value of a_n such that all the terms of the sequence are positive.
2. Solve the recurrence $d_{n+2} = d_n + 2d_n$ where $d_0 = 0$ and $d_1 = 1$.

3 Problems

1. (HMMT 2007) An infinite sequence of real numbers is defined by $a_0 = 1$ and $a_{n+2} = 6a_n - a_{n+1}$ for $n = 0, 1, 2, \ldots$. If $a_n \geq 0$ for all n, find all possible values of a_{2020}.
2. Compute $\left(\frac{1+\sqrt{13}}{2}\right)^{10} + \left(\frac{1-\sqrt{13}}{2}\right)^{10}$.
3. (HMMT 2016) An infinite sequence of real numbers $a_1, a_2, \ldots$ satisfies the recurrence

$$a_{n+3} = a_{n+2} - 2a_{n+1} + a_n$$

   for every positive integer n. Given that $a_1 = a_3 = 1$ and $a_{98} = a_{99}$, compute $a_1 + a_2 + \cdots + a_{100}$.
4. The Fibonacci numbers are defined by the recurrence $f_n = f_{n-1} + f_{n-2}$, with $f_0 = 0$ and $f_1 = 1$. Find a recurrence for the sequence $g_0, g_1, g_2, \ldots$ defined by $g_n = f_n^2$.
5. Solve the recurrence $g_{n+2} = g_{n+1} + g_n + 1$ where $g_0 = 0$.

6. (HMMT 2012) Let $a_0 = -2, b_0 = 1$, and for $n \geq 0$, let

$$a_{n+1} = a_n + b_n + \sqrt{a_n^2 + b_n^2},$$
$$b_{n+1} = a_n + b_n - \sqrt{a_n^2 + b_n^2}.$$

   Compute a_{2020}.

7. (HMMT 2017) Let f_n be the n-th Fibonacci number. Find the smallest positive integer m such that $f_m \equiv 0 \pmod{61}$ and $f_{m+1} \equiv 1 \pmod{61}$.

8. (PUMaC 2014) Given that

$$a_n a_{n-2} - a_{n-1}^2 + a_n - n a_{n-2} = -n^2 + 3n - 1$$

   and $a_0 = 1, a_1 = 3$, find a_{20}.