Inequalities

Varsity Practice 10/4/20

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Here's some inequalities in roughly increasing order of difficulty:

- AM-GM: Given $x_1, ..., x_n$, $\sum x_i/n \leq (\Pi x_i)^{1/n}$.
- Cauchy-Schwarz: Given two vectors u, v, $|u \cdot v|^2 \leq ||u||^2||v||^2$. Alternatively, given $x_1, ..., x_n$ and $y_1, ..., y_n$, you can find that $(\sum x_i y_i)^2 \leq (\sum x_i^2)(\sum y_i^2)$. This can also be expressed as $(\sum \sqrt{x_i y_i})^2 \leq (\sum x_i)(\sum y_i)$, assuming $x_i, y_i \geq 0$.
- Jensen: Let f be a convex function, $t \in (0,1)$. Then $tf(x_1)+(1-t)f(x_2) \geq f(tx_1+(1-t)x_2)$.

1 Warming Up

1. Prove $AM - GM$. Start with the $n = 2$ case.
2. Suppose a point satisfies $xyz^2 = 2$. Find the minimum distance of that point to the origin.
3. Prove Cauchy-Schwarz for when $u, v \in \mathbb{R}^2$.
4. Suppose $x + y + z = 3$. Find the minimum value of $(\frac{4}{x} + \frac{9}{y} + \frac{16}{z})$.

2 Problems Part 1

1. Let x, y, z be real numbers such that $x+2y+z = 4$. Find the maximum value of $xy+xz+yz$.
2. Let nonnegative $a + b + c + d = 1$. Find the maximum value of $ab + bc + cd$.
3. Let x, y, z be positive real numbers such that $x + y + z = 1$. Maximize x^3y^2z.
4. Find the number of ordered triples (x, y, z) such that $x^4 + y^4 + z^4 - 4xyz = 1$.
5. Find the maximum of $\sqrt{x+27} + \sqrt{13-x} + \sqrt{x}$ for $0 \leq x \leq 13$.
6. Let $x, y > 1$ such that $a^4 + b^4 + 8 = 8ab$. Calculate $2^{\sqrt{2}a} + 3^{\sqrt{3}b}$.
7. Let a be a real number. Find the minimum of the value $x + \frac{a}{x}$, for $x > 0$. Solve in terms of a.

3 Problems Part 2

These are pretty difficult.

1. Let x, y, z be distinct real numbers such that $x+y+z=0$. Find the maximum possible value of $\frac{xy+yz+xz}{x^2+y^2+z^2}$.
2. Let x be a positive real number. Find the maximum value of $\frac{x^2+2-\sqrt{x^4+4}}{x}$.
3. For $n \geq 2$ let a_1, a_2, ..., a_n be positive real numbers such that

$$(a_1 + a_2 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2$$

   Prove that $\max(a_1, a_2, ..., a_n) \leq 4\min(a_1, a_2, ..., a_n)$.

4. Let a, b, c be positive real numbers such that $a^2+b^2+c^2+(a+b+c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$