Matrices and Geometry II

Varsity Practice 1/31/21

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1 Background

1.1 Polar and Cartesian Conversions

The Cartesian representation represents a vector with an x-coordinate and a y-coordinate. The polar representation instead measures the angle between the vector and the positive x-axis, as well as the magnitude of the vector (distance between the endpoint and the origin), and uses this to represent a vector with an angle θ and a radius r.

Radians are an alternate unit to measure angles in. A full circle is 360 degrees, or 2π radians. You can use this to convert between the two.

1.2 Vectors

Cartesian representation of vector:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \times 1 \text{ vector}$$

Dot product:

$$\vec{v} \cdot \vec{w} = v_1 * w_1 + v_2 * w_2 = \text{ scalar}$$

Projection of $\vec{v}$ onto $\vec{w}$:

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||} * \frac{\vec{w}}{||\vec{w}||}$$

Finding the angle between two vectors:

$$||\vec{v}|| * ||\vec{w}|| * \cos(\theta) = \vec{v} \cdot \vec{w}$$

1.3 Matrices

Matrix index convention:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = 2 \times 2 \text{ matrix}$$

Matrix multiplication on a vector:

$$M\vec{v} = \begin{bmatrix} M_{11}v_1 + M_{12}v_2 \\ M_{21}v_1 + M_{22}v_2 \end{bmatrix} = 2 \times 1 \text{ vector}$$

Matrix multiplication:

$$MN = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} M_{11}N_{11} + M_{12}N_{21} & M_{11}N_{12} + M_{12}N_{22} \\ M_{21}N_{11} + M_{22}N_{21} & M_{21}N_{12} + M_{22}N_{22} \end{bmatrix} = 2 \times 2 \text{ matrix}$$

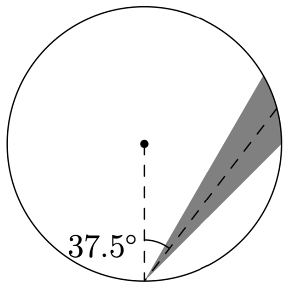
2 Problems

1. See Jamboards, Part 1 (Representing the compositions of reflections and rotations as single rotations or reflections)
2. See Jamboards, Part 2 (Identifying the symmetries of a series of 5 pairs of images, and comparing whether they are the same symmetries)

3 Further Problems

1. Pentagon $JAMES$ is such that $AM = SJ$ and the internal angles satisfy $J = A = E = 90\,\text{deg}$, and $M = S$. Given that there exists a diagonal of $JAMES$ that bisects its area, find the ratio of the shortest side of $JAMES$ to the longest side of $JAMES$.
2. Call a triangle nice if the plane can be tiled using congruent copies of this triangle so that any two triangles that share an edge (or part of an edge) are reflections of each other via the shared edge. How many dissimilar nice triangles are there?
3. Points E, F, G, H are chosen on segments AB, BC, CD, DA, respectively, of square $ABCD$. Given that segment EG has length 7, segment FH has length 8, and that EG and FH intersect inside $ABCD$ at an acute angle of $30\,\text{deg}$, then compute the area of square $ABCD$.
4. Draw an equilateral triangle with center O. Rotate the equilateral triangle 30 degrees, 60 degrees, and 90 degrees with respect to O so there would be four congruent equilateral triangles on each other. Look at the diagram. If the smallest triangle has area 1, the area of the original equilateral triangle could be expressed as $p + q\sqrt{r}$ where p, q, r are positive integers and r is not divisible by a square greater than 1. Find $p + q + r$.
5. Plot points A, B, C at coordinates $(0, 0)$, $(0, 1)$, and $(1, 1)$ in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let X_1 be the area swept out when Bobby rotates S counterclockwise 45 degrees about point A. Let X_2 be the area swept out when Calvin rotates S clockwise 45 degrees about point A. Find $\frac{X_1+X_2}{2}$.
6. Equilateral triangle ABC has circumcircle Ω. Points D and E are chosen on minor arcs AB and AC of Ω respectively such that $BC = DE$. Given that triangle ABE has area 3 and triangle ACD has area 4, find the area of triangle ABC.
7. A paper equilateral triangle of side length 2 on a table has vertices labeled A, B, C. Let M be the point on the sheet of paper halfway between A and C. Over time, point M is lifted upwards, folding the triangle along segment BM, while A, B, and C remain on the table. This continues until A and C touch. Find the maximum volume of tetrahedron $ABCM$ at any time during this process.
8. Let R be the rectangle in the Cartesian plane with vertices at $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$. R can be divided into two unit squares. Pro selects a point P uniformly at random in the interior of R. Find the probability that the line through P with slope $\frac{1}{2}$ will pass through both unit squares.

9. Let C be a circle in the xy plane with radius 1 and center $(0, 0, 0)$, and let P be a point in space with coordinates $(3, 4, 8)$. Find the largest possible radius of a sphere that is contained entirely in the slanted cone with base C and vertex P.



10. A mouse lives in a circular cage with completely reflective walls. At the edge of this cage, a small flashlight with vertex on the circle whose beam forms an angle of 15 degrees is centered at an angle of 37.5 degrees away from the center, as seen above. The mouse will die in the dark. What fraction of the total area of the cage can keep the mouse alive?