Transformations of the Plane

JV Practice 11/8/20

Elizabeth Chang-Davidson

1 Warmup

1.1 Vectors and Converting from Polar to Cartesian Coordinates

1. What is the vector $\vec{v}$ from the point $(1, 6)$ to the point $(-7, 2)$?
2. What is the angle θ (in radians) associated with $\vec{v}$? (This is the angle between the positive x-axis and $\vec{v}$)
3. What is the magnitude m associated with $\vec{v}$?

1.2 Transformations and Matrices

1. What is the reflection of $(3, 5)$ across the line $x = 0$?
2. What is the reflection of $(3, 5)$ across the line $y = -x$?
3. More generally, how would you reflect a vector across the line $x = 0$? Can you represent this in matrix form? What about for the line $y = -x$?
4. What is the length of the projection of $\vec{p} = (3, 5)$ onto the line $y = 2x$?
5. What is the translation of the point $(5, -1)$ by $\vec{v}$?

2 Some Useful Formulas

Cartesian representation of vector:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \times 1 \text{ vector}$$

Matrix index convention:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = 2 \times 2 \text{ matrix}$$

Matrix multiplication on a vector:

$$M\vec{v} = \begin{bmatrix} M_{11}v_1 + M_{12}v_2 \\ M_{21}v_1 + M_{22}v_2 \end{bmatrix} = 2 \times 1 \text{ vector}$$

Dot product:

$$\vec{v} \cdot \vec{w} = v_1 * w_1 + v_2 * w_2 = \text{ scalar}$$

Projection of $\vec{v}$ onto $\vec{w}$:

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{||\vec{w}||} * \frac{\vec{w}}{||\vec{w}||}$$

Finding the angle between two vectors:

$$||\vec{v}|| * ||\vec{w}|| * \cos(\theta) = \vec{v}\cdot\vec{w}$$

Matrix multiplication:

$$MN = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} M_{11}N_{11} + M_{12}N_{21} & M_{11}N_{12} + M_{12}N_{22} \\ M_{21}N_{11} + M_{22}N_{21} & M_{21}N_{12} + M_{22}N_{22} \end{bmatrix} = 2\times 2 \text{ matrix}$$

3 Formulas We Will Derive

3.1 Reflection

To reflect a point p across a line with normal vector $\vec{\ell}$, the resulting point q will be:

$$q = 2 * (\vec{p}\cdot\vec{\ell}) * \vec{\ell} - \vec{p}$$

Since $\vec{\ell}$ is a normal vector, we can represent it as $\ell_1 = \cos(\theta)$ and $\ell_2 = \sin(\theta)$. Our formula then becomes:

$$q = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

3.2 Rotation

To rotate a point p around the origin by angle θ, the resulting point q will be:

$$q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

3.3 Translation

To translate a point p by a vector $\vec{v}$, the resulting point q will be:

$$q = \begin{bmatrix} v_1 + p_1 \\ v_2 + p_2 \end{bmatrix}$$

3.4 Glide Reflection

A glide reflection is just a composition of a reflection and translation along the same line.

4 Problems

4.1 Exploring Properties of Transformations

1. Find the rotation matrix for $\theta = \frac{2\pi}{3}$. What is the result when $(2, -6)$ is rotated by $\frac{2\pi}{3}$?

2. Find the reflection matrix for the line $y = 2x$. What is the result when the point $(1, 5)$ is reflected across this line?

3. What is the transformation matrix for rotating by $\frac{3\pi}{4}$ and then reflecting across the line $y = 0$? Does this correspond to any simpler transformation? If so, what?

4. What is the transformation matrix for reflecting across the line at $\frac{2\pi}{3}$ and then across the line at $\frac{\pi}{3}$? Does this correspond to any simpler transformation? If so, what?

5. What is the transformation matrix for reflecting across the line $y = 4$? What if you then reflect across the line $y = 2$? Can you simplify that? What if you then reflect across the line $y = 1$? Can you simplify that?

6. Find the rotation matrix for rotating by $\frac{\pi}{3}$. Now find the rotation matrix for rotating by $-\frac{\pi}{3}$. What do you get when you multiply them?

7. What are the fixed points under a rotation? Under a translation? Under a reflection?

8. Prove that rotations, reflections, translations, and glide reflections preserve collinearity.

4.2 Applying Geometry Concepts to Problems

1. A triangle with vertices $A(0, 2)$, $B(-3, 2)$, and $C(-3, 0)$ is reflected about the x-axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by 90° to produce $\triangle A''B''C''$. What transformation will return $\triangle A''B''C''$ to $\triangle ABC$?

2. A piece of graph paper is folded once so that $(0, 2)$ is matched with $(4, 0)$, and $(7, 3)$ is matched with (m, n). Find (m, n).

3. In the Cartesian plane let $A = (1, 0)$ and $B = \left(2, 2\sqrt{3}\right)$. Equilateral triangle ABC is constructed so that C lies in the first quadrant. Let $P = (x, y)$ be the center of $\triangle ABC$. Then $x \cdot y$ can be written as $\frac{p\sqrt{q}}{r}$, where p and r are relatively prime positive integers and q is an integer that is not divisible by the square of any prime. Find p,q, and r.

4. The points $(0, 0)$, $(a, 11)$, and $(b, 37)$ are the vertices of an equilateral triangle. Find the value of ab.

5. A bee starts flying from point P_0. She flies 1 inch due east to point P_1. For $j \geq 1$, once the bee reaches point P_j, she turns 30° counterclockwise and then flies $j + 1$ inches straight to point P_{j+1}. When the bee reaches P_{2015} she is exactly $a\sqrt{b} + c\sqrt{d}$ inches away from P_0, where a, b, c and d are positive integers and b and d are not divisible by the square of any prime. What is $a + b + c + d$?

6. A parabola with equation $y = ax^2 + bx + c$ is reflected about the x-axis. The parabola and its reflection are translated horizontally five units in opposite directions to become the graphs of $y = f(x)$ and $y = g(x)$, respectively. What is the form of $y = (f + g)(x)$?

7. The parabola with equation $p(x) = ax^2 + bx + c$ and vertex (h, k) is reflected about the line $y = k$. This results in the parabola with equation $q(x) = dx^2 + ex + f$. What is $a + b + c + d + e + f$ in terms of h or k or both?

4.3 Other Geometry Problems

1. Let points $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 2, 0)$, and $D = (0, 0, 3)$. Points E, F, G, and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of rectangle $EFGH$?

2. Ajay is standing at point A near Pontianak, Indonesia, $0°$ latitude and $110°$ E longitude. Billy is standing at point B near Big Baldy Mountain, Idaho, USA, $45°$ N latitude and $115°$ W longitude. Assume that Earth is a perfect sphere with center C. What is the degree measure of $\angle ACB$?

3. Points A, B, C, D and E are located in 3-dimensional space with $AB = BC = CD = DE = EA = 2$ and $\angle ABC = \angle CDE = \angle DEA = 90^o$. The plane of $\triangle ABC$ is parallel to $\overline{DE}$. What is the area of $\triangle BDE$?

4. A rectangular box has width 12 inches, length 16 inches, and height $\frac{m}{n}$ inches, where m and n are relatively prime positive integers. Three faces of the box meet at a corner of the box. The center points of those three faces are the vertices of a triangle with an area of 30 square inches. Find $\frac{m}{n}$.