Logarithms

JV Practice

1 Warmup

1. (NYCIML F10B25) Compute $(\log_{125} 16)(\log_4 27)(\log_3 625)$.
2. (NYCIML F06B07) Compute
$$\frac{\log 8}{\log \frac{1}{8}}$$
3. (NYCIML S11B26) Let $\log_{10} 70 = m$ and $\log_{10} 20 = p$. Given that $\log_{10} 14 = Am + Bp + C$ where A, B, and C are integers, compute the ordered triple (A, B, C).
4. (NYCIML F06A19) If $\log_b(a)\log_c(a)\log_c(b) = 25$ and $\frac{a^2}{c^2} = c^k$, what is the sum of all possible values of k?

2 Log Rules

- Definition: If $b^x = a$ where $b > 0$ and $x > 0$, then $x = \log_b(a)$.
- Multiplication: $\log_b(xy) = \log_b(x) + \log_b(y)$
- Division: $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
- Exponentiation: $\log_b(x^y) = y\log_b(x)$
- Convention: Usually, $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- For any $b \neq 0$, $\log_b(1) = 0$ because $b^0 = 1$, and $\log_b(b) = 1$ because $b^1 = b$.
- $b^{\log_b(a)} = a$
- Change of Base: $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ (the **b**ase goes on the **b**ottom)
- Fun identity (follows from Change of Base): $\log_a(b) = \frac{1}{\log_b(a)}$

3 Problems 1

1. (2018 AMC 12B Problem 7) What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

2. (2002 AMC 12B Problem 22) For all integers n greater than 1, define $a_n = \frac{1}{\log_n 2002}$. Let $b = a_2 + a_3 + a_4 + a_5$ and $c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}$. What is $b - c$?

3. (2008 AMC 12A Problem 16) The numbers $\log(a^3b^7)$, $\log(a^5b^{12})$, and $\log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12^{th} term of the sequence is $\log b^n$. What is n?

4. (2019 AMC 12A Problem 15) Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b} = 100$$

and all four terms on the left are positive integers, where log denotes the base 10 logarithm. What is ab?

5. (1984 AIME Problem 5) Determine the value of ab given

$$\log_8 a + \log_4 b^2 = 5$$

$$\log_8 b + \log_4 a^2 = 7$$

.

6. (David Altizio, Mock AMC 10/12 2013) Suppose x and y are real numbers such that $\log_x(y) = 6$ and $\log_{2x}(2y) = 5$. What is $\log_{4x}(4y)$?

7. (2000 AIME II Problem 1) The number $\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

8. What is the value of a for which $\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1$?

9. (2015 AMC 12A Problem 14) Simplify $\frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \frac{1}{\log_4 N} + \cdots + \frac{1}{\log_{100} N}$ where $N = (100!)^3$

10. (David Altizio, Mock AMC 10/12 2013) Suppose x and y are real numbers such that $\log_x(y) = 6$ and $\log_{2x}(2y) = 5$. What is $\log_{4x}(4y)$?

11. (NYCIML F11) If $\log_{4n} 96 = \log_{5n} 75\sqrt{5}$, compute n^5.

4 Challenge Problems 1

1. The domain of the function $f(x) = \log_{\frac{1}{2}}(\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}} x))))$ is an interval of length $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

2. Let S be the set of ordered triples (x, y, z) of real numbers for which

$$\log_{10}(x + y) = z \text{ and } \log_{10}(x^2 + y^2) = z + 1.$$

There are real numbers a and b such that for all ordered triples (x, y, z) in S we have $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$. What is the value of $a + b$?

3. The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n?
4. Let $m > 1$ and $n > 1$ be integers. Suppose that the product of the solutions for x of the equation
$$8(\log_n x)(\log_m x) - 7\log_n x - 6\log_m x - 2013 = 0$$
is the smallest possible integer. What is $m + n$?
5. (David Altizio, Mock AIME I 2015) Suppose that x and y are real numbers such that $\log_x(3y) = \frac{20}{13}$ and $\log_{3x}(y) = \frac{2}{3}$. The value of $\log_{3x}(3y)$ can be expressed in the form $\frac{a}{b}$ where a and b are positive relatively prime integers. Find $a + b$.

5 Challenge Problems 2

1. (2006 AIME I Problem 9) The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).
2. (2013 AIME II Problem 2) Positive integers a and b satisfy the condition $\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0$. Find the sum of all possible values of $a + b$.
3. (1995 AIME Problem 2) Find the last three digits of the product of the positive roots of $\sqrt{1995}x^{log_{1995}x} = x^2$.
4. (1983 AIME Problem 1) Let x, y and z all exceed 1, and let w be a positive number such that $log_x w = 24$, $log_y w = 40$, and $log_{xyz} w = 12$. Find $log_z w$.
5. (2016 AIME II Problem 3) Let x, y, and z be real numbers satisfying the system $\log_2(xyz - 3 + \log_5 x) = 5$, $\log_3(xyz - 3 + \log_5 y) = 4$, $\log_4(xyz - 3 + \log_5 z) = 4$, Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.
6. (2009 AIME II Problem 2) Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find
$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$
7. (AIME II 2006) The lengths of the sides of a triangle with positive area are $\log_{10} 12$, $\log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values for n.
8. (Math Prize 2011) If $n > 10$, compute the greatest possible value of
$$\log n^{\log(\log(\log n))} - \log(\log n))^{\log(\log n)}$$
9. (2000 AIME I Problem 9) The system of equations
$$\begin{aligned} \log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) &= 4 \\ \log_{10}(2yz) - (\log_{10} y)(\log_{10} z) &= 1 \\ \log_{10}(zx) - (\log_{10} z)(\log_{10} x) &= 0 \end{aligned}$$
has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$.