Fermat's and Euler's Theorems

Varsity Practice 11/3/19

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1 Pre-Problems

1. Prove that $\binom{p}{i}$ is divisible by p for any prime p and $1 \leq i \leq p-1$.
2. Compute $10^{73} \pmod{19}$.
3. Prove that the equation $5y^2 + 2x^{100} = z^2$ doesn't have any nontrivial solutions in the integers.

2 Problems

1. Let $n = pq$, where p, q are distinct primes. Let a be an integer relatively prime to n. Prove that $a^{(p-1)(q-1)} \equiv 1 \pmod{n}$.
2. Compute $2^{98} \pmod{33}$.
3. (PUMaC 2008) If $f(x) = x^{x^{x^x}}$, find the last two digits of $f(17) + f(18) + f(19) + f(20)$.
4. A number n is said to be atrocious if there exists a prime p such that $p^n | (7^{p^n} + 1)$. Find all atrocious numbers.
5. How many prime numbers p are there such that $29^p + 1$ is a multiple of p?
6. Show that if p is a prime and there exists x such that $p | x^2 + 1$, then $p \equiv 1 \pmod{4}$.

   Very important statement in number theory
7. Prove that if p is a prime number, then $7p + 3^p - 4$ is not a perfect square.

   You might want to use the previous problem.
8. Prove or disprove: if p is a prime number, and k is an integer $2 \leq k \leq p$, then the sum of the products of each k-element subset of $\{1, 2, \ldots, p\}$ will be divisible by p?

   Hint: Note that $x^p - x \equiv x(x-1)(x-2)\ldots(x-(p-1)) \pmod{p}$ *and apply Vieta's formulas.*
9. Find all pairs (p, q) of prime numbers such that pq divides $(5^p + 5^q)$.
10. Find all primes p, q such that $\dfrac{(5^p - 2^p)(5^q - 2^q)}{pq} \in \mathbb{Z}$.
11. Solve in prime numbers the equation $x^y - y^x = xy^2 - 19$.