Modular Inverses

Varsity Practice 10/27/19

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1 Warm-Up

1. 11 girls and n boys picked n^2+9n-2 mushrooms, and each kid picked equal many mushrooms. Determine whether girls are more than boys or boys are more than girls.
2. (Modulo inverse) Let a, n be coprime integers. Show that
$$ax \equiv 1 \pmod{n}$$
has a unique solution $x \in \{0, 1, \cdots, n-1\}$. Solve for $a = 50$ and $n = 2019$.
3. (Wilson's theorem) Let p be a prime number. Show that
$$(p-1)! \equiv -1 \pmod{p}.$$

2 Problems

1. Find the number of pairs of positive integers (a, b) are there such that $\gcd(a, b) = 1$ and
$$\frac{a}{b} + \frac{14b}{9a} \in \mathbb{Z}.$$
2. Suppose x, y, z are integers such that
$$(x-y)(y-z)(z-x) = x+y+z.$$
Show that $27 \mid x+y+z$.
3. (AIME 1985) The numbers in the sequence $101, 104, 109, 116, \cdots$ are of the form $a_n = 100+n^2$, where $n = 1, 2, 3, \cdots$ For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.
4. Let $p > 5$ be a prime. Show that p^2 divides the numerator of the expression
$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}.$$
5. (ARML 2018) The increasing infinite arithmetic sequence $x_1, x_2, \cdots$ contains the terms 17! and 18!. Compute the greatest integer X for which $X!$ must also appear in the sequence.
6. (USAMO 2018) Let p be a prime, and let $a_1, \cdots, a_p$ be integers. Show that there exists an integer k such that the numbers
$$a_1 + k, a_2 + 2k, \cdots, a_p + pk$$
produce at least $\frac{p}{2}$ distinct remainders upon division by p.

7. (IMO 2019) Solve over positive integers the equation

$$k! = \prod_{i=0}^{n-1} (2^n - 2^i)$$

8. (China 2019) Let m be an integer with $|m| \geqslant 2$. If a sequence of integers $a_1, a_2, \cdots$ such that $a_1 \neq 0$ or $a_2 \neq 0$, and for every positive integer n, $a_{n+2} = a_{n+1} - ma_n$. If integers $r > s \geqslant 2$ such that $a_r = a_s = a_1$, show that $r - s \geqslant |m|$.