

Equations

Varsity 9/8/19

Matthew Shi

1 Useful Identities

$$\begin{aligned}x^2 - y^2 &= (x - y)(x + y) \\x^3 - y^3 &= (x - y)(x^2 + xy + y^2) \\x^3 + y^3 &= (x + y)(x^2 - xy + y^2) \\(x + y)^n &= \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}\end{aligned}$$

2 Warm-up Problems

- (SUMO 2019) David owns a parking lot for vehicles. A vehicle is either a motorcycle with two wheels or a car with four wheels. Today, there are 100 vehicles parked in his parking lot. The total number of wheels in David's parking lot is 326. If David collects \$1.00 from each motorcycle and \$2.00 from each car per day, how much money in dollars does David collect today?
- Suppose x, y are positive reals satisfying

$$\begin{aligned}x^3 + y^3 &= 1 \\x^4 + y^4 &= 1\end{aligned}$$

Solve for all possible pairs (x, y) .

- Alice and Bob are painting a house. If Alice and Bob do not take any breaks, they will finish painting the house in 20 hours. If, however, Bob stops painting once the house is half-finished, then the house takes 30 hours to finish. Given that Alice and Bob paint at a constant rate, compute how many hours it will take for Bob to paint the entire house if he does it by himself.
- (SMT 2013) A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?
- (HMMT 2008) Positive real numbers x, y satisfy $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Compute xy .
- (OMO Spring 2016) Let x, y, z be real numbers such that $x + y + z = 20$ and $x + 2y + 3z = 16$. What is the value of $x + 3y + 5z$?
- (SUMO 2019) Try to simplify $(350 + 90\sqrt{15})^{1/3} + (350 - 90\sqrt{15})^{1/3}$ as much as you can. The final answer is an integer.

3 More Problems

- Suppose $x + \frac{1}{x} = 15$. Compute the absolute value of the difference between $x^3 + \frac{1}{x^3}$ and $x^4 + \frac{1}{x^4}$.
- (OMO Fall 2016) A store offers packages of 12 pens for \$10 and packages of 20 pens for \$15. Using only these two types of packages of pens, find the greatest number of pens \$173 can buy at this store.
- (HMMT 2019) Let x, y be positive real numbers. Define $a = 1 + \frac{x}{y}$ and $b = 1 + \frac{y}{x}$. If $a^2 + b^2 = 15$, compute $a^3 + b^3$.

4. (BMT 2017) If $xy = 15$ and $x + y = 11$, calculate the value of $x^3 + y^3$.
5. (HMMT Nov. 2017) Compute $\frac{x}{w}$ if $w \neq 0$ and $\frac{x-6y+3z}{-3x+4w} = \frac{-2y+z}{x-w} = \frac{2}{3}$.
6. (HMMT 2019) Let a_1, a_2, \dots be an arithmetic sequence, and b_1, b_2, \dots be a geometric sequence. If $a_1b_1 = 20, a_2b_2 = 19, a_3b_3 = 14$, compute the maximum possible value for a_4b_4 .
7. (SUMO 2019) Let x, y be real numbers such that $x + y = 2$, and $x^4 + y^4 = 1234$. Find xy .
8. (CHMMC 2017) Two towns, A and B, are 100 miles apart. Every 20 minutes (starting at midnight) a bus traveling at 60 mph leaves town A for town B, and every 30 minutes (starting at midnight) a bus traveling at 20 mph leaves town B for town A. Dirk starts in Town A and gets on a bus leaving for town B at noon. However, Dirk is always afraid he has boarded a bus going in the wrong direction, so each time the bus he is in passes another bus, he gets out and transfers to that other bus. How many hours pass before Dirk finally reaches Town B?
9. (CHMMC 2015) A trio of lousy salespeople charge increasing prices on tomatoes as you buy more. The first charges you a dollars for the a th tomato you buy from him, the second charges b^2 dollars for the b th tomato, and the third charges c^3 dollars for the c th tomato. If you want to buy 100 tomatoes for as cheap as possible, how many should you buy from the first salesperson?
10. (NIMO 2017) The USAMO is a 6 question test. For each question, you submit a positive integer number p of pages on which your solution is written. On the i th page of this question, you write the fraction i/p to denote that this is the i th page out of p for this question. When you turned in your submissions for the 2017 USAMO, the bored proctor computed the sum of the fractions for all of the pages which you turned in. Surprisingly, this number turned out to be 2017. How many pages did you turn in?
11. (BMT 2019) Find the maximum value of $\frac{x}{y}$ if x, y are real numbers such that $x^2 + y^2 - 8x - 6y + 20 = 0$.