Complex Numbers

JV Practice 9/29/19

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1 Warmup

1. **Definition:** $i = \sqrt{-1}$ and $i^2 = -1$
2. **Definition:** The **standard form** of a complex number is $a + bi$.
3. **Definition:** The **complex conjugate** of a complex number $z = a + bi$ is $\overline{z} = a - bi$.
4. **DeMoivre's Theorem**: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all integers n.
5. Write each of the following expressions in standard form:
   - $(-4 + 7i) + (5 - 10i)$
   - $(1 - 5i)(-9 + 2i)$
   - $\frac{3-i}{2+7i}$
6. Find all the roots of $2x^3 + 2x^2 + x - 5 = 0$.
7. Find c if a, b, and c are positive integers which satisfy $c = (a + bi)^3 - 107i$
8. Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^\circ$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

2 Problems

1. Compute $|1 + 2i|^2$ and $(1 + 2i)^2$. Do the same for $|2 + 3i|^2$, $(2 + 3i)^2$. Do you notice anything special about the numbers you find?
2. If $\frac{(x+yi)}{i} = (7 + 9i)$, where x and y are real, what is the value of $(x + yi)(x - yi)$?
3. Determine all complex number z that satisfy the equation $z + 3z' = 5 - 6i$, where z' is the complex conjugate of z.
4. Find all complex numbers z such that $(4+2i)z+(8-2i)z' = -2+10i$, where z' is the complex conjugate of z.
5. Given that the complex number $z = -2+7i$ is a root to the equation: $z^3+6z^2+61z+106 = 0$, find the real root to the equation.
6. Prove that $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$ for all θ.
7. Find the number of ordered pairs of real numbers (a, b) such that $(a + bi)^{2002} = a - bi$.

8. Write the complex number $1-i$ in polar form. Then use DeMoivre's Theorem to write $(1-i)^{10}$ in the complex form $a+bi$, where a and b are real numbers and do not involve the use of a trigonometric function.

9. Find all of the solutions to the equation $x^3-1=0$.

10. (AMC 2017) There are 24 different complex numbers z such that $z^{24}=1$. For how many of these is z^6 a real number?

11. (AIME 2009) There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n}=4i.$$

Find n.