PIE and "margent needs"

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1 Derangements

1. Four people named Apple, Banana, Cantaloupe, and Durian are standing in a line. Each of them has their namesake fruit. How many ways are there for them to hold these fruits such that no one is holding their namesake fruit?

2. A Dolphin, an Elephant, a Fox, a Giraffe, and a Hamster are all wearing (very small, very different) hats. They throw them all into a dark room. Then, all of them go into the room, pick a hat at random, and exit. What is the probability that none of them exit wearing their original hat?

3. A permutation of $[n]$ is a function $f : [n] \to [n]$ such that every element in $[n]$ is attained exactly once. A derangement is a permutation such that $f(i) \neq i$ for all $i \in [n]$. Using inclusion/exclusion, find the number of derangements on $[n]$.

4. Write a recursive formula representing how many derangements there are for n integers (i.e, write the number of derangements of $[n]$ in terms of the number of derangements of $[n-1]$ and $[n-2]$).

5. Let $D(n)$ denote the number of derangements of $[n]$. Use a counting argument to show that

$$n! = D(n) + \binom{n}{1}D(n-1) + \binom{n}{2}D(n-2) + \cdots + \binom{n}{n-1}D(1) + 1$$

2 PIE (Principle of Inclusion/Exclusion)

1. C.J. has asked the class to vote on what pizza toppings they like. 24 people liked pepperoni, 11 liked chicken, and 8 liked mushrooms. 3 people like both mushrooms and chicken, 4 like both mushroom and pepperoni, and 9 like both chicken and pepperoni. 2 People wanted all 3 toppings. How many people voted?

2. How many multiples of 4 or 7 are found in [100]?

3. How many permutations of the 26 letters do not contain the words "heart", "artichokes", or "flower" ?

4. In order to decide which continents are allowed to survive under his glorious rule, overlord Misha rolls 10 7-sided dice. If a continent appears on the dice, the people on it are allowed to live. What is the probability that Misha doesn't get to use his evil squirrel laser (all 7 continents appear on the dice)?

5. In a normal poker deck (not the squirrel deck), how many 3-card hands can we have which contain at most 2 suits?

6. (2002 AIME) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

3 Challenge Problems

1. In Number Theory, Euler's totient function $\varphi : \mathbb{N} \to \mathbb{N}$ is defined by

$$\varphi(n) = \#\{i \leq n, i \in \mathbb{N} \mid i, n \text{ are relatively prime (share no factor which is not 1 or } -1)\}$$

   Given the prime factorization of $n = p_1^{r_1} \dots p_n^{r_n}$ where $p_1, \dots, p_n$ are distinct primes, find $\varphi(n)$.

2. (1972 USAMO) A random number selector can only select one of the nine integers 1, 2, ..., 9, and it makes these selections with equal probability. Determine the probability that after n selections ($n > 1$), the product of the n numbers selected will be divisible by 10.

3. (SUPER DUPER DUPER CHALLENGE) (1989 IMO) A permutation $\{x_1, \dots, x_{2n}\}$ of the set$\{1, 2, \dots, 2n\}$ where n is a positive integer, is said to have property T if $|x_i - x_{i+1}| = n$ for at least one $i \in \{1, 2, \dots, 2n-1\}$. Show that, for each n, there are more permutations with property T than without.

4 Background

Definition 1. $[n]$ denotes the set of natural numbers $\{1, 2, \dots, n\}$

Theorem 2 (Inclusion/Exclusion)**.** We are given a list of properties $p_1, p_2, \dots, p_k$ and a set of objects which can have these properties. We can find the total number of objects using

$$\begin{aligned}\text{Total number of objects} = &\text{ Number of objects with exactly 1 property}\\ &- \text{Number of objects with exactly 2 properties}\\ &+\\ &\vdots\\ &+ (-1)^k \times \text{Number of objects with all k properties}\end{aligned}$$