Polynomials

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Preliminaries

A *polynomial* p is a function of the form $p(x) = a_n x^n + a_{n-1} xn - 1 + \cdots + a_1 x + a_0$ (assume that $a_n \neq 0$). The number n is the *degree* of the p. The number r is a *root* of p if $p(r) = 0$. The polynomial p can always be written in the form $p(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n)$, where $r_1, \ldots, r_n$ are complex numbers, not necessarily distinct. $r_1, \ldots, r_n$ are roots of p, and they are the only roots.

Vieta's Formulas

The sum of the roots of p, namely $r_1 + r_2 + \cdots + r_n$ is given by the formula $\frac{-a_{n-1}}{a_n}$. The product of the roots of p, namely $r_1 r_2 \cdots r_n$ is $(-1)^n \frac{a_0}{a_n}$. A particularly common example is when $p(x) = ax^2 + bx + c$. In this case, the sum of the roots of p is $\frac{-b}{a}$ and the product of the roots of p is $\frac{c}{a}$.

Problems

1. (02 AMC 10B #10) Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. Compute the pair (a, b).

2. (05 AMC 10A # 10) There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. Compute the sum of those values of a.

3. (02 AMC 10A #14) Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. Compute the number of possible values of k.

4. (05 AMC 10B #16) The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, and p is zero. Compute n/p.

5. (03 AMC 10A #18, adapted) Compute the sum of the reciprocals of the roots of the equation $\frac{2017}{2018}x + 1 + \frac{1}{x} = 0$.

6. ('00 AMC 10 #24) Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

7. (07 PUMaC Algebra B #3) Compute all values of b such that the difference between the maximum and minimum values of $f(x) = x^2 - bx - 1$ on the interval $[0, 1]$ is 1.

8. (01 AIME I #3) Compute the sum of the roots, real and non-real, of the equation $x^{2017} + \left(\frac{1}{2} - x\right)^{2017} = 0$, given that there are no multiple roots.

9. (08 PUMaC Algebra B #5) Let $p(x) = x^5 + 3x^4 - 4x^3 - 8x^2 + 6x - 1$ and $q(x) = x^5 - 3x^4 - 2x^3 + 10x^2 - 6x + 1$. How many real numbers r are roots of both $p(x)$ and $q(x)$?

10. (07 PUMaC Algebra B #8) For how many rational numbers p is the area of the triangle formed by the intercepts and vertex of $f(x) = -x^2 + 4px - p + 1$ an integer?

11. (05 AIME I #6) Compute the product of the *nonreal* roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005$.

12. (07 PUMaC Algebra B #9) Compute all values of a such that $x^6 - 6x^5 + 12x^4 + ax^3 + 12x^2 - 6x + 1$ is nonnegative for all real x.

13. (03 AIME II #9) Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, compute $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

Challenge Problems

1. (Putnam '86 B5) Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. Let $p(x, y, z), q(x, y, z), r(x, y, z)$ be polynomials with real coefficients satisfying

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x, \pm y, \pm z$, where the number of minus signs is 0 or 2.

2. (Putnam '90 B5) Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?