Generating functions	Misha Lavrov

Combinatorial interpretations of generating functions

Western PA ARML Practice *March 29, 2015*

1 Useful facts

Definition. If you have a bunch of outcomes, each with its own weight (or cost, or value, or length, or...) then the *counting generating function* (c.g.f.) of those outcomes is

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where a_i is the number of outcomes with weight i.

Counting Lemma. Suppose you are making a sequence of independent choices one after the other; the choices have c.g.f.'s $A_1(x), A_2(x), \ldots, A_n(x)$. Then the c.g.f. for all the possible final outcomes of this sequence of choices (weighted by the sum of the weights in each choice) is

$$A(x) = A_1(x) \cdot A_2(x) \cdot A_3(x) \cdots A_{n-1}(x) \cdot A_n(x).$$

Example. The c.g.f. for rolling a 6-sided die is

$$D(x) = x + x^2 + x^3 + x^4 + x^5 + x^6.$$

The c.g.f. for rolling two 6-sided dice and adding their values is

$$(D(x))^2 = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.$$

2 Problems

(In this section, you don't need to simplify the expressions you get for these generating functions; in fact, this is often much harder than just coming up with the expressions.)

1. A marble store has 1000 marbles in stock: 100 marbles in each of 10 colors. Write an expression for the c.g.f. $A(x)$ such that the coefficient of x^n in $A(x)$ is the number of ways to buy n marbles from the marble store.
2. Five items that you're considering taking with you on a trip weigh 4 ounces, 7 ounces, 13 ounces, 15 ounces, and 17 ounces. Write an expression for the c.g.f. in which the coefficient of x^n is the number of ways you can choose some items with total weight exactly n ounces.
3. There are 15 people at ARML practice, each of whom eats 2, 3, or 4 slices of pizza. Write an expression for the *pizza generating function* in which the coefficient of x^n is the number of ways in which n slices of pizza can be split between everyone at ARML practice.

4. Let $D(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$ be the c.g.f. for rolling a 6-sided die. Express in terms of $D(x)$ the c.g.f. for the following way of generating a "super-random" number: you roll a 6-sided die, then you roll that many 6-sided dice and take the total value, then you roll that many 6-sided dice and take the total value.

3 Pirates and Gold

Newton's Binomial Theorem. For any complex numbers x and r with $|x| < 1$,

$$(1+x)^r = 1 + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \binom{r}{4}x^4 + \cdots$$

where we define $\binom{r}{k}$ to be $\frac{r(r-1)(r-2)(\cdots)(r-k+1)}{k!}$.

(In this section, you should express your answers in terms of binomial coefficients such as $\binom{25}{10}$ but don't bother simplifying them; this lets me use larger numbers than I would otherwise.)

0. Prove that for an integer $n > 0$, $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$. From this it follows that

$$\frac{1}{(1-x)^r} = 1 + \binom{r}{1}x + \binom{r+1}{2}x^2 + \binom{r+2}{3}x^3 + \cdots + \binom{r+k-1}{k}x^k + \cdots$$

1. Five pirates want to split a chest of 1000 gold pieces. In how many ways can they do this?
2. Five pirates want to split a chest of 1000 gold pieces; however, one of the pirates is fair-minded and will not accept more than a fair share of 200 gold pieces. In how many ways can the pirates make the split?
3. Five pirates want to split a chest of 1000 gold pieces; however, one of the pirates is greedy and will not accept less than 500 gold pieces. In how many ways can the pirates make the split?
4. Five pirates want to split a chest of 1000 gold pieces. Afterwards, each pirate will go drinking, and may spend any number of gold pieces on rum.
   (a) The c.g.f. for the number of ways in which a pirate can spend his/her gold is

   $$R(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$$

   (For instance, the coefficient of x^4 is 5 because a pirate who gets 4 gold pieces can spend 0, 1, 2, 3, or 4 of them on rum.) Simplify $R(x)$.

   (b) Find the number of ways in which the pirates can split and then spend the 1000 gold pieces.
5. A drunken pirate stumbles around the number line, starting at 0. Each step the pirate takes is either forward (from k to $k+1$) or backward (from k to $k-1$); these are chosen at random and are equally likely.

   It is known that $(-1)^n\binom{-1/2}{n}$ is the probability that after $2n$ steps, the drunken pirate will end up back at 0. Simplify this formula into something that makes more sense.