Chapter 3

Length and Area Techniques

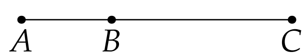
In this chapter we develop some basic length and area techniques. For lengths, we will cover the Pythagorean Theorem, adding and subtracting lengths of segments on the same line, the Two Tangent Theorem, and finding the distance between the centers of tangent circles.

For areas, we will use $\frac{bh}{2}$ to find the area of a triangle, explain how to find the area of a circle and a sector of a circle, and use addition and subtraction to find the area of composite figures.

3.1 Lengths

We start with a fundamental postulate that we will use for the rest of the chapter and the rest of the entire book.

Postulate 3.1. If point B lies on segment AC, then $AB + BC = AC$.



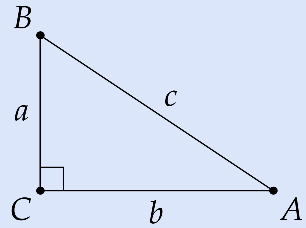
We can rearrange this into $AB = AC - BC$, so subtraction follows from this additive postulate as well. The other direction, which states that B lies on segment AC only if $AB + BC = AC$, follows from the Triangle Inequality, so we do not need to include it as part of this postulate.

3.1.1 The Pythagorean Theorem

We start by explicitly mentioning and proving the Pythagorean Theorem, which we have been implicitly using in the last couple of chapters.

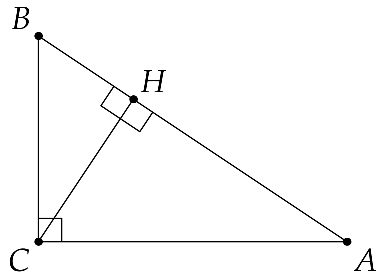
Theorem 3.2 (Pythagorean Theorem).

In $\triangle ABC$ with $\angle C = 90^\circ$, $a^2 + b^2 = c^2$.



There are two proofs that we will present: one with similar triangles and another with areas. Since we're starting with length techniques, you should consider the similarity proof the "canonical" one, but the area proof does not require much technical knowledge.

Proof. This proof uses similar triangles. Let H be the foot of the altitude from C to AB.



Note that $\triangle AHC \sim \triangle ACB \sim \triangle CHB$. Thus,

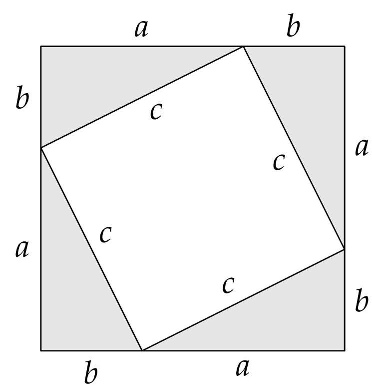
$$\frac{AH}{AC} = \frac{AC}{AB}$$
$$\frac{HB}{CB} = \frac{CB}{AB}$$

This implies that $AH = \frac{AC^2}{AB}$ and $HB = \frac{BC^2}{AB}$. But note that $AH + HB = AB$, so

$$AB = AH + HB = \frac{AC^2}{AB} + \frac{BC^2}{AB}.$$

Multiplying both sides by AB gives us our desired result. $\square$

Proof. This proof uses areas; refer to the diagram below.



Note that we can get the area of the large square by squaring the side length. This gives us an area of $(a+b)^2$.

But also note that we can add the area of the small square and the 4 triangles to get the area of the large square, which gives us an area of $c^2 + 2ab$.

Since $(a+b)^2 = c^2 + 2ab$, subtracting $2ab$ from both sides gives us $a^2 + b^2 = c^2$, as desired. $\square$

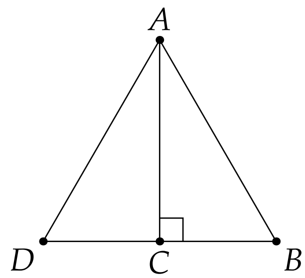
Let's start by discussing **special right triangles**. We know from **??** that all right triangles (indeed, all triangles) with the same angles have the same ratios of side lengths too. With the Pythagorean Theorem, we can explicitly find these ratios for triangles with certain angles.

Example 3.3 (30–60–90 Triangle).

In $\triangle ABC$, say $\angle C = 90°$ and $\angle A = 30°$. Prove that the side lengths a, b, c have a ratio of 1, $\sqrt{3}$, and 2. In other words, show that

$$a : b : c = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1.$$

Solution. Take equilateral triangle ABD and let C be the midpoint of BD.[1] By SSS congruence, $\triangle ADC \cong \triangle ABC$, implying that $DC = CB$. Since $DC + CB = DB$, $CB = \frac{1}{2}DB$.



Since $\triangle ABC$ is equilateral, $CB = \frac{1}{2}AB$. By the Pythagorean Theorem (1.2),

$$AC^2 = AB^2 - CB^2$$
$$AC^2 = AB^2 - \left(\frac{1}{2}AB\right)^2$$
$$AC^2 = \frac{3}{4}AB^2$$
$$AC = \frac{\sqrt{3}}{2}AB.$$

Thus,

$$a : b : c = BC : CA : AB = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1,$$

as desired. ■

Problem 3.4 (45–45–90 Triangle). In $\triangle ABC$, say $\angle C = 90°$ and $\angle A = 45°$. Find the ratio of the side lengths of $\triangle ABC$. **Hint:** 7

Some other right triangles are noteworthy because their sides are all integers. For instance, there exists a right triangle with lengths 3, 4, and 5.

From now on, we will refer to a right triangle with lengths a, b, and c as an a–b–c right triangle. The triangle mentioned previously would be a 3–4–5 right triangle.

If $a^2 + b^2 = c^2$, we call the integers (a, b, c) a **Pythagorean triple**. Here are a few common triples you might want to know:

- 3–4–5
- 5–12–13
- 7–24–25
- 8–15–17
- 9–40–41

It doesn't hurt to verify that each of these triples is indeed a Pythagorean triple. Don't bother memorizing them though; once you've solved enough problems, you'll naturally be able to recognize Pythagorean triples.

Now let's get down to some examples of the Pythagorean Theorem at play.

Example 3.5 (AIME I 2006/1).

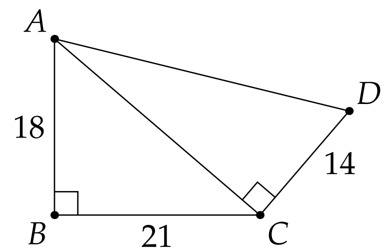
In quadrilateral $ABCD$, $\angle B$ is a right angle, diagonal $\overline{AC}$ is perpendicular to $\overline{CD}$, $AB = 18$, $BC = 21$, and $CD = 14$. Find the perimeter of $ABCD$.

[1] The way you'd construct $\triangle ABD$ is by reflecting B about C. However, we'll sacrifice a little bit of rigor in favor of not using a concept introduced much later.

Solution. We're missing one side length of $ABCD$: the length of DA. By the Pythagorean Theorem (1.2),

$$DA^2 = AC^2 + CD^2 = AB^2 + BC^2 + CD^2 = 18^2 + 21^2 + 14^2 = 961,$$

implying that $DA = 31$. Thus, the perimeter is $18 + 21 + 14 + 31 = 84$.

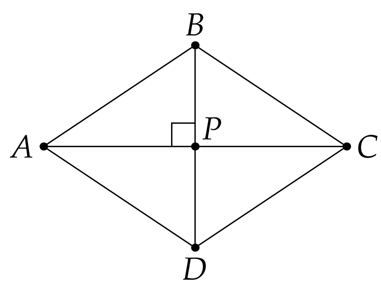


■

Example 3.6.

Show that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ for all parallelograms $ABCD$.

Solution. Let AC intersect BD at P. Note by **??** that $AP = PC$ and $BP = PD$. SSS implies $\triangle ABP \cong \triangle CBP \cong \triangle ADP \cong \triangle CDP$, so $\angle ABP = \angle CBP = \angle ADP = \angle CDP = 90°$. Also, $AC = 2AP$ and $BD = 2BP$.



By the Pythagorean Theorem (1.2),

$$\begin{aligned} AB^2 &= AP^2 + BP^2 \\ BC^2 &= BP^2 + CP^2 \\ CD^2 &= CP^2 + DP^2 \\ DA^2 &= DP^2 + AP^2. \end{aligned}$$

Adding this all up yields

$$\begin{aligned} AB^2 + BC^2 + CD^2 + DA^2 &= 2AP^2 + 2BP^2 + 2CP^2 + 2DP^2 \\ &= 4AP^2 + 4CP^2 \\ &= AC^2 + BD^2, \end{aligned}$$

as desired. ■

The "only if" direction is true too; all quadrilaterals satisfying the length condition are parallelograms. However, we will need tools from **??** — specifically, the Law of Cosines (**??**) — to prove it. Therefore, we will show the other direction in Example **??**.

Example 3.7 (13–14–15 Triangles).

Say in $\triangle ABC$ that $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to BC. What are the lengths AD, BD, and CD?

This example is important: 13–14–15 triangles, while not being right triangles themselves, often appear in geometry problems. (In fact, the next example features one.) You will see why when you know the answer.

Solution. Say $BD = x$, which implies that $CD = 14 - x$. By the Pythagorean Theorem (1.2) on $\triangle ABD$ and $\triangle ACD$ respectively,

$$AB^2 = BD^2 + AD^2$$
$$AC^2 = CD^2 + AD^2$$

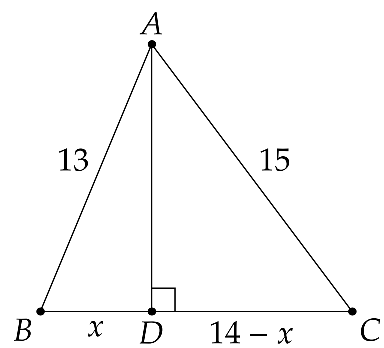
Substituting in the values we have gives

$$13^2 = x^2 + AD^2$$
$$15^2 = (14 - x)^2 + AD^2$$

Subtracting the first equation from the second gives

$$56 = 196 - 28x,$$

implying that $x = 5$. Thus $CD = 9$ and $AD = 12$.



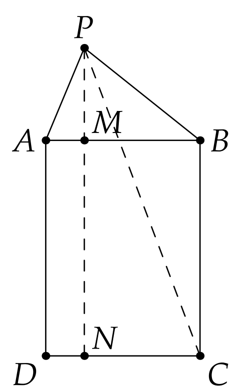
■

Sometimes it's not obvious that you can invoke the Pythagorean Theorem at all. Nor is it obvious which right triangle you should construct (i.e. where the bases should be). The next two examples demonstrate this.

Example 3.8 (NARML 2020/1).

Suppose $ABCD$ is a rectangle with $AB = 14$ and $BC = 28$. If point P outside $ABCD$ satisfies the conditions $PA = 13$ and $PB = 15$, compute the length of PC.

Solution. Let the line through P perpendicular to AB intersect AB at M and CD at N. Since AB and CD are parallel, PN is also perpendicular to CD.



Note that $PM = 12$ as $\triangle PAB$ is a $13-14-15$ triangle, so $BM = CN = 9$ and $PN = PM + MN = PM + BC = 12 + 28 = 40$. Thus by the Pythagorean Theorem (1.2),

$$PC = \sqrt{9^2 + 40^2} = 41.$$

■

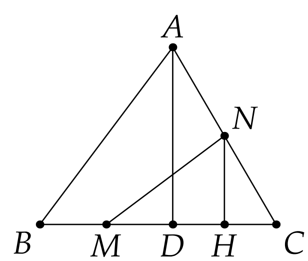
Example 3.9 (JMC 10 2020/17).

Let $\triangle ABC$ be acute and let D be the foot of the altitude from A to BC. If $AD = 24$ and $BC = 32$, what is the distance between the midpoints of BD and AC?

Solution. Let M be the midpoint of BD, N be the midpoint of AC, and H be the foot of the altitude from N to AC. Then note that $\triangle ADC \sim \triangle NHC$ with a scale factor of 2 (since $AC = 2NC$ by definition), so $NH = \frac{AD}{2} = 12$. Using the same pair of similar triangles, we can deduce that $MH = MD + HD = \frac{BD}{2} + \frac{CD}{2} = \frac{BC}{2} = 16$.

So by the Pythagorean Theorem (1.2) on $\triangle NHM$,

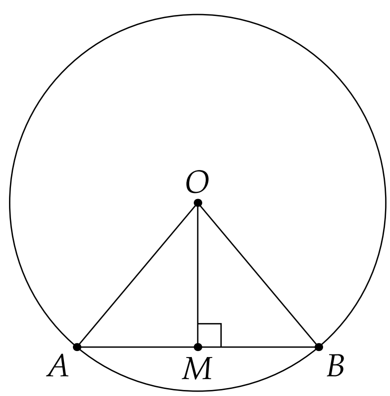
$$MN = \sqrt{MH^2 + NH^2} = \sqrt{16^2 + 12^2} = 20.$$



■

Notice we leverage the lengths we have: we want to use AD and BC somehow — AD especially, because the only way we can use AD is by dropping an altitude from N to AC.

The Pythagorean Theorem also works well with circles, particularly when chords are drawn. Say there is a circle centered at O passing through A and B. Then, if M is the foot of the altitude from O to AB (recall that M is the midpoint of AB by Problem **??**), then we can invoke the Pythagorean Theorem on $\triangle OAM$.

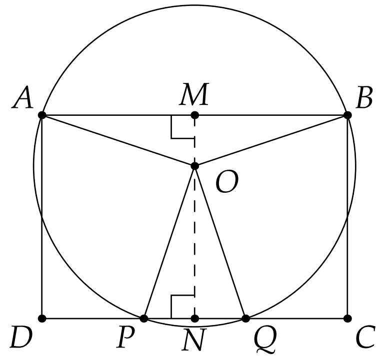


When we have multiple chords like AB, we may be able to set up a system of equations as this next example will show.

Example 3.10 (DMC 10B 2021/14).

Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. A circle passes through A and B and intersects side $\overline{CD}$ at two points which trisect the side. What is the area of the circle?

Solution. Let O be the center of this circle, and let the line through O perpendicular to AB intersect AB at M and CD at N.



Note that M and N are the midpoints of AB and CD, respectively, because $\triangle OAB$ and $\triangle OCD$ are isosceles. By the Pythagorean Theorem (1.2),

$$\begin{aligned} OA^2 &= AM^2 + MO^2 \\ OP^2 &= PN^2 + NO^2. \end{aligned}$$

Note that $AM = 3$ and $PN = 1$, and also note that $OA = OP$. Subtracting the second equation from the first gives

$$\begin{aligned} 0 &= 3^2 - 1^2 + MO^2 - NO^2 \\ 8 &= NO^2 - MO^2 \\ &= (NO + MO)(NO - MO) \\ &= 4(4 - 2MO). \end{aligned}$$

This implies that $MO = 1$ and $NO = 3$. Thus the area of the circle is

$$\pi OA^2 = \pi(AM^2 + MO^2) = \pi(3^2 + 1^2) = 10\pi.$$

■

3.1.2 The Triangle Inequality

With the Pythagorean Theorem, we unlock one of the most important length techniques: the Triangle Inequality.

Theorem 3.11 (Triangle Inequality).

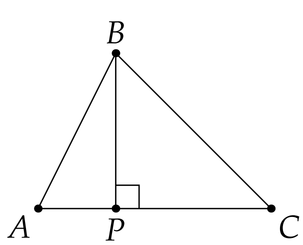
Given points A, B, and C,

$$AB + BC \geq AC,$$

with equality if and only if B lies on segment AC.

We say that lengths AB, BC, and CA satisfy the triangle inequality if $AB + BC \geq CA$, $BC + CA \geq AB$, and $CA + AB \geq BC$.

Proof. Let P be the foot of the altitude from P to AC. By the Pythagorean Theorem (1.2), $AB \geq AP$ and $BC \geq PC$, with equality if and only if $BP = 0$ in both cases.



Now note $AP + PC \geq AC$ with equality if and only if P lies on segment AC. (We can say "only if" because Postulate 1.1 will apply in some way since A, B, and C are collinear.)

Thus,

$$AB + BC \geq AP + PC \geq AC,$$

with both inequalities simultaneously reaching equality if and only if B lies on segment AC, as desired. □

As promised in the beginning of this section, we have used the Triangle Inequality to show that $AB + BC = AC$ only if B lies on segment AC.

The converse of the Triangle Inequality, which guarantees the existence of a triangle if its lengths satisfy the Triangle Inequality, is also useful. (We use the term "converse" a little loosely here.)

Theorem 3.12 (Converse of Triangle Inequality).

A triangle with lengths a, b, and c exists if and only if a, b, and c satisfy the Triangle Inequality (1.11).

If this is your first read through the chapter, the proof isn't particularly important; rather, it is important to know what assumptions we are allowed to make (e.g. triangles with "valid" lengths always exist), which is why this theorem is presented to begin with. Experienced geometers, on the other hand, may want to verify or even derive these equations on their own.

Proof. The "only if" direction is just the Triangle Inequality; we only include it for completeness.[2] Hence we focus solely on the "if" direction.

Let the triangle we will construct be $\triangle ABC$, where we want $BC = a$, $CA = b$, and $AB = c$, as is standard. Without loss of generality, say $a \geq b$ and $a \geq c$. Now we explicitly perform this construction.

Construct segment BC with length a. Now, pick point P on segment BC such that $BP = \frac{a^2-b^2+c^2}{2a}$, and now pick point H on the line through P perpendicular to BC such that $HP = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}$. An explicit computation with the Pythagorean Theorem (1.2) — which we omit — shows that $CA = b$ and $AB = C$, as desired. □

If you know Heron's Formula, seeing $h = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}$ should raise some eyebrows. Indeed, a calculation with $\frac{bh}{2}$ shows that the area of a triangle with sides a, b, and c is $\sqrt{s(s-a)(s-b)(s-c)}$. This is no coincidence: this proof is remarkably similar to the proof of Heron's Formula with the Pythagorean Theorem. Here we have omitted many details, but when we prove Heron's Formula in **??**, we will explicitly derive and perform all of the calculations.

Problem 3.13 (Semiperimeter Triangle Inequality). Show that the triangle inequality is equivalent to $AC \leq \frac{AB+BC+CA}{2}$.

This form of the triangle inequality is known as the Semiperimeter Triangle Inequality because the right side is the **semiperimeter**, i.e. half of the perimeter.

As a corollary, if the triangle inequality is true for the largest length of three lengths, it is true for all three lengths. This can be very useful when checking if a triangle with particular lengths can exist. For instance, a triangle with sides 3, 4, 5 exists because $5 \leq \frac{3+4+5}{2}$. However, a "triangle" with sides 3, 4, and 9 does not exist, because $8 \geq \frac{3+4+9}{2}$.

It is also worth noting that a triangle whose vertices all lie on the same line sometimes may not be considered a triangle. For instance, it is possible for $AB = 3$, $BC = 4$, and $CA = 7$ — this is because $7 = \frac{3+4+7}{2}$ and occurs when B lies on segment AC — but you'd be hard-pressed to call this a proper triangle. Indeed, when a triangle's vertices all lie on the same line, it is called a **degenerate triangle**.

The Triangle Inequality can also be extended to more than two intermediate segments, which further reinforces the idea that the shortest path between two points is a line.[3]

[2] "Converse" is a bit of a misnomer for this theorem, and its formulation is not exactly the same as the Triangle Inequality.

[3] This is not a proof of the concept, just a tool for building intuition. If two segments isn't shorter than a straight line, then intuitively, neither should three segments be, or four...

Example 3.14 (Extended Triangle Inequality).

Prove that given points $P_0, P_1, \ldots, P_{n+1}$,

$$P_0P_1 + P_1P_2 + \cdots + P_nP_{n+1} \geq P_0P_{n+1}$$

with equality if and only if P_i lies on segment $P_{i-1}P_{i+1}$ for all $1 \leq i \leq n$.

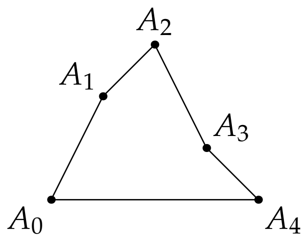


Figure 3.15: The Extended Triangle Inequality with 4 segments; in this case, $A_0A_1 + A_1A_2 + A_2A_3 + A_3A_4 \leq A_0A_4$.

The equality condition is equivalent to P_0, P_1, . . ., P_{n+1} lying on the same line, **in that order**.

Solution. Note that in general, because of the Triangle Inequality 1.11, $P_{i-1}P_i + P_iP_{i+1} \geq P_{i-1}P_{i+1}$ with equality when P_i lies on segment $P_{i-1}P_{i+1}$, implying that

$$P_0P_1 + \cdots + P_{i-1}P_i + P_iP_{i+1} \geq P_0P_1 + \cdots + P_{i-1}P_{i+1}$$

with the same equality condition. Applying this repeatedly gives

$$P_0P_1 + \cdots + P_iP_{i+1} \geq P_0P_1 + \cdots + P_{i-1}P_{i+1} \geq \cdots \geq P_0P_{n+1}.$$

(We've done a little bit of implicit relabeling of points when repeatedly applying the inequality, but the general concept is identical each time.)

Each of these inequalities reaches equality when P_i lies on segment $P_{i-1}P_{i+1}$, and iterating yields:

- P_n lies on $P_{n-1}P_{n+1}$
- P_{n-1} lies on $P_{n-2}P_{n+1}$
- ...and so on.

In the end, this is equivalent to P_i lying on segment $P_{i-1}P_{i+1}$, as desired. ■

If the general case is confusing, you should try proving the inequality for $n = 2$. This should give you some intuition for how the general argument works. (The original Triangle Inequality is equivalent to $n = 1$, so the next step would be $n = 2$.)

A useful example of the Extended Triangle Inequality is determining the furthest and closest possible distances between two points on two different circles.

Example 3.16.

Consider two circles ω_1 and ω_2 with radii r_1, r_2 and centers O_1, O_2, respectively. If $O_1O_2 \geq r_1 + r_2$, point P is on ω_1, and point Q is on ω_2, what is the minimum length of segment PQ? The maximum?

Solution. We first find the maximum length of PQ. Note that $PQ \leq P_1O_1 + O_1O_2 + O_2P_2 = r_1 + O_1O_2 + r_2$ by the Extended Triangle Inequality (1.14), with equality when P, O_1, O_2, and Q lie on a line in that order.

The equality condition.

Figure 3.17: $PQ \le PO_1 + O_1O_2 + O_2Q$.

Now we find the minimum length of PQ. Note that $O_1O_2 \le O_1P + PQ + PO_2$ with equality when O_1, P, Q, and O_2 lie on a line in that order, implying that $O_1O_2 - (r_1 + r_2) \le PQ$ with the same equality condition.

The equality condition.

Figure 3.18: $O_1O_2 \le O_1P + PQ + PO_2$.

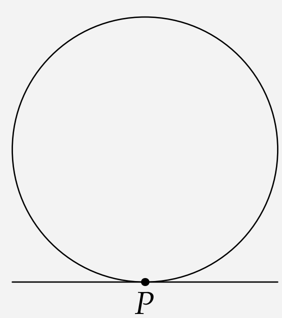
■

3.1.3 Tangents

In this subsection we develop some theory for tangents: both lines tangent to circles and circles tangent to circles will be covered. Heavy use of the Pythagorean Theorem follows.

Definition 3.19 (Tangent Line).

We say line ℓ is tangent to circle ω if and only if it intersects ω at exactly one point. If, for example, we denote the point of intersection as P, then we say ℓ is tangent to ω at point P.



We start by establishing an important relation between the tangent and radius of a circle.

Theorem 3.20 (Tangent Perpendicularity).

If line ℓ is tangent to circle ω with center O at P, then OP is perpendicular to ℓ.

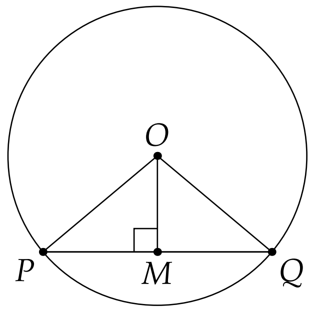
This theorem is the basis for this entire subsection, much like the postulates of previous chapters were the basis for said chapters. The only reason it isn't marked as a postulate is because we can prove this vital result.

Even without a proof, this result should intuitively feel true. The shortest path from a point O to a line ℓ is the segment from O perpendicular to ℓ because of the Pythagorean Theorem (1.2). Therefore, if the circle and the line “barely intersect”, then the radius should “barely” be long enough to reach the line, meaning it should be perpendicular to the line. Of course, this is just intuition to convince you the theorem should be true. It does not replace the proof below.

Proof. Assume for the sake of contradiction that OP is not perpendicular to ℓ. Then let M be the point on ℓ such that $OM \perp \ell$. Now reflect P across M to get Q, and note that

$$OQ^2 = OM^2 + MQ^2 = OM^2 + MP^2 = OP^2.$$

Thus $OQ = OP$, implying that Q lies on ω. Since P and M are distinct points, P and Q must also be distinct points, contradiction.



□

Problem 3.21. Prove the converse of Theorem 1.20: if line ℓ intersects ω at point P and OP is perpendicular to ℓ, then ℓ is tangent to ω. **Hint:** 3

Now we will use the Triangle Inequality to show that two circles can intersect at most twice, assuming that they are distinct.

Theorem 3.22 (Intersections of Two Circles).

If two circles intersect a finite number of times, they intersect at most twice.

While this theorem is important, it is the proof that is really illuminating, particularly the earlier parts that mention the Triangle Inequality. Concepts like internally tangent and externally tangent circles are naturally derived from it, so this proof is one you should pay close attention to.

Proof. Say the circles are ω_1 and ω_2 with radii r_1, r_2 and centers O_1, O_2, respectively. (If ω_1 and ω_2 intersect a positive finite number of times, this implies that O_1 and O_2 are distinct points. In any case, the case where $O_1 = O_2$ is just a minor detail.) Then the number of times ω_1 and ω_2 intersect is based on the Triangle Inequality (1.11) for lengths r_1, r_2, and O_1O_2.

If there is some point P that lies on both ω_1 and ω_2, then $\triangle PO_1O_2$ must satisfy the Triangle Inequality (**??**. Thus, for example, if we have some case like $O_1O_2 > r_1 + r_2$ or $r_1 > O_1O_2 + r_2$, then the circles would not intersect, because no point P would satisfy these length conditions.

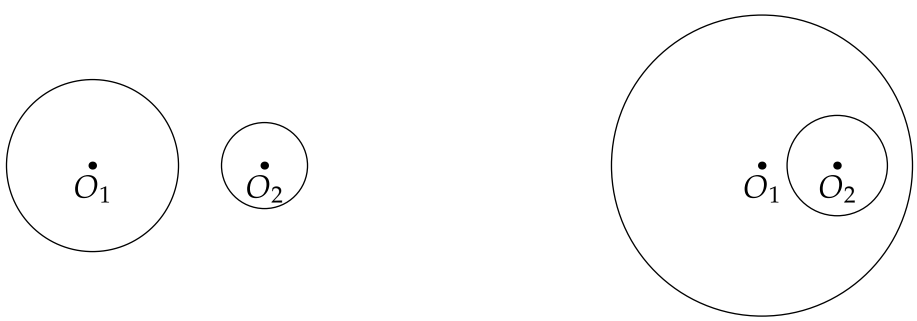
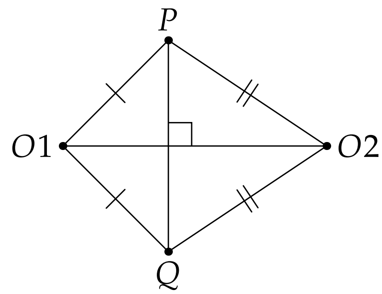


Figure 3.23: $O_1O_2 > r_1 + r_2$ and $r_1 > O_1O_2 + r_2$, respectively.

If P lies on line O_1O_2, then by Postulate 1.1, this point P is unique. Otherwise, reflecting P about O_1O_2 gives another intersection point Q. Now we claim that for any point Q on ω_1 and ω_2 such that $Q \neq P$, PQ is perpendicular to O_1O_2.

Since $\triangle PQO_1$ and $\triangle PQO_2$ are isosceles, the line through O_1 perpendicular to PQ passes through the midpoint of PQ, and same for the line through O_2 perpendicular to PQ. Thus these two lines are identical, implying $PQ \perp O_1O_2$. The Pythagorean Theorem (1.2) shows that the distance from Q to O_1O_2 can only take one value, so only two intersection points exist.



$\square$

When $r_1 + r_2 = O_1O_2$, we say that ω_1 and ω_2 are **externally tangent**, and when $r_1 - r_2 = O_1O_2$, we say that ω_1 and ω_2 are **internally tangent**.[4] Note in these cases that the tangency point P lies on O_1O_2, as described by the proof. The diagrams below show why we use these terms.

Figure 3.24: On the left, two externally tangent circles. On the right, two internally tangent circles. Note that in both cases, the tangency point P lies on line O_1O_2. This is because of Postulate 1.1.

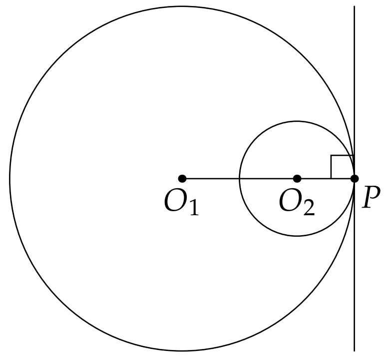
Next we present a corollary of the proof of Theorem 1.22 that ties tangent lines and circles together.

Theorem 3.25.

Say that circles ω_1 and ω_2 are tangent at P, whether externally or internally. Then there is a line ℓ passing through P tangent to both ω_1 and ω_2.

Proof. Because O_1, O_2, and P are collinear, the line through P perpendicular to O_1P is also perpendicular to O_2P. Thus, said line is tangent to both ω_1 and ω_2 by Theorem 1.20.

[4]Strictly, the definition of internal tangency is $|r_1 - r_2| = O_1O_2$.



□

Problem 3.26. Show the converse of Theorem 1.25: if there is some line ℓ passing through P tangent to both ω_1 and ω_2, then ω_1 and ω_2 must be tangent at P.

Another powerful corollary of the proof of Theorem 1.22 is the Two Tangent Theorem, a crucial tool for length chasing.

Theorem 3.27 (Two Tangent Theorem).

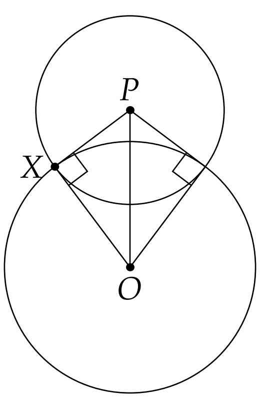
Consider a circle ω centered at O with radius r. Let P be a point outside ω; that is, let P be a point such that $OP > r$. Then:

- There are exactly two lines ℓ_1 and ℓ_2 that pass through P and are tangent to ω.
- Say ℓ_1, ℓ_2 are tangent to ω at A and B respectively. Then $PA = PB$.

Proof. Let ℓ be any line tangent to ω at X. Note that $PX^2 = PO^2 - OX^2 = PO^2 - r^2$, implying that $PX = \sqrt{PO^2 - r^2}$, as $PO > r$.

Since a line is uniquely determined by two points, the possible lines ℓ are uniquely determined by the possible points X, since ℓ must also pass through P. The locus of points X that satisfy $PX = \sqrt{PO^2 - r^2}$ is a circle. Thus ω and the locus of points X intersect at most twice.

Because both radii are less than PO and $(PX+r)^2 > PX$, which implies $PX + r > PO$, these circles intersect exactly twice by the Triangle Inequality (1.11). If we refer to these intersection points A and B, then $PA = \sqrt{PO^2 - r^2} = PB$ by definition, as desired.

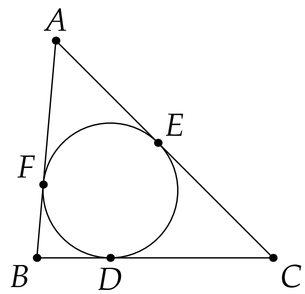


□

With our theory established, we now turn to some examples. First, we will explore some properties of the **incircle**, the circle tangent to **segments** AB, BC, and CA.

Example 3.28.

Say the incircle of $\triangle ABC$ is tangent to segments BC, CA, and AB at D, E, and F, respectively, and say that half the perimeter — also known as the **semiperimeter** — of $\triangle ABC$ is s. Then show that $AE = AF = s - a$, and analogously, $BF = BD = s - b$ and $CD = CE = s - c$.

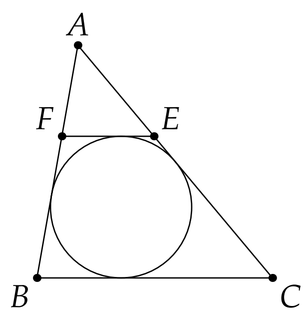


Solution. Say $AE = AF = x$, $BF = BD = y$, and $CD = CE = z$.[5] Also note that by definition, $2x + 2y + 2z = AB + BC + CA = 2s$, implying $x + y + z = s$.

Now note that $y + z = BD + CD = BC$, so $(x + y + z) - (y + z) = s - a$ as desired.[6] ∎

Example 3.29.

In $\triangle ABC$, let E and F be points on segments CA and AB such that $EF \parallel BC$ and EF is tangent to the incircle of $\triangle ABC$. What is the length of EF?



Solution. Say that the incircle intersects EF, CA, and AB at P, X, and Y, respectively. Note that $EX = EP$ and $FY = FP$ by the Two Tangent Theorem (1.27).

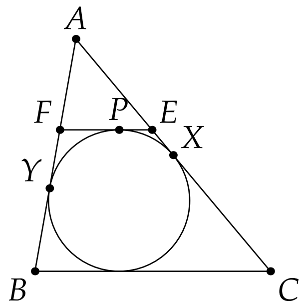
Since $\triangle AFE \sim \triangle ABC$, our strategy will now be to find the perimeter of $\triangle AFE$ and compare it to the perimeter of $\triangle ABC$ to find the factor of similarity. Then, we can find EF by comparing it to BC.

Now note that $EF = EP + FP = EX + FY$, implying that the perimeter of $\triangle AFE$ is $AE + AF + EF = (AE + EX) + (AF + FY) = AX + AY$. By Example 1.28, $AX + AY = 2(s - a) = b + c - a$. Thus,

$$EF = \frac{b + c - a}{a + b + c} BC = \frac{a(b + c - a)}{a + b + c}.$$

[5] These lengths are equal by the Two Tangent Theorem (1.27).

[6] Note that $BF = BD = s - b$ and $CD = CE = s - c$ by symmetry.



■

The formula for the previous example is quite complicated. Trying to blindly memorize it is counterproductive. Instead, remember the main steps of the solution:

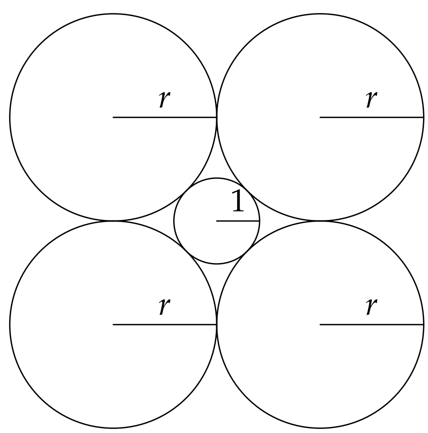
- The perimeter of $\triangle AFE$ is $AX + AY$ by the Two Tangent Theorem (1.27).
- You should remember what the length of AX (and AY) is; Example 1.28 is something you should know by heart.
- Use $\triangle AFE \sim \triangle ABC$ to find EF.

It is also worth noting that in the last example, there is no need to know all three of the values a, b, and c. We only need to know a and s to find the length of EF.

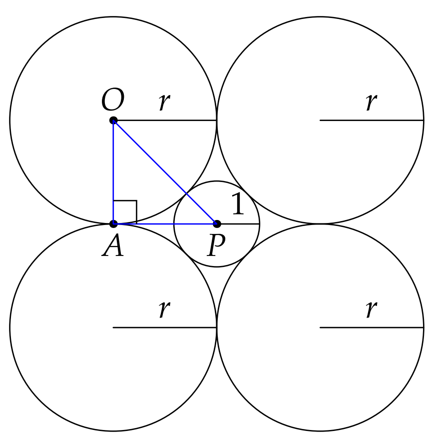
Now we move on from the incircle and show how all of our length techniques can be used together.

Example 3.30 (AMC 10B 2007/18).

A circle of radius 1 is surrounded by 4 circles of radius r as shown. What is r?



Solution. Refer to the figure below for point names.



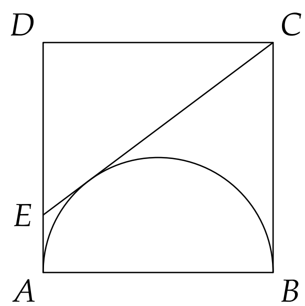
Since the circles are symmetrically placed, OAP is a right triangle, where A is a point of tangency. Because the circle with radius r is tangent to the circle with radius 1, $OP = 1 + r$. By the Pythagorean Theorem (1.2),

$$\begin{aligned} OA^2 + AP^2 &= OP^2 \\ 2r^2 &= (r+1)^2 \\ r^2 - 2r + 1 &= 2 \\ (r-1)^2 &= 2 \\ r - 1 &= \sqrt{2} \\ r &= 1 + \sqrt{2}. \end{aligned}$$

■

Example 3.31 (AMC 10A 2004/22).

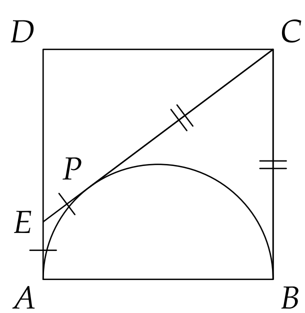
Square $ABCD$ has side length 2. A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at E. What is the length of $\overline{CE}$?



Solution. Say EC intersects the semicircle at P. By the Two Tangent Theorem (1.27), $EA = EP$ and $CP = CB = 2$. Then by the Pythagorean Theorem (1.2),

$$\begin{aligned} DE^2 + DC^2 &= CE^2 \\ (2 - EA)^2 + 2^2 &= (2 + EP)^2 \\ 4 &= 8EA \\ \frac{1}{2} &= EA. \end{aligned}$$

Thus $CE = 2 + \frac{1}{2} = \frac{5}{2}$.



■

3.2 Areas

Because the proofs of our main area formulas and the definition of area itself are heavily predicated on calculus, much rigor will be omitted in this section. Instead, our "proofs" will mostly be focused on intuition, answering the question “why should this feel right?” rather than “how do we know for sure?”

3.2.1 Triangles

You likely already know the formula $\frac{bh}{2}$. To introduce a little more rigor, we will first formally define what the base and height of a triangle (as lengths of segments, rather than segments) are with respect to a vertex A.

Definition 3.32 (Base and Height).

Pick some vertex A of $\triangle ABC$. Then the base of $\triangle ABC$ with respect to A is the length of side BC, and the height of $\triangle ABC$ with respect to A is $\delta(A, BC)$.

Recall that $\delta(A, BC)$ is the distance from point A to line BC.

Theorem 3.33 ($\frac{bh}{2}$).

Let b and h be the base and height of $\triangle ABC$ with respect to any vertex. Then the area of $\triangle ABC$ is $\frac{bh}{2}$.

This should be true because the area of a right triangle should be half the area of a rectangle with the same leg lengths, because two such right triangles form said rectangle. (We take for granted that the area of a $w \times l$ rectangle is wl.)

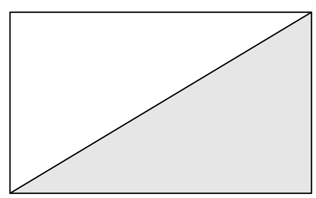
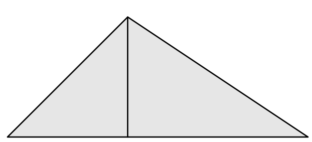


Figure 3.34: Why $\frac{bh}{2}$ is true for right triangles.

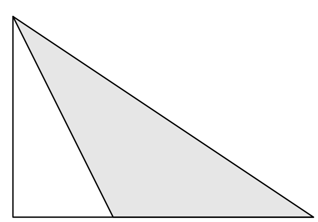
For what follows, we are taking the base and height of $\triangle ABC$ with respect to A. To show $\frac{bh}{2}$ is still true when

- $\angle B$ and $\angle C$ are both acute,
- or one of $\angle B$ and $\angle C$ are obtuse,

we should add and subtract areas, respectively. (Postulate 1.48, which is further ahead, is relevant here.)



Addition of areas when $\angle B$ and $\angle C$ are both acute.



Subtraction of areas when one of $\angle B$ or $\angle C$ are obtuse.

Figure 3.35: $\frac{bh}{2}$ is also true for non-right triangles.

Problem 3.36. Show that an equilateral triangle with side length x has area $\frac{\sqrt{3}x^2}{4}$. **Hint:** 8

$\frac{bh}{2}$ can also be used in ways besides direct area calculations. Say you have two triangles with bases b_1, b_2 and heights h_1, h_2, respectively. If you know the area of the first triangle (i.e. $\frac{b_1 h_1}{2}$) and the ratios $\frac{b_2}{b_1}$ and $\frac{h_2}{h_1}$, you can find the area of the second triangle (i.e. $\frac{b_2 h_2}{2}$). Note that you do not need to know any of b_1, b_2, h_1, or h_2 for this technique to work.

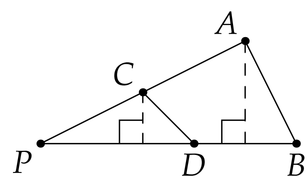
Example 3.37.

Say lines AC and BD intersect at P. Show that

$$\frac{[ABP]}{[CDP]} = \frac{AP \cdot BP}{CP \cdot DP}.$$

Solution. By Problem **??**, $\frac{AP}{CP} = \frac{\delta(A,BP)}{\delta(C,DP)}$. Combining the previous observation with $\frac{bh}{2}$,

$$\frac{AP \cdot BP}{CP \cdot DP} = \frac{\delta(A,BP) \cdot BP}{\delta(C,DP) \cdot DP} = \frac{[ABP]}{[CDP]}.$$



■

Example 3.38 (AMC 10A 2008/20).

Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$ and diagonals intersecting at K. Suppose that $AB = 9$, $DC = 12$, and the area of $\triangle AKD$ is 24. What is the area of trapezoid $ABCD$?

Solution. By Postulate **??**, $\angle ABK = \angle CDK$ and $\angle BAK = \angle DCK$. Thus $\triangle ABK \sim \triangle CDK$, and since $\frac{AB}{CD} = \frac{3}{4}$, the ratio of similarity is $3 : 4$.

Since $AK : KC = 3 : 4$, adding KC to the left side of the ratio yields $AC : KC = 7 : 4$, implying that

$$[ADC] = \frac{7}{4}[KDC].$$

By Postulate 1.48[7],

$$[ADK] + [KDC] = [ADC],$$

implying that

$$[ADK] = \frac{3}{4}[KDC],$$

or

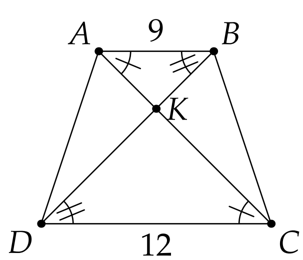
$$[KDC] = \frac{4}{3}[ADK] = 32.$$

Thus $[ADC] = 24 + 32 = 56$.

Now note that the height of $\triangle DAC$ with respect to A is the same as the height of $\triangle CAB$ with respect to C. However, the base of $\triangle DAC$ is 12, while the base of $\triangle CAB$ is 9. Therefore,

$$[CAB] = \frac{9}{12}[DAB] = 42.$$

This yields a total area of $56 + 42 = 98$.



[7]This is introduced later in the chapter. For the purposes of this example, you can just treat it as "areas add".

■

3.2.2 Circles

You likely also know the formula πr^2. However, as with triangles, we will first introduce a bit of rigor by defining π.

Definition 3.39.

We say π is the ratio of the circumference of a circle to its diameter.

We will take for granted that this ratio π is the same for each circle. Note that 2π is the ratio of the circumference of a circle to its radius.

Theorem 3.40 (Area of a Circle).

The area of a circle with radius r is πr^2.

We can approximate the area of a circle with a regular polygon whose vertices are all on its circumference.

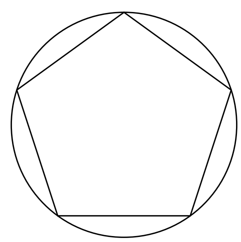


Figure 3.41: Approximating a circle with a pentagon.

As we increase the sides of the polygon, its area becomes closer to the area of the circle. Now here's the kicker: if we draw segments from the center of the polygon to its vertices, then we can use $\frac{bh}{2}$ to find its area.

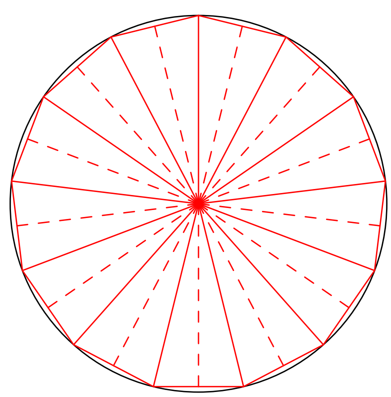


Figure 3.42: The solid red lines enclose the triangles whose areas we will sum. The dotted red lines indicate the heights of these triangles.

As the polygon has more sides, the sum of the bases will approach the circumference of the circle and the height will approach the radius of the circle. Since all the triangles have the same height, we can multiply the sum of the bases with the height and then divide by two to get the total area. Doing this, we find that the area of a circle is

$$\frac{1}{2} \cdot 2\pi r \cdot r = r^2.$$

We can also find the area of a sector of a circle. A sector is exactly what it sounds like: a slice of a circle through the center, similar to a pizza slice.

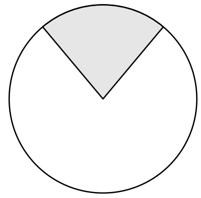
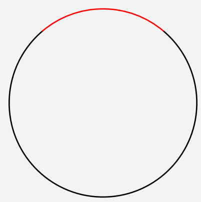


Figure 3.43: The sector encloses the shaded area.

More precisely, it is the figure formed by an arc and the corresponding radii. For additional rigor, we should properly define what an arc and a sector are.

Definition 3.44 (Arc of a Circle).

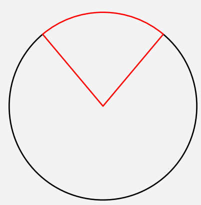
An **arc** of a circle is a path along the circumference of a circle.



If the length of the arc is s and the circumference is $2\pi r$, then we say the measure of the arc is $360° \cdot \frac{s}{2\pi r}$.

Definition 3.45 (Sector of a Circle).

A **sector** of a circle is formed by taking an arc and joining its endpoints with the center of the circle.



The measure of a sector is the same as the measure of its corresponding arc.

If the length of a sector's corresponding arc is s, then the fraction of the total circumference it covers is $\frac{s}{2\pi r}$. Thus, the fraction of the total area it covers should be the same.

Theorem 3.46 (Area of a Sector).

A sector with arc length s in a circle of radius r has area

$$\frac{s}{2\pi r} \cdot \pi r^2 = \frac{sr}{2}.$$

Problem 3.47. If a sector of a circle with area A has measure m, show that the sector has area $\frac{m}{360} \cdot A$.

3.2.3 The Principle of Inclusion-Exclusion

Our fundamental postulate for this subsection is that areas can be added, and conversely, they can be subtracted as well. We've already implicitly used this idea in our intuitive "proof" of $\frac{bh}{2}$.

Postulate 3.48. Given two **disjoint**[8] figures A and B, the area of the figure attained by combining A and B is the same as the sum of the areas of figures A and B.

[8] Alternatively, non-overlapping.

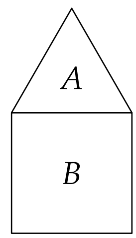
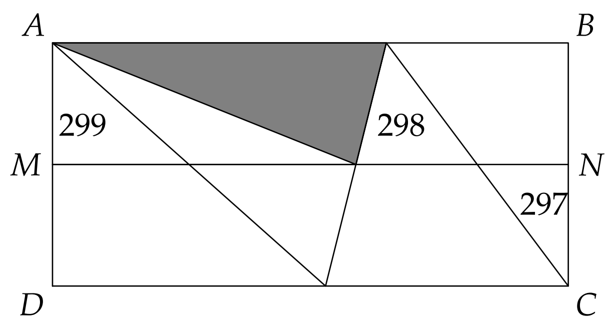


Figure 3.49: $[AB] = [A] + [B]$.

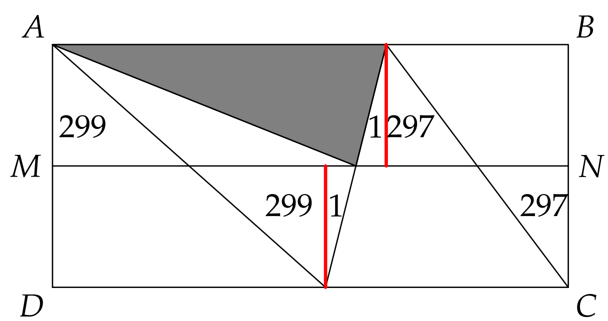
Of course, we must first determine the areas of figures if we want to be able to meaningfully add them. Therefore, this postulate never appears on its own in a problem; in some way or another, an area formula will also have to be utilized.

Example 3.50 (Winter MAT 2022/5).

Let $ABCD$ be a rectangle and M and N be midpoints on the sides of the rectangle, as shown in the diagram below. The areas of the different triangles within the rectangle are shown to be 299, 298, and 297. What is the area of the shaded region?



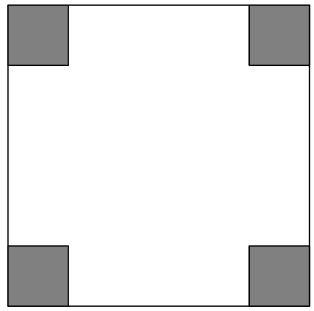
Solution. After dropping the altitudes from the points on AB and CD, we end up with three pairs of congruent right triangles. The pair on the left has area 299, and the pair on the right has area 297. Then the pair of congruent triangles in the center has area $298 - 297 = 1$. The bottom triangle with area $299 + 1 = 300$ and the shaded triangle have equal heights, but the bottom triangle's base is half that of the shaded triangle. The shaded triangle then has area $2 \cdot 300 = 600$.



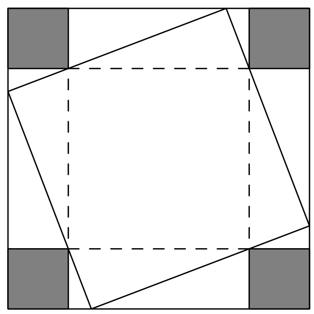
■

Example 3.51 (AMC 8 2015/25).

One-inch squares are cut from the corners of this 5 inch square. What is the area in square inches of the largest square that can be fitted into the remaining space?



Solution. The figure below is the best configuration.



Note that it can be decomposed into a square of side length 3 and four triangles with base 3 and height 1. Thus, its area is

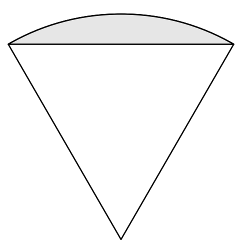
$$3^2 + 4 \cdot \frac{1}{2} \cdot 3 \cdot 1 = 15.$$

■

Though Postulate 1.48 only explicitly mentions addition, it can be used for subtraction too. And it is subtraction, rather than addition, that often makes for not so straightforward applications.

Example 3.52.

In the figure below is a 60° sector of a circle with radius 1. Find the shaded area.



Solution. By Postulate 1.48, the area of the sector is the sum of the area of the shaded area and the triangle. Denote the area of the shaded area as K.

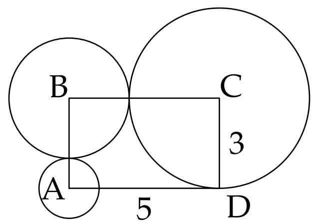
Note the unshaded triangle is an equilateral triangle with side length 1, and note the sector has area $\frac{1}{6} \cdot 1^2\pi$, so

$$K + \frac{\sqrt{3}}{4} = \frac{\pi}{6}.$$

Thus $K = \frac{\pi}{6} - \frac{\sqrt{3}}{4}$. ■

Example 3.53 (AMC 8 2014/20).

Rectangle $ABCD$ has sides $CD = 3$ and $DA = 5$. A circle of radius 1 is centered at A, a circle of radius 2 is centered at B, and a circle of radius 3 is centered at C. What is the area of the region inside the rectangle but outside all three circles?



Solution. We subtract the area of the circles in the rectangle from the area of the rectangle. Since $\frac{1}{4}$ of each circle is in the rectangle, our desired area is

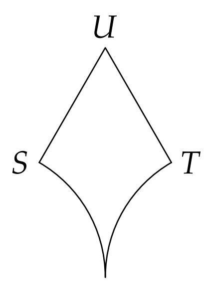
$$3 \cdot 5 - \frac{1}{4}\pi(1^2 + 2^2 + 3^2) = 15 - \frac{7\pi}{2}.$$

■

These past two examples were fairly straightforward because both the total regions and the regions to subtract were both clearly drawn. However, sometimes you will only be given a strange figure and have to complete it into something more ordinary yourself. The next example demonstrates this.

Example 3.54 (AMC 8 2017/25).

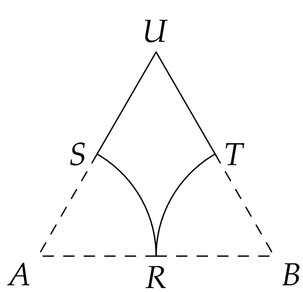
In the figure shown, $\overline{US}$ and $\overline{UT}$ are line segments each of length 2, and $m\angle TUS = 60°$. Arcs TR and SR are each one-sixth of a circle with radius 2. What is the area of the region shown?



Solution. Let A be the center of arc SR and B be the center of arc TR. Observe that because AB is parallel to ST and $\angle UST = \angle SAR$, S lies on UA by Postulate **??**. (More precisely, $US \parallel SA$, implying US and SA are the same line since they intersect.)

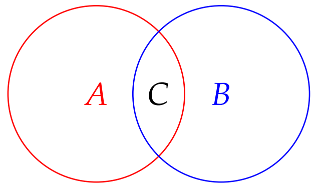
Now draw $\triangle UAB$. The area of our desired region is the area of $\triangle UAB$ minus the area of the arcs, or

$$\frac{\sqrt{3} \cdot 4^2}{4} - 2 \cdot \frac{1}{6} \cdot 2^2\pi = 4\sqrt{3} - \frac{4\pi}{3}.$$



■

So far we have only been doing addition and subtraction of disjoint regions, and if the subsection ended here, “The Principle of Inclusion-Exclusion” would be a poor choice of a title. However, the final — and in my opinion, hardest — application of Postulate 1.48 is using it to find the area of the union of **overlapping** figures.



Take the figure above. Regions A and B are circles, and region C is the overlap of regions A and B. There are two ways to think about the area of $[AB]$:

- Add the area of A and the area of B together. Then, note that we counted the area in C twice, so subtract it.
- Make two new regions $A-C$ and $B-C$, where $A-C$ is A with C removed and likewise with $B-C$. Then we add the areas of disjoint figures $A-C$, $B-C$, and C.

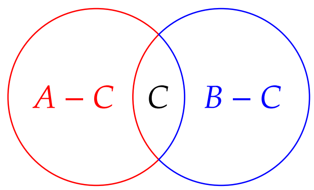


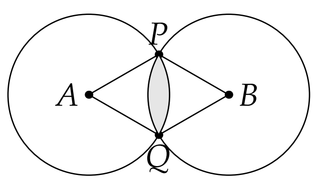
Figure 3.55: Here we use the second approach.

Either way, we get the same area in the end: $[AB] = [A] + [B] - [C]$.

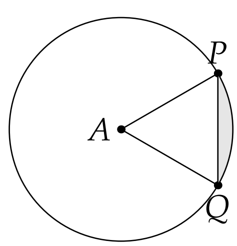
Example 3.56.

There are two circles with radius 1 and centers A and B such that $AB = \sqrt{3}$. Find the area of the union of the circles.

Solution. Let our regions A and B be the circles with centers A and B. We can easily see that $[A] + [B] = 2\pi$. To find the union of the areas, we just have to subtract the area of the region shaded below from 2π.



Instead of finding the entire shaded area, we find half of it. Note that $PQ = 1$, so $\triangle APQ$ is equilateral.



Observe now that finding this shaded area is identical to Example 1.52, so we skip the calculations and take for granted that it has area $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$. Then the area of the union of the circles is

$$2\pi - 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{5\pi}{3} + \frac{\sqrt{3}}{2}.$$

■

It is possible to extend this concept out to more than two figures, just as the combinatorial Principle of Inclusion-Exclusion extends beyond two sets. However, this seldom appears in problems, so we will not demonstrate nor require it in the text.

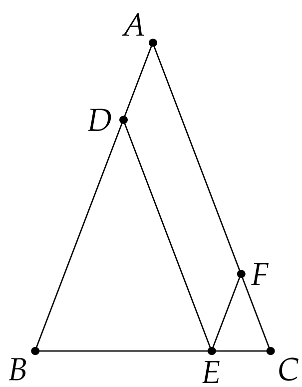
3.3 Problems

Good problems are not mindless applications of one concept. Rather, they will weave multiple concepts together under a general idea or theme. Do not expect any of the problems to only use one concept, and expect many of them to require ideas and facts found in previous chapters.

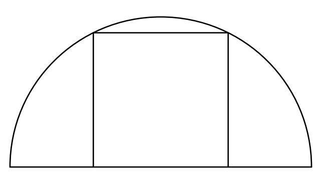
Problem 3.57 (Intersection of a Circle and a Line). Show that a circle and line intersect at most twice. **Hints:** 1 6 5

Problem 3.58 (Inradius of a Right Triangle). A right triangle has legs of lengths a and b and hypotenuse of length c. Find, in terms of a, b, and c, the inradius of this triangle. **Hint:** 4

Problem 3.59 (AMC 10A 2013/12). In $\triangle ABC$, $AB = AC = 28$ and $BC = 20$. Points D, E, and F are on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of parallelogram $ADEF$?

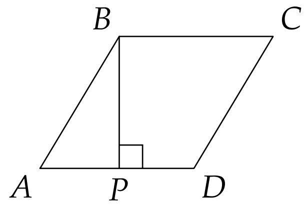


Problem 3.60 (Amador Valley Geometry Bee 2022/2). A square is inscribed in a semicircle of radius 1 such that one of its sides lays entirely on its diameter. If the area of this square can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$.



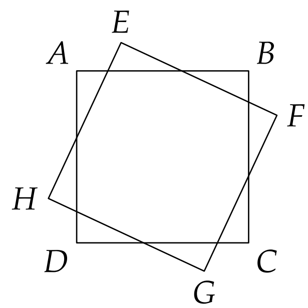
Problem 3.61 (AAMC 10A 2021/6). Alexis starts at the park, walks 80 meters south, 340 meters in some direction, and then 160 meters east. Given that her ending location is k meters directly north from the park, what is the value of k?

Problem 3.62 (AMC 10B 2022/2). In rhombus $ABCD$, point P lies on segment $\overline{AD}$ so that $\overline{BP} \perp \overline{AD}$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$? (Note: The figure is not drawn to scale.)

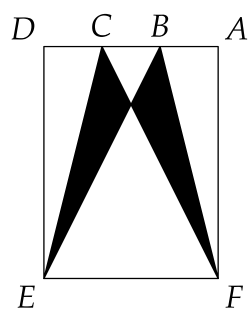


Problem 3.63 (AMC 10A 2022/5). Square $ABCD$ has side length 1. Points P, Q, R, and S each lie on a side of $ABCD$ such that $APQCRS$ is an equilateral convex hexagon with side length s. What is s?

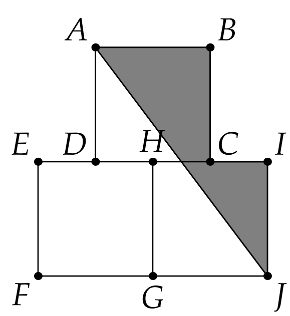
Problem 3.64 (AMC 10B 2023/7). Square $ABCD$ is rotated 20° clockwise about its center to obtain square $EFGH$, as shown below. What is the degree measure of $\angle EAB$?



Problem 3.65 (AMC 8 2016/22). Rectangle $DEFA$ below is a 3×4 rectangle with $DC = CB = BA$. What is the area of the "bat wings"?

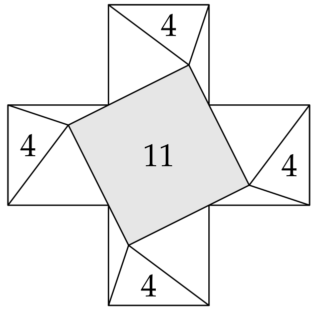


Problem 3.66 (AMC 8 2013/24). Squares $ABCD$, $EFGH$, and $GHIJ$ are equal in area. Points C and D are the midpoints of sides IH and HE, respectively. What is the ratio of the area of the shaded pentagon $AJICB$ to the sum of the areas of the three squares?



Problem 3.67 (AMC 12B 2022/10). Regular hexagon $ABCDEF$ has side length 2. Let G be the midpoint of $\overline{AB}$, and let H be the midpoint of $\overline{DE}$. What is the perimeter of $GCHF$?

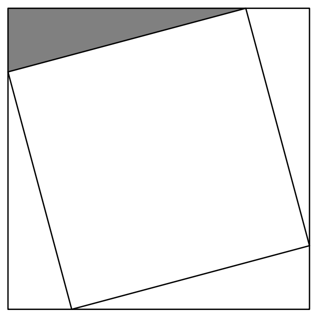
Problem 3.68 (Summer MAT 2023/1). A plus sign is formed with five congruent squares. A gray square with area 11 is placed inside the plus sign such that the inner vertices of the plus sign lie on the sides of the square. The indicated triangles all have area 4. Find the area of the plus sign.



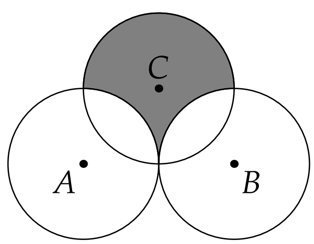
Problem 3.69 (HMMT Nov. General 2021/3). Let $ABCD$ be a unit square. A circle with radius $\frac{32}{49}$ passes through point D and is tangent to side AB at point E. Then $DE = \frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$. Find $100m + n$.

Problem 3.70 (AAMC 10A 2021/12). Let $ABCD$ be a rectangle with $AB = 5$ and $BC = 8$. There exists a circle tangent to sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ of the rectangle that meets side $\overline{DA}$ at points X and Y. What is XY^2?

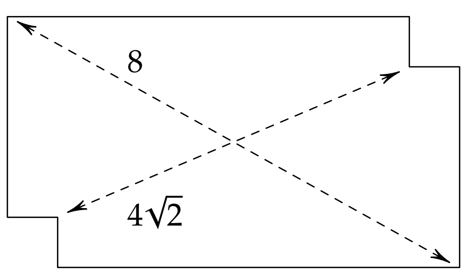
Problem 3.71 (AMC 10A 2023/11). A square of area 2 is inscribed in a square of area 3, creating four congruent triangles, as shown below. What is the ratio of the shorter leg to the longer leg in the shaded right triangle?



Problem 3.72 (AMC 10A 2011/18). Circles A, B, and C each have radius 1. Circles A and B share one point of tangency. Circle C has a point of tangency with the midpoint of $\overline{AB}$. What is the area inside circle C but outside circle A and circle B?

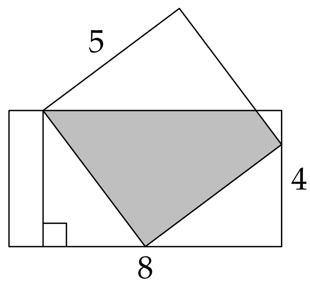


Problem 3.73 (AMC 10A 2022/10). Daniel finds a rectangular index card and measures its diagonal to be 8 centimeters. Daniel then cuts out equal squares of side 1 cm at two opposite corners of the index card and measures the distance between the two closest vertices of these squares to be $4\sqrt{2}$ centimeters, as shown below. What is the area of the original index card?



Problem 3.74 (AMC 12A 2023/18). Circle C_1 and C_2 each have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externally tangent to C_3. What is the radius of C_4?

Problem 3.75 (AMC 10B 2022/16). The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



Problem 3.76 (AMC 12A 2009/20). Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE?

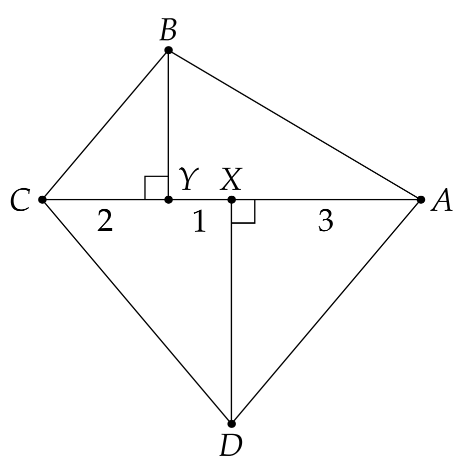
Problem 3.77 (AIME II 2007/9). Rectangle $ABCD$ is given with $AB = 63$ and $BC = 448$. Points E and F lie on AD and BC respectively, such that $AE = CF = 84$. The inscribed circle of triangle BEF is tangent to EF at point P, and the inscribed circle of triangle DEF is tangent to EF at point Q. Find PQ.

Problem 3.78 (AMC 10A 2017/10). Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod? **Hint:** 2

Challenge Problem 3.79. Consider trapezoid $ABCD$ with bases AB and CD, and let diagonals AC and BD intersect at P. Prove that $[ABP] + [CDP] \geq \frac{1}{2}[ABCD]$. When does equality occur?

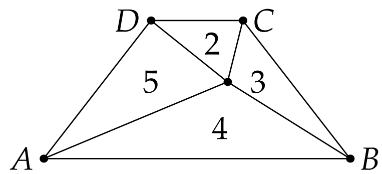
Challenge Problem 3.80. Theorem 1.12 states that given any set of 3 lengths that satisfies the Triangle Inequality, there exists a (possibly degenerate) triangle with those lengths. Does this extend beyond three sides? That is, given a set of lengths $l_1, \ldots, l_n$ that satisfy the Extended Triangle Inequality, must there exist an n-sided polygon with lengths $l_1, \ldots, l_n$? (We say a set of lengths satisfies the Extended Triangle Inequality if $l_i \leq \frac{l_1+\cdots+l_n}{2}$ for all $1 \leq i \leq n$.)

Challenge Problem 3.81 (AMC 12A 2021 Fall/21). Let $ABCD$ be an isosceles trapezoid with $\overline{BC} \parallel \overline{AD}$ and $AB = CD$. Points X and Y lie on diagonal $\overline{AC}$ with X between A and Y, as shown in the figure. Suppose $\angle AXD = \angle BYC = 90°$, $AX = 3$, $XY = 1$, and $YC = 2$. What is the area of $ABCD$?



Challenge Problem 3.82 (AMC 10B 2019/23). Points $A = (6, 13)$ and $B = (12, 11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

Challenge Problem 3.83 (AMC 12B 2021/17). Let $ABCD$ be an isoceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB > CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are 2, 3, 4, and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below. What is the ratio $\frac{AB}{CD}$?



Appendix A

Hints

1. Let the center of the circle be O and the line be ℓ. Draw the foot of the altitude from O to ℓ. (For subsequent hints, we will refer to it as H.)
2. The claim in Problem 1.80 is true.
3. Let X be a point on ℓ distinct from P. Using the Pythagorean Theorem (1.2), what can you say about the length of OX compared to the length of OP?
4. Call our right triangle $\triangle ABC$ and say $\angle C = 90^\circ$. Then the tangent length from C is equal to the inradius. (Why?)
5. Essentially, the value of PH is fixed (if it exists at all). If H is fixed and ℓ passes through H, then at most how many points P exist such that $PH = \sqrt{OP^2 - OH^2}$? (Assume PH is a real number; otherwise, the circle and line never intersect and we have nothing left to prove.)
6. Say P lies on the circle and the line. By the Pythagorean Theorem (1.2), $PH^2 = OP^2 - OH^2$.
7. Note that $\angle A = \angle B$, so $a = b$.
8. Refer to Example 1.3.

Appendix B

Solutions

Chapter 3

3.4. Note that $a = b$ because $\angle A = \angle B$. By the Pythagorean Theorem (1.2),

$$a^2 + b^2 = c^2.$$

Since $a = b$, this implies $2a^2 = c^2$, or $c = a\sqrt{2}$. Thus

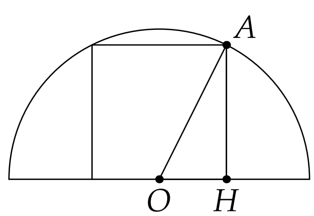
$$a : b : c = 1 : 1 : \sqrt{2}.$$

(If we scale c to 1 in the ratio, which will be useful when trigonometry is introduced, then $a : b : c = \frac{\sqrt{2}}{2} : \frac{\sqrt{2}}{2} : 1$.)

3.21. No point X on ℓ distinct from P lies on ω because by the Pythagorean Theorem (1.2), $OX^2 = OP^2 + PX^2 > OP^2$, implying that $OX > OP$.

3.26. This is a direct result of Theorem 1.20: if O_1 and O_2 are the centers of ω_1 and ω_2, then $O_1P \perp \ell \perp O_2P$, implying that O_1P and O_2P are the same line.

3.60. Let the center of the semicircle be O, a vertex of the square on the semicircle be A, and the foot of the altitude from A to the diameter be H.

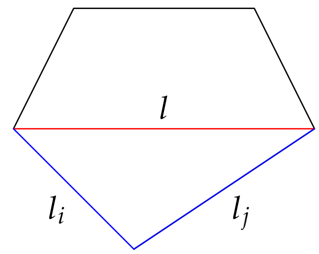


Note that $OH = \frac{1}{2}AH$, and $OH^2 + AH^2 = 1$. This implies that $AH = \frac{2}{\sqrt{5}}$, so the area of the square is $\frac{4}{5}$. Thus our answer is $4 + 5 = 9$.

3.80. Yes, the converse of the Extended Triangle Inequality is true. We implicitly assume that $n \geq 3$ since no polygon with fewer sides exist, and we proceed by induction. (Refer to Section **??** for a brief introduction to induction.)

Our base case of $n = 3$ is already known: this is just Theorem 1.12. So we move on to the inductive step.

Assume this is true for n. Now we want to show it is true for $n + 1$. In general, we can replace two lengths l_i and l_j with some other length l as long as l_i, l_j and l satisfy the Triangle Inequality. This is because we can just construct an n-gon with length l replacing lengths l_i and l_j. If the vertices of the segment of length l are A and B, then we can construct our $n + 1$-gon by constructing a point P such that $AP = l_i$ and $PB = l_j$, and then replacing segment AB with segments AP and PB.



The replacement process described is contingent on the new set of lengths satisfying the Extended Triangle Inequality as well; that way, we can construct the initial n-gon to begin with. We now show this is possible.

Take the longest length l_i out of $l_1, \ldots, l_{n+1}$ and any other length l_j, and let l_k be the longest length remaining. Now if $l_i - l_j \geq l_k$, replacing l_i and l_j with $l_i - l_j$ still preserves the Triangle Inequality, as

$$l_i - l_j \leq \frac{l_1 + \cdots + l_n - 2l_j}{2}.$$

(Note that $l_i - l_j$ is still the longest length in our new set of l's, and replacing l_i and l_j with $l_i - l_j$ decreases the sum of the l's by $2l_j$.)

And if $l_i - l_j \leq k$, then replacing l_i and l_j with l_k also preserves the Triangle Inequality, as there are two sides of length l_k, meaning the perimeter is at least $2l_k$, so the semiperimeter is at least l_k.