Bosonic string modes in 2+1 dimensions

Colin Morningstar

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1 Introduction

The purpose of these notes is to outline the spectrum and nature of the expected string modes for both the toroidal and fixed end cases in 2+1 dimensions. Very general properties will be used to deduce the expected pattern of degeneracies and level orderings.

2 Fixed end string levels

The first step in determining the energies of the stationary states of gluons in the presence of a static quark and antiquark, fixed in space some distance r apart, is to classify the levels in terms of the symmetries of the problem. In two spatial dimensions, such a system has a reflection symmetry about the axis $\hat{\mathbf{r}}$ passing through the quark and the antiquark (the molecular axis). Here, we shall denote the eigenvalue of this transformation by Λ which can take values $\Lambda = \pm 1$. States which are symmetry ($\Lambda = +1$) under such a reflection will be labeled as S states, and antisymmetric states ($\Lambda = -1$) will be labeled as A states. Another symmetry is the combined operation of charge conjugation and spatial inversion about the midpoint between the quark and the antiquark. Here, we denote the eigenvalue of this transformation by η which can take values ± 1. States which are even (odd) under this parity–charge-conjugation operation are indicated by the subscripts g (u). Hence, the low-lying gluon levels Λ_η are labeled S_g, S_u, A_g, and A_u.

Next, assume that the fixed ends of the string of flux lie along the y-axis. The location of the string can be specified in terms of displacements $x(y,t)$ in the x direction from the y-axis at time t. The boundary conditions are $x(0,t) = 0$ and $x(L,t) = 0$ where $L = r$. Furthermore, we assume that the displacements (and their first derivatives with respect to y and t) are continuous and single-valued for each value of y and t; in other words, string configurations which double-back on themselves or overhang the ends are disallowed (although this assumption can be removed by parametrizing the string displacements differently).

The effective string action, without interactions, is taken to be

$$S = \int dt \int_0^L dy \left[\tfrac{1}{2}\rho\ \dot{x}^2 - \tfrac{1}{2}\kappa x'^2 \right], \qquad (1)$$

where ρ is the linear mass density of the string, κ is the string tension, and

$$\dot{x} = \frac{\partial x}{\partial t}, \qquad x' = \frac{\partial x}{\partial y}. \tag{2}$$

The momentum canonically conjugate to $x(y,t)$ is

$$\pi(y,t) = \frac{\partial L}{\partial \dot{x}} = \rho\, \dot{x}, \tag{3}$$

so that the Hamiltonian is

$$H = \int_0^L dy \left\{ \frac{1}{2\rho}\, \pi^2 + \frac{\kappa}{2}\, x'^2 \right\}, \tag{4}$$

and the equal-time commutation relations are

$$[\ x(y,t),\ \pi(y',t)\ ] = i\delta(y-y'). \tag{5}$$

The system is solved by expressing the displacements in terms of their normal modes. For fixed ends, the normal modes are standing waves $\sin(m\pi y/L)$ having energy $m\omega$ for positive integer m. Using such modes, we can introduce ladder operators:

$$x(y,t) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m\omega\rho L}} \sin\left(\frac{m\pi y}{L}\right) \left(a_m\, e^{-im\omega t} + a_m^\dagger\, e^{im\omega t}\right), \qquad \omega = \frac{\pi}{L}\sqrt{\frac{\kappa}{\rho}}. \tag{6}$$

For fixed ends, these are standing waves having energy $m\omega$. Note that the displacement operators are Hermitian, as they should be. The ladder operators satisfy the commutation relations

$$[a_m, a_{m'}] = 0, \qquad [a_m, a_{m'}^\dagger] = \delta_{mm'}. \tag{7}$$

In order to show that the above commutation relations are consistent with the commutators of Eq. 5, we need the Fourier series of the periodic Dirac δ-function. Recall the definition of the Fourier series for a periodic function with period T:

$$f(y) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos\left(\frac{2\pi m y}{T}\right) + b_m \sin\left(\frac{2\pi m y}{T}\right) \right], \tag{8}$$

$$a_m = \frac{2}{T}\int_c^{c+T} dy\ f(y) \cos\left(\frac{2\pi m y}{T}\right), \tag{9}$$

$$b_m = \frac{2}{T}\int_c^{c+T} dy\ f(y) \sin\left(\frac{2\pi m y}{T}\right). \tag{10}$$

Here, the modes are $\sin(m\pi y/L)$ so we need $T = 2L$ and can choose $c = -L$, even though we are only interested in the range $0 \le y \le L$. Each of the modes is *odd* in y, so any linear combinations of the normal modes will be odd, so we can use a Fourier sine series:

$$f(y) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{\pi m y}{L}\right), \tag{11}$$

$$b_m = \frac{2}{L}\int_0^L dy\ f(y) \sin\left(\frac{\pi m y}{L}\right). \tag{12}$$

If $f(y) = \text{sgn}(y)\delta(|y| - y')$, which is odd in y, then $b_m = (2/L)\sin(m\pi y'/L)$ if $0 < y' \leq L$. Hence,

$$\delta(y - y') = \frac{2}{L}\sum_{m=1}^{\infty} \sin\left(\frac{\pi m y}{L}\right)\sin\left(\frac{\pi m y'}{L}\right), \quad \text{for } 0 < y \leq L. \tag{13}$$

The Hamiltonian is then given by, discarding an irrelevant (but infinitely large) constant,

$$H = \sum_{m=1}^{\infty} m\omega\ a_m^\dagger a_m, \qquad \omega = \frac{\pi}{L}\sqrt{\frac{\kappa}{\rho}}. \tag{14}$$

Let $|0\rangle$ denote the ground state of the string, then the string eigenmodes are

$$\prod_{m=1}^{\infty} \frac{(a_m^\dagger)^{n_m}}{\sqrt{n_m!}}\ |0\rangle, \tag{15}$$

where n_m are the occupation numbers which take values $0, 1, 2, \ldots$.

We now wish to determine the symmetry properties of these states. Let $\mathcal{P}_{L/2}$ denote spatial inversion about the point midway between the quark and the antiquark, and C denote charge conjugation. The flux in the effective string has a direction associated with it (except in $SU2$), so that charge conjugation simply effects a reversal of this direction. This direction is also reversed under $\mathcal{P}_{L/2}$ so that $C\mathcal{P}_{L/2}$ is a symmetry of the system. Also, let σ_y denote a reflection in the molecular axis (along the y-axis). The ground state satisfies

$$C\mathcal{P}_{L/2}\ |0\rangle \ = \ |0\rangle, \tag{16}$$

$$\sigma_y\ |0\rangle \ = \ |0\rangle. \tag{17}$$

The string displacement transform according to

$$C\mathcal{P}_{L/2}\ x(y,t)\ \mathcal{P}_{L/2}^\dagger C^\dagger \ = \ -x(L-y,t), \tag{18}$$

$$\sigma_y\ x(y,t)\ \sigma_y^\dagger \ = \ -x(y,t). \tag{19}$$

Using Eq. 6 and the above transformation properties, one easily determines

$$C\mathcal{P}_{L/2}\ a_m^\dagger\ \mathcal{P}_{L/2}^\dagger C^\dagger \ = \ (-1)^m a_m^\dagger, \tag{20}$$

$$\sigma_y\ a_m^\dagger\ \sigma_y^\dagger \ = \ -a_m^\dagger, \tag{21}$$

also using $\sin(m\pi(L-y)/L) = -(-1)^m \sin(m\pi y/L)$.

Now act with the Hamiltonian on the string eigenstates:

$$\begin{aligned}
H\prod_{m=1}^{\infty}(a_m^\dagger)^{n_m}|0\rangle &= \sum_{m'=1}^{\infty} m'\omega a_{m'}^\dagger a_{m'} \prod_{m=1}^{\infty}(a_m^\dagger)^{n_m}|0\rangle \\
&= \sum_{m'=1}^{\infty}\left(\prod_{m\neq m'}(a_m^\dagger)^{n_m}\right) m'\omega a_{m'}^\dagger a_{m'}(a_{m'}^\dagger)^{n_{m'}}|0\rangle \\
&= \sum_{m'=1}^{\infty}\left(\prod_{m\neq m'}(a_m^\dagger)^{n_m}\right) n_{m'}m'\omega(a_{m'}^\dagger)^{n_{m'}}|0\rangle \\
&= \left(\sum_{m'=1}^{\infty} n_{m'}m'\omega\right)\left(\prod_{m=1}^{\infty}(a_m^\dagger)^{n_m}\right)|0\rangle,
\end{aligned}$$

which tells us the energy of each eigenstate. Next, act with the two symmetry operators on the string eigenstates:

$$
\begin{aligned}
\sigma_y \prod_{m=1}^{\infty} (a_m^\dagger)^{n_m}|0\rangle &= \prod_{m=1}^{\infty} (\sigma_y\, a_m^\dagger\, \sigma_y^\dagger)^{n_m}|0\rangle = \prod_{m=1}^{\infty} (-a_m^\dagger)^{n_m}|0\rangle \\
&= \left(\prod_{m'=1}^{\infty} (-1)^{n_{m'}}\right) \prod_{m=1}^{\infty} (a_m^\dagger)^{n_m}|0\rangle, \\
C\mathcal{P}_{L/2} \prod_{m=1}^{\infty} (a_m^\dagger)^{n_m}|0\rangle &= \prod_{m=1}^{\infty} (C\mathcal{P}_{L/2} a_m^\dagger \mathcal{P}_{L/2}^\dagger C^\dagger)^{n_m}|0\rangle \\
&= \prod_{m=1}^{\infty} ((-1)^m a_m^\dagger)^{n_m}|0\rangle = \left(\prod_{m'=1}^{\infty} (-1)^{m' n_{m'}}\right) \prod_{m=1}^{\infty} (a_m^\dagger)^{n_m}|0\rangle.
\end{aligned}
$$

Hence, if E_0 denotes the energy of the ground state (with the above Hamiltonian, it has been defined to be zero), then the eigenvalues E (energy), Λ, and η associated with the string eigenstates are given by

$$
E = E_0 + \frac{N\pi}{r}\sqrt{\frac{\kappa}{\rho}}, \tag{22}
$$

$$
N = \sum_{m=1}^{\infty} m\, n_m, \tag{23}
$$

$$
n = \sum_{m=1}^{\infty} n_m, \tag{24}
$$

$$
\Lambda = (-1)^n, \tag{25}
$$

$$
\eta = (-1)^N. \tag{26}
$$

Using these properties, the orderings and degeneracies of the Goldstone string energy levels and their symmetries are as shown in Table 1. Hence, for $\kappa = \rho$, the $N\pi/r$ behavior and a well-defined pattern of degeneracies and level orderings among the different channels form a very distinctive signature of the onset of the Goldstone modes for the effective QCD string.

3 Toroidal string levels

A string without fixed ends which winds around a box with periodic (toroidal) boundary conditions has different symmetry properties. Here we shall assume that the string loop winds around the torus in the y-direction, and let L be the circumference of the torus in this direction. Let the position of the string in the x direction be specified by $x(y,t)$. Once again, assume that the string is stiff enough that $x(y,t)$ are single valued (no configurations which double back on themselves). Of course, this assumption could be relaxed by labeling the position along the string by some parameter other than y, but this is an unnecessary complication for our purposes.

The effective string action is taken to be

$$
S_T = \int dt \int_0^L dy \left[\tfrac{1}{2}\rho\, \dot{x}^2 - \tfrac{1}{2}\kappa x'^2\right], \tag{27}
$$

Table 1: Low-lying string levels for fixed ends. The $N = 2$ level is two-fold degenerate, and the $N = 3, 4$ levels are 3, 5-fold degenerate, respectively. The $S(A)$ states are even (odd) under reflections in the molecular axis. Subscripts $g(u)$ indicate evenness (oddness) under $C\mathcal{P}_{L/2}$, charge conjugation combined with spatial inversion about the midpoint between the quark and the antiquark.

$N = 0$:	S_g	$\|0\rangle$
$N = 1$:	A_u	$a_1^\dagger\|0\rangle$
$N = 2$:	S_g'	$(a_1^\dagger)^2\|0\rangle$
	A_g	$a_2^\dagger\|0\rangle$
$N = 3$:	A_u'	$(a_1^\dagger)^3\|0\rangle$
	S_u	$a_1^\dagger a_2^\dagger\|0\rangle$
	A_u''	$a_3^\dagger\|0\rangle$
$N = 4$:	S_g''	$(a_1^\dagger)^4\|0\rangle$
	A_g'	$(a_1^\dagger)^2 a_2^\dagger\|0\rangle$
	S_g'''	$a_1^\dagger a_3^\dagger\|0\rangle$
	S_g''''	$(a_2^\dagger)^2\|0\rangle$
	A_g''	$a_4^\dagger\|0\rangle$

where ρ is the linear mass density of the string and κ is the string tension. The momentum canonically conjugate to x is

$$\pi = \frac{\partial L}{\partial \dot{x}} = \rho\, \dot{x}, \tag{28}$$

so that the Hamiltonian is

$$H = \int_0^L dy \left\{ \frac{1}{2\rho}\pi^2 + \frac{\kappa}{2} x'^2 \right\}, \tag{29}$$

and the equal-time commutation relations are

$$[\ x(y,t),\ \pi(y',t)\ ] = i\delta(y-y'). \tag{30}$$

Now define the "center of mass" and the total transverse momentum by

$$Q(t) \;=\; \frac{1}{L}\int_0^L dy\ x(y,t), \tag{31}$$

$$P(t) \;=\; \int_0^L dy\ \pi(y,t), \tag{32}$$

which satisfy the equal-time commutation relations

$$[\ Q(t),\ P(t)\ ] = i. \tag{33}$$

The Hamiltonian can be diagonalized by expressing the string location and momentum in terms of the normal modes, introducing ladder operators:

$$x(y,t) = Q + \frac{t}{\rho L}P + \sum_{m\neq 0} \frac{1}{\sqrt{2\rho L \Omega_m}} \left(a_m\, e^{-i\Omega_m t + ik_m y} + a_m^{\dagger}\, e^{i\Omega_m t - ik_m y} \right), \tag{34}$$

where $Q = Q(0)$, $P = P(0) = P(t)$ since P is conserved, and

$$k_m = \frac{2\pi}{L}\ m, \qquad \Omega_m = \frac{2\pi}{L}\sqrt{\frac{\kappa}{\rho}}\ |m|. \tag{35}$$

These operators satisfy the commutation relations

$$[a_m, a_{m'}] = 0, \qquad [a_m, a_{m'}^{\dagger}] = \delta_{mm'}, \tag{36}$$

$$[a_m, P] = 0, \qquad [a_m, Q] = 0, \qquad [Q, P] = i. \tag{37}$$

In order to show that the above commutation relations are consistent with the commutators of Eq. 30, set $c = 0$ and $T = L$ in Eqs. 8-10 to show that

$$\delta(y-y') = \frac{1}{L} + \frac{2}{L}\sum_{m=1}^{\infty} \cos\left(\frac{2\pi m}{L}(y-y')\right). \tag{38}$$

Note that $x(y,t)$ are Hermitian operators and satisfy the boundary conditions $x(0,t) = x(L,t)$ and $x'(0,t) = x'(L,t)$. Satisfying both of these equations results in the 2ω energy quantization, instead of $\omega = (\pi/L)\sqrt{\kappa/\rho}$ as with fixed ends. With periodic boundary conditions, the normal modes are traveling plane waves having energy Ω_m. Also note that Eq. 34 is consistent with Eq. 31 given that

$$Q(t) = Q + \frac{t}{\rho L} P, \tag{39}$$

since $\int_0^L dy \ \exp(2\pi i m y/L) = 0$ for non-zero integer m.

In terms of the ladder operators, the Hamiltonian is given by, discarding an irrelevant constant,

$$H = \frac{1}{2\rho L} P^2 + \sum_{m\neq 0} \Omega_m a_m^\dagger a_m, \qquad \Omega_m = \frac{2\pi}{L}\sqrt{\frac{\kappa}{\rho}}\ |m|. \tag{40}$$

The ground state satisfies

$$P|0\rangle = 0. \tag{41}$$

Since we are not interested in the simple transverse-translational modes, we work in the zero transverse momentum sector and consider only the eigenstates

$$\prod_{m\neq 0} \frac{(a_m^\dagger)^{n_m}}{\sqrt{n_m!}}\ |0\rangle. \tag{42}$$

Next, consider how these operators transform under translations along the longitudinal y-direction. Let $T_y(b)$ denote the translation in the y-direction by length b. We require that

$$T_y(b)\ x(y,t)\ T_y^\dagger(b) = x(y+b,t). \tag{43}$$

From Eq. 34, we see that this means

$$T_y(b)\ a_m^\dagger\ T_y^\dagger(b) \ = \ e^{-ibk_m}\ a_m^\dagger. \tag{44}$$

Let P_y denote the generator of such longitudinal translations $T_y(b) = \exp(-ibP_y)$, then

$$P_y\ a_m^\dagger\ P_y^\dagger \ = \ k_m\ a_m^\dagger, \tag{45}$$

which shows that each phonon mode has longitudinal momentum k_m.

Let $\mathcal{P}_{L/2}$ denote spatial inversion about the point $(0, L/2)$ and C denote charge conjugation. The flux in the effective QCD string has a direction associated with it, so that charge conjugation simply effects a reversal of this direction. This direction is also reversed under $\mathcal{P}_{L/2}$ so that $C\mathcal{P}_{L/2}$ is a possible symmetry of the system. Also, let σ_y denote a reflection in the y-axis. The ground state satisfies

$$P_y\ |0\rangle \ = \ 0, \tag{46}$$

$$C\mathcal{P}_{L/2}\ |0\rangle \ = \ |0\rangle, \tag{47}$$

$$\sigma_y\ |0\rangle \ = \ |0\rangle. \tag{48}$$

To determine the behavior of the operators $a_m^\dagger$ under these symmetry operations, one uses Eq. 34 and the following transformation properties of the string coordinates:

$$C\mathcal{P}_{L/2}\ x(y,t)\ \mathcal{P}_{L/2}^\dagger C^\dagger \ = \ -x(L-y,t), \tag{49}$$

$$\sigma_y\ x(y,t)\ \sigma_y^\dagger \ = \ -x(y,t). \tag{50}$$

Furthermore, we know that

$$\begin{aligned} C\mathcal{P}_{L/2}\ Q\ \mathcal{P}_{L/2}^\dagger C^\dagger &= -Q, &(51)\\ C\mathcal{P}_{L/2}\ P\ \mathcal{P}_{L/2}^\dagger C^\dagger &= -P, &(52)\\ C\mathcal{P}_{L/2}\ P_y\ \mathcal{P}_{L/2}^\dagger C^\dagger &= -P_y, &(53)\\ \sigma_y\ Q\ \sigma_y^\dagger &= -Q, &(54)\\ \sigma_y\ P\ \sigma_y^\dagger &= -P. &(55)\end{aligned}$$

Using Eq. 34 with the above transformation properties, one easily determines

$$\begin{aligned} C\mathcal{P}_{L/2}\ a_m^\dagger\ \mathcal{P}_{L/2}^\dagger C^\dagger &= -a_{-m}^\dagger, &(56)\\ \sigma_y\ a_m^\dagger\ \sigma_y^\dagger &= -a_m^\dagger. &(57)\end{aligned}$$

The symmetries of the system are, thus, as follows. For states with zero total longitudinal momentum, the symmetries are exactly the same as for the fixed end case. We denote these levels using $S_g(0)$, $S_u(0)$, $A_g(0)$, $A_u(0)$, where the zero in parentheses indicates that these levels correspond to states having zero longitudinal momentum. For non-zero longitudinal momentum, $C\mathcal{P}_{L/2}$ is no longer a symmetry since it reverses the longitudinal momentum. Hence, these levels may be labeled $S(p)$, $A(p)$. Here, $p = \pm 1, \pm 2, \pm 3, \dots$ and corresponds to longitudinal momentum $2\pi p/L$. Note that the energy is independent of the sign (direction) of the longitudinal momentum.

Hence, if E_0 denotes the energy of the ground state (with the above Hamiltonian, it has been defined to be zero), then the eigenvalues E (energy), longitudinal momentum k_L, Λ, and η (for the $k_L\!=\!0$ states) associated with the string eigenmodes are given by

$$\begin{aligned} E &= E_0 + \frac{2N\pi}{L}\sqrt{\frac{\kappa}{\rho}}, &(58)\\ k_L &= \frac{2M\pi}{L}, &(59)\\ N &= \sum_{m\neq 0} |m|\ n_m, &(60)\\ M &= \sum_{m\neq 0} m\ n_m, &(61)\\ n &= \sum_{m\neq 0} n_m, &(62)\\ \Lambda &= (-1)^n. &(63)\end{aligned}$$

For zero-momentum states, we make even and odd $C\mathcal{P}_{L/2}$ states using symmetric and anti-symmetric superpositions, respectively, under $m \to -m$.

Using these properties, the orderings and degeneracies of the Goldstone string energy levels and their symmetries are as shown in Tables 2–3. Hence, for $\kappa = \rho$, the $2N\pi/L$ behavior and a well-defined pattern of degeneracies and level orderings among the different channels form a very distinctive signature of the onset of the Goldstone modes for the effective QCD string.

Table 2: Low-lying torelon string levels. Note that the signed integers refer to the phonon mode. A positive integer indicates a mode with longitudinal momentum in the positive y-direction, whereas a negative integer indicates a mode with oppositely directed longitudinal momentum. The total longitudinal momentum of each level, in terms of the fundamental quantum $2\pi/L$, is indicated in parentheses. For the states having zero longitudinal momentum, the levels which are even and odd under $C\mathcal{P}_{L/2}$ are indicated by subscripts g and u, respectively. States which are even (odd) under reflection in the molecular axis are indicated by $S(A)$. The $N = 1, 2, 3$ energies are 2, 5, 10-fold degenerate, respectively.

$N = 0$:	$S_g(0)$	$\vert 0\rangle$
$N = 1$:	$A(1)$	$a_1^\dagger \vert 0\rangle$
	$A(-1)$	$a_{-1}^\dagger \vert 0\rangle$
$N = 2$:	$A(2)$	$a_2^\dagger \vert 0\rangle$
	$A(-2)$	$a_{-2}^\dagger \vert 0\rangle$
	$S(2)$	$(a_1^\dagger)^2 \vert 0\rangle$
	$S(-2)$	$(a_{-1}^\dagger)^2 \vert 0\rangle$
	$S_g(0)$	$a_1^\dagger a_{-1}^\dagger \vert 0\rangle$
$N = 3$:	$A(3)$	$a_3^\dagger \vert 0\rangle$
	$A(-3)$	$a_{-3}^\dagger \vert 0\rangle$
	$S(3)$	$a_1^\dagger a_2^\dagger \vert 0\rangle$
	$S(-3)$	$a_{-1}^\dagger a_{-2}^\dagger \vert 0\rangle$
	$A(3)$	$(a_1^\dagger)^3 \vert 0\rangle$
	$A(-3)$	$(a_{-1}^\dagger)^3 \vert 0\rangle$
	$S(1)$	$a_{-1}^\dagger a_2^\dagger \vert 0\rangle$
	$S(-1)$	$a_1^\dagger a_{-2}^\dagger \vert 0\rangle$
	$A(1)$	$(a_1^\dagger)^2 a_{-1}^\dagger \vert 0\rangle$
	$A(-1)$	$a_1^\dagger (a_{-1}^\dagger)^2 \vert 0\rangle$

Table 3: The $N = 4$ torelon string levels. See Table 2 for a description of the notation used. The $N = 4$ level is 20-fold degenerate.

$N = 4$:	$A(4)$	$a_4^\dagger\|0\rangle$
	$A(-4)$	$a_{-4}^\dagger\|0\rangle$
	$S(4)$	$a_1^\dagger a_3^\dagger\|0\rangle$
	$S(-4)$	$a_{-1}^\dagger a_{-3}^\dagger\|0\rangle$
	$S(4)$	$(a_2^\dagger)^2\|0\rangle$
	$S(-4)$	$(a_{-2}^\dagger)^2\|0\rangle$
	$A(4)$	$(a_1^\dagger)^2 a_2^\dagger\|0\rangle$
	$A(-4)$	$(a_{-1}^\dagger)^2 a_{-2}^\dagger\|0\rangle$
	$S(4)$	$(a_1^\dagger)^4\|0\rangle$
	$S(-4)$	$(a_{-1}^\dagger)^4\|0\rangle$
	$S(2)$	$a_{-1}^\dagger a_3^\dagger\|0\rangle$
	$S(-2)$	$a_1^\dagger a_{-3}^\dagger\|0\rangle$
	$A(2)$	$a_1^\dagger a_{-1}^\dagger a_2^\dagger\|0\rangle$
	$A(-2)$	$a_1^\dagger a_{-1}^\dagger a_{-2}^\dagger\|0\rangle$
	$S(2)$	$(a_1^\dagger)^3 a_{-1}^\dagger\|0\rangle$
	$S(-2)$	$a_1^\dagger (a_{-1}^\dagger)^3\|0\rangle$
	$S_g(0)$	$a_2^\dagger a_{-2}^\dagger\|0\rangle$
	$S_g(0)$	$(a_1^\dagger)^2 (a_{-1}^\dagger)^2\|0\rangle$
	$A_u(0)$	$((a_1^\dagger)^2 a_{-2}^\dagger + (a_{-1}^\dagger)^2 a_2^\dagger)\|0\rangle$
	$A_g(0)$	$((a_1^\dagger)^2 a_{-2}^\dagger - (a_{-1}^\dagger)^2 a_2^\dagger)\|0\rangle$