Sponsoring Mobile Data: Analyzing the Impact on Internet Stakeholders

Carlee Joe-Wong, *Member, IEEE,* Soumya Sen, *Senior Member, IEEE* and Sangtae Ha, *Senior Member, IEEE*

***Abstract*—As demand for mobile data increases, end users increasingly experience higher costs for consuming data. Sponsored data is a new pricing model that allows content providers (CPs) to subsidize some of this cost. It potentially offers benefits to multiple Internet stakeholders: users can enjoy lower data costs, CPs can attract more users by subsidizing their data access, and Internet service providers (ISPs) can create new revenue streams by charging CPs for sponsored data. However, the distribution of these benefits between different users, CPs, and the ISP remains unclear. Although concerns have been raised that sponsored data disproportionately benefits larger, less cost-sensitive CPs, little attention has been paid to analyzing sponsored data's impact on end users. This work does so by first formulating an analytical model of user, CP, and ISP interactions for heterogeneous users and CPs and deriving their optimal behaviors. We then show that while all three parties can benefit from sponsored data, sponsorship benefits users more than CPs. These disproportionate benefits are more pronounced for more cost-sensitive users when they receive sponsorship from less cost-sensitive CPs, indicating that sponsored data may help to bridge the digital divide between users who can afford the cost of mobile data and those who cannot. We then show that sponsorship disproportionately benefits less cost-sensitive CPs and more cost-sensitive users, exacerbating disparities among CPs but reducing disparities among users. We finally illustrate these results through numerical simulations with data from a commercial pricing trial.**

I. Introduction

As demand for mobile data grows, ISPs are using various smart data pricing (SDP) techniques to manage demand and increase revenue [2]. While much of SDP research has introduced various ways to charge end-users for data access [3]–[5], *sponsored data* instead introduces a new party to data pricing: content providers (CPs). Examples of sponsored data include *zero rating*, in which CPs pay for all of users' data costs [6]. Facebook's Free Basics data plan, which zero-rates Facebook-sponsored content, is offered by more than 50 operators [7]. AT&T and Verizon both offer sponsored data in the US [8], [9]. The startups Syntonic Wireless and DataMi offer marketplaces for CPs to sponsor data for different apps. Recent moves by the U.S. Federal Communications Commission that approves zero-rating practices are likely to drive more adoption of sponsored data in the near future [10].

A. Sponsoring Mobile Data

Under AT&T's and Verizon's sponsored data plans, CPs can subsidize part of the user's cost of using mobile data

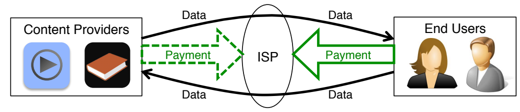


Fig. 1: Data and payment flows between users, CPs, and ISPs. The dashed arrow represents sponsorship.

traffic. Figure 1 illustrates the resulting data and payment flows between end users, CPs, and ISPs. Sponsorship is represented by the dashed arrow showing payments from CPs to the ISP.

Data sponsorship has the potential to benefit users, CPs, and ISPs: users are charged lower prices due to CP subsidies, CPs can attract more traffic as users increase their demands in response to lower prices, and ISPs can enjoy an additional revenue stream. Yet such plans have raised concerns over the possible advantage they give to larger CPs that can better afford to sponsor data, possibly violating network neutrality principles [11]. These concerns echo the controversy over CPs paying for higher quality-of-service (QoS) for their users, thereby creating a "tiered" Internet in which only a select few CPs will be able to provide superior QoS, attracting more user demand [11]. Unlike the previous works that consider this issue of differential QoS (cf. Section II), we show that even *without* a QoS component, sponsored data can exacerbate profit and demand disparities between different CPs, but can also even out demand (usage) differences among different types of users and benefit users relatively more than CPs.

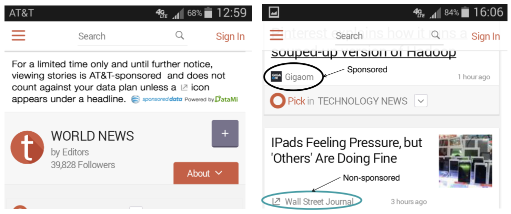
Our study thus reflects the real-world sponsored data implementations, in which CPs do not receive differentiated QoS. Instead, CPs decide which content to sponsor and mark it as such in their apps, as in Figure 2's example of sponsored news stories on Trove, a news aggregator application. Thus, CPs can choose to sponsor different amounts of data for different users.[1] We therefore model these decisions as CPs deciding how much data to sponsor for each user. Sponsored data's impact on the mobile data market will be determined by these CP decisions, given the access prices set by the ISPs. In this work, we examine this decision and the resulting social welfare distribution among different market stakeholders.

B. Modeling Content Provider Sponsorship

Different types of content providers may have different motivations to sponsor data: while their primary motivation

[1]To realize such user-dependent sponsorship, the CP app routes all sponsored traffic through per-app proxies or VPNs implemented on users' devices, allowing ISPs to identify sponsored traffic and bill the CP. Since the VPNs are implemented on individual user devices, the CP can sponsor different amounts of data for different users. Both iOS [12] and Android [13] platforms now support per-app VPNs.

Manuscript received March 21, 2017; revised October 25, 2017.
C. Joe-Wong is with Carnegie Mellon University, S. Sen is with the University of Minnesota, and S. Ha is with the University of Colorado–Boulder. A previous version appeared at IEEE INFOCOM 2015 [1].



(a) Sponsored data banner. (b) In-app sponsorship.

Fig. 2: Trove's sponsored data website on LTE.

TABLE I: CP benefit from increased demand.

CP Type	Benefit source	Benefit from usage	Example
Revenue	Ad revenue Subscriptions	Linear in usage Linear in usage	Pandora Vimeo
Promotion	Goodwill Usage	Concave in usage Concave in usage	Promotions Enterprise

will likely be to attract more users and traffic to their content, CPs will derive quantitatively different benefits (utilities) from this increased traffic depending on their cost sensitivities. They are thus willing to sponsor different amounts of data, which also affects the benefits that their users derive from their data sponsorship. We model this heterogeneity in user and CP behavior by accounting for their cost sensitivities: those CPs that derive greater benefit from increased user demand will be less sensitive to the price of sponsoring data. Similarly, users will also vary in their cost sensitivity [14]. Moreover, to better model the full heterogeneity of CP behavior, we consider two qualitatively different types of CPs: revenue-seeking ("revenue") CPs and promotion-seeking ("promotion") CPs. Table I shows examples of these CP types; we note that within each type, CPs may have different cost sensitivities.

We define "revenue CPs" as those whose benefit or utility grows linearly with an increase in usage. Most revenue CPs rely on either ads or freemium subscriptions to make money [15]. For apps that rely on ads like Pandora or Facebook [16], [17], revenue grows linearly with usage, as the number of ads shown is often proportional to the amount of content used (e.g., ads at regular intervals between songs). Other apps charge users per unit of content, e.g., Vimeo's per-video fees.

We define "promotion CPs" as those for which the CP's benefit from increased demand is concave rather than linear. These CPs might benefit from user goodwill or exposure (eyeballs) instead of revenue, e.g., new mobile apps might sponsor data to attract more users. Thus, in contrast to revenue CPs, promotion CPs experience decreasing marginal utility from additional usage for a single user.

C. *Implications of Sponsored Data*

After reviewing related work in Section II, we derive the optimal behaviors for users, CPs, and ISPs in Section III and analyze the implications in Section IV. Our main contributions are the following results that data sponsorship can:

- *Change user demand patterns (Section IV-A).* For instance, we find that user demands are smaller for less compared to more price-elastic users, though in a non-sponsored market user demand can be larger for less price-elastic users (Proposition 4). This effect is due to CPs' sponsoring less data for less price-elastic users, leading to lower demands. We also would generally expect user utility to decrease as CPs show more ads, but revenue CPs may sponsor more data if they show more ads, increasing users' utility overall (Proposition 5).
- *Benefit users more than CPs (Section IV-B).* Data sponsorship increases the overall social welfare, benefiting users, CPs, and ISPs (Proposition 6). However, these benefits are not distributed evenly. For a given user-CP pair under reasonable conditions, user utility increases proportionally more than a CP's when the CP chooses the amount of data sponsored so as to maximize its utility (Proposition 7), and may increase in absolute terms as well (Proposition 8). This imbalance in the benefits allocated to users and CPs increases for less cost-sensitive CPs (Corollary 3), as well as for more cost-sensitive users with promotion (but not revenue) CPs (Corollary 4). Thus, it is important to consider different CP revenue models, as they lead to different benefits for users. These findings somewhat alleviate concerns that sponsored data will primarily benefit larger, less cost-sensitive CPs [11]: it also significantly benefits users.
- *Disproportionately benefit more cost-sensitive users and less cost-sensitive CPs (Section IV-C).* When directly comparing the benefits of sponsored data for CPs with different levels of cost sensitivity, our results justify the concern that sponsored data will exacerbate the advantage of larger, less cost-sensitive CPs compared to more cost-sensitive ones [11] (Proposition 10). However, sponsored data's benefit to more cost-sensitive users (Proposition 9) implies that it can help to reduce the digital divide between less and more cost-sensitive users' ability to access and pay for mobile data.

In Section V, we consider sponsored data in practice, illustrating our results with data from a commercial pricing trial and proposing a framework for CPs to decide which pieces of content to sponsor. We conclude in Section VI.

II. Related Work

Much of the prior work on sponsored data either focuses on ISPs' optimal actions in splitting costs between CPs and users or includes QoS prioritization and studies the impact on network neutrality. For instance, the literature on Internet "fast lanes" or traffic prioritization models scenarios in which CPs pay ISPs extra fees for higher QoS [18]–[20]. But such payments do not subsidize the amount that users still have to pay to their ISP. These works mostly use game theory to identify ISPs' and CPs' optimal actions from a two-sided market perspective [21]–[23], and studies whether or not the outcome supports network neutrality [24]. When user and CP demands are defined in terms of bandwidth speed (i.e., QoS), [25] considers the optimal amounts that monopolistic and perfectly competitive ISPs charge CPs and end users, while [26] proposes a similar model that includes transit and user-facing ISPs. Other works consider ways in which users can decide to subsidize others' data plans, e.g., by sharing data quotas [27], [28]. However, in the sponsored data context, heterogenous CPs decide on the amount of data sponsorship, which we capture in our model.

TABLE II: Notation used in our model.

Variable	Definition
i, j	User and CP indices, respectively
s $x_{i,j}$	User i's demand for CP j
$\alpha_{i,j}$	Inverse price elasticity of user demand
$c_{i,j}$	Inverse cost sensitivity of user i for CP j
$U_{i,j}\left(x_{i,j}\right)$	User i's utility from CP j as a function of demand
p_u, p_c	Unit price ISP charges for user and CP data respectively
$\gamma_{i,j}$	Fraction of data sponsored per unit of content
s_j	Fraction of ads offered per unit of content
$r_{i,j}$	Utility scaling factor for ads compared to content
$W_{i,j}(\gamma_{i,j})$	CP utility as a function of the fraction of data sponsored
$d_{i,j}$	Inverse cost sensitivity of CP j for user i
$\beta_{i,j}$	Inverse price elasticity of CP demand
X	ISP network capacity
$R^b_{i,j}$, $R^a_{i,j}$	Ratio of CP to user utility (before, after) sponsorship

CP sponsorship of advertisements is explicitly addressed in [29], which accounts for users' probabilities of viewing different ads and monthly data caps, and [30], which considers CPs' incentives to report truthful parameters to the ISP. Implementation challenges of sponsored data are discussed in [31], and [32] considers the special case of zero-rating, or fully subsidizing data, with competing CPs. However, these works focus on CP and ISP decisions and do not consider the distribution of sponsored data's benefits to different types of users. Other works study the effects of sponsored data on different CPs, examining CPs' market shares when they sponsor data for homogeneous users [14] or when CPs pay fixed side payments to ISPs [33]. A similar model of interactions between users, CPs, and an ISP is studied in [34], but without a detailed examination of heterogeneity in CP incentives to sponsor data or of the relative benefits that sponsored data provides to different types of end users and CPs.

III. Sponsored Data Model

In a traditional non-sponsored setting, the data that users consume consists of content and advertisements shown by the CP, and users pay the ISP to transmit both types of data. In our sponsored model, data similarly consists of content and advertisements, and CPs can subsidize users' costs of transmitting either or both types of data.

We consider three players in the sponsored data ecosystem: users, CPs, and ISPs. They make decisions in three stages, as shown in Figure 3: first, the ISP chooses the prices to charge users and CPs. CPs then decide how much data to sponsor, and finally users choose how much content to consume from each CP, depending on the amount of data sponsored and ISP price. CPs can sponsor different amounts of data for different types of users. Each party maximizes its own utility subject to others' decisions. We solve this three-stage sequential decision process through backwards induction to find the optimal, Nash equilibrium outcomes. Following prior works [14], [23], [30], we model a monopoly ISP scenario to analyze the impact of sponsored data on users and CPs.[2] In Section VI, we discuss possible extensions to multiple ISPs. Table II summarizes the notation used in our model.

[2]While many ISPs, e.g., in the US, are oligopolists, due to low churn rates they are often effective monopolies [35].

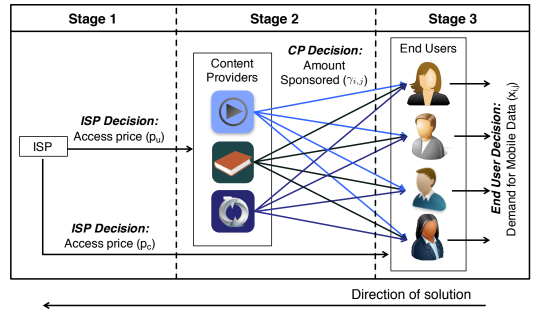


Fig. 3: Three-stage decision hierarchy for users, CPs, and ISPs. We model these interactions as a sequential game, with the ISP moving first to decide the access prices it charges users and CPs. Given these prices, CPs follow the ISP in deciding how much content to sponsor. Finally, users follow the CP by realizing their demands for each CP according to the ISP's prices and CPs' amounts sponsored. We solve this model using backwards induction.

A. End Users' Decisions

Suppose that N users and M CPs exchange data over the ISP's network. Throughout this work, we use "data" to denote content plus advertisements. Each user i receives a utility $V_{i,j}(x_{i,j}, p_u, \gamma_{i,j})$ from CP j, where $x_{i,j}$ is the monthly volume of content that user i consumes from CP j, p_u is the unit price of data that the ISP charges users, and $\gamma_{i,j}$ is the fraction of data, which includes content and advertisements, sponsored by CP j for user i per volume of content over one month.[3] One could equivalently optimize the absolute volume of data sponsored instead of optimizing $\gamma_{i,j}$. We assume that users incur a linear data cost with no volume discounts, as in [14], [25], [29]. For instance, several tiered data plans with different monthly data caps in the US translate to an effective rate of \$10/GB [36]; others like Google Fi in the U.S. impose true usage-based plans [37]. In Appendix B, we show that similar results hold if users instead attempt to keep their expenditures within a monthly cap, without linear data costs (i.e., flat-fee pricing). In this model, the amount of sponsored data would not count towards the user's data cap.

Users are affected by two factors: the volume of ads per volume of content, s_j, and the fraction of data sponsored, $\gamma_{i,j}$. We assume that s_j is constant for all of the CP's content; for instance, Pandora plays ads at regular intervals between songs. We use $\gamma_{i,j}$ to denote the fraction of data sponsored per volume of content; thus, $\gamma_{i,j} \in [0, 1+s_j]$ (ref. to Figure 4). Since users do not necessarily know whether the ads or content is sponsored, we do not distinguish between sponsoring these two types of data. The scenario of Pandora playing ads to users but not sponsoring data corresponds to taking $\gamma_{i,j} = 0$ for all users i and $s_j > 0$; if Pandora decides to sponsor enough data to cover the cost of streaming its advertisements, we would instead have $\gamma_{i,j} = s_j > 0$. With sponsored data, the user pays $p_u\left(1 - \gamma_{i,j} + s_j\right)x_{i,j}$ for $\left(1 + s_j\right)x_{i,j}$ amount of data, including content and advertisements.[4]

We suppose that, absent the data cost, each user i derives utility $U_{i,j}(x_{i,j}(1 + r_{i,j}s_j))$ from consuming $x_{i,j}$ amount of content from CP j, where $U_{i,j}$ is a concave utility function.

[3]In practice, a user i can represent a group of users with similar behavior, and a CP j can represent a group of CPs with similar behavior.

[4]We assume that users cannot skip viewing ads.

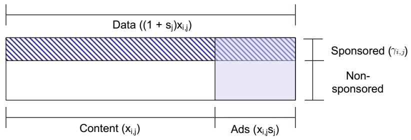


Fig. 4: Relationship of data, content, advertisements, and sponsorship for a given CP-user pair. Data, represented by the entire rectangle, has volume $x_{i,j}(1+s_j)$, divided into content (with volume $x_{i,j}$) and ads (with volume $s_j x_{i,j}$). The CP sponsors a fraction $\gamma_{i,j}$ of data (content and ads) per unit of content; $\gamma_{i,j} \in [0, 1+s_j]$.

The factor $r_{i,j}$ scales user i's utility from viewing ads for CP j relative to viewing content. For instance, while users rarely derive utility from ads, clicking on an ad indicates that they find it entertaining or useful, so we might approximate $r_{i,j}$ with the ad click-through rate. We assume that $r_{i,j}$ is independent of sponsorship[5] (users' marginal utility from ads is more related to their interest in the ads' content than the data cost [38], which we consider separately). Note that if users derive disutility from viewing ads that they have not clicked on, then we may take $r_{i,j} < 0$; however, we suppose that users' disutility from advertisements does not exceed their utility from content, so that $r_{i,j} > -1$: users' net utility from consuming data, $U_{i,j}(x_{i,j}(1+r_{i,j}s_j))$, is then positive.

Each user i's utility function for CP j is then

$$V_{i,j} = \frac{c_{i,j}\left(x_{i,j}\left(1+r_{i,j}s_j\right)\right)^{1-\alpha_{i,j}}}{1-\alpha_{i,j}} - p_u\left(1-\gamma_{i,j}+s_j\right)x_{i,j}, \tag{1}$$

where we take $U_{i,j}(x) = c_{i,j}x^{1-\alpha_{i,j}}/(1-\alpha_{i,j})$, the isoelastic utility function with $\alpha_{i,j} \in [0,1)$ and a scaling factor $c_{i,j} > 0$ following [39], [40].[6] This scaling factor models the user's relative valuation of his or her data consumption relative to the cost of consuming data: for instance, lower-income users might have lower $c_{i,j}$ values, as their overall utility from data consumption is determined more by the cost than their valuations of data consumption. We can thus interpret $c_{i,j}$ as a measure of users' cost sensitivities. A user's total utility is the sum of his utilities from each type of CP: usage of one CP does not affect the utility from others, e.g., browsing Facebook does not affect the utility of watching Hulu. We thus follow [34] in not directly considering competing CPs, i.e., those that offer substitutable content; we discuss the implications of this decision in Section VI. This assumption follows the marketing literature in assuming that users' utility from different CPs, i.e., different brands, is decoupled [41]. User i's optimal demand for data from each CP j is then

$$x^*_{i,j}\left(p_u\left(1-\gamma_{i,j}+s_j\right)\right) = \left(\frac{p_u\left(1-\gamma_{i,j}+s_j\right)}{c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}}. \tag{2}$$

We primarily characterize users by two attributes:

Definition 1 (Price elasticity): User i's *price elasticity* for CP j is defined in the usual economic manner as the % change in usage in response to a 1% change in price.

[5]Our main results hold even when we consider $r_{i,j}$ as a function of s_j; we state our results for the case $r_{i,j}$ independent of s_j for clarity of presentation.

[6]We note that with this form of utility function, one could alternatively interpret the scaling of advertisement content as a scaling of the utility itself by a factor $(1+r_{i,j}s_j)^{1-\alpha_{i,j}}$, rather than scaling the content demand $x_{i,j}$.

In our model, each user i's price elasticity for CP j is a constant $\alpha_{i,j}^{-1}$. As $\alpha_{i,j}$ increases, users have lower price elasticity and their demands are less sensitive to price changes.

Definition 2 (Cost sensitivity): A user i's *cost sensitivity* for CP j is $c_{i,j}^{-1}$, the reciprocal of the $U_{i,j}$ scaling factor in (1).[7]

As $c_{i,j}$ increases, the user becomes less cost-sensitive: the utility function's cost term is weighted less compared to the utility valuation term $U_{i,j}$ in the utility function (1).

B. Content Provider Sponsorship

As discussed in Section I-B, the amount of data that a CP sponsors depends on the CP's benefit from user demand, i.e., whether the CP is a "revenue" or "promotion" CP. These two scenarios can be viewed as special cases of a general CP utility model with linear and concave functional forms. As with end users, we suppose that CPs' utility functions include a utility and a cost component. We use $W_{i,j}$ to denote CP j's overall utility function for user i:

$$W_{i,j}\left(\gamma_{i,j}\right) = \overline{U}_{i,j}(x^*_{i,j}) - p_c\gamma_{i,j}x^*_{i,j}, \tag{3}$$

where $\overline{U}_{i,j}(x) = d_{i,j}x^{1-\beta_{i,j}}/(1-\beta_{i,j})$, with $\beta_{i,j} \in [0,1)$ and $d_{i,j}$ a positive scaling factor, specifies the CP's utility from data usage. Here $x^*_{i,j}$ is user i's optimal demand for CP j as in (2). Note that this utility is also in the isoelastic form, with $\beta_{i,j}^{-1}$ denoting the CP's price elasticity and $d_{i,j}^{-1}$ its cost sensitivity (cf. Definitions 1 and 2 for users). Here $x^*_{i,j}$ is the user demand (2). The term $p_c\gamma_{i,j}x^*_{i,j}$ represents the CP's sponsorship cost for each user i; p_c is the unit data price that ISPs charge CPs. Substituting (2) into (3), we have

$$\begin{aligned} W_{i,j}\left(\gamma_{i,j}\right) =& \frac{d_{i,j}}{1-\beta_{i,j}}\left(\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}}\right)^{1-\beta_{i,j}} \\ &- \frac{p_c\gamma_{i,j}}{\left(1+r_{i,j}s_j\right)^{1-\frac{1}{\alpha_{i,j}}}}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}}. \end{aligned}$$

We assume that s_j, the fraction of ads per content volume is exogenously determined. This is because in practice, the fraction of ads offered is generally constrained by physical factors like the size or form-factor of the user device and users' tolerance for advertisements; industry best practices suggest specific fractions of ads to offer [42]. Therefore, in our main model, the CP does not optimize $s_j \in [0, \overline{s}]$, where $\overline{s}$ is a maximum fraction of ads determined by device and user constraints. In Appendix C, we show that the CP would offer either no ads or as many ads as possible, and our key results still hold when s_j is optimized. The CP chooses $\gamma_{i,j}$ to maximize $\sum_{i=1}^{N} W_{i,j}$, with the constraint $\gamma_{i,j} \in [0, 1+s_j]$.

We now consider the CP's optimization problem for Section I-B's revenue and promotion CPs.

1) Revenue CPs: We first consider a CP whose utility $\overline{U}_{i,j}$ is its revenue. The CP's revenue is a linear function of user demand, as discussed in Section I-B and Table I. Aside from sponsorship costs paid to the ISP, we do not explicitly consider

[7]Optimizing $V_{i,j}$ is equivalent to optimizing $V_{i,j}/c_{i,j}$, for which the utility scaling factor equals one and the cost term is scaled by $c_{i,j}^{-1}$.

CPs' costs of producing content; these may be included by reducing the per-unit revenue by a constant marginal cost.

We now consider the case of $\beta_{i,j} = 0$ and $d_{i,j}$ as the marginal revenue per unit of content in (3), making (3) the CP's revenue less the cost of sponsoring data. For instance, CPs deriving revenue from a cost-per-click advertising model would take $d_{i,j} = ar_{i,j}s_j$, where a is the revenue per volume of ads clicked on.[8] Thus, we find the CP utility function

$$\begin{aligned}&\left(d_{i,j} - p_c\gamma_{i,j}\right) x^*_{i,j}\left(p_u\left(1-\gamma_{i,j}+s_j\right)\right)\\&= \left(\frac{d_{i,j} - p_c\gamma_{i,j}}{(1+r_{i,j}s_j)^{1/\alpha_{i,j}-1}}\right)\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}}, \quad (4)\end{aligned}$$

yielding the optimization problem

$$\max_{\gamma_{i,j}} \sum_{i=1}^{N}\left(\frac{d_{i,j} - p_c\gamma_{i,j}}{(1+r_{i,j}s_j)^{\frac{1}{\alpha_{i,j}}-1}}\right)\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}} \quad (5)$$

$$\text{s.t. } \gamma_{i,j} \in [0, 1+s_j]. \quad (6)$$

Proposition 1: Suppose that $d_{i,j} < p_c(1+s_j)$ for all users i. Then (5–6) has the optimal solution

$$\gamma^*_{i,j}\left(p_c, p_u\right) = \max\left\{0, \frac{d_{i,j}}{p_c(1-\alpha_{i,j})} - \frac{\alpha_{i,j}\left(1+s_j\right)}{1-\alpha_{i,j}}\right\}. \quad (7)$$

For instance, if $d_{i,j} = ar_{i,j}s_j$, users' click-through rates $r_{i,j}$ are generally small ($< 5\%$ [43]) and Prop. 1's condition $d_{i,j} < p_c(1+s_j)$ easily holds. By setting $\gamma^*_{i,j} > 0$ in (7), we can find conditions under which a CP wishes to sponsor data:

Corollary 1: CP j sponsors data for user i ($\gamma^*_{i,j} > 0$) if and only if $d_{i,j} > \alpha_{i,j}p_c\left(1+s_j\right)$.

Thus, if a CP's marginal revenue $d_{i,j}$ is sufficiently small compared to its price of sponsoring data p_c, it will not sponsor any data. Figure 5 illustrates some parameter values under which CPs sponsor data, assuming that $d_{i,j} = ar_{i,j}s_j$ in Corollary 1. We see that even when a/p_c is large, indicating that the CP earns significant advertising revenue relative to the price for sponsoring data, p_c, the CP may not sponsor data if it only runs a few ads (s_j is small); in this case, the marginal revenue $ar_{i,j}s_j$ per unit of content may not be sufficient to make up for the marginal price, $p_c(1+s_j)$. As user utility becomes more concave (i.e., $\alpha_{i,j}$ increases and the user becomes less price-elastic), the CP becomes less likely to sponsor data: its sponsorship has less effect on user demand and is thus less profitable. However, $\gamma^*_{i,j}$ is independent of $c_{i,j}$, indicating that revenue CPs will not need to discriminate in the sponsorship level among users with different cost sensitivities.

2) Promotion CPs: Promotion CPs' benefits are concave in usage: for instance, emerging CPs that wish to attract users to their app will experience diminishing marginal utilities from this increased usage. We can model this usage with the same isoelastic utilities that we use for user demands in (1) [34]. For example, enterprise CPs - a type of promotion CPs - will have the same utility function as their employees when sponsoring company apps: the benefit of increased usage for both is proportional to employee productivity. For promotion CPs, we thus see that user and CP utility components likely have the same shape: $\beta_{i,j} = \alpha_{i,j}$ in (3).[9] By taking different CP and user cost sensitivities $d_{i,j}$ and $c_{i,j}$, we can introduce different weights on the utility, e.g., if CPs care less about cost relative to gaining demand than users do. We then solve for the optimal $\gamma_{i,j}$:

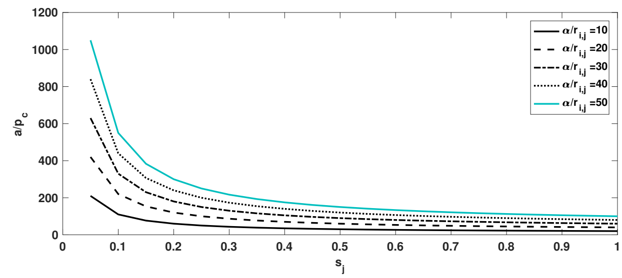


Fig. 5: Contour plot of a/p_c vs. s_j for Corollary 1. We take $d_{i,j} = ar_{i,j}s_j$ and $r_{i,j} = 0.02$, as in a 2% clickthrough rate, for each contour line shown. CPs sponsor data for a/p_c above the contours.

Proposition 2: Suppose that $\beta_{i,j} = \alpha_{i,j} > 0$. Then $W_{i,j}$ is maximized with respect to $\gamma_{i,j} \in [0, 1+s_j]$ at

$$\gamma^*_{i,j} = \max\left\{0, \frac{(1+s_j)\left(d_{i,j}p_u - c_{i,j}\alpha_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}p_c\right)}{d_{i,j}p_u + (1-\alpha_{i,j})p_c c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right\}. \quad (8)$$

Promotion CPs thus sponsor data only if the user price p_u and the inverse CP cost sensitivity $d_{i,j}$ are sufficiently high:

Corollary 2: CP j sponsors data for user i if only if $d_{i,j}\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}-1}p_u > \alpha_{i,j}c_{i,j}p_c$.

Intuitively, if users' data is already inexpensive (small p_u) or the CP cares more about cost than promoting usage (small $d_{i,j}$), the CP has no incentive to sponsor data. In contrast to revenue CPs, however, $\gamma^*_{i,j}$ depends on users' utility scaling factors $c_{i,j}$. Promotion CPs experience diminishing marginal utility with usage volume, so they will be less likely to sponsor data for users with higher $c_{i,j}$, whose demand $x^*_{i,j}$ is already large without sponsorship (cf. (2)). Figure 6 illustrates parameter values for which CP j sponsors a nonzero amount of data. Unlike for revenue CPs (Figure 5), if $d_{i,j}/p_c$ is sufficiently large, then the promotion CP always sponsors a nonzero amount of data, even if it does not show any ads ($s_j = 0$). For promotion CPs, the marginal benefit from sponsoring data is independent of s_j. However, by comparing Figures 5 and 6, we see that both types of CPs sponsor less data as $\alpha_{i,j}$ increases (the user becomes less price-elastic).

C. ISP Price Optimization

Like the CPs, the ISP chooses the prices p_c and p_u so as to maximize its profit. We suppose that the ISP has a finite amount of available capacity in its network, e.g., spectrum license availability, limited number of LTE base stations, and

[8] Similarly, $d_{i,j} = as_j$ if the CP receives cost-per-mille ad revenue.

[9] While this appears to be a restrictive assumption, in Appendix A we show numerically that our results qualitatively hold for arbitrary $\beta_{i,j}$.

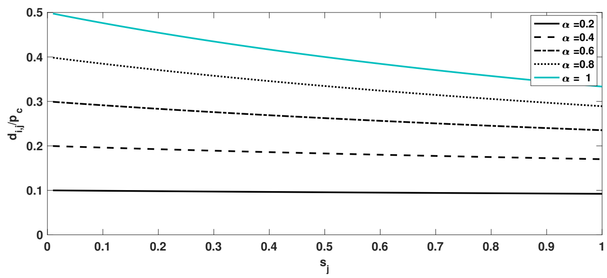


Fig. 6: Contour plot of $d_{i,j}/p_c$ vs. s_j for Corollary 2. We fix $r_{i,j} = 0.02$, corresponding to a 2% clickthrough rate, p_u =\$10/GB, and $c_{i,j} = 5$. CPs sponsor data for $d_{i,j}/p_c$ above the contour lines.

backhaul/middle-mile lease capacity limit the amount of traffic that the ISP can handle at any given time. We translate this instantaneous capacity into a maximum monthly demand for data, X, by supposing that the peak demand over time is a function of the total demand for data. Including this capacity constraint then allows us to assume that the increase in demand due to sponsored data does not materially affect the congestion experienced by users, which could change the utility functions $U_{i,j}$. We introduce the constraint $\sum_{i,j}(1+s_j)\,x^*_{i,j}\left(\pi^*_{i,j}\right) \leq X$, where $(1+s_j)\,x^*_{i,j}$ is the total volume of data pushed over the ISP's network by user i for CP j, and $\pi^*_{i,j}(p_c,p_u) = p_u\left(1-\gamma^*_{i,j}(p_c,p_u)+s_j\right)$ denotes user i's *effective data price* for each CP j. The ISP then wishes to maximize its total profit subject to this capacity constraint, i.e., to solve

$$\max_{p_c,p_u\geq 0} \sum_{i=1}^{N}\sum_{j=1}^{M}\left(\pi^*_{i,j}+p_c\gamma^*_{i,j}\right)x^*_{i,j}\left(\pi^*_{i,j}\right) \tag{9}$$

$$\text{s.t.} \sum_{i=1}^{N}\sum_{j=1}^{M}(1+s_j)\,x^*_{i,j}\left(\pi^*_{i,j}\right) \leq X \tag{10}$$

We can solve (9–10) by noting that both $x^*_{i,j}\left(\pi^*_{i,j}\right)$ and $\left(\pi^*_{i,j}+p_c\gamma^*_{i,j}\right)x^*_{i,j}\left(\pi^*_{i,j}\right)$ are decreasing in p_c:

Proposition 3: If each CP optimally chooses $\gamma_{i,j}$ so as to maximize (3), $x^*_{i,j}\left(\pi^*_{i,j}\right)$ and (9) are both decreasing functions of p_c. For any given p_u, the optimal p_c is the unique minimum value of p_c for which either (10) is satisfied with equality or $d_{i,j} = p_c(1+s_j)$ for some revenue CP j. The optimal value of p_u can be found by a bounded line search.

IV. Impact on Users and CPs

We now consider the implications of user, CP, and ISP behavior.[10] We show that sponsorship can qualitatively alter the relationship between user demand and different system parameters (Section IV-A) as well as change the distribution of social welfare in favor of users, compared to CPs (Section IV-B). We finally find that sponsorship favors more cost-sensitive users and less cost-sensitive CPs relative to other users and CPs respectively (Section IV-C). We use the term "user-CP pair" to refer to a given user's demand for and utility derived from a given CP, as well as the CP's utility from that user. Unless otherwise stated, our results hold for any ISP prices p_u and p_c. We use p_u and p^*_u to respectively denote the users' data prices chosen by the ISP before and after sponsorship.

A. Variation in Demand and Utility

We first show some numerical examples of interesting user behaviors before deriving conditions under which they are observed. In all simulations in the paper, unless otherwise noted we use the following parameters: ISP prices are $p_u = p_c =$ \$10/GB, which approximates current ISP data prices (e.g., AT&T's data plans vary from \$7.50/GB to \$25/GB [36]). Revenue CPs are assumed to make money from advertising with $a =$ \$1800 per GB of ads, based on a \$2 revenue per ad click [44] and an 880 KB average ad size, e.g., a short video. We assume that CPs carry an additional 15% of ads per content volume ($s_j = 0.15$), e.g., a 30-second ad for a 200-second video, and that users' ad click-through rate is 2% ($r_{i,j} = 0.02$) [43], [44]. Revenue CPs then have $d_{i,j} = ar_{i,j}s_j = 5.4$. In all figures, we use "before" and "after" to respectively denote the scenarios before (i.e., without) and after (with) sponsorship.

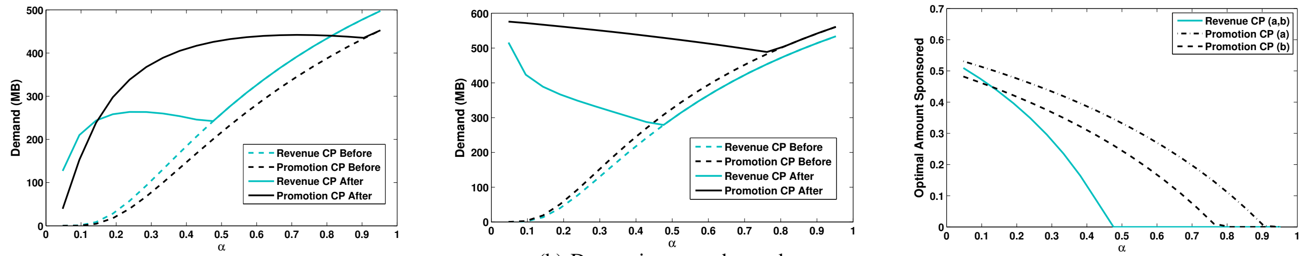
Without sponsorship, user demand $x^*_{i,j} = (p_u(1+s_j)/c_{i,j})^{\frac{-1}{\alpha_{i,j}}}/(1+r_{i,j}s_j)$; thus, if $p_u(1+s_j) > c_{i,j}$, users' demands increase as they become less price-elastic (i.e., $\alpha_{i,j}$ increases). We can explain this result by noting that at the price $p_u = c_{i,j}/(1+s_j)$, users' demands are independent of $\alpha_{i,j}$. All users would then increase their demands for higher prices $p_u > c_{i,j}/(1+s_j)$, with a lower increase for less price-elastic users. However, as Figure 7a shows, with sponsorship demand can both increase and decrease as $\alpha_{i,j}$ increases. We can derive sufficient conditions under which sponsorship changes the relationship between user demand and price elasticity:

Proposition 4 (Demand and price elasticity): User demand increases as the CP sponsors more data: $\frac{\partial x^*_{i,j}}{\partial \gamma_{i,j}} \geq 0$. Moreover, a revenue CP with $\gamma^*_{i,j} > 0$ will experience smaller demand as users become less price-elastic if $p^*_u\left(d_{i,j}-p_c(1+s_j)\right) \geq p_c c_{i,j}(1+r_{i,j}s_j)$: $x^*_{i1,j1} \leq x^*_{i2,j2}$ if $\alpha_{i1,j1} \geq \alpha_{i2,j2}$ and the user-CP pairs $(i1,j1)$ and $(i2,j2)$ differ only in $\alpha_{i,j}$.

Intuitively, as users become less price-elastic ($\alpha_{i,j}$ increases), they do not increase their demands as much in response to CPs' sponsoring data to lower prices. CPs thus do not benefit as much from sponsorship and sponsor less data. In Figure 7c, we see that $\gamma^*_{i,j}$ indeed decreases to zero as $\alpha_{i,j}$ increases. As $\gamma^*_{i,j}$ decreases, user demand $x^*_{i,j}$ also eventually decreases in Figures 7a and 7b. Proposition 4 shows that if a CP's price p_c is low enough, the decrease in $\gamma^*_{i,j}$ will cause user demand to decrease at any $\alpha_{i,j}$ with $\gamma^*_{i,j} > 0$, as in Figure 7b. Under these conditions, the effect of a decrease in sponsorship, which would tend to decrease users' demands, outweighs the effect of decreased price elasticity, which would tend to increase user demands.

We next consider the amount of ads sponsored, s_j. Without sponsorship, user utility decreases as s_j increases, since users must pay for the ads' data and gain little utility from ads. However, Figure 8 shows that with sponsorship, user utility can increase with s_j if $\gamma^*_{i,j} > 0$ for revenue CPs. This increase is due to revenue CPs sponsoring more data as s_j increases:

[10] Appendix A numerically shows that our results hold for more general CP models ($\beta_{i,j} \neq \alpha_{i,j}$ for promotion CPs).



(a) Non-monotonic user demand. (b) Decreasing user demand. (c) Optimal amounts sponsored.

Fig. 7: Demand can be (a) non-monotonic and (b) decreasing as users become less price-elastic for a revenue and promotion CP with (c) decreasing optimal amounts sponsored $\gamma^*_{i,j}$. Parameters used are (a) $c_{i,j} = 5.8$ for the revenue CP and $c_{i,j} = 5.3$, $d_{i,j} = 4.8$ for the promotion CP; (b) $c_{i,j} = 6.2$ for the revenue CP and $c_{i,j} = 6.5$, $d_{i,j} = 5$ for the promotion CP.

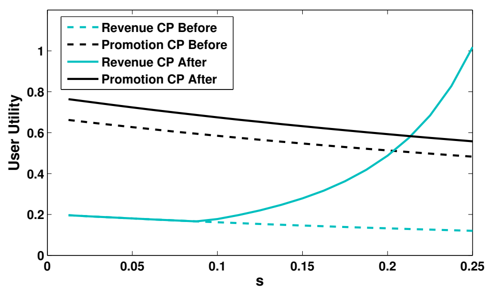


Fig. 8: User utilities increase with the fraction of ads shown s_j for a user of revenue CPs ($\alpha_{i,j} = 0.3$, $c_{i,j} = 4$) and decrease for a user of promotion CPs ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$, $d_{i,j} = 2$).

Proposition 5 (Utility of advertisements): Consider a revenue CP j with marginal revenue $d_{i,j} = ar_{i,j}s_j$. Then $\partial V_{i,j}(x^*_{i,j}\left(\pi^*_{i,j}\right), p_u, \gamma^*_{i,j})/\partial s_j \geq 0$, i.e., user utility increases with ads shown, if $ar_{i,j} > p_c(1 - r_{i,j})$ and $\gamma^*_{i,j} > 0$.

Revenue CPs' marginal revenue $ar_{i,j}s_j$ increases with s_j. Thus, if the expected revenue per volume of ads, $ar_{i,j}$, is sufficiently high, CPs sponsor more data as s_j increases. The resulting decrease in users' effective prices $\pi^*_{i,j}$ is enough to offset their lower utilities from ads. The a, $r_{i,j}$, and p_c values used in Figure 8 satisfy Proposition 5's condition, indicating that users will gain more utility from CPs that show more ads if those CPs also sponsor data. For instance, if one news app shows more ads than another, it can attract more usage by sponsoring more data.

B. Social Welfare under Sponsored Data

Allowing CPs to sponsor data not only alters users' demand patterns as in Section IV-A, but also affects the overall social welfare that users and CPs experience. We find that sponsored data increases the overall social welfare, benefiting users, CPs, and ISPs; and that it increases users' utilities proportionally more than CPs' utilities. These results indicate that the network neutrality debate over which CPs benefit the most from sponsored data is missing an important dimension: sponsored data's greater benefit to users.

We first show that if demand increases with data sponsorship as in Proposition 4, user utility also increases:

Lemma 1: User i's utility $V_{i,j}$ from CP j increases with sponsorship if and only if the CP sponsors data $(\gamma_{i,j} > 0)$.

Note that Lemma 1 does not require the amount sponsored to be chosen optimally; it is enough for CPs to sponsor any amount of data. We use this result to show that users, CPs, and ISPs can all benefit from data sponsorship:

TABLE III: CP-to-user utility ratios before $\left(R^b_{i,j}\right)$ and after $\left(R^a_{i,j}\right)$ sponsorship.

	Revenue CP	Promotion CP
$R^b_{i,j}$	$\frac{d_{i,j}(1-\alpha_{i,j})}{\alpha_{i,j}p_u(1+s_j)}$	$\frac{d_{i,j}}{\alpha_{i,j}c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}}$
$R^a_{i,j}$	$\frac{p_c(1-\alpha_{i,j})}{p^*_u}$	$\frac{d_{i,j}}{c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}} + \frac{(1-\alpha_{i,j})p_c}{p^*_u}$

Proposition 6 (Social welfare increase): Suppose the ISP's chosen price p^*_u is not greater than users' data price p_u before sponsorship (e.g., due to constraints from market competition). Then users and CPs do not decrease their utilities and the ISP does not decrease its profit with sponsorship.

If users' data price p_u does not increase, their effective prices $\pi^*_{i,j}$ will decrease with sponsorship; user demands then increase, which benefits users, CPs, and ISPs. ISPs may be constrained from charging $p^*_u > p_u$ to avoid user dissatisfaction; our numerical results in Section V-A show that in practice $p^*_u \leq p_u$. CPs and users, however, do not benefit equally. We now consider how social welfare is distributed among users and CPs. To do so, we find the ratio of CP to user utility when $\gamma^*_{i,j} > 0$ (i.e., some data is sponsored) and show that it decreases with sponsorship, meaning that *CPs experience lower increases in utility compared to users*. We use $R^a_{i,j}$ to denote this ratio after sponsorship, and $R^b_{i,j}$ to denote the ratio before sponsorship.

Proposition 7 (User and CP utilities): Table III gives the ratio of CP to user utility for each user-CP pair before and after sponsorship. For promotion CPs, this ratio is always lower after sponsorship: $R^a_{i,j} \leq R^b_{i,j}$ for all users i and CPs j. For revenue CPs, $R^a_{i,j} \leq R^b_{i,j}$ if $p^*_u d_{i,j} \leq \alpha_{i,j}p_u(1 + s_j)$.

Proposition 7's result is surprising, as the $\gamma^*_{i,j}$ are chosen to maximize CP, not user, utility: thus, we would expect CPs to benefit more than users. However, while both users and CPs benefit from higher user demand with sponsorship, CPs must drive this demand by paying subsidies to ISPs. These subsidies lower CPs' overall utility gains relative to users'. The exception to this reasoning occurs with revenue CPs: a much higher price p^*_u compared to p_u may lead to a higher CP-to-user utility ratio with sponsorship. Unlike promotion CPs, the revenue CP's utility is linear in user demand, so the revenue CP would have greater incentive to sponsor more data in order to offset the higher p_u^*, leading to higher CP utility.

We can further use Table III's results to examine how the CP-to-user utility ratio changes with different user and CP parameters. We see that surprisingly, CPs benefit more relative to users (i.e., the CP-to-user utility ratio decreases less) as their data price p_c increases: CPs then sponsor less data, bringing the utility ratio closer to that before sponsorship.

We now consider which types of users and CPs experience the greatest changes in their utility ratios before and after sponsorship. We first consider the effect of CP cost sensitivity:

Corollary 3 (User and CP utilities for different CP cost-sensitivity): The CP-to-user utility ratio with optimal sponsorship, divided by that before sponsorship decreases as CP cost sensitivity decreases ($\partial \left(R_{i,j}^a / R_{i,j}^b\right) / \partial d_{i,j} \leq 0$).

We thus see that the CP-to-user utility ratio decreases more with sponsorship for less cost-sensitive CPs: less cost-sensitive CPs experience a greater redistribution of utility towards users and away from CPs. This result is likely due to the fact that less cost-sensitive CPs sponsor more data, benefiting users more; Corollary 3 shows that this user benefit is not only proportionally larger than the CP benefit, as shown in Proposition 7, but that this user-CP imbalance grows as CPs become less cost-sensitive. This result holds for both promotion and revenue CPs. The effect of user cost sensitivity, however, differs from revenue and promotion CPs:

Corollary 4: The CP-to-user utility ratio with optimal sponsorship, divided by that before sponsorship decreases as user cost sensitivity increases ($\partial \left(R_{i,j}^a / R_{i,j}^b\right) / \partial c_{i,j} \geq 0$) for promotion CPs, but is independent of $c_{i,j}$ for revenue CPs.

For revenue CPs, the CP-to-user utility ratio itself is independent of user cost sensitivity, likely due to the fact that the optimal sponsorship amount is independent of user cost sensitivity (Proposition 1). For promotion CPs, however, cost-sensitive users benefit proportionally more relative to CPs ($R_{i,j}^a$ grows relative to $R_{i,j}^b$), likely because promotion CPs sponsor more data for more cost-sensitive users (Proposition 2). As CPs sponsor more data, the imbalance of users' and CPs' utility increases with sponsored data is exacerbated: users benefit more from sponsorship, but CPs must still pay a portion of their benefit back to the ISP.

Corollaries 3 and 4 can be interpreted as alleviating some concerns that sponsored data will unduly or primarily benefit larger CPs. While larger, less cost-sensitive CPs will sponsor more data, in doing so they benefit users proportionally more, and this effect grows as CPs become less cost-sensitive and users become more cost-sensitive, i.e., they can likely afford less mobile data. In the next subsection, we consider the distribution of CP and user utilities across different levels of cost sensitivity in more detail.

We finally derive conditions under which the *absolute* improvement in users' utilities with sponsorship is larger than the absolute improvement in CP utilities. For clarity, we define $V_{i,j}^a$ to denote user i's utility from CP j with optimal sponsorship and $V_{i,j}^b$ as the utility before sponsorship, and similarly for $W_{i,j}^a$ and $W_{i,j}^b$ with CP utility.

Proposition 8 (Absolute difference in utilities): Consider a given user-CP pair and suppose that $\gamma_{i,j}^* > 0$, i.e., that CP j sponsors a positive amount of data for user i. When the CP is a revenue CP, the absolute improvement in user utilities with sponsorship exceeds that of CP utilities $\left(V_{i,j}^a - V_{i,j}^b + W_{i,j}^b - W_{i,j}^a \geq 0\right)$ if

$$d_{i,j}(1-\alpha_{i,j}) \geq p_u \alpha_{i,j}(1+s_j),\ p_u^* \geq (1-\alpha_{i,j})p_c. \quad (11)$$

When the CP is a promotion CP, the absolute improvement in user utilities with sponsorship exceeds that of users if

$$\begin{aligned} &c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}\left(1-\frac{(1-\alpha_{i,j})p_c}{p_u^*}\right) \\ &\geq d_{i,j} \geq \alpha_{i,j} c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}. \end{aligned} \quad (12)$$

Thus, as long as $d_{i,j}$ is sufficiently large, i.e., the CP experiences a sufficiently large marginal benefit from an increase in user demand, users experience a larger increase in utility than CPs. Even though a larger $d_{i,j}$ implies a larger gain in CP utility, it also results in a larger $\gamma_{i,j}^*$ (Propositions 1 and 2). Thus, CPs sponsor more data, leading to a larger benefit for users that outweighs that experienced by CPs. As in Proposition 7's result that users benefit proportionally more than CPs, this increase in sponsorship leads to a larger absolute increase in user utility than in CP utility.

C. Distributions of Demand and Utility

We next characterize the distributions of demand and utility with and without sponsorship, focusing on how evenly they are spread across different users and CPs:

Definition 3 (Fairness): Let the *fairness* of a set of user-CP variables $\{y_{i,j}\}$ (e.g., demand $x_{i,j}$) be given by a function $F\left(\{y_{i,j}\}\right)$, where F is a homogeneous Schur-concave fairness measure satisfying the axioms in [45].

Evaluating the fairness of user and CP demands and utilities thus allows us to quantitatively compare sponsored data's effects for different types of users and CPs.

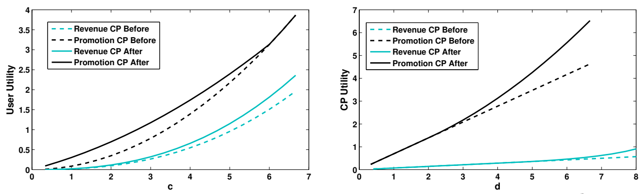
Table IV shows the fairness of demands and utilities before and after sponsorship for the numerical examples given in this section. We use Jain's index [45] as the fairness function; an index nearer to 1 indicates greater fairness. We can compare the fairness of user and CP utilities before and after sponsorship, and thus the benefits accrued to different types of users and CPs, using the following metric:

Definition 4 (Relative benefit): User i's *relative benefit* from CP j's sponsorship is the ratio of his or her utility $V_{i,j}\left(x_{i,j}^*(\pi_{i,j}^*), p_u, \gamma_{i,j}^*\right)$ with optimal sponsorship to the utility $V_{i,j}\left(x_{i,j}^*\left(p_u(1+s_j)\right), p_u, 0\right)$ before sponsorship. Similarly, a CP j's relative benefit from user i is the ratio of its utility $W_{i,j}$ with optimal sponsorship to that before sponsorship.

The *relative demand* of a user-CP pair is analogously defined as the ratio of demand with optimal sponsorship $x_{i,j}^*\left(\pi_{i,j}^*\right)$ to that without sponsorship. Table V shows the relative benefit and demand for both types of CPs and users in the special case that $p_u^* = p_u$, i.e., the ISP does not change

TABLE IV: Jain's fairness index for distributions of (demands, user utilities, CP utilities) before and after sponsorship.

Variable	Before (Revenue CP)	After (Revenue CP)	Before (Promotion CP)	After (Promotion CP)
$\alpha_{i,j}$ (Figure 7a)	(0.674, 0.214, 0.674)	(0.912, 0.219, 0.719)	(0.65, 0.21, 0.256)	(0.922, 0.228, 0.267)
$\alpha_{i,j}$ (Figure 7b)	(0.694, 0.218, 0.694)	(0.961, 0.224, 0.757)	(0.709, 0.221, 0.276)	(0.998, 0.237, 0.289)
$c_{i,j}$ (Figure 9a)	(0.503, 0.503, 0.503)	(0.503, 0.503, 0.503)	(0.57, 0.57, 0.768)	(0.856, 0.676, 0.868)
$d_{i,j}$ (Figure 9b)	(1, 1, 0.769)	(0.643, 0.859, 0.673)	(1, 1, 0.769)	(0.769, 0.927, 0.705)



(a) User utilities, varied $c_{i,j}$.

(b) CP utilities, varied $d_{i,j}$.

Fig. 9: (a) User utility with a revenue ($\alpha_{i,j} = 0.4$) and promotion ($\alpha_{i,j} = 0.5$, $d_{i,j} = 3$) CP; (b) CP utility for a user of revenue ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$) and promotion ($\alpha_{i,j} = 0.5$, $c_{i,j} = 4$) CPs.

its user price p_u with sponsorship. Similar expressions can be derived in the more general case.

To analyze the fairness of different demand and utility distributions, we will make use of the following lemma:

Lemma 2: Consider two vectors $\vec{x}$ and $\vec{y}$ of length K with elements in increasing order (i.e., $\vec{x}_k > \vec{x}_l$ for $k > l$), such that $\vec{y}_k/\vec{x}_k \leq \vec{y}_l/\vec{x}_l$ $\forall$ indices $k \leq l$. Then $F(\vec{y}) \leq F(\vec{x})$.

For instance, $\vec{x}$ and $\vec{y}$ could be vectors of different users' demands or utilities. Our first result shows that *sponsorship favors more cost-sensitive users, allowing them to disproportionately increase their utility and demand*:

Proposition 9 (Fairness across user cost sensitivity): Consider a set of users who vary only in their cost sensitivities $c_{i,j}^{-1}$ and a CP j with the same $d_{i,j}$ for all users. If CP j is a promotion CP, the distributions of user demands and utilities across different cost sensitivities become more fair with sponsorship ($F\left(\left\{{x_{i,j}^*}^b\right\}\right) \leq F\left(\left\{{x_{i,j}^*}^a\right\}\right)$ and $F\left(\left\{V_{i,j}^b\right\}\right) \leq F\left(\left\{V_{i,j}^a\right\}\right)$). Users' relative demands and benefits increase as $c_{i,j}^{-1}$ increases.

If CP j is a revenue CP, relative demand and benefit is independent of cost sensitivity, so fairness does not change.

User demand increases as users become less cost-sensitive, with and without sponsorship. Thus, we might expect less cost-sensitive users to benefit more from sponsorship, since their overall demand and utility levels are larger. Indeed, with revenue CPs, relative benefit is independent of cost sensitivity. However, promotion CPs experience diminishing marginal utility from greater user demand, and we see from (8) that they sponsor less data for less cost-sensitive (high $c_{i,j}$) users. Promotion CPs thus increase the effective prices $\pi_{i,j}^*$ as $c_{i,j}$ increases, dampening user demand enough to ensure that relative demand decreases as $c_{i,j}$ increases. This result is consistent with that of Corollary 4, which shows that more cost-sensitive users benefit proportionally more than promotion CPs: combined with Proposition 9's result, we then find that more cost-sensitive users benefit proportionally more from sponsorship compared to either less cost-sensitive users or promotion CPs.

Figure 9a illustrates Proposition 9's result; user utility not only increases as $c_{i,j}$ increases, but is never less than the utility before sponsorship. Table IV shows that the demand and user utility distributions become more fair for the promotion CP; fairness does not change for the revenue CP.

If we instead consider homogeneous users and vary CP cost sensitivity, we find the opposite effect: the distributions of demand and utility always become less fair.

Proposition 10 (Fairness across CP cost sensitivity): Consider a set of homogeneous users and either revenue or promotion CPs varying only in their cost sensitivity $d_{i,j}^{-1}$. The demand and CP utility distributions across different CP cost sensitivities become less fair with sponsorship ($F\left(\left\{{x_{i,j}^*}^b\right\}\right) \geq F\left(\left\{{x_{i,j}^*}^a\right\}\right)$ and $F\left(\left\{W_{i,j}^b\right\}\right) \geq F\left(\left\{W_{i,j}^a\right\}\right)$).

Sponsored data favors less cost-sensitive CPs, as some have feared [11]. Less cost-sensitive CPs receive higher utilities even without sponsorship, due to their greater valuation of user demand. As we would expect, they can also sponsor more data, disproportionately increasing their utility. However, Corollary 3 somewhat moderates this finding: though less cost-sensitive CPs do benefit proportionally more than other CPs, as indicated by Proposition 10, Corollary 3 shows that they benefit proportionally less than their users. Figure 9b illustrates Proposition 10 by showing CP utilities as $d_{i,j}$ varies. Sponsorship increases the disparities between these utilities, making their distribution more unfair (Table IV).

V. Sponsored Data in Practice

We illustrate Section IV's results with data from a small-scale commercial pricing trial. We then turn to a practical question of sponsorship and provide a framework for CPs to decide which content to sponsor, e.g., specific videos.

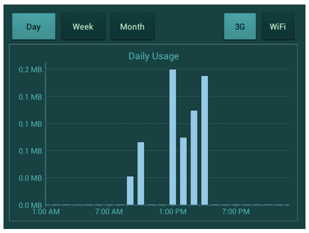
A. Numerical Evaluation

1) Trial Data: We estimate user utility parameters $(\alpha_{i,j}, c_{i,j})$ from a small-scale pricing trial with a U.S. ISP. We recruited 18 users and offered them different prices for their data usage at different times during June 2013; the offered prices were randomized and ranged from \$10/GB to \$20/GB. We then recorded users' per-app cellular data usage during the month of the trial. Figure 10 shows screenshots of the trial app: users could see their hourly and monthly usage of different apps. Since our trial only ran for one month, we did not vary the monthly data plan prices for individual users, but different users experienced different data prices due to different prevailing prices at the times of their data usage.

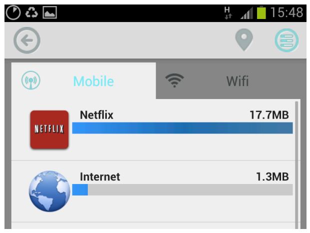
We consider sixty user-CP pairs in our simulation, consisting of six CPs and ten users. We group the apps from our trial data into three revenue CPs (for social networking, browsing, and video) and three promotion CPs (social

TABLE V: Relative demands and benefits for users and CPs with optimal sponsorship.

	Revenue CP	Promotion CP
Relative demand	$\left(\frac{1}{1-\alpha_{i,j}} - \frac{d_{i,j}}{p_c(1-\alpha_{i,j})(1+s_j)}\right)^{\frac{-1}{\alpha_{i,j}}}$	$\left(1-\alpha_{i,j} + \frac{d_{i,j}p_u}{c_{i,j}p_c(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{1}{\alpha_{i,j}}}$
Relative user benefit	$\left(\frac{1}{1-\alpha_{i,j}} - \frac{d_{i,j}}{p_c(1-\alpha_{i,j})(1+s_j)}\right)^{1-\frac{1}{\alpha_{i,j}}}$	$\left(1-\alpha_{i,j} + \frac{d_{i,j}p_u}{c_{i,j}p_c(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{1}{\alpha_{i,j}}-1}$
Relative CP benefit	$\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\left(\frac{p_c(1+s_j)}{d_{i,j}} - 1\right)\left(\frac{1}{1-\alpha_{i,j}} - \frac{d_{i,j}}{p_c(1-\alpha_{i,j})(1+s_j)}\right)^{\frac{-1}{\alpha_{i,j}}}$	$\left(1-\alpha_{i,j} + \frac{d_{i,j}p_u}{p_c c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{1}{\alpha_{i,j}}} \times \frac{\alpha_{i,j}p_c c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}{p_u d_{i,j}}$



(a) Total usage over time.



(b) Per-app monthly usage.

Fig. 10: Partial screenshots of the trial app.

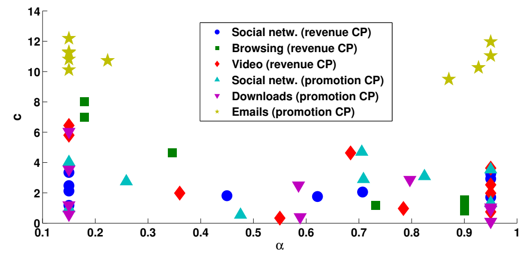


Fig. 11: Estimated user utility parameters for each user-CP pair in simulations with trial data.

networking, downloads, and email). We then estimate the $(\alpha_{i,j}, c_{i,j})$ parameters for each app by fitting the demand curve $x^*_{i,j}(p) = (p/c_{i,j})^{-1/\alpha_{i,j}}$ to all users' monthly usage of the app, where p is the user's \$/GB data plan price. We generate parameters for ten different users of each CP by assuming they are normally distributed around the parameters estimated for apps of this CP type. Figure 11 shows the resulting $\alpha_{i,j}$ and $c_{i,j}$ values; while the $\alpha_{i,j}$ values cluster towards the very high and very low, a more comprehensive user base would likely exhibit greater behavioral variation.

2) Simulation Results: We assume that ISPs charge their optimal data prices p^*_u = \$17/GB and p^*_c = \$6.19/GB, which are close to ISPs' currently offered prices [36]; the maximum network capacity in (10) is $X = 30$ GB. Revenue CPs take $d_{i,j} = ar_{i,j}s_j = 5.4$ as calculated in Section IV-A.

Figure 12a shows the distribution of the optimal amounts sponsored $\gamma^*_{i,j}$ for each user-CP pair. While $\gamma^*_{i,j} = 0$ for 28% of the user-CP pairs, a few CPs sponsor >100% of users' content (i.e., they also sponsor some ads, which add 15% to the content volume). This scenario would correspond to zero-rating an application; for instance, some enterprises may sponsor all traffic on a corporate email app. Despite the $\alpha_{i,j}$ estimates clustering around very low and very high values, except for the concentration at 0 the $\gamma_{i,j}$ values are approximately evenly distributed between 0 and 1.15, likely due to different cost sensitivities $c^{-1}_{i,j}$ and $d^{-1}_{i,j}$.

Figure 12b shows the ratios of CP to user utility for each user-CP pair before and after sponsorship; as in Proposition 7, the ratios decrease with sponsorship, indicating that users benefit more than CPs. The decrease is most apparent for larger CP-to-user ratios, likely because these correspond to users with very low utility. The users and CPs with the lowest utility values experience the greatest increase in utility with sponsorship, as shown in Figure 12c, though both CP and user utilities generally increase. In fact, the fairness of the CP and user utility distributions over all user-CP pairs increases with sponsorship. Jain's index for user utilities increases from 0.158 to 0.172, as is consistent with Proposition 9 for users facing one CP, and Jain's index for CP utilities increases from 0.126 to 0.134. While sponsorship makes CP utilities more unfair with homogeneous users (Proposition 10), in reality CPs face different user demands, allowing the distribution of CP utility to become (slightly) more fair in our simulation.

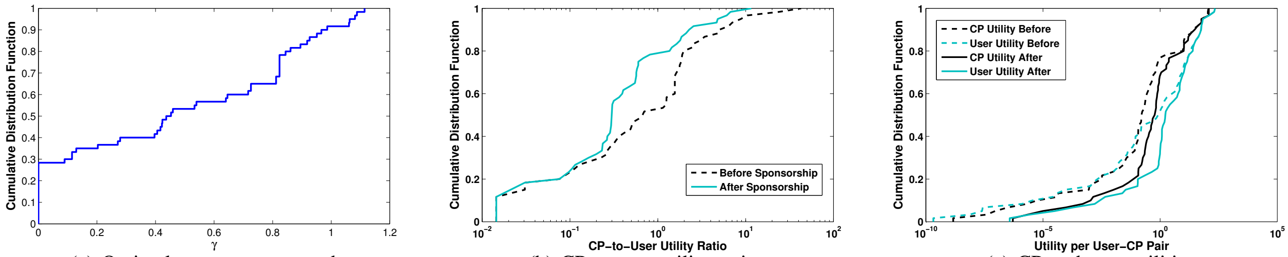
Not only do user and CP utilities become more fair with sponsorship, but user demand also increases (Figure 13a). However, different CPs experience different changes in the distribution of these demands. Figure 13b shows that some CPs experience less and some more disparate user demands with sponsorship. Similarly, each individual user faces all six CPs. While the CPs have the same cost sensitivities $d^{-1}_{i,j}$ and fractions of ads sponsored s_j for each user, users' price elasticities $\alpha^{-1}_{i,j}$ and cost sensitivities $c^{-1}_{i,j}$ vary for each CP. Thus, the distribution of a given user's demands across the CPs changes from user to user, as shown in Figure 13c.

B. Which Content to Sponsor?

While Props. 1 and 2 give CPs the optimal amount of data to sponsor, they do not help decide *which content or associated ads* to sponsor. Since most users choose the content they view (e.g., which videos to watch), sponsorship can influence demand for different content.[11] If the CP sponsors more popular content, more users will benefit, but CPs may wish to promote less popular content by sponsoring it.

We suppose that a CP j has K_j types of content (e.g., videos), each with a probability $v^k_{i,j}\left(p^k_{i,j}\right) \in [0,1]$ of being viewed by type i users, where $p^k_{i,j}$ is the fraction of content type k that is sponsored. The CP chooses the $p^k_{i,j}$ so as to maximize an objective function $G\left(\left\{v^k_{i,j}\left(p^k_{i,j}\right)\right\}\right)$. For instance, to spread demand across different types of content the CP can equalize the $v^k_{i,j}\left(p^k_{i,j}\right)$ (i.e., make G a fairness function [45]).

[11] Since we assume that CPs fix the ratio of ads to content, sponsoring ads will affect overall demand but not which content users choose to view.



(a) Optimal amount sponsored. (b) CP-to-user utility ratios. (c) CP and user utilities.

Fig. 12: (a) Optimal amount sponsored and the resulting distributions over all user-CP pairs of (b) CP-to-user utility ratios and (c) CP and user utilities. User and CP parameters are taken from the trial data (Figure 11), with $d_{i,j} = 6.295,\ 0.607,\ 3$ for the three promotion CPs.

To target more popular content, it can maximize the sum of all viewing probabilities: $G = \sum_k v_{i,j}^k\left(p_{i,j}^k\right)$.

The CP maximizes G subject to two constraints. First, the total fraction of data sponsored should equal $\gamma_{i,j}^*$, as given by (7) or (8): $\sum_k p_{i,j}^k z_k = \gamma_{i,j}^* \sum_k z_k$, where z_k is the volume of type k content. Second, users' overall demand should be $x_{i,j}^*(\pi_{i,j}^*)$, i.e., $\sum_k v_{i,j}^k\left(p_{i,j}^k\right) z_k = x_{i,j}^*(\pi_{i,j}^*)$. If the CP cannot sponsor fractions of each content type (e.g., a content type is a single video), we constrain $p_{i,j}^k \in \{0,1\}$. We then have a knapsack problem with multiple constraints [46].

VI. Conclusion

In this work, we consider sponsored data's benefits for heterogeneous users, CPs, and ISPs. We first derive the optimal sponsored data behaviors for users, CPs, and ISPs in a three-stage backwards induction model, and we then consider their implications for heterogeneous CPs and users. In particular, we find that sponsored data disproportionately benefits users over CPs, and that this disparity is further skewed towards users when less cost-sensitive CPs sponsor data. Moreover, while less cost-sensitive CPs benefit more than cost-sensitive ones from sponsoring data, more cost-sensitive users benefit more compared to other users. Thus, the concern that sponsored data disproportionately benefits larger, less cost-sensitive CPs over other CPs is a valid one, but misses the point that such sponsorship also benefits more cost-sensitive users. It thus offers a way to help bridge the digital divide between those who are more and less able to afford the cost of mobile data.

Though sponsored data offers a seemingly simple new choice to CPs, its effect will be felt throughout the mobile data market: as we show in this work, CPs' newfound market power over mobile data can significantly change the distributions and relative values of both CP and user utilities. Extensions of our work to multiple ISPs may show that offering sponsored data leads to a competitive advantage over other ISPs, due to its benefits for users. Indeed, it may even encourage ISPs to subsidize data themselves in order to attract more users, e.g., as in T-Mobile's BingeOn initiative zero-rating some video streaming [47]. An open question, however, is the effect of competition between CPs. Our work in Appendix B shows that if this competition manifests as budget constraints on users' data consumption of different CPs, then our results qualitatively hold. However, if CPs directly compete with each other, they may be tempted to subsidize even more data, further benefiting users at the expense of CPs. Our work is necessarily an approximation of user, CP, and ISP behavior, but nevertheless provides a guide towards understanding sponsored data's implications for stakeholders in the mobile data market.

References

[1] C. Joe-Wong, S. Ha, and M. Chiang, "Sponsoring mobile data: An economic analysis of the impact on users and content providers," in *Proc. of IEEE INFOCOM*. IEEE, 2015, pp. 1499–1507.

[2] S. Sen, C. Joe-Wong, S. Ha, and M. Chiang, "A survey of smart data pricing: Past proposals, current plans, and future trends," *ACM Computing Surveys*, vol. 46, no. 2, p. 15, 2013.

[3] ——, "Smart data pricing (SDP): Economic solutions to network congestion," in *Recent Advances in Networking*, H. Haddadi and O. Bonaventure, Eds. ACM SIGCOMM, 2013, pp. 221–274.

[4] ——, "Incentivizing time-shifting of data: A survey of time-dependent pricing for internet access," *IEEE Communications Magazine*, vol. 50, no. 11, 2012.

[5] ——, "Smart data pricing: Using economics to manage network congestion," *Communications of the ACM*, vol. 58, no. 12, pp. 86–93, 2015.

[6] D. Brake, "Mobile zero rating: The economics and innovation behind free data," in *Net Neutrality Reloaded: Zero Rating, Specialised Service, Ad Blocking and Traffic Management*, L. Belli, Ed., 2016, p. 132.

[7] J. Malcolm, C. McSherry, and K. Walsh, "Zero rating: What it is and why you should care," Electronic Frontier Foundation, 2016. [Online]. Available: https://www.eff.org/deeplinks/2016/02/zero-rating-what-it-is-why-you-should-care

[8] AT&T, "AT&T sponsored data," 2017. [Online]. Available: http://www.att.com/att/sponsoreddata/en/index.html#fbid=fNjHshoHkg_

[9] Verizon, "FreeBee data," 2017. [Online]. Available: http://freebee.verizonwireless.com/business/freebeedata

[10] J. Kastrenakes, "Republicans ready to take down the FCC," The Verge, 2017. [Online]. Available: http://www.theverge.com/2017/2/10/14577404/republicans-fcc-reform-modernization-remove-consumer-protections

[11] M. Reardon, "AT&T says 'sponsored data' does not violate net neutrality," CNet, 2014. [Online]. Available: http://www.cnet.com/news/at-t-says-sponsored-data-does-not-violate-net-neutrality/

[12] Apple, "iOS 7 and business," July 2014. [Online]. Available: https://www.apple.com/ios/business/

[13] S. M. Kerner, "Juniper debuts new per-app VPN tech," Enterprise Networking Planet, 2013. [Online]. Available: http://www.enterprisenetworkingplanet.com/netsecur/juniper-debuts-new-per-app-vpn-tech.html

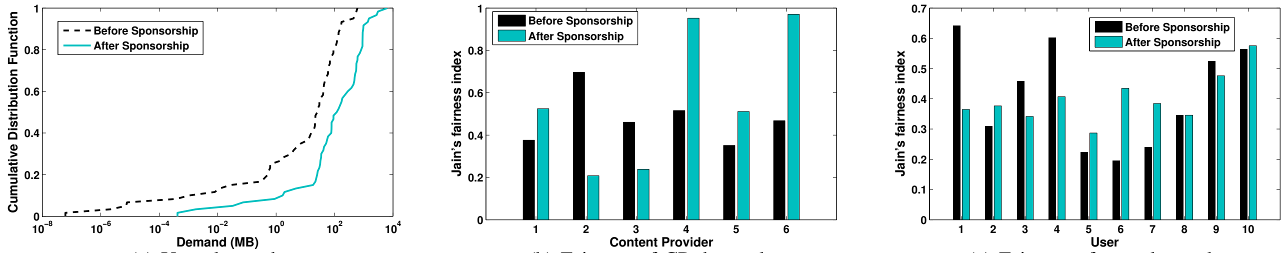
[14] L. Zhang and D. Wang, "Sponsoring content: Motivation and pitfalls for content service providers," in *Proc. of the SDP Workshop*. IEEE, 2014.

[15] App Annie, "App Annie & IDC mobile app advertising and monetization trends 2012-2017: The economics of free," 2014. [Online]. Available: http://inbound.appannie.com/app-annie-idc-mobile-app-advertising-and-monetization-trends-2012-2017

[16] Pandora, "SEC Form 10-K." [Online]. Available: http://investor.pandora.com/phoenix.zhtml?c=227956&p=proxy

[17] J. Constine, "Facebook beats in Q1 with $2.5B in revenue, 59% of ad revenue from mobile, 1.28B users," TechCrunch, 2014. [Online]. Available: http://techcrunch.com/2014/04/23/facebook-q1-2014-earnings/

[18] N. Economides and J. Tåg, "Network neutrality on the Internet: A two-sided market analysis," *Information Economics and Policy*, vol. 24, no. 2, pp. 91–104, 2012.



(a) User demands.

(b) Fairness of CP demands.

(c) Fairness of user demands.

Fig. 13: (a) Demand for all user-CP pairs and the resulting fairness of demands for (a) each CP and (b) each user (parameters from Figures 11 and 12). CP types from 1 to 6 are social networking (revenue CP), browsing, video, social networking (promotion CP), downloads, email.

[19] H. H. Gharakheili, A. Vishwanath, and V. Sivaraman, "An economic model for a new broadband ecosystem based on fast and slow lanes," *IEEE Network*, vol. 30, no. 2, pp. 26–31, 2016.

[20] M. H. Lotfi, K. Sundaresan, M. A. Khojastepour, and S. Rangarajan, "The economics of quality sponsored data in wireless networks," in *Proc. of WiOpt*. IEEE, 2015, pp. 395–402.

[21] J.-C. Rochet and J. Tirole, "Platform competition in two-sided markets," *J. of the European Economic Association*, vol. 1, no. 4, pp. 990–1029, 2003.

[22] P. Njoroge, A. Ozdaglar, N. E. Stier-Moses, and G. Y. Weintraub, "Investment in two-sided markets and the net neutrality debate," *Review of Network Economics*, vol. 12, no. 4, pp. 355–402, 2013.

[23] X. Wang, R. T. Ma, and Y. Xu, "On optimal two-sided pricing of congested networks," *Proc. of ACM SIGMETRICS*, vol. 1, no. 1, p. 7, 2017.

[24] E. Altman, J. Rojas, S. Wong, M. K. Hanawal, and Y. Xu, "Net neutrality and quality of service," in *Game Theory for Networks*. Springer, 2012, pp. 137–152.

[25] P. Hande, M. Chiang, R. Calderbank, and S. Rangan, "Network pricing and rate allocation with content provider participation," in *Proc. of IEEE INFOCOM*. IEEE, 2009, pp. 990–998.

[26] Y. Wu, H. Kim, P. H. Hande, M. Chiang, and D. H. Tsang, "Revenue sharing among ISPs in two-sided markets," in *Proc. of IEEE INFOCOM*. IEEE, 2011, pp. 596–600.

[27] L. Zheng, C. Joe-Wong, C. W. Tan, S. Ha, and M. Chiang, "Secondary markets for mobile data: Feasibility and benefits of traded data plans," in *Proc. of IEEE INFOCOM*. IEEE, 2015, pp. 1580–1588.

[28] S. Sen, C. Joe-Wong, and S. Ha, "The economics of shared data plans," in *Proc. of WITS*, 2012.

[29] M. Andrews, U. Ozen, M. I. Reiman, and Q. Wang, "Economic models of sponsored content in wireless networks with uncertain demand," in *Proc. of the SDP Workshop*. IEEE, 2013, pp. 3213–3218.

[30] M. Andrews, Y. Jin, and M. I. Reiman, "A truthful pricing mechanism for sponsored content in wireless networks," in *Proc. of IEEE INFOCOM*. IEEE, 2016, pp. 1–9.

[31] M. Andrews, "Implementing sponsored content in wireless data networks," in *Proc. of Allerton*. IEEE, 2013, pp. 1208–1212.

[32] R. Somogyi *et al.*, "The economics of zero-rating and net neutrality," Université catholique de Louvain, Center for Operations Research and Econometrics (CORE), Tech. Rep., 2016.

[33] S. Caron, G. Kesidis, and E. Altman, "Application neutrality and a paradox of side payments," in *Proc. of the Re-Arch Workshop*. ACM, 2010, p. 9.

[34] L. Zhang, W. Wu, and D. Wang, "Sponsored data plan: A two-class service model in wireless data networks," in *ACM SIGMETRICS Performance Evaluation Review*, vol. 43, no. 1. ACM, 2015, pp. 85–96.

[35] S. E. Ante and R. Knutson, "Good luck leaving your wireless phone plan," Wall Street Journal, July 2013. [Online]. Available: http://online.wsj.com/news/articles/SB10001424127887323971204578630202425653338

[36] AT&T, "Family cell phone plans," July 2014. [Online]. Available: http://www.att.com/shop/wireless/data-plans.html#fbid=bOFEaeX6Gbp

[37] Google, "Google fi data plan," 2017. [Online]. Available: https://fi.google.com/about/plan/

[38] A. Ghose, A. Goldfarb, and S. P. Han, "How is the mobile internet different? search costs and local activities," *Information Systems Research*, vol. 24, no. 3, pp. 613–631, 2012.

[39] L. Zheng and C. Joe-Wong, "Understanding rollover data," in *Proc. of the Fifth Workshop on Smart Data Pricing*. IEEE, 2016, pp. 662–667.

[40] P. Hande, M. Chiang, R. Calderbank, and J. Zhang, "Pricing under constraints in access networks: Revenue maximization and congestion management," in *Proc. of IEEE INFOCOM*. IEEE, 2010, pp. 1–9.

[41] T. Erdem and M. P. Keane, "Decision-making under uncertainty: Capturing dynamic brand choice processes in turbulent consumer goods markets," *Marketing science*, vol. 15, no. 1, pp. 1–20, 1996.

[42] Mobile Marketing Association, "Mobile advertising guidelines," 2011. [Online]. Available: http://www.mmaglobal.com/files/mobileadvertising.pdf

[43] B. Osman, "Digital advertising: Click-through rates," Coull.com, 2014. [Online]. Available: http://coull.com/our-blog/digital-advertising-click-through-rates

[44] J. Lee, "Mobile cost per click down for PPC in retail, business services," Search Engine Watch, 2014. [Online]. Available: http://searchenginewatch.com/article/2325359/Mobile-Cost-Per-Click-Down-for-PPC-in-Retail-Business-Services

[45] T. Lan, D. Kao, M. Chiang, and A. Sabharwal, "An axiomatic theory of fairness in network resource allocation," *Proc. of IEEE INFOCOM*, pp. 1–9, 2010.

[46] H. Kellerer, U. Pferschy, and D. Pisinger, *Knapsack Problems*. Springer, 2004.

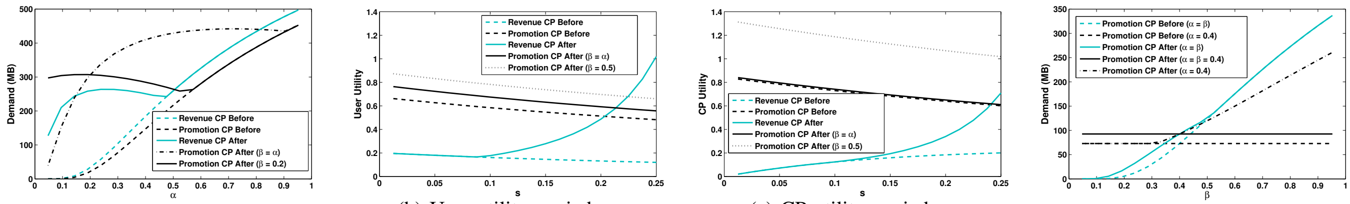
[47] T-Mobile, "Binge on," 2017. [Online]. Available: https://www.t-mobile.com/offer/binge-on-streaming-video.html

APPENDIX A
ADDITIONAL SIMULATIONS

In this appendix, we present numerical evidence that our conclusions in Section IV hold for $\beta_{i,j} \neq \alpha_{i,j}$ in the CP utility function (3). In particular, we plot analogues of Figures 7–9, as well as evaluating the fairness of the demand, user utility, and CP utility distributions when $\beta_{i,j} = 0.5$ is fixed (Table IV).

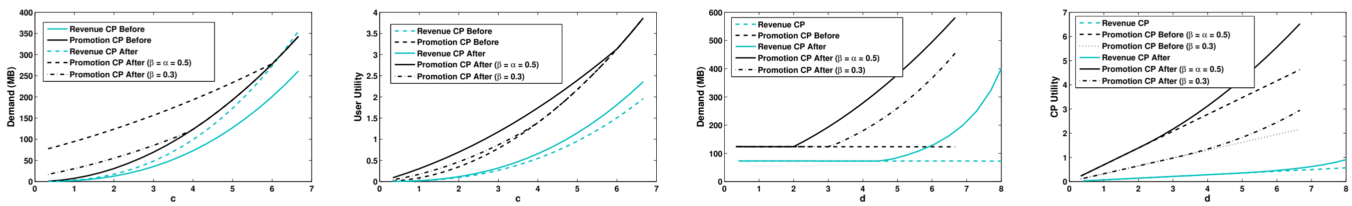
Figure 14 shows analogues of Figures 7 and 8 for $\beta_{i,j} \neq \alpha_{i,j}$ in the CP utility function (3). We see in Figure 14a that when $\beta_{i,j} = 0.2$, user demand can both increase and decrease with $\alpha_{i,j}$, just as it does when $\beta_{i,j} = \alpha_{i,j}$ in Figure 7b (Proposition 4). Similarly, when the amount of ads s_j decreases, user utility for promotion CPs decreases for $\beta_{i,j} = 0.4$ and for $\beta_{i,j} = \alpha_{i,j}$ in Figure 14b. We see a similar trend in CP utility in Figure 14c. Finally, we observe that as $\beta_{i,j}$ varies in Figure 14d, we see qualitatively similar user demands for $\beta_{i,j} = \alpha_{i,j}$ and $\alpha_{i,j}$ fixed at 0.4 as $\beta_{i,j}$ varies. We also plot demand when $\beta_{i,j} = \alpha_{i,j}$ is fixed at 0.4; the demand then does not change with $\beta_{i,j}$, but is of similar magnitude.

Figure 15 expands Figure 9's results for a fixed $\beta_{i,j} = 0.3$. User utility (Figure 15b) and CP utility (Figure 15d) show qualitatively similar trends as their cost awareness varies for both $\beta_{i,j} = \alpha_{i,j}$ and $\beta_{i,j} = 0.3$, as do the demands (Figures 15a and 15c). However, $\gamma^*_{i,j} = 0$, i.e., the CP does not sponsor content, for lower values of $c_{i,j}$ and higher values of $d_{i,j}$ with $\beta_{i,j} = 0.3$ compared to $\beta_{i,j} = \alpha_{i,j}$. Intuitively, for $\beta_{i,j} < \alpha_{i,j}$, the CP's utility function is less concave, so as



(a) User demand, varied $\alpha_{i,j}$. (b) User utility, varied s_j. (c) CP utility, varied s_j. (d) Demand, varied $\beta_{i,j}$.

Fig. 14: With sponsorship, (a) demand decreases as price elasticity decreases for users of a revenue ($c_{i,j} = 5.8$, $d_{i,j} = 5.4$) and promotion ($c_{i,j} = 5.3$, $d_{i,j} = 4.8$) CP; (b) user and (c) CP utility increases with ads for a user of revenue CPs ($\alpha_{i,j} = 0.3$, $c_{i,j} = 4$) and decreases for a user of promotion CPs ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$); (d) demand increases as CP price elasticity decreases for a user of revenue CPs ($\alpha_{i,j} = 0.3$, $c_{i,j} = 4$) or a user of promotion CPs ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$, $d_{i,j} = 2$).



(a) Demand, varied $c_{i,j}$. (b) User utility, varied $c_{i,j}$. (c) Demand, varied $d_{i,j}$. (d) CP utility, varied $d_{i,j}$.

Fig. 15: (a) User utility and (b) demand increases as user cost awareness decreases; similarly, (c) demand and (d) CP utility increases as CP cost awareness decreases. We use (a,b) one revenue ($\alpha_{i,j} = 0.4$) and one promotion ($\alpha_{i,j} = 0.5$, $d_{i,j} = 3$) CP; (c,d) one user of revenue ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$, $r_{i,j} \in [0.002, 0.03]$) and one of promotion CPs ($\alpha_{i,j} = 0.5$, $c_{i,j} = 4$, $d_{i,j} = 2$).

users' cost awareness shrinks, the CP benefits more from the resulting increase in demand and need not sponsor as much data. Similarly, as the CP's cost awareness decreases, a less concave utility function means that CPs already derive more utility from user demand and can sponsor less data.

As we vary $\alpha_{i,j}$ and hold $\beta_{i,j} = 0.5$ constant, the fairness of user demand and user utility increases with sponsorship, just as it does with $\beta_{i,j} = \alpha_{i,j}$[12]. Similarly, as user cost awareness $c_{i,j}$ varies, the distributions of user demand and utility become more fair, though to a lesser degree than for $\beta_{i,j} = \alpha_{i,j}$ (Proposition 9). As CP cost awareness varies, the distributions of demand and user utility both become more unfair for $\beta_{i,j} = \alpha_{i,j}$ and for $\beta_{i,j} = 0.3$. Before sponsorship, the fairness of CP utilities was 0.793 for $\beta_{i,j} = 0.3$, so we see that the distribution of CP utility becomes more unfair as well for both values of $\beta_{i,j}$ (Proposition 10).

Finally, in Figure 16, we show the ratios of CP to user utility before and after sponsorship for all simulations varying $\alpha_{i,j}$, s_j, $c_{i,j}$, and $d_{i,j}$ in Figures 14a–14c and 15. We see that for all types of CPs (revenue, promotion with $\beta_{i,j} = \alpha_{i,j}$, and promotion with $\beta_{i,j} \neq \alpha_{i,j}$), the utility ratio decreases with sponsorship (Proposition 7).

Appendix B
Sponsored Data with User Budgets

In this appendix, we consider a model in which users have a total budget cap of B amount of data per month, and sponsored data is not counted towards this cap. This model corresponds to a scenario in which users do not wish to exceed their data cap, e.g., if their data flows will be automatically throttled above a given cap, as is done by T-Mobile and AT&T.

[12]Note that we cannot directly compare the fairness of CP utilities with fixed $\beta_{i,j}$ after sponsorship with that before sponsorship, as the fairness before sponsorship depends on $\beta_{i,j}$.

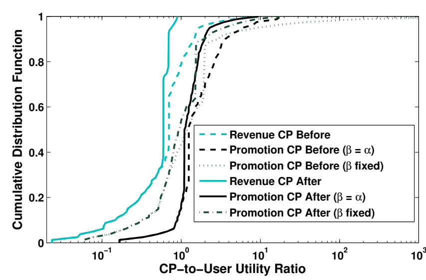


Fig. 16: Distribution of CP-to-user utility ratios over all CP-user pairs for all simulations in Figures 14a–14c and 15.

A. User Demands

Using the same utility model as in Section III-A, we can then express each user i's decision of how much data to consume at each CP in the following optimization problem:

$$\begin{aligned} \max_{x_{i,j}} \ & \frac{c_{i,j}\left(x_{i,j}\left(1 + r_{i,j}s_j\right)\right)^{1-\alpha_{i,j}}}{1-\alpha_{i,j}} \\ \text{s.t.} \ & \sum_j \left(1 - \gamma_{i,j} + s_j\right) x_{i,j} \leq B, \end{aligned} \tag{13}$$

where we have normalized the monetary units so that $p_u = 1$. We can then solve for user i's optimal demands:

Proposition 11: Each user i solves (13) at

$$x^*_{i,j} = \left(\frac{c_{i,j}^{\frac{1}{\alpha_{i,j}}}\left(1 - \gamma_{i,j} + s_j\right)^{\frac{-1}{\alpha_{i,j}}}}{1 + r_{i,j}s_j}\right) \times \frac{B}{\sum_k c_{i,j}^{\frac{1}{\alpha_{i,j}}}\left(1 + r_{i,k}s_k\right)^{-1}\left(1 - \gamma_{i,k} + s_k\right)^{1-\frac{1}{\alpha_{i,k}}}}. \tag{14}$$

We might expect that with this user model, a CP j that sponsors some of its data might not only increase its own demand, $x^*_{i,j}$, but also user i's demand for data at other CPs: since the data $\gamma_{i,j}$ no longer counts towards the user's total budget B, the user can put this "saved" data towards consuming data on all CPs, not just on CP j. However, we find from Proposition 11 that this substitution does not occur:

Corollary 5: User i's optimal demand (14) for CP j is increasing in $\gamma_{i,j}$ but decreasing in $\gamma_{i,k}$ for $k \neq j$. Moreover, if $\alpha > 1/5$, $x^*_{i,j}$ is a convex function of $\gamma_{i,j}$.

Given this characterization of user demand, we can now turn to CP' optimal sponsorship decisions.

B. CP Sponsorship

Just as in Section III-B, each CP now chooses the optimal amount of data to sponsor so as to maximize its utility from all users $i = 1, \ldots, N$:

$$W_{i,j}(\gamma_{i,j}) = \frac{d_{i,j} {x^*}_{i,j}^{1-\beta_{i,j}}}{1-\beta_{i,j}} - p_c \gamma_{i,j} {x^*}_{i,j}, \tag{15}$$

where $x^*_{i,j}$ is given by (14). We note that, since users' demands (14) depend on *all* CPs' sponsorship decisions, each CP j's optimal sponsorship decision for user i depends on other CPs' sponsorship decisions. However, since we assume all CPs act independently, we take all other CPs' decisions as given and consider only CP j's decision. We make the assumption that $d_{i,j} < p_c(1+s_j)$ for all users i to be consistent with Proposition 1, which makes the same assumption, and we further assume that no CP sponsors all $1+s_j$ amount of its data, as in Proposition 12. We then find conditions under which a revenue CP j sponsors a nonzero amount of data:

Proposition 12: A revenue CP j sponsors a nonzero amount of data $\gamma^*_{i,j}$ for user i if

$$p_c > \frac{d_{i,j}\left(B + (1+s_j)(\alpha_{i,j}-1)x^*_{i,j}(0)\right)}{\alpha_{i,j}(1+s_j)B}. \tag{16}$$

Under this condition, the optimal $\gamma^*_{i,j}$ satisfies

$$p_c = \frac{d_{i,j}\left(B + (1-\gamma_{i,j}+s_j)(\alpha_{i,j}-1)x^*_{i,j}\right)}{(1-\gamma_{i,j}+s_j)\left(\alpha_{i,j}B + \gamma_{i,j}(\alpha_{i,j}-1)x^*_{i,j}\right) + \gamma_{i,j}B}. \tag{17}$$

However, $\gamma^*_{i,j} < 1+s_j$: CP j never sponsors all data.

A revenue CP's optimal sponsorship decision under this user model is thus analogous to that given in Proposition 1 and Corollary 1: a CP sponsors data for user i if and only if its marginal utility of doing so, $d_{i,j}$, is sufficiently large compared to the cost p_c. However, no revenue CP sponsors all $1+s_j$ fraction of its data.

Promotion CPs' optimal decisions are similarly analogous to their decisions when users experience a linear cost of consuming data, as given in Proposition 2:

Proposition 13: A promotion CP j sponsors a nonzero amount of data $\gamma^*_{i,j} > 0$ if

$$p_c > \frac{d_{i,j}x^*_{i,j}(0)^{-\alpha_{i,j}}\left(B + (1+s_j)(\alpha_{i,j}-1)x^*_{i,j}(0)\right)}{\alpha_{i,j}(1+s_j)B}. \tag{18}$$

If (18) holds, then $\gamma^*_{i,j} < 1+s_j$, and $\gamma^*_{i,j}$ satisfies

$$p_c = \frac{d_{i,j}{x^*}_{i,j}^{-\beta_{i,j}}\left(B + (1-\gamma_{i,j}+s_j)(\alpha_{i,j}-1)x^*_{i,j}\right)}{(1-\gamma_{i,j}+s_j)\left(\alpha_{i,j}B + \gamma_{i,j}(\alpha_{i,j}-1)x^*_{i,j}\right) + \gamma_{i,j}B}. \tag{19}$$

C. User and CP Benefits from Sponsored Data

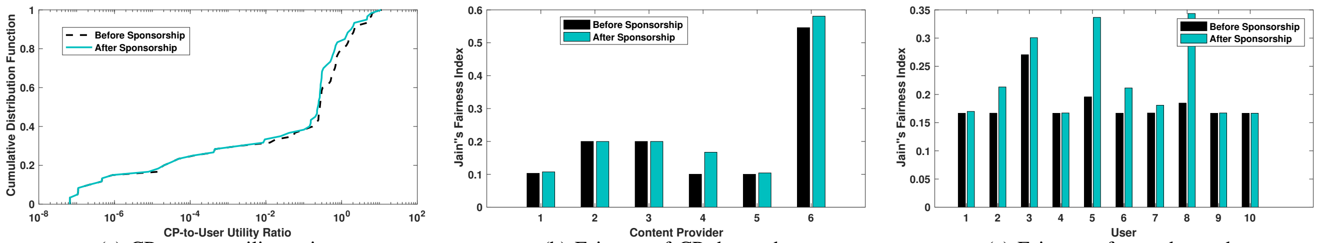
We evaluate the distribution of CP and user benefits from sponsored data by solving for users' and CPs' optimal behavior as characterized in Propositions 11–13. We use the parameters estimated from our pricing trial as in Section V-A and find qualitatively similar results to those in Section V-A. Users overall benefit more than CPs from sponsoring data. However, with realistic variation in CP and user utility parameters, we find that the distribution of demands for one user across multiple CPs becomes more fair with optimal sponsorship, again alleviating network neutrality concerns.

Figure 17 summarizes the results of our simulation. From Figure 17a, we observe that the CP-to-user utility ratios decrease with sponsorship, which parallels our result for linear data costs in Proposition 7 and Figure 12b. Thus, users benefit proportionally more than CPs, though the change in CP-to-user utility ratios is relatively small. CPs may have less incentive to sponsor data when users experience a budget constraint, since the increase in demand due to their sponsorship can be undermined by other CPs' sponsorship decisions.

We compare the distributions of CP and user demands in Figures 17b and 17c respectively. While the fairness indices generally take lower values than those in Figures 13b and 13c for linear data costs, we find that they generally increase with sponsorship. Four out of six CPs experience less disparate user demands (higher fairness index), indicating that more cost-sensitive users realize a disproportionate increase in demand compared to less cost-sensitive users. The results for CP demands are even more striking: all users exhibit less disparate demands for different CPs with sponsorship. Thus, unlike in the case of linear data costs, CPs are able to leverage sponsorship to even out disparities in their experienced demands. When users have a budget constraint, sponsorship by more cost-sensitive CPs not only increases these CPs' demands, but also lowers user demand at less cost-sensitive CPs, further reducing demand disparities. These results thus suggest that concerns over larger CPs benefiting more from sponsored data may be misplaced: in some realistic scenarios, sponsored data not only disproportionately benefits users, but can also reduce disparities in the demands experienced by different CPs.

Appendix C
Optimal Advertising Levels

In this appendix, we consider the effect of each CP j optimizing not only the fraction of data sponsored $\gamma_{i,j}$, but also the advertisement level s_j. As is consistent with industry practice, we suppose that the advertisement level s_j is upper-bounded by a given parameter $\overline{s}$ [42]. We then find that the optimal sponsorship levels are as follows:



(a) CP-to-user utility ratios. (b) Fairness of CP demands. (c) Fairness of user demands.

Fig. 17: (a) CP-to-user utility ratios, and (b,c) Jain's fairness index values for CP and user utilities before and after sponsorship, when users experience a budget constraint instead of linear data costs. We observe qualitatively similar results as in the linear data cost case (Figures 12 and 13). User and CP parameters are as in Figure 11. We assume users have a fixed budget $B = p_u$.

Proposition 14 (Optimal advertisement levels): Consider a revenue CP j with $d_{i,j} = ar_{i,j}s_j$, as in Section III-B1. Then the optimal (i.e., utility-maximizing) advertisement level for a revenue CP with optimal sponsorship is s_j^* is $s_j^* = 0$ if $ar_{i,j} \leq p_c(1 - s_j)$, and $s_j^* = \overline{s}$ otherwise.

If the CP is a promotion CP, then the CP-utility-maximizing ad level with optimal sponsorship is $s_j^* = 0$.

Revenue CPs would thus choose the maximum advertisement level, since $ar_{i,j} \geq p_c \geq p_c(1 - s_j)$, e.g., as in the numerical example from Section III-B1. Thus, revenue and promotion CPs would usually offer the same amount of advertisements, as is generally done in practice [42]. We further observe that users', CPs', and ISPs' optimal actions, as derived in Propositions 1–3, still hold when s_j is optimized, since they hold for any value of s_j. We can similarly show that Proposition 4, analyzing the change in user demand in response to changes in price elasticity, still holds as well.

To assess our remaining results in the case of optimal sponsorship, we first consider Proposition 7, i.e., that users benefit proportionally more than CPs with sponsorship. We can show that Proposition 7 continues to hold when the CP's optimal advertisement level changes with sponsorship as in Proposition 14. Similarly, Corollaries 3 and 4 still hold, as does Proposition 9. Thus, in most cases users benefit more from sponsorship than do CPs, and this disproportionate benefit is larger for more cost-sensitive users and less cost-sensitive CPs. Sponsorship also leads to a distribution of utilities that is more fair across user cost sensitivities.

If $s_j^* \leq s_j$, i.e., the CP offers fewer advertisements with than without sponsorship, then sponsorship also leads to a distribution of utilities that is less fair across CP cost sensitivities, analogous to Proposition 10:

Proposition 15 (Fairness across CP cost sensitivity with optimal advertisement): Consider a set of homogeneous users and CPs varying only in their cost sensitivity $d_{i,j}^{-1}$. Then the demand distribution across different CP cost sensitivities becomes less fair with sponsorship ($F\left(\left\{x_{i,j}^*\right\}\right)$ decreases). If the CPs are revenue CPs or promotion CPs with $s_j^* \leq s_j$, then the CP utility distributions across different CP cost sensitivities become less fair ($F\left(\{W_{i,j}\}\right)$ decreases) with sponsorship.

From Proposition 14, $s_j^* = 0$ for all promotion CPs, so the conditions in Proposition 15 will hold in practice.

Appendix D
Proofs

A. Proof of Proposition 1

We first note that since (5–6) is additively separable with respect to the $\gamma_{i,j}$ variables, we can simply choose each $\gamma_{i,j}$ so as to maximize $W_{i,j}\left(\gamma_{i,j}\right)$. Moreover, $W_{i,j}$ is maximized at one of three possible values: $\gamma_{i,j} = 0$, $\gamma_{i,j} = 1 + s_j$, or a critical point of $W_{i,j}(\gamma_{i,j})$. Since we assume that $d_{i,j} < p_c(1+s_j)$, taking $\gamma_{i,j} = 1+s_j$ yields a negative profit for the CP. Thus, either the optimal value of $\gamma_{i,j}$, which we denote as $\gamma_{i,j}^*$, is either 0 or a critical point of $W_{i,j}$.

We find a unique critical point of $W_{i,j}(\gamma_{i,j})$ by taking the first derivative of (5) and setting it equal to zero:

$$p_u \left(\frac{d_{i,j} - p_c\gamma_{i,j}}{(1 + r_{i,j}s_j)}\right)^{\frac{1}{\alpha_{i,j}}-1} \left(p_u\left(1 - \gamma_{i,j} + s_j\right)\right)^{\frac{-1}{\alpha_{i,j}}-1} - \alpha_{i,j}\left(\frac{p_c}{(1 + r_{i,j}s_j)}\right)^{\frac{1}{\alpha_{i,j}}-1} \left(p_u\left(1 - \gamma_{i,j} + s_j\right)\right)^{\frac{-1}{\alpha_{i,j}}} = 0. \tag{20}$$

We then solve (20) for

$$\gamma_{i,j}^* = \frac{d_{i,j}}{p_c(1 - \alpha_{i,j})} - \frac{\alpha_{i,j}(1 + s_j)}{1 - \alpha_{i,j}}. \tag{21}$$

Note that (21) satisfies the constraint $\gamma_{i,j}^* \leq 1+s_j$ if and only if $d_{i,j} \leq 1 + s_j$, which is true by assumption.

We now show that taking $\gamma_{i,j}^* = 0$ yields a higher CP profit than (21) if and only if (21) is negative, i.e., $d_{i,j} < \alpha_{i,j}p_c(1+s_j)$. We find the ratio of CP utility with $\gamma_{i,j}^*$ as in (21) to that with $\gamma_{i,j}^* = 0$ to be

$$\sigma_{i,j} = \frac{\alpha_{i,j}}{1 - \alpha_{i,j}}\left(\mu_{i,j} - 1\right)\left(\frac{1}{1 - \alpha_{i,j}} - \frac{1}{\mu_{i,j}\left(1 - \alpha_{i,j}\right)}\right)^{\frac{-1}{\alpha_{i,j}}}$$

where for notational convenience we define $\mu_{i,j} = p_c(1 + s_j)/\left(d_{i,j}\right) > 1$. Setting $\sigma_{i,j} \geq 1$, we find the equivalent condition

$$\left(\mu_{i,j} - 1\right)^{1 - \frac{1}{\alpha_{i,j}}} \geq \frac{\mu_{i,j}^{\frac{-1}{\alpha_{i,j}}}}{\alpha_{i,j}}\left(1 - \alpha_{i,j}\right)^{1 - \frac{1}{\alpha_{i,j}}}.$$

Raising both sides to the power $\alpha_{i,j}/(\alpha_{i,j} - 1)$, we find the equation

$$\alpha_{i,j}^{\frac{\alpha_{i,j}}{1-\alpha_{i,j}}}\left(1 - \alpha_{i,j}\right)\mu_{i,j}^{\frac{1}{1-\alpha_{i,j}}} - \mu_{i,j} + 1 \geq 0.$$

At $\mu_{i,j} = \alpha_{i,j}^{-1}$, we see by inspection that this inequality is satisfied with equality. Taking the derivative with respect to $\mu_{i,j}$, we find the expression

$$\alpha_{i,j}^{\frac{\alpha_{i,j}}{1-\alpha_{i,j}}} \mu_{i,j}^{\frac{\alpha_{i,j}}{1-\alpha_{i,j}}} - 1,$$

which is positive if and only if $\mu_{i,j} \geq \alpha_{i,j}^{-1}$. Thus, $\sigma_{i,j} \geq 1$ and $\gamma_{i,j} = 0$ yields lower profit than (21) for all $\mu_{i,j} \geq \alpha_{i,j}^{-1}$, or equivalently, $\alpha_{i,j} p_c(1 + s_j) \geq d_{i,j}$. ■

B. Proof of Proposition 2

As in Proposition 1, we first observe that it suffices to consider the $\gamma_{i,j}$ separately, and that the optimal value of each $\gamma_{i,j}$ is either 0, $1 + s_j$, or a critical point of $W_{i,j}(\gamma_{i,j})$. We now observe that

$$\begin{aligned}\lim_{\gamma_{i,j}\to 1+s_j} W_{i,j}(\gamma_{i,j}) =& \frac{d_{i,j}}{1-\beta_{i,j}} \left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{\beta_{i,j}-1}{\alpha_{i,j}}} \\ &- p_c\gamma_{i,j}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}} \\ &\to -\infty.\end{aligned}$$

Thus, we first find the unique critical point of $W_{i,j}$ and then derive conditions under which this critical point yields greater CP utility than taking $\gamma_{i,j} = 0$. For simplicity, we first define the constants $C_1 = \frac{d_{i,j}}{1-\beta_{i,j}}\left(\frac{p_u}{c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{1-\frac{1}{\alpha_{i,j}}}$ and $C_2 = \frac{p_c p_u^{\frac{-1}{\alpha_{i,j}}}}{\left(c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}}}$. We then take the derivative of $W_{i,j}$ and find

$$\begin{aligned}\frac{dW_{i,j}}{d\gamma_{i,j}} =& \left(\frac{1-\beta_{i,j}}{d_{i,j}}\right) C_1 \left(1-\gamma_{i,j}+s_j\right)^{\frac{\beta-\alpha_{i,j}-1}{\alpha_{i,j}}} \\ &- C_2\left(1-\gamma_{i,j}+s_j\right)^{\frac{-1}{\alpha}} - \frac{C_2\gamma_{i,j}}{\alpha_{i,j}}\left(1-\gamma_{i,j}+s_j\right)^{\frac{-1}{\alpha_{i,j}}-1}\end{aligned}$$

After multiplying through by common factors, we find the equation

$$\gamma_{i,j}\left(1-\frac{1}{\alpha_{i,j}}\right)(C_1+C_2) = \left(1-\frac{1}{\alpha_{i,j}}\right)C_1(1+s_j)+C_2(1+s_j),$$

which simplifies to

$$\gamma_{i,j}^* = \frac{d_{i,j}(1+s_j)p_u - c_{i,j}\alpha_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}p_c(1+s_j)}{d_{i,j}p_u + (1-\alpha_{i,j})p_c c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}. \tag{22}$$

It is easy to see that $\gamma_{i,j}^* < 1 + s_j$.

We now observe that, using similar arguments as above, $dW_{i,j}/d\gamma_{i,j}$ is negative for $\gamma_{i,j} < \gamma_{i,j}^*$ and otherwise positive. Thus, $\gamma_{i,j}^*$ in (22) represents the optimal value of $\gamma_{i,j}$ unless it is negative, in which case the optimal amount sponsored is $\gamma_{i,j} = 0$. ■

C. Proof of Proposition 3

We first note that since each CP optimally chooses $\gamma_{i,j}$, $dW_{i,j}(\gamma_{i,j})/d\gamma_{i,j}|_{\gamma_{i,j}^*}$ must equal zero if $\gamma_{i,j} \in (0, 1+s_j)$.

$$\begin{aligned}\frac{dW_{i,j}}{d\gamma_{i,j}} \propto& \left(\frac{1-\beta_{i,j}}{\alpha_{i,j}}\right) C_1\left(1-\gamma_{i,j}+s_j\right)^{\frac{\beta}{\alpha}} \\ &- C_2 p_c\left(1+\gamma_{i,j}\left(\frac{1}{\alpha_{i,j}}-1\right)-s_j\right),\end{aligned} \tag{23}$$

where we define the constants

$$C_1 = \frac{d_{i,j}}{1-\beta_{i,j}}\left(\frac{p_u}{c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{1-\frac{1}{\alpha_{i,j}}}$$

$$C_2 = \frac{p_u^{\frac{-1}{\alpha_{i,j}}}}{\left(c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}}}.$$

Thus, by taking $\gamma_{i,j}^*$ as a function of p_c and taking the total derivative of (23) with respect to p_c, we find that

$$\frac{d\gamma_{i,j}^*}{dp_c} = \frac{C_2\left(1+s_j+\left(\frac{1}{\alpha_{i,j}}-1\right)\gamma_{i,j}\right)}{\frac{\beta_{i,j}(\beta_{i,j}-1)}{\alpha_{i,j}^2}C_1\left(1-\gamma_{i,j}+s_j\right)^{\frac{\beta_{i,j}}{\alpha_{i,j}}-1} + C_2p_c\left(1-\frac{1}{\alpha_{i,j}}\right)},$$

which is < 0. We now find that

$$\begin{aligned}\frac{dx_{i,j}^*}{dp_c} =& \frac{1}{\alpha_{i,j}}\left(\frac{p_u}{c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}} \\ &\times\left(1-\gamma_{i,j}^*+s_j\right)^{\frac{-1}{\alpha_{i,j}}-1}\frac{d\gamma_{i,j}^*}{dp_c} < 0.\end{aligned}$$

We now define $R_{i,j} = \left(\pi_{i,j}^* + p_c\gamma_{i,j}^*\right)x_{i,j}^*$ and compute

$$\begin{aligned}\frac{dR_{i,j}}{dp_c} =& \left({x_{i,j}^*}^{-\alpha_{i,j}}c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}} + p_c\gamma_{i,j}^*\right)\frac{\partial x_{i,j}^*}{\partial p_c} \\ &+ \gamma_{i,j}^*x_{i,j}^* + (p_c-p_u)x_{i,j}^*\frac{\partial\gamma_{i,j}^*}{\partial p_c} \end{aligned} \tag{24}$$

$$\begin{aligned}&\propto \gamma_{i,j}^* + \frac{p_u}{\alpha_{i,j}}\frac{d\gamma_{i,j}^*}{dp_c} \\ &\left(\frac{p_c\gamma_{i,j}^*p_u{x_{i,j}^*}^{\alpha_{i,j}}}{\alpha_{i,j}c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}} + p_c - p_u\right)\frac{d\gamma_{i,j}^*}{dp_c} \\ &\propto \left(p_c + p_u\left(\frac{1}{\alpha_{i,j}}-1\right) + \frac{p_cp_u\gamma_{i,j}^*{x_{i,j}^*}^{\alpha_{i,j}}}{\alpha_{i,j}c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right) \\ &\times\left(1+s_{i,j}+\left(\frac{1}{\alpha_{i,j}}-1\right)\gamma_{i,j}^*\right) + \gamma_{i,j}^*p_c\left(1-\frac{1}{\alpha_{i,j}}\right) \\ &- \frac{\beta_{i,j}d_{i,j}\gamma_{i,j}^*}{\alpha_{i,j}^2}{x_{i,j}^*}^{\alpha_{i,j}-\beta_{i,j}}\left(\frac{p_u}{c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right).\end{aligned} \tag{25}$$

We now use the fact that (23) equals zero at $\gamma_{i,j}^*$ to solve for

$$x_{i,j}^*d_{i,j} = \alpha_{i,j}p_c\left(1+s_j+\left(\frac{1}{\alpha_{i,j}}-1\right)\gamma_{i,j}^*\right),$$

and substituting into (25), we find

$$\begin{aligned}\frac{dR_{i,j}}{dp_c} \propto& -\alpha_{i,j}(1+s_j)p_c - \left(1+s_j+\left(\frac{1}{\alpha_{i,j}}-1\right)\gamma_{i,j}^*\right) \\ &\times\left(p_u\left(1-\alpha_{i,j}\right) + (1-\beta_{i,j})\frac{p_cp_u\gamma_{i,j}^*{x_{i,j}^*}^{\alpha_{i,j}}}{c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right),\end{aligned}$$

which is < 0. If (23) cannot be exactly satisfied by $\gamma \in [0, 1 + s_j]$, then either $\gamma_{i,j} = 0$ or $\gamma_{i,j} = 1 + s_j$. We first suppose that $\gamma_{i,j}^* = 0$; it is easy to see by inspection that $x_{i,j}^*$ is then independent of p_c, as is $R_{i,j} = \pi_{i,j}^* x_{i,j}^* + p_c \gamma_{i,j} x_{i,j}^*$. If $\gamma_{i,j}^* = 1 + s_j$, then $\pi_{i,j}^* = 0$ and $x_{i,j}^* \to \infty$, which violates the constraint (10).

We now see that for a given p_u, since (9) is decreasing in p_c, the optimal value of p_c is the minimal one for which the constraints are all satisfied. Neglecting the constraints $d_{i,j} < p_c(1 + s_j)$ from revenue CPs, this optimal value of p_c is the unique one for which (10) is satisfied with equality: for all higher values of p_c, (10) is satisfied without equality since $x_{i,j}^*$ is decreasing in p_c. If we include the constraints from revenue CPs, we may need to increase p_c until all constraints are satisfied, and one is satisfied with equality.

It remains to show that we can perform a line search for the optimal value of p_u on a bounded range. Clearly, we have a lower bound of $p_u \geq 0$. We see that there is an upper bound by noting that $\gamma_{i,j}^*$ is increasing with p_u if (23) equals zero:

$$\begin{aligned}\frac{d\gamma_{i,j}^*}{dp_u} =& d_{i,j}\beta_{i,j} {x_{i,j}^*}^{\frac{\beta_{i,j}}{\alpha_{i,j}}} \Bigg/ \left(d_{i,j}\beta_{i,j} {x_{i,j}^*}^{\frac{\beta_{i,j}}{\alpha_{i,j}}-1} \frac{p_u^2}{c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}}} \right. \\ &\left. + \left(\frac{1}{\alpha_{i,j}} - 1\right) p_u p_c \alpha_{i,j} \left(1 + s_j + \left(\frac{1}{\alpha_{i,j}} - 1\right)\gamma_{i,j}^*\right)\right)\end{aligned}$$

Since $\gamma_{i,j}^*$ is upper-bounded by $1 + s_j$, $\gamma_{i,j}^*(p_u)$ must converge as $p_u \to \infty$. As $\gamma_{i,j}^*$ converges, we then have

$$\begin{aligned}\frac{dR_{i,j}}{dp_u} =& \left((1 - \alpha_{i,j})c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}} {x_{i,j}^*}^{-\alpha_{i,j}} + p_c\gamma_{i,j}^*\right) \\ & \times \frac{\partial x_{i,j}^*}{\partial p_u} + p_c x_{i,j}^* \frac{\partial \gamma_{i,j}^*}{\partial p_u} \\ \to& \left((1 - \alpha_{i,j})c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}} {x_{i,j}^*}^{-\alpha_{i,j}} + p_c\gamma_{i,j}^*\right) \\ & \times \frac{{x_{i,j}^*}^{1+\alpha_{i,j}} \left(\gamma_{i,j}^* - 1 - s_j\right)}{c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}}\alpha_{i,j}} \\ <& 0,\end{aligned}$$

since

$$\begin{aligned}\frac{\partial x_{i,j}^*}{\partial p_u} =& \frac{1}{\alpha_{i,j}} \left(\frac{p_u(1 - \gamma_{i,j} + s_j)}{c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}-1} \\ & \times \left(\frac{\gamma_{i,j} - 1 - s_j + p_u\frac{\partial\gamma_{i,j}}{\partial p_u}}{c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}}}\right). \quad \blacksquare\end{aligned}$$

D. Proof of Proposition 4

The first part of the proposition follows directly from observing that $x_{i,j}^* \left(p_u(1 + \gamma_{i,j} - s_j)\right) = c_{i,j}^{1/\alpha_{i,j}}/(1 + r_{i,j}s_j) \left(p_u\left(1 - \gamma_{i,j} + s_j\right)\right)^{-1/\alpha_{i,j}}$.

We now consider promotion CPs and compute

$$x_{i,j}^*(\pi_{i,j}^*) = \left(\frac{d_{i,j}}{p_c(1 + s_j)} + \frac{(1 - \alpha_{i,j})c_{i,j}\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}}{p_u(1 + s_j)}\right)^{\frac{1}{\alpha}} \tag{26}$$

By inspection, $x_{i,j}^*$ decreases when $\alpha_{i,j}$ increases as long as $d_{i,j}p_u + (1 - \alpha_{i,j}c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}}p_c \geq p_c p_u(1 + s_j)$, which holds for all α if it holds at $\alpha = 0$. Thus, we obtain the condition $d_{i,j} \geq p_c(1 + s_j)$, which always holds under the condition stated in the proposition.

Finally, we consider revenue CPs when $\gamma_{i,j}^* > 0$ and compute

$$x_{i,j}^*(\pi_{i,j}^*) = \left(\frac{p_u c_{i,j}(1 - \alpha_{i,j})(1 + r_{i,j}s_j)^{\alpha_{i,j}}}{p_u\left(p_c(1 + s_j) - d_{i,j}\right)}\right)^{\frac{1}{\alpha_{i,j}}}. \tag{27}$$

We then find that whenever $p_u c_{i,j}(1 - \alpha_{i,j})(1 + r_{i,j}s_j)^{\alpha_{i,j}} \geq p_u\left(p_c(1 + s_j) - d_{i,j}\right)$, $x_{i,j}^*$ decreases when $\alpha_{i,j}$ increases. Thus, we find the condition $p_u d_{i,j} \geq p_c\left(c_{i,j}(1 + r_{i,j}s_j) + p_u(1 + s_j)\right)$.

E. Proof of Proposition 5

We first find that user i's utility for a revenue CP j, $V_{i,j}^*$, is

$$\frac{\alpha_{i,j}c_{i,j}^{\frac{1}{\alpha_{i,j}}}}{1 - \alpha_{i,j}} \left(\frac{p_u\left(p_c(1 + s_j) - ar_{i,j}s_j\right)}{p_c(1 - \alpha_{i,j})(1 + r_{i,j}s_j)}\right)^{1-\frac{1}{\alpha_{i,j}}} \tag{28}$$

when $\gamma_{i,j}^* > 0$. It thus suffices to show that,

$$\frac{\partial}{\partial s_j}\left(\frac{p_u\left(p_c(1 + s_j) - ar_{i,j}s_j\right)}{p_c(1 - \alpha_{i,j})(1 + r_{i,j}s_j)}\right) < 0,$$

is increasing in s_j if $ar_{i,j} > p_c(1 - r_{i,j})$. Equivalently, we must show that

$$p_c - ar_{i,j} - r_{i,j}p_c < 0,$$

which occurs exactly under the condition stated in the proposition. $\blacksquare$

F. Proof of Lemma 1

Suppose that CP j sponsors $\gamma_{i,j}$ amount of content for user i. Then users' utility is

$$\left(\frac{p_u(1 - \gamma_{i,j} + s_j)}{1 + r_{i,j}s_j}\right)^{1-\frac{1}{\alpha_{i,j}}} c_{i,j}^{\frac{1}{\alpha}} \frac{\alpha}{1 - \alpha} \tag{29}$$

Since $\alpha_{i,j} < 1$, we see that (29) is increasing with $\gamma_{i,j}$. Thus, the user will always benefit if a CP decides to sponsor some data. $\blacksquare$

G. Proof of Proposition 6

We first show that ISPs benefit relative to before sponsorship. Suppose that the ISP chooses $p_u^* = p_c^*$ to be the same as p_u before sponsorship. Then the ISP price per unit of content is independent of the $\gamma_{i,j}^*$. Moreover, since user demand can only increase with sponsorship (Proposition 4), ISP revenue can only increase. If the constraint on total demand $\sum_{i,j} x_{i,j}^* \leq X$ does not hold with $p_u^* = p_c^*$ equal to the price before sponsorship, then the ISP can increase p_c until it holds. Its marginal price $p_u(1 + s_j) + \gamma_{i,j}^*\left(p_c - p_u\right)$ must then increase relative to that before sponsorship, while its demand is the maximum possible and therefore at least that before sponsorship; its total revenue is then at least that before sponsorship. Without the constraint $p_u^* = p_c^*$, ISPs have more flexibility to choose their prices, so their revenue can only increase further.

We now consider users and CPs. If p_u^* decreases relative to that before sponsorship, users' effective data prices $p_u(1-\gamma_{i,j}^*+s_j)$ must decrease (for any $\gamma_{i,j}^*>0$). Lemma 1 then shows the result for end users. Finally, we show that CPs benefit since they optimally choose $\gamma_{i,j}^*$. Thus, their utilities must be at least as large as that obtained by choosing $\gamma_{i,j}=0$, i.e., no sponsorship. If CPs choose not to sponsor data, their cost of sponsorship is zero, as it is before sponsorship. However, the data component $\overline{U}_{i,j}$ of their utility function (3) increases since user demand increases due to lower p_u^*. Thus, CP utility also increases relative to that without sponsorship. ■

H. Proof of Proposition 7

We first compute promotion CP utility before sponsorship, which is

$$\frac{d_{i,j}}{1-\alpha_{i,j}}\left(\frac{p_u(1+s_j)}{c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}}\right)^{1-\frac{1}{\alpha_{i,j}}}.$$

With sponsorship, promotion CP utility $W_{i,j}\left(\gamma_{i,j}^*\right)$ is

$$\left(\frac{p_u p_c(1+s_j)}{d_{i,j}p_u+(1-\alpha_{i,j})p_c c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}}\right)^{1-\frac{1}{\alpha_{i,j}}} \times\left(\frac{d_{i,j}\alpha_{i,j}}{1-\alpha_{i,j}}+\frac{c_{i,j}\alpha_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}p_c}{p_u}\right).$$

We can then compute the desired ratios $R_{i,j}^b$ and $R_{i,j}^a$ upon noting that user utility without sponsorship is

$$\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\left(p_u(1+s_j)\right)^{1-\frac{1}{\alpha_{i,j}}}\left(c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}\right)^{\frac{1}{\alpha_{i,j}}}, \tag{30}$$

and that with sponsorship is

$$\frac{\alpha_{i,j}c_{i,j}\left(1+r_{i,j}s_j^*\right)^{1-\alpha_{i,j}}}{1-\alpha_{i,j}} \times\left(\frac{p_u^*p_c(1+s_j^*)}{d_{i,j}p_u^*+(1-\alpha_{i,j})p_c c_{i,j}\left(1+r_{i,j}s_j^*\right)^{1-\alpha_{i,j}}}\right)^{1-\frac{1}{\alpha_{i,j}}}. \tag{31}$$

Similarly, we find that revenue CP utility before sponsorship is $d_{i,j}\left(p_u(1+s_j)/\left(c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}\right)\right)^{\frac{-1}{\alpha_{i,j}}}$, while that after sponsorship is given by

$$\alpha_{i,j}\left(\frac{p_u^*}{p_c c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}}\left(\frac{p_c(1+s_j^*)-d_{i,j}}{(1-\alpha_{i,j})\left(1+r_{i,j}s_j^*\right)}\right)^{1-\frac{1}{\alpha_{i,j}}}.$$

We can then compute the desired ratios with user utility, which after sponsorship is

$$\frac{\alpha_{i,j}c_{i,j}^{\frac{1}{\alpha_{i,j}}}}{1-\alpha_{i,j}}\left(\frac{p_u^*\left(p_c(1+s_j^*)-d_{i,j}\right)}{p_c(1-\alpha_{i,j})(1+r_{i,j}s_j^*)}\right)^{1-\frac{1}{\alpha_{i,j}}}.$$

To see that the CP-to-user utility ratio for revenue CPs always decreases with sponsorship, we set

$$\frac{(1-\alpha_{i,j})d_{i,j}}{\alpha_{i,j}p_u\left(1+s_j\right)}\geq\frac{p_c\left(1-\alpha_{i,j}\right)}{p_u^*},$$

or equivalently $d_{i,j}\geq p_c p_u\alpha_{i,j}(1+s_j)/p_u^*$, which is always the case when $\gamma_{i,j}^*>0$ and under the conditions stated in the proposition. With promotion CPs, we have

$$\frac{d_{i,j}\alpha_{i,j}p_u^*+c_{i,j}\alpha_{i,j}(1-\alpha_{i,j})p_c\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}}{p_u^*c_{i,j}\alpha_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}} \leq\frac{d_{i,j}}{\alpha_{i,j}c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}},$$

which simplifies to

$$c_{i,j}\alpha_{i,j}p_c\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}\leq d_{i,j}p_u^*,$$

which is always the case when $\gamma_{i,j}^*>0$. ■

I. Proof of Proposition 8

We first consider a user-CP pair given that the CP is a revenue CP. In that case, the difference between the improvements in user and CP utilities is given by

$$\left(\frac{p_u^*}{c_{i,j}p_c}\right)^{\frac{-1}{\alpha_{i,j}}}\left(\frac{p_c(1+s_j^*)-d_{i,j}}{(1-\alpha_{i,j})\left(1+r_{i,j}s_j^*\right)}\right)^{1-\frac{1}{\alpha_{i,j}}}\left(\frac{\alpha_{i,j}p_u^*}{(1-\alpha_{i,j})p_c}-\alpha_{i,j}\right) +c_{i,j}^{\frac{1}{\alpha_{i,j}}}\left(\frac{p_u(1+s_j)}{(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}}\left(d_{i,j}-\frac{\alpha_{i,j}p_u(1+s_j)}{1-\alpha_{i,j}}\right).$$

In that case, a sufficient condition for this difference to be positive is that $p_u^*\geq(1-\alpha_{i,j})p_c$ and $d_{i,j}(1-\alpha_{i,j})\geq\alpha_{i,j}p_u(1+s_j)$, as stated in the proposition.

When the CP is a promotion CP, this difference in utility improvements is

$$\left(\frac{p_u^*p_c(1+s_j^*)}{d_{i,j}p_u^*+(1-\alpha_{i,j})p_c c_{i,j}\left(1+r_{i,j}s_j^*\right)^{1-\alpha_{i,j}}}\right)^{1-\frac{1}{\alpha_{i,j}}} \times\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\left(c_{i,j}\left(1+r_{i,j}s_j^*\right)^{1-\alpha_{i,j}}\left(1-\frac{p_c}{p_u^*}\right)-d_{i,j}\right) +\frac{(p_u(1+s_j))^{1-\frac{1}{\alpha_{i,j}}}}{1-\alpha_{i,j}}\left(\frac{d_{i,j}}{\left(c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}\right)^{1-\frac{1}{\alpha_{i,j}}}} -\alpha_{i,j}\left(c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}\right)^{\frac{1}{\alpha_{i,j}}}\right).$$

A sufficient condition for this difference to be positive is then that $d_{i,j}\geq\alpha_{i,j}c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\alpha_{i,j}}$ and $d_{i,j}p_u^*\leq c_{i,j}\left(1+r_{i,j}s_j^*\right)^{1-\alpha_{i,j}}\left(p_u^*-(1-\alpha_{i,j})p_c\right)$ ■.

J. Proof of Lemma 2

Since F is Schur-concave and homogeneous, it is sufficient to show that $\vec{x}/\left|\vec{x}\right|$ can be obtained from $\vec{y}/\left|\vec{y}\right|$ via a finite set of Robin-Hood operations. We wish to find a threshold index k^* such that for $l<k^*$, $\vec{y}_l/\left|\vec{y}\right|\leq\vec{x}_l/\left|\vec{x}\right|$, and for $l\geq k^*$, $\vec{y}_l/\left|\vec{y}\right|\geq\vec{x}_l/\left|\vec{x}\right|$. If such a k^* exists, we can easily obtain $\vec{x}/\left|\vec{x}\right|$ from $\vec{y}/\left|\vec{y}\right|$ by noting that for $l\geq k^*$, we need to reduce $\vec{y}_l/\left|\vec{y}\right|$, and for $l<k^*$, we must increase $\vec{y}_l/\left|\vec{y}\right|$. We do so by starting from $\vec{y}_K$ (the largest element) and reducing it by the required amount, distributing this amount to $\vec{y}_l$ with $l<k^*$. Since $\vec{y}/\left|\vec{y}\right|=\vec{x}/\left|\vec{x}\right|=1$, this procedure will recover $\vec{x}/\left|\vec{x}\right|$.

We thus need to show that such a threshold k^* exists from our assumption that $\vec{y}_k/\vec{x}_k \leq \vec{y}_{k+1}/\vec{x}_{k+1}$ for all k. It suffices to show that $\vec{y}_1/\vec{x}_1 \leq |\vec{y}|/|\vec{x}| \leq \vec{y}_K/\vec{x}_K$. We do so by induction on K: clearly, the assertion is true for $K = 1$. Assuming it is true for $K = n$, we suppose that $\vec{y}_n/\vec{x}_n \leq \vec{y}_{n+1}/\vec{x}_{n+1}$. Then

$$\frac{\sum_{k\leq n}\vec{x}^*_k + \vec{x}^*_{n+1}}{\sum_{k\leq n}\vec{x}_k + \vec{x}_{n+1}} \leq \frac{\vec{x}^*_{n+1}}{\vec{x}_{n+1}}$$

which we find by multiplying out the terms and using the fact that $\sum_{k\leq n}\vec{y}_k/\sum_{k\leq n}\vec{x}_n \leq \vec{y}_n/\vec{x}_n \leq \vec{y}_{n+1}/\vec{x}_{n+1}$. Similarly, $\left(\sum_{k\leq n}\vec{x}^*_k + \vec{x}^*_{n+1}\right)/\left(\sum_{k\leq n}\vec{x}_k + \vec{x}_{n+1}\right) \geq \sum_{k\leq n}\vec{x}^*_k/\sum_{k\leq n}\vec{x}_k \geq \vec{x}^*_1/\vec{x}_1$. ■

K. Proof of Proposition 9

For simplicity, we consider the case in which $p_u = p_u^*$. A similar argument can be made for arbitrary p_u^*.

We first consider a promotion CP and observe by inspection that user utility with optimal sponsorship, $U_{i,j}\left(x^*_{i,j}\left(\pi^*_{i,j}\right)\right)$, given by (31), is increasing in $c_{i,j}$ if $d_{i,j}p_u \geq \alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}$. If $d_{i,j}p_u < \alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}$, then user utility is given by (30), which is clearly increasing in $c_{i,j}$.

From (26), $x^*_{i,j}(\pi^*_{i,j})$ is also increasing in $c_{i,j}$ if $d_{i,j}p_u \geq \alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}$; if $d_{i,j}p_u < \alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}$, user demand is that before sponsorship and is also increasing in $c_{i,j}$.

We then observe from Table V that relative user benefit and relative demand decrease with $c_{i,j}$ when $d_{i,j}p_u \geq \alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}$; if $d_{i,j}p_u < \alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}$, $\gamma^*_{i,j} = 0$ and relative user benefit and relative demand equal 1. Thus, both relative user benefit and relative demand are non-increasing in $c_{i,j}$.

Given this result, we now let $\vec{x}$ denote a K-element vector containing the demands before sponsorship and $\vec{x}^*$ that after sponsorship, both sorted in increasing order (i.e., increasing $c_{i,j}$). We have shown that $\vec{x}^*_k/\vec{x}_k \geq \vec{x}^*_{k+1}/\vec{x}_{k+1}$ for all k, so the result follows from Lemma 2.

We now observe that since $\gamma^*_{i,j}$ is independent of $c_{i,j}$ for revenue CPs (Proposition 1), either $\gamma^*_{i,j} = 0$ for all users or $\gamma^*_{ij} > 0$ for all users. Since relative demand and relative user benefit is independent of $c_{i,j}$ for revenue CPs if $\gamma^*_{i,j} > 0$ (Table V), content sponsorship merely multiplies users' demands and utilities by a constant. Since our fairness function F is homogeneous, fairness does not change. ■

L. Proof of Proposition 10

For simplicity, we consider the case in which $p_u = p_u^*$ (i.e., the ISP does not change its user price with sponsorship). A similar argument can be made for arbitrary p_u^*.

1) Relative Demands: We first note that, from Table V, relative demand for revenue and promotion CPs increases with $d_{i,j}$ if $d_{i,j} \geq \alpha_{i,j}p_c(1 + s_j)$ (for revenue CPs) or $d_{i,j}p_u \geq \alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}$ (for promotion CPs). If $d_{i,j} < \alpha_{i,j}p_c(1 + s_j)$ (revenue CPs) or $d_{i,j}p_u < \alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}$ (promotion CPs), $\gamma^*_{i,j} = 0$ and relative demand is a constant 1. Thus, relative demand is nondecreasing in $d_{i,j}$.

Following the method used in the proof of Proposition 9, it now suffices to show that user demand without sponsorship also increases in $d_{i,j}$. Since user demand is independent of $d_{i,j}$ this follows immediately.

2) Relative Benefits: We first consider revenue CPs. CP utility before sponsorship is then $d_{i,j}x^*_{i,j}\left(p_u(1 + s_j)\right)$, which is clearly increasing in $d_{i,j}$.

We find that the CP's relative benefit when $d_{i,j} \geq \alpha_{i,j}p_c(1+s_j)$ (then $\gamma^*_{i,j} > 0$ unless $d_{i,j} = \alpha_{i,j}p_c(1+s_j)$, from Corollary 1) is

$$\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\left(\frac{p_c(1+s_j)}{d_{i,j}} - 1\right)\left(\frac{1}{1-\alpha_{i,j}}\left(1 - \frac{d_{i,j}}{p_c(1+s_j)}\right)\right)^{\frac{-1}{\alpha_{i,j}}},$$

whose derivative with respect to $d_{i,j}$ is proportional to

$$\frac{-p_c(1+s_j)}{\left(d_{i,j}\right)^2(1-\alpha_{i,j}}\left(1 - \frac{d_{i,j}}{p_c(1+s_j)}\right) + \left(\frac{p_c(1+s_j)}{d_{i,j}} - 1\right)\left(\frac{1}{\alpha_{i,j}p_c(1+s_j)(1-\alpha_{i,j})}\right),$$

which is positive if

$$(1 + \alpha_{i,j})\,d_{i,j}p_c(1+s_j) > p_c^2\,(1+s_j)^2\,\alpha_{i,j} + \left(d_{i,j}\right)^2.$$

Gathering terms and factoring, we find the condition $\left(d_{i,j} - \alpha_{i,j}p_c(1+s_j)\right)\left(p_c(1+s_j) - d_{i,j}\right) < 0$, which holds by assumption. Again, if $\gamma_{i,j} = 0$, then relative benefit is 1, and independent of $d_{i,j}$. Thus, both CP utility before sponsorship and relative benefit are nondecreasing in $d_{i,j}$, which by the same argument in Proposition 9 proves the desired fairness result.

We now consider promotion CPs. CP utility before sponsorship, $d_{i,j}x^*_{i,j}\left(p_u(1+s_j)\right)^{1-\alpha_{i,j}}/(1-\alpha_{i,j})$, is increasing in $d_{i,j}$; thus, it suffices to show that CPs' relative benefit increases with $d_{i,j}$. We take the derivative of the relative benefit (Table V if $d_{i,j}p_u \geq \alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}$) with respect to $d_{i,j}$ to find that it is proportional to

$$\frac{-1}{d_{i,j}^2}\left(1 - \alpha_{i,j} + \frac{d_{i,j}p_u}{p_c c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{1}{\alpha_{i,j}}} + \frac{1}{d_{i,j}\alpha_{i,j}}\left(\frac{p_u}{p_c c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}}}\right) \times \left(1 - \alpha_{i,j} + \frac{d_{i,j}p_u}{p_c c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}}}\right)^{\frac{1}{\alpha_{i,j}}-1}.$$

Upon multiplying out this expression, we find the equivalent condition

$$\frac{d_{i,j}p_u}{\alpha_{i,j}c_{i,j}p_c\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}} > 1 - \alpha_{i,j} + \frac{d_{i,j}p_u}{c_{i,j}p_d\left(1 + r_{i,j}s_j\right)^{1-\alpha_{i,j}}},$$

or equivalently $d_{i,j}p_u > \alpha_{i,j}p_c c_{i,j}(1 + r_{i,j}s_j)^{1-\alpha_{i,j}}$, which holds by assumption. ■

M. Proof of Proposition 11

We first note that (13) is a convex optimization problem, so we may solve it by applying the Karush-Kuhn-Tucker optimality conditions. Letting λ denote the Lagrange multiplier for the budget constraint $\sum_j (1-\gamma_{i,j}+s_j)\, x_{i,j} \leq B$, we can then solve for

$$x^*_{i,j} = \left(\frac{\lambda\,(1-\gamma_{i,j}+s_j)}{c_{ij}\,(1+r_{i,j}s_j)^{-\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}}.$$

Noting that the budget constraint $\sum_j (1-\gamma_{i,j}+s_j)\, x_{i,j} \leq B$ is tight at optimality, we can then solve for λ and substitute back into the optimal demand (14). ■

N. Proof of Corollary 5

By inspection, $x^*_{i,j}$ decreases as $\gamma_{i,k}$ increases for $k \neq j$, since $\alpha_{i,j} < 1$ and therefore $1 - \frac{1}{\alpha_{i,k}} < 0$. We must then show that $x^*_{i,j}$ increases as $\gamma_{i,j}$ increases. It suffices to show that the function $f(x) = x^{\frac{-1}{\alpha}} / \left(c + x^{1-\frac{1}{\alpha}}\right)$ is a decreasing function of x, which is convex when $\alpha > 1/5$. To do so, we compute the first derivative

$$f'(x) = \frac{-\alpha x^{\frac{-2}{\alpha}} - c x^{-1-1/\alpha}}{\alpha\left(c + x^{1-1/\alpha}\right)^2} < 0.$$

When $\alpha > 1/5$, we find that

$$f''(x) = \frac{2\alpha x^{\frac{-3}{\alpha}} + c\left(5 - \frac{1}{\alpha}\right) x^{\frac{-2}{\alpha}-1} + c^2\left(\frac{1}{\alpha}+1\right) x^{\frac{-1}{\alpha}-2}}{\alpha\left(c + x^{1-\frac{1}{\alpha}}\right)^3} > 0$$

if $\alpha > 1/5$. Thus, $x^*_{i,j}$ is a convex function of $\gamma_{i,j}$ when $\alpha > 1/5$. ■

O. Proof of Proposition 12

The revenue CP's optimal sponsorship level occurs either at $\gamma_{i,j} = 0$, $\gamma_{i,j} = 1 + s_j$, or when $W'_{i,j}(\gamma_{i,j}) = 0$. We show that, if (16) holds, then optimal sponsorship cannot occur at either $\gamma_{i,j} = 1 + s_j$ or $\gamma_{i,j} = 0$. To do so, we first take the derivative

$$\begin{aligned}\frac{dW_{i,j}}{d\gamma_{i,j}} &= \frac{dx^*_{i,j}}{d\gamma_{i,j}}\left(d_{i,j}{x^*}_{i,j} - p_c\gamma_{i,j}\right) - p_c x^*_{i,j} \\ &= \frac{d_{i,j}\left(B + (1-\gamma_{i,j}+s_j)(\alpha_{i,j}-1)\,x^*_{i,j}\right)}{(1-\gamma_{i,j}+s_j)\left(\alpha_{i,j}B + \gamma_{i,j}(\alpha_{i,j}-1)\,x^*_{i,j}\right) + \gamma_{i,j}B} - p_c\end{aligned}$$

and find that, as $\gamma_{i,j} \to 1 + s_j$,

$$\frac{dW_{i,j}}{d\gamma_{i,j}} \to \frac{d_{i,j}\left(B + B\,(\alpha_{i,j}-1)\right)}{(1+s_j)B + \gamma_{i,j}(\alpha_{i,j}-1)\,B} - p_c = \frac{d_{i,j}}{1+s_j} - p_c,$$

which is negative since we assume that $d_{i,j} < p_c(1+s_j)$. Thus, the optimal $\gamma^*_{i,j} < 1 + s_j$: there exists $\gamma_{i,j} < 1 + s_j$ that yields a higher CP utility $W_{i,j}$.

Similarly, at $\gamma_{i,j} = 0$, we find that

$$\frac{dW_{i,j}}{d\gamma_{i,j}} = \frac{d_{i,j}\left(B + (1+s_j)(\alpha_{i,j}-1)x^*_{i,j}(0)\right)}{\alpha_{i,j}(1+s_j)B} - p_c,$$

which is positive if (16) holds. Thus, $\gamma^*_{i,j} > 0$. ■

P. Proof of Proposition 13

The proof is analogous to that of Proposition 12: we first find that

$$\frac{dW_{i,j}}{d\gamma_{i,j}} = \frac{d_{i,j}{x^*}_{i,j}^{-\beta_{i,j}}\left(B + (1-\gamma_{i,j}+s_j)(\alpha_{i,j}-1)\,x^*_{i,j}\right)}{(1-\gamma_{i,j}+s_j)\left(\alpha_{i,j}B + \gamma_{i,j}(\alpha_{i,j}-1)\,x^*_{i,j}\right) + \gamma_{i,j}B} - p_c.$$

When (18) holds, this derivative is positive at $\gamma_{i,j} = 0$, indicating that the optimal $\gamma^*_{i,j} > 0$. We must then show that $W_{i,j}$ is decreasing at $\gamma_{i,j} = 1 + s_j$; by inspection, $dW_{i,j}/d\gamma_{i,j} \to -p_c < 0$ as $\gamma_{i,j} \to 1 + s_j$. ■

Q. Proof of Proposition 14

We first consider a revenue CP's utility after sponsorship, which is given by:

$$W^*_{i,j} = \alpha_{i,j}\left(\frac{p^*_u}{p_c c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}}\left(\frac{p_c(1+s_j) - a r_{i,j}s_j}{(1-\alpha_{i,j})(1+r_{i,j}s_j)}\right)^{1-\frac{1}{\alpha_{i,j}}}.$$

Our goal is to show that this utility is an increasing function of s_j, which we do by taking the derivative with respect to s_j and discarding constants. We find that $W^*_{i,j}$ is an increasing function of s_j if and only if

$$\frac{(1+r_{i,j}s_j)(p_c - a r_{i,j}) - (p_c + s_j(p_c - a_{i,j}r_j))}{(1+r_{i,j}s_j)^2} \leq 0.$$

Thus, under the conditions $a r_{i,j} \geq p_c(1 - r_{i,j})$ stated in the proposition, $s^*_j = \overline{s}$.

Finally, we consider a promotion CP and show that its utility after sponsorship is always a decreasing function of s_j. To do so, we note that the derivative is a positive multiple of

$$\begin{aligned}&\left(1 - \frac{1}{\alpha_{i,j}}\right)(1+s_j)^{\frac{-1}{\alpha_{i,j}}} \\ &\times\left(d_{i,j}p_u + (1-\alpha_{i,j})p_c c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}\right)^{\frac{1}{\alpha_{i,j}}-1} \\ &\times\Bigg[d_{i,j}p_u + (1-\alpha_{i,j})p_c c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}} \\ &-(1+s_j)r_{i,j}(1+r_{i,j}s_j)^{-\alpha_{i,j}}\Bigg]. \\ &\propto d_{i,j}p_u + (1-\alpha_{i,j})p_c c_{i,j}(1-r_{i,j})(1+r_{i,j}s_j)^{1-\alpha_{i,j}} \geq 0.\end{aligned}$$

R. Proof of Proposition 15

Following the proof of Proposition 10, it suffices to show that the relative CP demand and utility are increasing functions of $d_{i,j}$. The relative utility of a promotion CP is

$$\begin{aligned}&\left(\frac{c_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}}{p_c(1+s_j)}\right)^{1-\frac{1}{\alpha_{i,j}}} \\ &\times\left(\alpha_{i,j} + \frac{c_{i,j}\alpha_{i,j}(1+r_{i,j}s_j)^{1-\alpha_{i,j}}\,p_c(1-\alpha_{i,j})}{d_{i,j}p^*_u}\right) \\ &\times\left(\frac{d_{i,j}}{p_c(1+s^*_j)} + \frac{(1-\alpha_{i,j})c_{i,j}\left(1+r_{i,j}s^*_j\right)^{1-\alpha_{i,j}}}{p^*_u(1+s^*_j)}\right)^{\frac{1}{\alpha_{i,j}}-1}.\end{aligned}$$

We then take the derivative of the relative utility with respect to $d_{i,j}$ and find that if $s_j^* \leq s_j$, it is proportional to

$$
\begin{aligned}
& - \alpha_{i,j} \left(1 - \alpha_{i,j}\right) p_c^2 c_{i,j}^2 \left(1 + r_{i,j} s_j\right)^{1-\alpha_{i,j}} \left(1 + r_{i,j} s_j^*\right)^{1-\alpha_{i,j}} \\
& + d_{i,j}^2 {p_u^*}^2 + \left(1 - 2\alpha_{i,j}\right) d_{i,j} p_u^* c_{i,j} p_c \left(1 + r_{i,j} s_j\right)^{1-\alpha_{i,j}} \\
& \geq \left(d_{i,j} p_u^* - \alpha_{i,j} c_{i,j} p_c \left(1 + r_{i,j} s_j\right)^{1-\alpha_{i,j}}\right) \\
& \times \left(d_{i,j} p_u^* + (1 - \alpha_{i,j}) c_{i,j} p_c \left(1 + r_{i,j} s_j\right)^{1-\alpha_{i,j}}\right) \\
& \geq 0.
\end{aligned}
$$