# COMPACTNESS VS. SEQUENTIAL COMPACTNESS
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## Definitions

**Definition.** A *topological space* is a pair $(\mathcal{X}, \tau)$ where $\mathcal{X}$ is a set and $\tau$ is a collection of subsets of $\mathcal{X}$ that satisfies the following properties:

(1) $\mathcal{X} \in \tau$
(2) $\varnothing \in \tau$
(3) If $\{U_\alpha\}_{\alpha \in A}$ is a collection of sets from $\tau$ then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
(4) If $U, V \in \tau$ then $U \cap V \in \tau$.

The sets in $\tau$ are refered to as the *open* sets.

We often decline to explicitly name the collection of open sets when working with topological spaces.

**Definition.** A topological space $\mathcal{X}$ is *compact* if whenever $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets and

$$\mathcal{X} = \bigcup_{\alpha \in A} U_\alpha$$

then there exist $\alpha_1, \ldots, \alpha_k \in A$ such that

$$\mathcal{X} = \bigcup_{i=1}^{k} U_{\alpha_i}$$

This definition is usually stated succinctly as "$\mathcal{X}$ is compact if every open cover of $\mathcal{X}$ admits a finite subcover".

**Definition.** A topological space $\mathcal{X}$ is *sequentially compact* if whenever $(x_n)_{n=1}^\infty$ is a sequence in $\mathcal{X}$ then $(x_n)_{n=1}^\infty$ has a convergent subsequence. (Recall that a sequence $(x_n)_{n=1}^\infty$ converges to a point $x \in \mathcal{X}$ if for any open set $U$, $x \in U$ implies that there exists $N_U$ such that $x_n \in U$ for all $n \geq N_U$.)

## Sequentially Compact but not Compact

**Definition.** Given a linearly ordered set $(P, <)$, define the *order topology* on $P$ to be the topology generated by the collection of sets of the form

$$\{x \in P \mid x < a\} \text{ and } \{x \in P \mid x > a\}$$

where $a \in P$. It follows that the open sets of this topology are unions of *open intervals*.

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The order topology on $(\mathbb{R}, <)$ corresponds with the usual metric topology on the real line. This is the topology studied in elementary calculus courses.

Let $\Omega$ denote the first uncountable ordinal, so

$$\Omega = \{\alpha \mid \alpha \text{ is a countable ordinal}\}$$

Imbue $\Omega$ with the order topology. Consider the collection of open sets

$$\{\{x \in \Omega \mid x < \alpha\}\}_{\alpha \in \Omega}$$

Then clearly

$$\Omega = \bigcup_{\alpha \in \Omega} \{x \in \Omega \mid x < \alpha\}$$

Given any countable collection of ordinals $\{\alpha_1, \alpha_2, \dots\} \subseteq \Omega$, the union

$$\bigcup_{n=1}^{\infty} \{x \in \Omega \mid x < \alpha_n\}$$

is not equal to $\Omega$ since it is countable (each of the sets in the union is countable). It follows that no finite collection of these sets will cover $\Omega$, so $\Omega$ is not compact.

**Definition.** Let $(x_n)_{n=1}^{\infty}$ be a sequence in a linearly ordered set $(P, <)$. An index $n \geq 1$ is a *peak point* of $(x_n)_{n=1}^{\infty}$ if $x_n \geq x_k$ for all $k \geq n$.

**Lemma.** (Peak Point Lemma) *Let $(x_n)_{n=1}^{\infty}$ be a sequence in a linearly ordered set $(P, <)$. Then $(x_n)_{n=1}^{\infty}$ has a monotone subsequence.*

See Spivak's *Calculus*.

*Proof.* If $(x_n)_{n=1}^{\infty}$ has infinitely many peak points $n_1, n_2, \dots$ then $(x_{n_k})_{k=1}^{\infty}$ is a monotone non-increasing subsequence. Otherwise suppose that $(x_n)_{n=1}^{\infty}$ has finitely many peak points $n_1 < \cdots < n_k$. Let $m_1 > n_k$. Then $m_1$ is not a peak point, so there is $m_2 > m_1$ such that $x_{m_1} \leq x_{m_2}$. Similarly, $m_2$ is not a peak point, so there is $m_3 > m_2$ such that $x_{m_2} \leq x_{m_3}$. Continuing this this process we obtain a monotone non-decreasing subsequence $(x_{m_k})_{k=1}^{\infty}$. $\square$

Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence in $\Omega$. I will show that $(\alpha_n)_{n=1}^{\infty}$ has a convergent subsequence, and thus $\Omega$ is sequentially compact. By the Peak Point lemma $(\alpha_n)_{n=1}^{\infty}$ has a monotone subsequence $(\alpha_{n_k})_{k=1}^{\infty}$. If it is non-decreasing then let

$$\alpha = \bigcup_{k=1}^{\infty} \alpha_{n_k}$$

otherwise let $\alpha$ be the smallest element of the set $\{\alpha_{n_1}, \alpha_{n_2}, \dots\}$ (there is a smallest element because $\Omega$ is well-ordered). Then $(x_{n_k})_{k=1}^{\infty}$ converges to $\alpha$, since for any open interval around $\alpha$, say $\{x \in \Omega \mid \gamma_1 < x < \gamma_2\}$, by defintion of $\alpha$ and because $\{x_{n_k}\}_{k=1}^{\infty}$ is monotone, the sequence will eventually be inside of the interval. Thus $\Omega$ is sequentially compact.

## Compact but not Sequentially Compact

**Definition.** Given a collection $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ of topological spaces, the *product space* is

$$\prod_{\alpha \in A} \mathcal{X}_\alpha = \{(x_\alpha)_{\alpha \in A} \mid \alpha \in A \text{ and } \alpha \in \mathcal{X}_\alpha\}$$

The *product topology* on the product space is the topology generated by the sets

$$\{p_\alpha^{-1}(U) \mid \alpha \in A \text{ and } U \subseteq \mathcal{X}_\alpha \text{ is open}\}$$

The only fact about the product topology that we will need for this talk is that a sequence $(x^{(n)})_{n=1}^\infty$ converges in the product topology if and only if the sequence $(x_\alpha^{(n)})_{n=1}^\infty$ converges for each $\alpha \in A$.

**Theorem.** (Tychonov) *Let $\{\mathcal{X}_\alpha\}_{\alpha\in A}$ be a collection of compact topological spaces. Then*

$$\mathcal{X} := \prod_{\alpha\in A} \mathcal{X}_\alpha$$

*is compact in the product topology.*

Let $\mathcal{X} = [0,1]^{[0,1]}$, the set of all funtions from $[0,1]$ to $[0,1]$ (with the usual topology on $[0,1]$ inherited from $\mathbb{R}$). $[0,1]$ is compact by the Heine-Borel Theorem, so $\mathcal{X}$ is compact in the product topology by Tychonov's Theorem.

Define a sequence $(f_n)_{n=1}^\infty$ in $\mathcal{X}$ by

$$f_n(x) = \text{the } n^{\text{th}} \text{ digit in the binary expansion of } x$$

(If a number could have 2 different binary expansions, favor the one that ends in an infinite string of zeroes over the one that ends in an infinite string of one's.) Suppose that $(f_{n_k})_{k=1}^\infty$ is a convergent subsequence of $(f_n)_{n=1}^\infty$. Then $(f_{n_k}(x))_{k=1}^\infty$ converges for each $x \in [0,1]$. Define $y \in [0,1]$ by setting

$$n^{\text{th}} \text{ digit of } y = \begin{cases} 1 & \text{if } n = n_k \text{ and } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Then $(f_{n_k}(y))_{k=1}^\infty$ does not converge. This contradiction shows that $\mathcal{X}$ is not sequentially compact.