Carnegie Mellon University

A Model for the Limit Order Book in Heavy Traffic

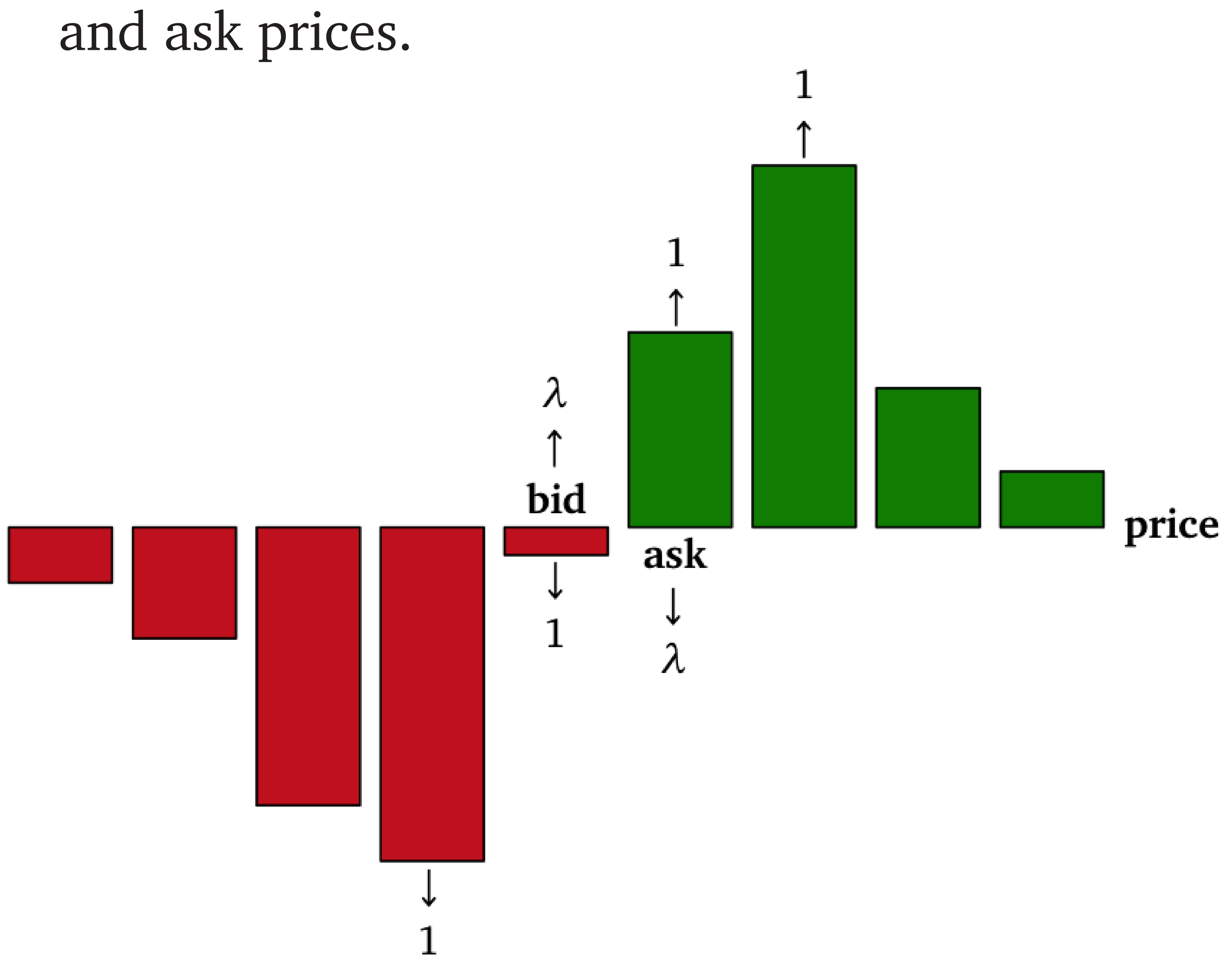
Christopher Almost

Department of Mathematical Sciences

Model

Our LOB model follows Cont et al. [1]:

- Discrete price levels.
- Orders of equal size.
- Arrivals are independent and Markovian.
- Rates depend on LOB state only via current bid and ask prices.



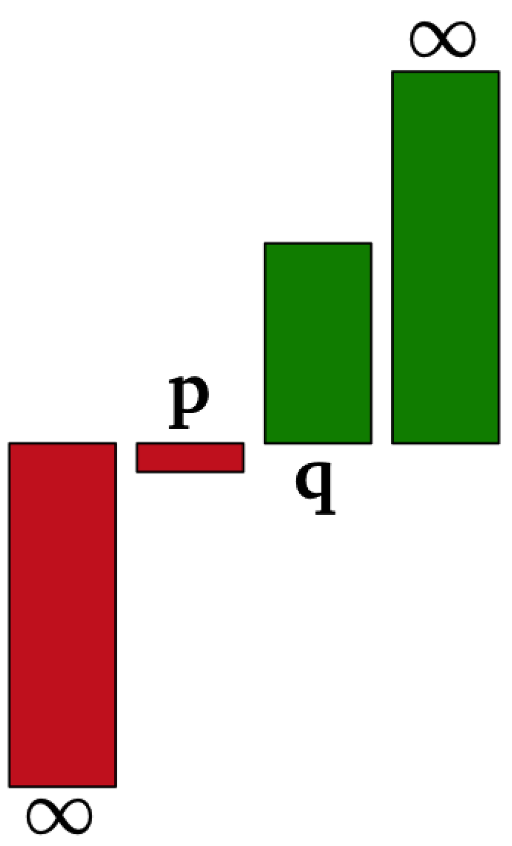
We investigate a sequence of such models:

- Order arrival rates are scaled by n
- Order size is scaled by $1/\sqrt{n}$.
- Scale factors come from queueing theory.

Simplifying assumptions for this investigation:

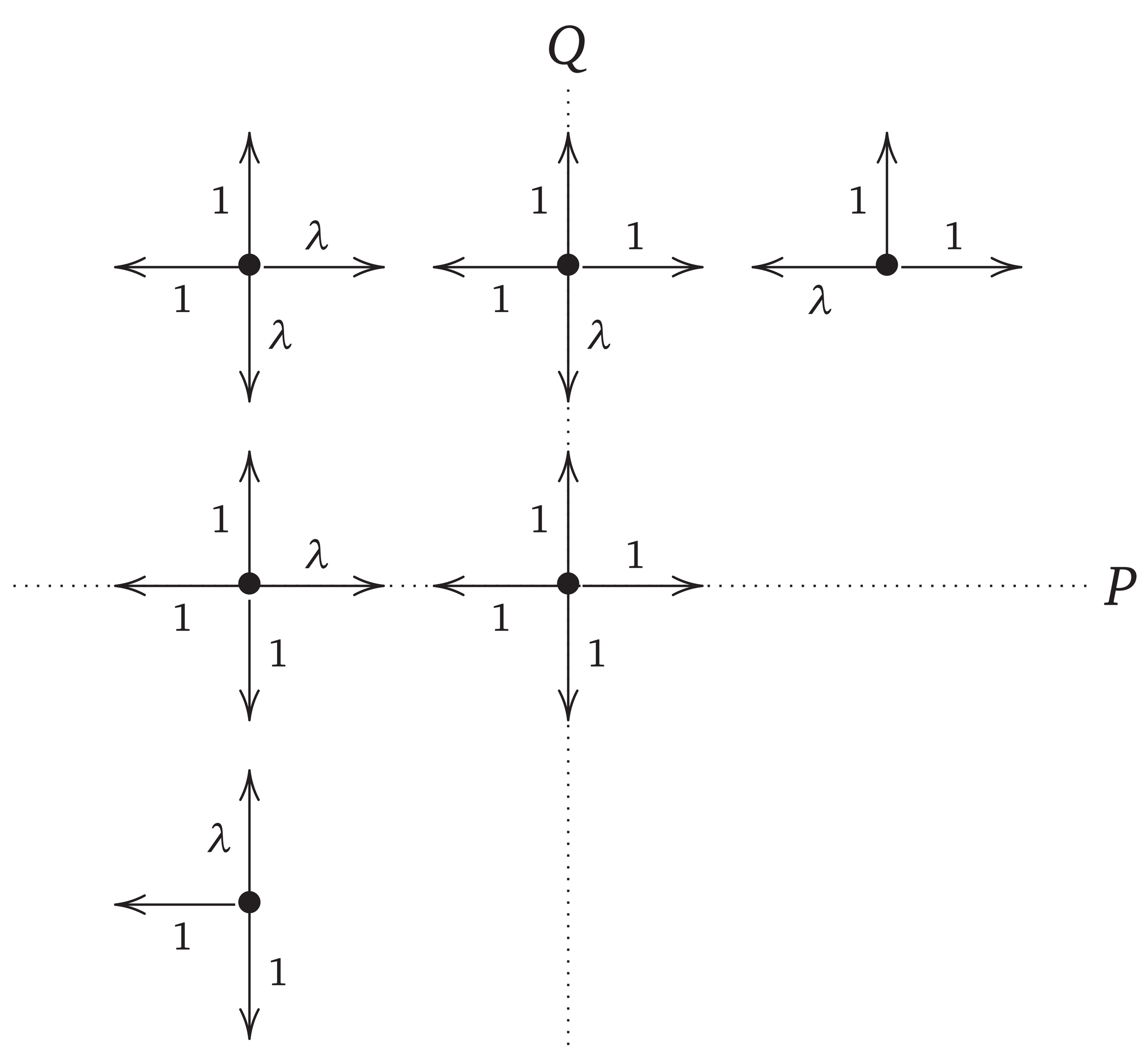
- Combined market/limit order rate is $\lambda > 1$, arriving at opposite best price.
- Limit order rates of 1 at next two prices.
- Interesting but tractable behavior.

Heuristic



Assume there are "large" queues of buy and sell orders between which the action takes place.

- Simulation suggests this is often the case.
- With simplified arrival rates, there are only two intermediate prices, p and q.
- In this case the signed order quantities P and Q form a CTMC on $\mathbb{Z}^2$.



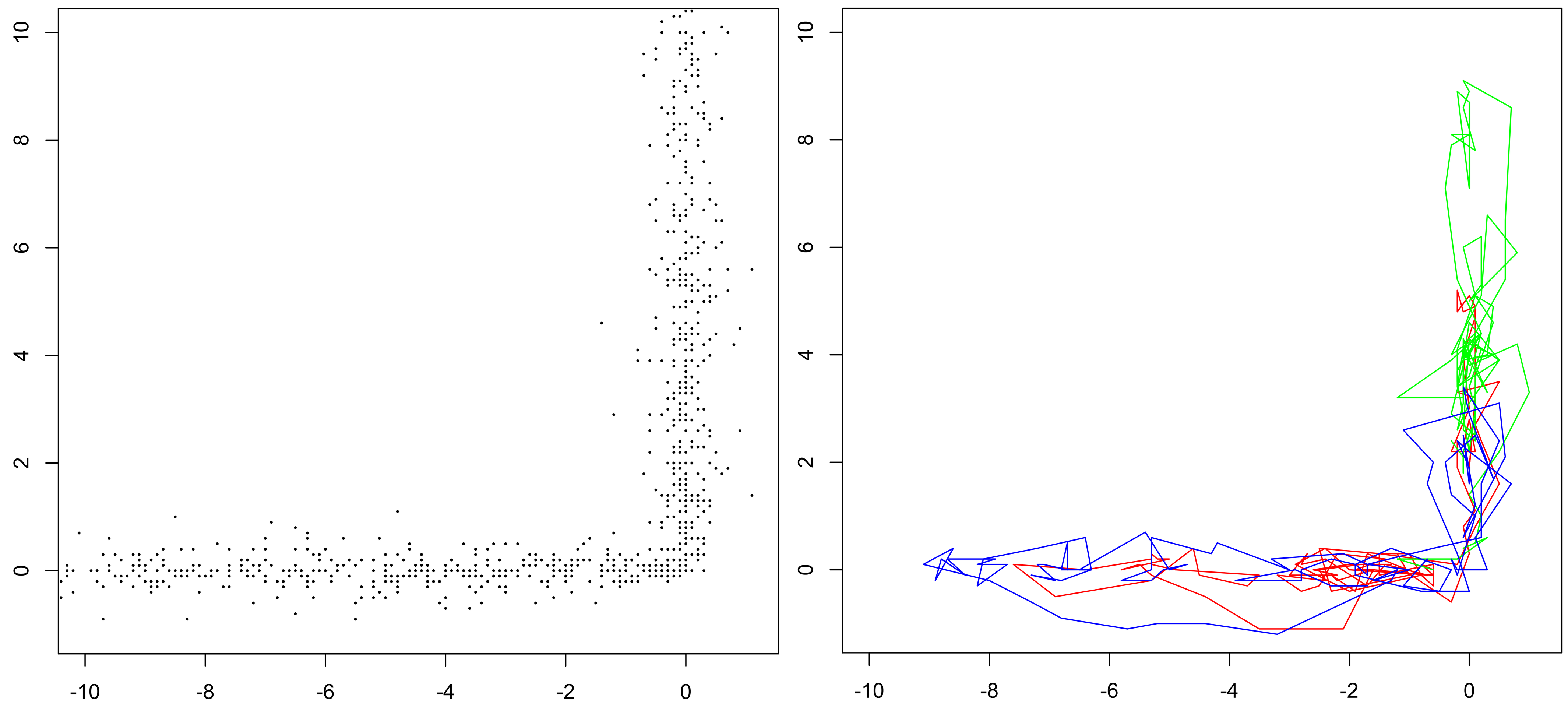
Advisors: Steven Shreve and John Lehoczky

Theorem

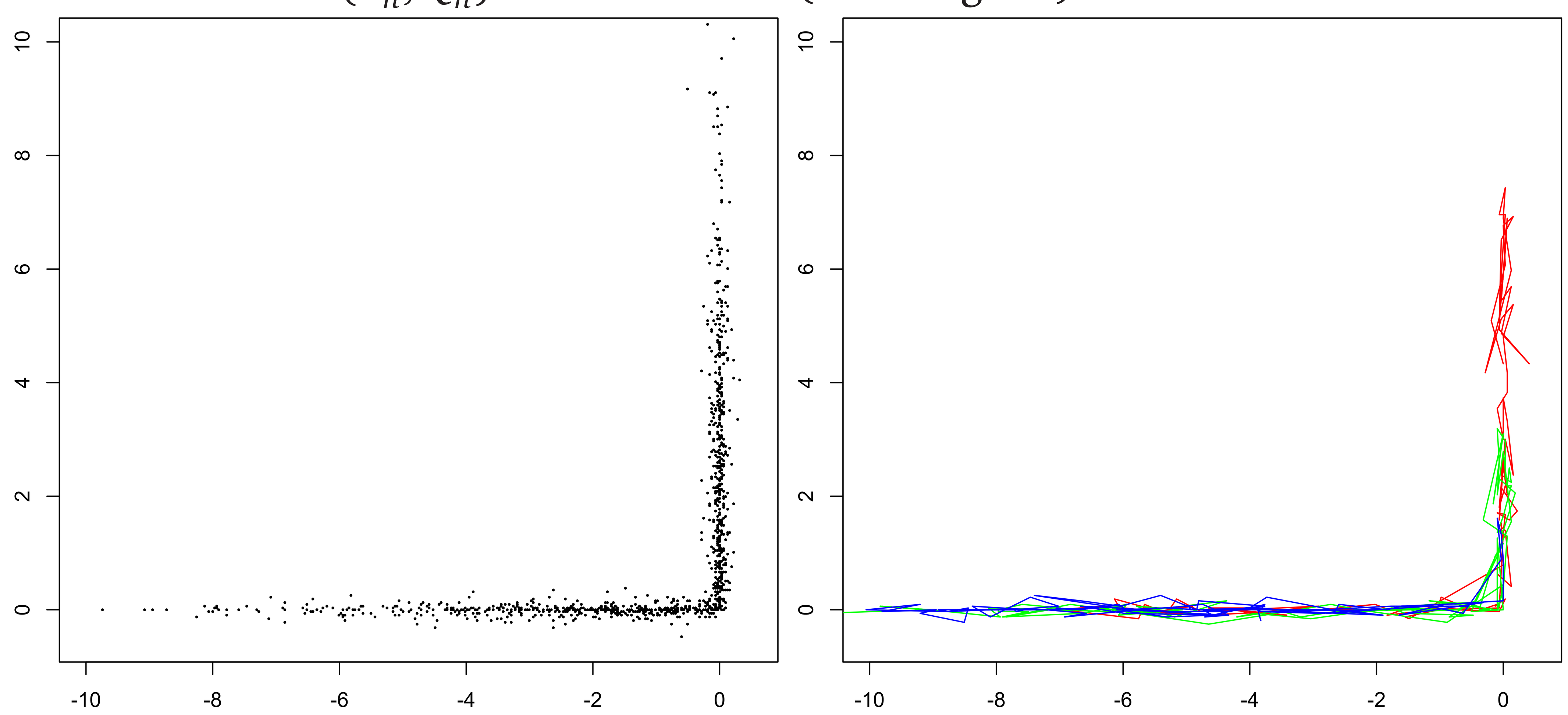
If $\lambda = (1+\sqrt{5})/2$ then $(\widehat{P}_n, \widehat{Q}_n) \Rightarrow (-B_-, B_+)$ in $\mathscr{D}[0,\infty)^2$, where B is a one dimensional Brownian motion with variance parameter 4λ. Here $\widehat{P}_n(t) := P(nt)/\sqrt{n}$ and $\widehat{Q}_n(t) := Q(nt)/\sqrt{n}$ are the diffusion scaled queue length processes.

Evidence

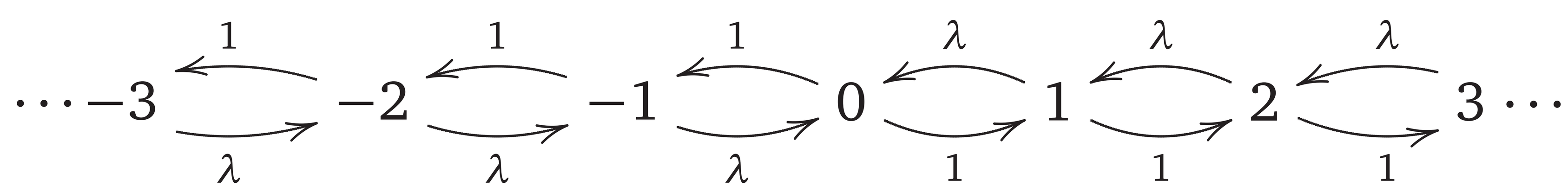
- Simulation of $(\widehat{P}_n, \widehat{Q}_n)$ with $n = 100$:
  - Black dots are final points of 1000 independent trials run for 2 scaled time units.
  - Coloured lines are independent paths 10 scaled time units long, sampled every 0.1 scaled times units.



- Simulation of $(\widehat{P}_n, \widehat{Q}_n)$ with $n = 1000$ (same legend):



- Distance to "backwards L" is crushed because it behaves like the CTMC:



Sketch of proof

- Transform coordinates to "straighten out" the problem.

$$U := \begin{cases} P & P > 0 \\ \max\{P, -Q\} & P \le 0, Q \ge 0 \\ -Q & Q < 0. \end{cases} \qquad V := \begin{cases} \lambda P + Q & P \ge 0, Q \ge 0 \\ P + Q & P \le 0, Q \ge 0 \\ P + \lambda Q & P \le 0, Q \le 0. \end{cases}$$

- U is the distance to the "backwards L," and is shown to be crushed ($U \Rightarrow 0$) by comparison with a sub-critical queueing system.
- V is a martingale, and $V \Rightarrow \sqrt{4\lambda} W$ is shown using martingale methods, which rely on semimartingale characteristics.
- It is here that the requirement $\lambda^2 - \lambda - 1 = 0$ comes out.
- Interestingly, infinitesimal generator methods will not work for this problem because λ does not scale down to 1 as $n \to \infty$.

References

[1] R. Cont, S. Stoikov, R. Talreja. A stochastic model for order book dynamics. *Operations research*, 58:549–563, 2010.