Finite Dimensional Lie Algebras and their Representations
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Prof. I. Grojnowski[*]

Chris Almost[†]

The exam questions will consist of a subset of the in-class exercises.

Contents

[*]groj@dpmms.cam.ac.uk
[†]cda23@cam.ac.uk

1 Motivation and Some Definitions

1.1 Lie Groups and Lie Algebras

This class is about the structure and representations of the so-called simple Lie groups

$$SL_n = \{A \in \mathrm{Mat}_n \mid \det A = 1\}, SO_n = \{A \in SL_n \mid AA^T = I\}, Sp_{2n}$$

and exactly five others. The only technology we will use is linear algebra. These groups are slightly complicated, and they have a topology on them. To study them we pass to a "linearization," the Lie algebra, and study these instead.

1.1.1 Definition. A *linear algebraic group* is a subgroup of GL_n, for some n, defined by polynomial equations in the matrix coefficients.

1.1.2 Example. All of the groups mentioned so far are linear algebraic groups, as is the collection of upper triangular matrices.

This definition requires an embedding; it is desirable to come up with one which does not. It turns out the linear algebraic groups are exactly the affine algebraic groups, i.e. affine algebraic varieties such that multiplication and inversion are regular functions.

1.1.3 Definition. The *Lie algebra* associated with G is $\mathfrak{g} = T_1 G$, the tangent space to G at the identity.

1.1.4 Example. Let $g = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] + \varepsilon \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, where $\varepsilon^2 = 0$. Then

$$\det g = (1 + \varepsilon a)(1 + \varepsilon d) - bc\varepsilon^2 = 1 + \varepsilon(a + d)$$

Therefore $\det g = 1$ if and only if $\operatorname{tr} g = 0$.

We are working in the ring $E = \mathbb{C}[\varepsilon]/\varepsilon^2 = 0$ (the so-called *dual numbers*). If $G \subseteq GL_n$ is a linear algebraic group, we can consider

$$G(E) = \{A \in \mathrm{Mat}_n(E) \mid A \text{ satisfies the equations defining } G\}.$$

There is a natural map $\pi : G(E) \to G(\mathbb{C})$ induced by the ring homomorphism $\pi : \varepsilon \mapsto 0$.

1.1.5 Definition. $G(E)$ is the *tangent bundle* to G, denoted TG, so

$$\mathfrak{g} = T_1 G = \pi^{-1}(I) = \{A \in \mathrm{Mat}_n \mathbb{C} \mid I + \varepsilon A \in G(E)\}.$$

1.1.6 Exercise. Show that this definition is equivalent to the usual definition of TG (say, from differential geometry).

1.1.7 Examples.

1. Let $G = GL_n = \{A \in \mathrm{Mat}_n \mid A^{-1} \in \mathrm{Mat}_n\}$. Then

$$G(E) = GL_n \oplus \mathrm{Mat}_n \varepsilon = \{A + B\varepsilon \mid A^{-1} \in \mathrm{Mat}_n\},$$

since $(A + B\varepsilon)(A^{-1} - A^{-1}BA^{-1}\varepsilon) = I$. Therefore

$$\mathfrak{gl}_n := T_1 GL_n = \mathrm{Mat}_n \cong \mathbb{R}^{n^2}.$$

2. As an exercise, show that $\det(I + X\varepsilon) = 1 + \mathrm{tr}(X)\varepsilon$. As a corollary, $\mathfrak{sl}_n := T_1 SL_n = \{X \in \mathrm{Mat}_n \mid \mathrm{tr}(X) = 0\}$.

3. Let $G = O_n = \{A \mid A^T A = I\}$. Then $(I + X\varepsilon)^T(I + X\varepsilon) = I + (X^T + X)\varepsilon$, so $\mathfrak{o}_n := T_1 O_n = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid X^T + X = 0\}$.

Remark. If $X \in \mathfrak{o}_n$ then $\mathrm{tr} X = 0$ (when $2 \neq 0$), so the Lie algebra of O_n is contained in the Lie algebra of SL_n. Also note that $\mathfrak{o}_n = \mathfrak{so}_n$, so the Lie algebra does not keep track of whether the group is connected.

What structure on $\mathfrak{g}$ is there coming from the group structure on G? Notice that

$$(I + A\varepsilon)(I + B\varepsilon) = I + (A + B)\varepsilon,$$

so traditional multiplication only sees the vector space structure. Instead we use the commutator map

$$G \times G \to G : (P, Q) \mapsto PQP^{-1}Q^{-1},$$

and consider it infinitesimally. If $P = I + A\varepsilon$ and $Q = I + B\delta$, then $P^{-1} = I - A\varepsilon$ and $Q^{-1} = I - B\delta$, so

$$PQP^{-1}Q^{-1} = I + (AB - BA)\varepsilon\delta.$$

Let $[A, B]$ denote the map $(A, B) \mapsto AB - BA$.

1.1.8 Exercises.

1. Show that $(PQP^{-1}Q^{-1})^{-1} = QPQ^{-1}P^{-1}$ implies $[X, Y] = -[Y, X]$

2. Show that the fact that multiplication is associative in G implies

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

1.1.9 Definition. Let K be a field of characteristic not 2 or 3. A *Lie algebra* $\mathfrak{g}$ over K is a vector space over K equipped with a bilinear map $[\cdot, \cdot] : \Lambda^2 \mathfrak{g} \to \mathfrak{g}$, the *Lie bracket*, such that for all $X, Y, Z \in G$,

1. $[X, Y] = -[Y, X]$ (skew-symmetry); and

2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity).

1.1.10 Example. In most of the following the Lie bracket is inherited from $\mathfrak{gl}_n$. The following are Lie algebras.

1. $\mathfrak{sl}_n = \{A \in \mathfrak{gl}_n \mid \mathrm{tr} A = 0\}$;

2. $\mathfrak{so}_n = \{A \in \mathfrak{gl}_n \mid A^T + A = 0\}$;

3. $\mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid JA^T J^{-1} + A = 0\}$, where

$$J = \text{anti-diag}(\underbrace{-1, \ldots, -1}_{n}, \underbrace{1, \ldots, 1}_{n}).$$

4. Upper triangular matrices;

5. Strictly upper triangular matrices;

6. Let V be any vector space and define the Lie bracket to be the zero map. This example is known as the *Abelian Lie algebra*.

1.1.11 Exercises.

1. Check directly that $\mathfrak{gl}_n = \mathrm{Mat}_n$ is a Lie algebra with Lie bracket $[A,B] = AB - BA$, and in fact this works for any K-algebra;

2. Check that all of the above are truly subalgebras, i.e. vector subspaces and closed under the Lie bracket.

1.1.12 Exercise. Classify all Lie algebras $\mathfrak{g}$ over K with $\dim_K \mathfrak{g} \leq 3$.

1.2 Representations

1.2.1 Definition. A *Lie algebra homomorphism* $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map such that

$$\varphi([x,y]_1) = [\varphi(x), \varphi(y)]_2.$$

A *Lie algebra representation* (or simply *representation*) of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}_V$.

1.2.2 Example. If $\mathfrak{g} \subseteq \mathfrak{gl}_V$ then the inclusion is a representation of $\mathfrak{g}$ on V. For example, K^n is a representation of $\mathfrak{so}_n \subseteq \mathfrak{gl}_n$.

1.2.3 Definition. Let $x \in \mathfrak{g}$ and define the *adjoint* of x to be

$$\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g} : y \mapsto [x,y],$$

a linear map on $\mathfrak{g}$.

1.2.4 Lemma. *The adjoint map* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is linear and is in fact* ad *is a representation of* $\mathfrak{g}$ *on itself called the adjoint representation.*

PROOF: We must check that $\mathrm{ad}[x,y] = \mathrm{ad}\, x\, \mathrm{ad}\, y - \mathrm{ad}\, y\, \mathrm{ad}\, x$. For any z,

$$\begin{aligned} 0 &= [[x,y],z] + [[y,z],x] + [[z,x],y] && \text{Jacobi identity} \\ 0 &= [[x,y],z] - [x,[y,z]] + [y,[x,z]] && \text{skew-symmetry} \\ [[x,y],z] &= [x,[y,z]] - [y,[x,z]] \end{aligned}$$

so $\mathrm{ad}[x,y](z) = (\mathrm{ad}\, x\, \mathrm{ad}\, y - \mathrm{ad}\, y\, \mathrm{ad}\, x)(z)$ for every z, implying the result. □

1.2.5 Definition. The *center* of $\mathfrak{g}$ is

$$Z_{\mathfrak{g}} = \{x \in \mathfrak{g} \mid [x,y] = 0 \text{ for every } y \in \mathfrak{g}\} = \ker(\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})).$$

Warning: the notation $Z_{\mathfrak{g}}$ is not used in class (in fact, no notation for center is used in class).

By the definition of center, the adjoint homomorphism embeds $\mathfrak{g}$ into $\mathfrak{gl}_{\mathfrak{g}}$ if and only if $\mathfrak{g}$ has trivial center.

1.2.6 Example. The adjoint homomorphism maps $\mathfrak{g} = \left[\begin{smallmatrix} 0 & K \\ 0 & 0 \end{smallmatrix}\right]$ to zero in $\mathfrak{gl}_\mathfrak{g}$.

1.2.7 Theorem (Ado). *Any finite dimensional Lie algebra $\mathfrak{g}$ has a finite dimensional faithful representation, i.e. $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ for some n.*

PROOF: Omitted. □

1.2.8 Example. Recall that $\mathfrak{sl}_2 = \{\left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]\}$. Its standard basis is

$$\left\{ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Show that $[h,e] = 2e$, $[h,f] = -2f$, and $[e,f] = h$. Whence, a representation of $\mathfrak{sl}_2$ on $\mathbb{C}^n$ is a triple of matrices E,H,F such that $[H,E] = 2E$, $[H,F] = -2F$, and $[E,F] = H$. How do we find such a thing?

1.2.9 Definition. If G is an algebraic group then an *algebraic representation* of G is a group homomorphism $\rho : G \to GL_V$ defined by polynomial equations.

We pull the same trick and substitute $E = \mathbb{C}[\varepsilon]/\varepsilon^2$ for $\mathbb{C}$ to get a homomorphism of groups $G(E) \to GL_V(E)$. As $\rho(I) = I$,

$$\rho(I + A\varepsilon) = I + \varepsilon(\text{some function of } A).$$

Call this function $d\rho$, so that $\rho(I + A\varepsilon) = I + \varepsilon d\rho(A)$, giving $d\rho : \mathfrak{g} \to \mathfrak{gl}_V$.

1.2.10 Exercises.

1. Show that if $\rho : G \to GL_V$ is a group homomorphism then $d\rho : \mathfrak{g} \to \mathfrak{gl}_V$ is a Lie algebra homomorphism.
2. Show that $d\rho$ truly is the differential of ρ at the identity when ρ is considered as a smooth map between manifolds.

We have a functor $\rho \mapsto d\rho$ from the algebraic representations of G to the Lie algebra representations of $\mathfrak{g} = T_1 G$. We will discuss this further, but first an example.

1.2.11 Example. Let $G = SL_2$, and let $L(n)$[1] be the collection of homogeneous polynomials of degree n in variables x and y, so $\dim_K L(n) = n+1$. Indeed, it has a basis $\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$. Now GL_2 acts on $L(n)$ via

$$\rho_n : GL_2 \to GL(L(n)) = GL_{n+1}$$

by $(\rho_n(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right])f)(x,y) = f(ax+cy, bx+dy)$. Then ρ_0 is the trivial representation, ρ_1 is the standard 2-dimensional representation (i.e. ρ_1 is the identity map on GL_2), and $\rho_2 \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has matrix

$$\begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}.$$

[1] $L(n) = \Gamma(\mathbb{P}^1, \mathcal{O}(n))$ and $H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0$ for all $n \geq 0$.

Now of course $SL_2 < GL_2$ so the restriction is a representation of SL_2. To compute $d\rho_n(e)(x^i y^j)$ (recall $e = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$) we compute:

$$\rho_n(1+\varepsilon e)x^i y^j = \rho_n \left[\begin{smallmatrix} 1 & \varepsilon \\ 0 & 1 \end{smallmatrix}\right] x^i y^j = x^i(\varepsilon x + y)^j = x^i y^j + \varepsilon j x^{i+1} y^{j-1}$$

Therefore $d\rho_n(e)(x^i y^j) = j x^{i+1} y^{j-1}$. Similarly,

$$\rho_n(1+\varepsilon f)x^i y^j = \rho_n \left[\begin{smallmatrix} 1 & 0 \\ \varepsilon & 1 \end{smallmatrix}\right] x^i y^j = (x + \varepsilon y)^i y^j = x^i y^j + \varepsilon i x^{i-1} y^{j+1}$$

and

$$\rho_n(1+\varepsilon f)x^i y^j = \rho_n \left[\begin{smallmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{smallmatrix}\right] x^i y^j = (1+\varepsilon)^i x^i (1-\varepsilon)^j y^j = x^i y^j + \varepsilon(i-j) x^i y^j,$$

so

$$e \cdot x^i y^j = j x^{i+1} y^{j-1} \qquad f \cdot x^i y^j = i x^{i-1} y^{j+1} \qquad h \cdot x^i y^j = (i-j) x^i y^j.$$

Upon closer inspection, these are operators that we know and love,

$$d\rho_n(e) = x\frac{\partial}{\partial y}, \qquad d\rho_n(f) = y\frac{\partial}{\partial x}, \qquad d\rho_n(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \qquad (1)$$

1.2.12 Exercises.

1. Check directly that defining the action of e, f, and h on $L(2)$ by (1) gives a representation of $\mathfrak{sl}_2$.

2. Check that $L(2)$ is the adjoint representation of $\mathfrak{sl}_2$ (notice that they are both 3-dimensional).

3. If the characteristic of K is zero, check that $L(n)$ is an irreducible representation of $\mathfrak{sl}_2$, and hence of the group SL_2.

1.2.13 Example. Let's compare representations of G and representations of $\mathfrak{g}$ when $G = \mathbb{C}^*$ (and hence $\mathfrak{g} = \mathbb{C}$ with $[x, y] = 0$ for all $x, y \in \mathbb{C}$). Recall that the irreducible algebraic representations of G are one dimensional. $z \in \mathbb{C}^*$ acts as multiplication by z^n ($n \in \mathbb{Z}$). Moreover, any representation of G splits up into a finite direct sum of one dimensional representations ("finite Fourier series").

In contrast, a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of $\mathfrak{g} = \mathbb{C}$ on V is any matrix $A = \rho(1) \in \mathrm{End}(V)$ ($\rho : \lambda \mapsto \lambda A$). A subrepresentation $W \subseteq V$ is just an A-stable subspace (so $AW \subseteq W$). Any linear transformation has an eigenvector, so there is always a 1-dimensional subrepresentation, which implies that all irreducible representations are the 1-dimensional ones. V breaks up into a direct sum of irreducibles (i.e. is completely reducible) if and only if A is diagonalizable. e.g. if A is the Jordan block of size n, then the invariant subspaces are $\langle e_1 \rangle, \langle e_1, e_2 \rangle, \ldots, \langle e_1, \ldots, e_n \rangle$, and does not break up into a direct sum of subspaces. Representations up to isomorphism of $\mathfrak{gl}_1$ are given by the theory of Jordan normal forms.

Representations of G and $\mathfrak{g}$ look "really different" in this case. Even the irreducibles are different, since for G they are in 1-1 correspondence with $\mathbb{Z}$ ($z \mapsto z^n$), while for $\mathfrak{g}$ they are in 1-1 correspondence with $\mathbb{C}$ ($\lambda \mapsto$ "$1 \mapsto \lambda$"). (Here $\mathbb{Z} \hookrightarrow \mathbb{C} : z \mapsto 2\pi i z$.)

In contrast, the map from representations of G to representation of $\mathfrak{g}$ is an equivalence of categories if G is a *simply connected simple* algebraic group. In particular, every representation of the Lie algebra $\mathfrak{sl}_2$ is a direct sum of irreducibles, and the irreducibles are exactly $L(n)$.

2 Representations of $\mathfrak{sl}_2$

2.1 Classification

From now on, all representations and Lie algebras are over $\mathbb{C}$ unless otherwise specified.

2.1.1 Theorem.

1. *There is a unique irreducible representation of* $\mathfrak{sl}_2$ *of dimension* $(n+1)$ *for every* $n \geq 0$*.*
2. *Every finite dimensional representation of* $\mathfrak{sl}_2$ *is isomorphic to a direct sum of irreducible representations.*

PROOF (FIRST PART): Let V be a representation of $\mathfrak{sl}_2$. Define

$$V_\lambda := \{v \in V \mid hv = \lambda v\},$$

the λ-*weight space*. It is the space of eigenvectors of h with eigenvalue λ. For example, $L(n)_\lambda = \mathbb{C}x^i y^j$ if $i - j = \lambda$, and in particular $L(n)_n = \mathbb{C}x^n$. Suppose that $v \in V_\lambda$, and consider ev.

$$h(ev) = (he - eh + eh)v = [h,e]v + ehv = 2ev + \lambda ev = (2+\lambda)ev$$

Whence $v \in V_\lambda$ if and only if $ev \in V_{\lambda+2}$. Similarly, $v \in V_\lambda$ if and only if $fv \in V_{\lambda-2}$.

A *highest weight vector* of weight λ is an element of $\ker(e) \cap V_\lambda$, i.e. a $v \in V$ such that $ev = 0$ and $hv = \lambda v$.

Claim. If v is a highest weight vector then $W = \text{span}\{v, fv, f^2v, \dots\}$ is an $\mathfrak{sl}_2$-invariant subspace of V.

We need to show that W is stable by e, f, h. It is clear that $fW \subseteq W$. As $hf^kv = (\lambda - 2k)f^kv$, it is clear that $hW \subseteq W$. By definition, $ev = 0$, so

$$efv = (ef - fe)v + fev = hv = \lambda v \in W.$$

Similarly,

$$\begin{aligned} ef^{k+1}v &= (ef - fe + fe)f^kv \\ &= hf^kv + fef^kv \\ &= (\lambda - 2k)f^kv + k(\lambda - k + 1)f^kv && \text{by induction} \\ &= (k+1)(\lambda - k)f^kv, \end{aligned}$$

so $ef^kv = k(\lambda - k + 1)f^{k-1}v \in W$, by induction, so $eW \subseteq W$ and the claim is proved.

Claim. If v is a highest weight vector of weight λ then $\lambda \in \{0, 1, 2, \dots\}$ if V is finite dimensional.

Indeed, the vectors $f^k v$ all lie in different eigenspaces of h, and hence are linearly independent if they are non-zero. If V is finite dimensional then $f^k v = 0$ for some k. Let k be minimal with this property. Then

$$0 = ef^k v = k(\lambda - k + 1)f^{k-1}v,$$

and $k \geq 1$, so $\lambda = k - 1 \in \{0, 1, 2, \dots\}$ since $f^{k-1}v \neq 0$.

Claim. There exists a highest weight vector if V is finite dimensional.

Let $v \in V$ be any eigenvector for h with eigenvalue λ (such things exist because $\mathbb{C}$ is algebraically closed). As before, $v, ev, e^2v, \dots$ are all eigenvectors for h with eigenvalues $\lambda, \lambda+2, \lambda+4, \dots$. Hence they are linearly independent, so there is some $k+1$ minimal such that $e^{k+1}v = 0$. Clearly $e^k v \in V_{\lambda+2k}$ is a highest weight vector, proving the claim.

If V is irreducible and $\dim_{\mathbb{C}} V = n+1$ then there is a basis $\{v_0, \dots, v_n\}$ such that

$$hv_i = (n-2i)v_i, \qquad fv_i = \begin{cases} v_{i+1} & \text{if } i < n \\ 0 & \text{if } i = n \end{cases}, \qquad ev_i = i(n-i+1)v_{i-1}.$$

Indeed, let v_0 be a highest weight vector with weight $\lambda \in \{0, 1, 2, \dots\}$. The *string* $\{v, fv, \dots, f^\lambda v\}$ is a subrepresentation of V. As V is irreducible it must be all of V, so $\lambda = \dim_{\mathbb{C}} V - 1$. Take $v_i = f^i v$. This proves the first part of the theorem. □

2.1.2 Exercise. $\mathbb{C}[x, y]$ is a representation of $\mathfrak{sl}_2$ by the formulae in 1.2.11. Show that for $\lambda, \mu \in \mathbb{C}$, $x^\lambda y^\mu \mathbb{C}[x, y]$ is also a representation, infinite dimensional, and analyze its submodule structure.

PROOF (SECOND PART): We will show that strings of different lengths "do not interact," and then that strings of the same length "do not interact."

Let V be a finite dimensional representation of $\mathfrak{sl}_2$. Let

$$\Omega := ef + fe + \tfrac{1}{2}h^2 \in \operatorname{End}(V),$$

the *Casimir* of $\mathfrak{sl}_2$. Also notice that $\Omega = (\frac{1}{2}h^2 + h) + 2fe$.

Claim. (1) Ω is central, i.e. $\Omega e = e\Omega$, $\Omega f = f\Omega$, and $\Omega h = h\Omega$.

The proof of this is an exercise. For example,

$$\begin{aligned}
\Omega e &= e(ef + fe + \tfrac{1}{2}h^2) \\
&= e(ef - fe) + 2efe + \tfrac{1}{2}heh + \tfrac{1}{2}(eh - he)h \\
&= eh + 2efe + \tfrac{1}{2}heh - eh \\
&= 2efe + \tfrac{1}{2}heh \\
&= (ef + fe + \tfrac{1}{2}h^2)e
\end{aligned}$$

Claim. (2) If V is an irreducible representation with highest weight n then Ω acts on V by multiplication by $\frac{1}{2}n^2 + n$.

(It follows from Schur's Lemma that if V is an irreducible representation of $\mathfrak{sl}_2$ then Ω acts on V as multiplication by a scalar.) Let v be a highest weight vector, so $hv = nv$ and $ev = 0$, so

$$\Omega v = ((\tfrac{1}{2}h^2 + h) + 2fe)v = (\tfrac{1}{2}n^2 + n)v.$$

Hence $\Omega(f^k v) = f^k(\Omega v) = (\frac{1}{2}n^2 + n)f^k v$, so Ω acts by multiplication by $\frac{1}{2}n^2 + n$ since $\{v, fv, \dots, f^n v\}$ is a basis for V.

Claim. (3) If $L(n)$ and $L(m)$ are both irreducible representations and Ω acts on them by the same scalar ω then $n = m$ (so $L(n) \cong L(m)$) and $\omega = \frac{1}{2}n^2 + n$.

Indeed, $L(n)$ and $L(m)$ are irreducible representations with highest weight n and m, respectively, so by the last claim $\frac{1}{2}n^2 + n = \omega = \frac{1}{2}m^2 + m$. But $f(x) = \frac{1}{2}x^2 + x$ is a strictly increasing function for $x > -1$, so $m = n$ and the representations are isomorphic.

Let $V^\omega = \{v \in V \mid (\Omega - \omega I)^{\dim_{\mathbb{C}} V} v = 0\}$ be the generalized eigenspace of Ω associated with ω. The Jordan decomposition for Ω implies that $V \cong \bigoplus_\omega V^\omega$ (as vector spaces).

Claim. (4) Each V^ω is a subrepresentation of V, i.e. the above is a direct sum decomposition as representations of $\mathfrak{sl}_2$.

Indeed, if $x \in \mathfrak{sl}_2$ and $v \in V^\omega$ then since Ω is central,

$$(\Omega - \omega)^{\dim_{\mathbb{C}} V} xv = x(\Omega - \omega)^{\dim_{\mathbb{C}} V} v = 0.$$

Aside: For a $\mathfrak{g}$-module W, a *composition series* is a sequence of submodules

$$0 = W_0 < W_1 < \cdots < W_r = W$$

such that each quotient module W_i/W_{i-1} is an irreducible $\mathfrak{g}$-module. For example, with $\mathfrak{g} = \mathbb{C}$ and $W = \mathbb{C}^2$, acting as $1 \mapsto \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, then $\langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle$ is a composition series (and this is the only one). If the action is instead $1 \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ then any line in $\mathbb{C}^2$ gives rise to a composition series.

Recall that if W is finite dimensional then it has a composition series, and the subquotients are unique up to permutations.

Claim. (5) If $V^\omega \neq 0$ then $\omega = \frac{1}{2}n^2 + n$ for some n, and all the subquotients of a composition series for V^ω are $L(n)$, i.e. V^ω is a bunch of copies of $L(n)$, possibly stuck together in a complicated way.

Indeed, suppose W is an irreducible subrepresentation of V^ω. Then Ω acts on W as multiplication by ω, since ω is the only possible generalized eigenvalue and Ω acts on irreducible representations as scalar mulitplication by Schur's Lemma. Thus $\omega = \frac{1}{2}n^2 + n$ for some n with $W \cong L(n)$, by claim (3). By the same reasoning, Ω will act on any irreducible subrepresentation of V^ω/W as multiplication by ω, so any such irreducible subrepresentation must be $L(n)$, for the same n. Whence every subquotient in a composition series is isomorphic to $L(n)$.

To finish the proof we must show that each V^ω is a direct sum of $L(n)$'s. The key point will be to show that h is "diagonalizable." At this stage we may assume that $V = V^\omega$.

Notice that h acts on V with eigenvalues $\{n, n-2, \dots, 2-n, -n\}$. (It is a fact from linear algebra that if h acts on a vector space W and $W' \leq W$ is an h-invariant subspace then the eigenvalues of h on W are the eigenvalues of h on W' together with the eigenvalues of h on W/W'. The remark follows from this since V has a composition series with subquotients all isomorphic to $L(n)$, and we know from the first part that the eigenvalues of h on $L(n)$ are $\{n, n-2, \dots, 2-n, -n\}$.) Hence $V_m = 0$ if $m \notin \{n, n-2, \dots, 2-n, -n\}$. It follows that h has only one generalized eigenvalue on $\ker(e)$, namely n.

Why does this follow?

Claim. $hf^k = f^k(h-2k)$ and $ef^{n+1} = f^{n+1}e + (n+1)f^n(h-n)$.

The first equation is trivial, and the second is proved by induction as follows. Clearly $ef = fe + ef - fe = fe + h$, and

$$\begin{aligned} ef^{n+1} &= (fe + ef - fe)f^n \\ &= f^{n+1}e + nf^n(h-n+1) + f^n(h-2n) \qquad \text{by induction} \\ &= f^{n+1}e + (n+1)f^n(h-n). \end{aligned}$$

Claim. h acts as multiplication by n on $\ker(e)$, and further, $V_n = \ker(e)$.

Indeed, if $v \in V_n$ (so $hv = nv$) then $ev \in V_{n+2} = 0$, so $V_n \subseteq \ker(e)$. For the converse, consider these statements:

1. If $x \in \ker(e)$, then $(h-n)^{\dim V} x = 0$ since n is the only generalized eigenvalue of h on $\ker(e)$.
2. If $x \in \ker(e)$, then $(h-n+2k)^{\dim V} f^k x = f^k (h-n)^{\dim V} x = 0$ by the previous claim.
3. If $y \in \ker(e)$ and $y \neq 0$ then $f^n y \neq 0$. (Indeed, let (W_i) be a composition series for V, as above. Then there is i such that $y \in W_i \setminus W_{i-1}$. Let $\bar{y} = y + W_i \in W_i/W_{i-1} \cong L(n)$. Then $\bar{y}$ is a highest weight vector of $L(n)$, so $f^n \bar{y} \neq 0$, hence $f^n y \neq 0$.)
4. $f^{n+1}x = 0$, as $f^{n+1}x$ is in the generalized eigenspace with eigenvalue $-n-2$, which is zero.

Hence

$$0 = ef^{n+1}x = (n+1)f^n(h-n)x + f^{n+1}ex = (n+1)f^n(h-n)x$$

implying $f^n((h-n)x) = 0$, so by statement 3, $(h-n)x = 0$, or $hx = nx$.

Finally, choose a basis $\{w_1, \dots, w_\ell\}$ of $V_n = \ker(e)$. Then

$$\operatorname{span}\{w_1, fw_1, \dots, f^n w_1\} \oplus \operatorname{span}\{w_2, fw_2, \dots, f^n w_2\} \oplus \dots \oplus \operatorname{span}\{w_\ell, fw_\ell, \dots, f^n w_\ell\}$$

is a direct sum decomposition of V into subrepresentations, each isomorphic to $L(n)$. □

2.2 Consequences

2.2.1 Exercise. If V and W are representations of a Lie algebra $\mathfrak{g}$ then the map

$$\mathfrak{g} \to \operatorname{End}(V) \otimes \operatorname{End}(W) = \operatorname{End}(V \otimes W) : x \mapsto x \otimes 1 + 1 \otimes x$$

is a homomorphism of Lie algebras. (Recall that if a group G acts on V and W then G acts on $V \otimes W$ as $g \mapsto g \otimes g$. An action $\mathfrak{g} \to \operatorname{End}(V \otimes W)$ is obtained from this by differentiating.)

2.2.2 Corollary. *If V and W are representations of $\mathfrak{g}$ then so it $V \otimes W$.*

Remark. If A is an associative algebra and V and W are representations of A then $V \otimes W$ is a representation of $A \otimes A$ and *not* of A. So to make it a representation of A you need additional structure, an algebra map $A \to A \otimes A$ (making A a *Hopf algebra*, apparently).

2.2.3 Example. Now take $\mathfrak{g} = \mathfrak{sl}_2$. Then $L(n) \otimes L(m)$ is also a representation of $\mathfrak{sl}_2$, so according to our theorem it should decompose into representations $L(k)$ with some multiplicities, but how?

The naive method is to try and find all the highest weight vectors in $L(n) \otimes L(m)$. As an exercise, find all highest weight vectors in $L(1) \otimes L(n)$ and $L(2) \otimes L(n)$. If v_r denotes a highest weight vector in $L(r)$ then $v_n \otimes v_m$ is a highest weight vector in $L(n) \otimes L(m)$. Indeed,

$$h(v_n \otimes v_m) = (hv_n) \otimes v_m + v_n \otimes (hv_m) = (n+m)v_n \otimes v_m$$

and

$$e(v_n \otimes v_m) = (ev_n) \otimes v_m + v_n \otimes (ev_m) = 0.$$

This is a start, but $L(n) \otimes L(m) = L(n+m) +$ lots of other stuff, since

$$\dim(L(n) \otimes L(m)) = (n+1)(m+1) > n+m+1 = \dim L(n+m).$$

It is possible to write down explicit formulae for all the highest weight vectors in $L(n) \otimes L(m)$; they are messy but interesting (compute them as a non-compulsory exercise).

A better method is to use the weight space decomposition for h, which we will see later.

2.3 Characters

2.3.1 Definition. Let V be a finite dimensional representation of $\mathfrak{sl}_2$. Then

$$\operatorname{ch} V = \sum_{n \in \mathbb{Z}} \dim V_n z^n \in \mathbb{Z}^+[z, z^{-1}]$$

is the *character* of V.

2.3.2 Proposition (Properties of Characters).

1. $\operatorname{ch} V|_{z=1} = \dim V$
2. $\operatorname{ch} L(n) = z^n + z^{n-1} + \cdots + z^{2-n} + z^{-n} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} =: [n+1]$.

3. *$\operatorname{ch} V = \operatorname{ch} W$ implies that $V \cong W$.*

4. *$\operatorname{ch}(V \otimes W) = \operatorname{ch}(V)\operatorname{ch}(W)$.*

PROOF:

1. By our theorem, h acts diagonalizably on V, so $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with integer eigenvalues.

2. We saw this in the proof of our theorem.

3. $\{\operatorname{ch} L(n) \mid n \geq 0\}$ forms a basis for $\mathbb{Z}[z, z^{-1}]^{\mathbb{Z}/2\mathbb{Z}}$ (the action of $\mathbb{Z}/2\mathbb{Z}$ is that of swapping z and z^{-1}). Since any V uniquely decomposes as a direct sum $\bigoplus_{n\geq 0} a_n L(n)$, we have $\operatorname{ch} V = \sum_{n\geq 0} a_n[n+1]$, which uniquely determines the a_n.

4. $V_n \otimes W_m \subseteq (V \otimes W)_{n+m}$, so $(V \otimes W)_p = \sum_{n+m=p} V_n \otimes W_m$. □

Back to the example. For example we compute

$$\begin{aligned} \operatorname{ch}(L(1) \otimes L(3)) &= \operatorname{ch} L(1) \operatorname{ch} L(3) \\ &= (z + z^{-1})(z^3 + z + z^{-1} + z^{-3}) \\ &= (z^4 + z^2 + 1 + z^{-2} + z^{-4}) + (z^2 + 1 + z^{-2}) \end{aligned}$$

so $L(1) \otimes L(3) \cong L(4) + L(2)$. In general, the *Clebsch-Gordan rule* says

$$L(n) \otimes L(m) = \bigoplus_{\substack{k=|n-m| \\ k\equiv n+m \bmod 2}}^{k=|n+m|} L(k).$$

3 Semi-Simple Lie Algebras: Structure and Classification

3.0.3 Definition. A Lie algebra $\mathfrak{g}$ is *simple* if

1. $\mathfrak{g}$ is non-Abelian;

2. $\dim \mathfrak{g} > 1$; and

3. the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$.

$\mathfrak{g}$ is *semi-simple* if it is a direct sum of simple Lie algebras.

3.0.4 Examples.

1. $\mathbb{C}$ is not a simple or semi-simple Lie algebra.

2. $\mathfrak{sl}_2$ is a simple Lie algebra.

3.1 Solvable and Nilpotent Lie Algebras

3.1.1 Definition. Let $\mathfrak{g}$ be a Lie algebra. The *commutator algebra* of $\mathfrak{g}$ is $[\mathfrak{g},\mathfrak{g}] = \text{span}\{[x,y] \mid x,y \in \mathfrak{g}\}$.

3.1.2 Exercises.

1. $[\mathfrak{g},\mathfrak{g}]$ is an ideal of $\mathfrak{g}$; and
2. $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is Abelian.

3.1.3 Definition. The *central series* of $\mathfrak{g}$ is defined inductively by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^n = [\mathfrak{g}^{n-1},\mathfrak{g}]$ for $n \geq 1$. The *derived series* is defined by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)},\mathfrak{g}^{(n-1)}]$ for $n \geq 1$.

It is clear that $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$ for all n.

3.1.4 Definition. A Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if $\mathfrak{g}^n = 0$ for some $n > 0$ (i.e. if the central series ends at zero, eventually), and *solvable* if $\mathfrak{g}^{(n)} = 0$ for some $n > 0$ (i.e. if the derived series ends at zero, eventually).

Clearly nilpotent implies solvable.

3.1.5 Example. The Lie algebra $\mathfrak{n}$ of strictly upper triangular matrices is nilpotent, while the Lie algebra $\mathfrak{b}$ of upper triangular matrices is solvable but not nilpotent. Abelian Lie algebras are nilpotent.

3.1.6 Exercises.

1. Compute the derived and central series of the upper triangular and strictly upper triangular matrices and confirm the claims made in the example.
2. Compute the center of the upper triangular and strictly upper triangular matrices.
3. Let L be a vector space. Show that $W = L \oplus L^*$ is a symplectic vector space (i.e. a vector space together with a non-degenerate alternating form) with the alternating form $\langle \cdot,\cdot \rangle$ defined by $\langle L,L \rangle = \langle L^*,L^* \rangle = 0$ and $\langle v,f \rangle = f(v)$ for $v \in L$ and $f \in L^*$ (defined to be alternating).
4. Let W be a symplectic vector space. Define the *Heisenberg Lie algebra*, *Heis*$_W$, to be $W \oplus Kc$, where c is a fresh variable, with bracket defined by $[c,W] = 0$ and $[x,y] = \langle x,y \rangle c$ for $x,y \in W$. Show that Heis_W is a nilpotent Lie algebra. Is it Abelian?
5. Show that the simplest Heisenberg algebra (taking $L = \mathbb{C}p$ and $L^* = \mathbb{C}q$) has a representation on $\mathbb{C}[x]$ by $p \mapsto \frac{\partial}{\partial x}$, $q \mapsto x$, and $c \mapsto 1$.

3.1.7 Proposition.

1. *Subalgebras and quotient algebras of solvable (resp. nilpotent) Lie algebras are also solvable (resp. nilpotent).*
2. *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, then $\mathfrak{g}$ is solvable if and only if both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable*
3. *$\mathfrak{g}$ is nilpotent if and only if the center of $\mathfrak{g}$ is non-zero and $\mathfrak{g}/Z_{\mathfrak{g}}$ is nilpotent.*

PROOF: Exercise. (For the third part, suppose $\mathfrak{g} \supsetneq \mathfrak{g}^1 \supsetneq \cdots \supsetneq \mathfrak{g}^n = 0$. Then $[\mathfrak{g}^{n-1}, \mathfrak{g}] = 0$, so $\mathfrak{g}^{n-1} \neq 0$ is contained in the center of $\mathfrak{g}$.) □

3.1.8 Corollary. *In particular, $\mathfrak{g}$ is nilpotent if and only if $\operatorname{ad}\mathfrak{g} \subseteq \mathfrak{gl}_{\mathfrak{g}}$ is nilpotent.*

PROOF: $\operatorname{ad}\mathfrak{g} = \mathfrak{g}/Z_{\mathfrak{g}}$. □

3.1.9 Theorem (Lie). *Let $\mathfrak{g} \subseteq \mathfrak{gl}_V$ (V finite dimensional) be a solvable Lie algebra. Suppose K has characteristic zero and is algebraically closed. Then there is a basis $\{v_1, \dots, v_n\}$ of V such that, with respect to this basis, the matrices of all elements of $\mathfrak{g}$ are upper triangular.*

PROOF: Omitted. □

Equivalently, Lie's theorem says that there is a common eigenvector for $\mathfrak{g}$, i.e. there is $\lambda : \mathfrak{g} \to K$ linear and $v \in V$ such that $xv = \lambda(x)v$ for all $x \in \mathfrak{g}$. This is the same as saying that V has a 1-dimensional subrepresentation.

3.1.10 Exercises.

1. Prove the equivalence of the two formulations of Lie's Theorem (essential exercise).
2. Show that it is necessary that K is algebraically closed and that the characteristic is zero. (Hint: let $K = \mathbb{F}_p$ and $\mathfrak{g}$ be the 3-dimensional $\operatorname{Heis}(\frac{\partial}{\partial x}, x, c)$. Show that $K[x]/x^p$ is an irreducible representation of $\mathfrak{g}$.)

3.1.11 Corollary. *Suppose K has characteristic zero. If $\mathfrak{g}$ is a solvable Lie algebra then $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra.*

PROOF: Apply Lie's Theorem to the adjoint representation $\operatorname{ad}\mathfrak{g} \subseteq \mathfrak{gl}_{\mathfrak{g}}$. Then $\operatorname{ad}\mathfrak{g}$ is solvable, and Lie's theorem implies that it sits inside the collection of upper triangular matrices for some choice of basis. Hence $[\operatorname{ad}\mathfrak{g}, \operatorname{ad}\mathfrak{g}]$ consists of strictly upper triangular matrices, so it is nilpotent. But $\operatorname{ad}[\mathfrak{g}, \mathfrak{g}] = [\operatorname{ad}\mathfrak{g}, \operatorname{ad}\mathfrak{g}]$, so $\operatorname{ad}[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, so $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent by 3.1.8. □

3.1.12 Exercise. Construct a counterexample in characteristic p.

3.1.13 Theorem (Engel). *Let $\mathfrak{g} \subseteq \mathfrak{gl}_V$ (V finite dimensional), and suppose that $x : V \to V$ is a nilpotent transformation for all $x \in \mathfrak{g}$. Then there is a common eigenvector v for $\mathfrak{g}$.*

PROOF: Omitted. □

Here "nilpotent" means that all the generalized eigenvalues are 0. Whence Engel's theorem says that V has a 1-dimensional subrepresentation, or that $xv = 0$ for all $x \in \mathfrak{g}$.

3.1.14 Corollary. *$\mathfrak{g}$ is nilpotent if and only if $\operatorname{ad}\mathfrak{g}$ consists of nilpotent endomorphisms of $\mathfrak{g}$.*

PROOF: Exercise. □

Cryptic Remark: Solvable and nilpotent Lie algebras are garbage.

3.1.15 Definition. A symmetric bilinear form (a.k.a. inner product) $(\cdot,\cdot) : \mathfrak{g}\times\mathfrak{g} \to K$ is *invariant* if $([x, y], z) = (x, [y, z])$ for all $x, y, z \in \mathfrak{g}$.

3.1.16 Exercise. If $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal and $(\cdot,\cdot)$ is an invariant inner product on $\mathfrak{g}$ then

$$\mathfrak{h}^\perp := \{x \in \mathfrak{g} \mid (x, \mathfrak{h}) = 0\}$$

is an ideal. In particular, $\mathfrak{g}^\perp$ is an ideal.

3.1.17 Example (Trace forms). The most important class of invariant inner products are the so-called *trace forms*, defined as follows. If (V, ρ) is a representation of $\mathfrak{g}$, define $(\cdot,\cdot)_V : \mathfrak{g} \times \mathfrak{g} \to K$ by

$$(x, y)_V = \operatorname{tr}(\rho(x)\rho(y) : V \to V).$$

Clearly $(\cdot,\cdot)_V$ is symmetric and bilinear. Show that it is invariant (you will need to use the fact that ρ is a representation).

The *Killing form* is $(x, y)_{\mathrm{ad}} = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y : \mathfrak{g} \to \mathfrak{g})$.

3.1.18 Theorem (Cartan's Criterion). *Let V be a finite dimensional representation of $\mathfrak{g}$ (i.e. $\mathfrak{g} \hookrightarrow \mathfrak{gl}_V$) and K have characteristic zero. Then $\mathfrak{g}$ is solvable if and only if $(x, y)_V = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$.*

PROOF: The forward direction is immediate from Lie's Theorem. The other direction is where the content lies, and we omit the proof. □

3.1.19 Corollary. *$\mathfrak{g}$ is solvable if and only if $(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}])_{\mathrm{ad}} = 0$.*

PROOF: The forward direction follows from Lie's Theorem. For the reverse direction, Cartan's Criterion implies that $\mathfrak{g}/Z_{\mathfrak{g}} = \operatorname{ad}\mathfrak{g}$ is solvable. But then $\mathfrak{g}$ is solvable as well. □

If $\mathfrak{g}$ is a solvable non-Abelian Lie algebra then all trace forms are degenerate (as $\mathfrak{g}^\perp \supseteq [\mathfrak{g}, \mathfrak{g}]$). Warning/Exercise: Not every invariant inner product is a trace form.

3.1.20 Exercise. Let $\tilde{\mathfrak{h}} = \langle c, p, q, d\rangle$ be defined by $[c, \tilde{\mathfrak{h}}] = 0$, $[p, q] = c$, $[d, p] = p$, and $[d, q] = -q$ (we think of $p = \frac{\partial}{\partial x}$, $q = z$, and $d = -x\frac{\partial}{\partial x}$). Then $\tilde{\mathfrak{h}}$ is a Lie algebra. Show that it is solvable, and construct a non-degenerate invariant inner product on it.

3.1.21 Definition. The *radical* of $\mathfrak{g}$ is the maximal solvable ideal of $\mathfrak{g}$, denoted $R(\mathfrak{g})$.

3.1.22 Exercises.

1. Show that the sum of solvable ideals is solvable, and hence $R(\mathfrak{g})$ is the sum of all solvable ideals of $\mathfrak{g}$ (so there is none of this nasty Zorn's Lemma crap).
2. $R(\mathfrak{g}/R(\mathfrak{g})) = 0$.

Recall that $\mathfrak{g}$ is semi-simple (s.s.) if it is a direct sum of simple (non-Abelian) Lie algebras.

3.1.23 Theorem. *Suppose the characteristic of K is zero. The following are equivalent.*

1. *$\mathfrak{g}$ is semi-simple;*
2. *$R(\mathfrak{g}) = 0$;*
3. *The Killing form $(\cdot,\cdot)_{\mathrm{ad}}$ is non-degenerate (ironically, this is the "Killing criterion").*

Moreover, if $\mathfrak{g}$ is s.s. then every derivation $D : \mathfrak{g} \to \mathfrak{g}$ (i.e. map D satisfying $D[a,b] = [Da,b] + [a,Db]$) is inner (i.e. $D = \operatorname{ad} x$ for some $x \in \mathfrak{g}$), but not conversely.

PROOF: Begin by observing that $R(\mathfrak{g}) = 0$ if and only if $\mathfrak{g}$ has no non-zero commutative ideals.

(i) implies (ii): If $\mathfrak{g}$ is s.s. then it has no non-zero commutative ideals, by definition.

(iii) implies (ii): It suffices to show that if $\mathfrak{a}$ is a commutative ideal then it is contained in the kernel of the Killing form, i.e. $\mathfrak{g}^{\perp}$. Write $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$ for $\mathfrak{h}$ some vector space complement. If $a \in \mathfrak{a}$ then $\operatorname{ad} a$ has matrix $\left[\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right]$, as $\mathfrak{a}$ is Abelian and an ideal. For $x \in \mathfrak{g}$, $\operatorname{ad} x$ has matrix $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$, so $(a,x)_{\mathrm{ad}} = \operatorname{tr}(\operatorname{ad}(a)\operatorname{ad}(x)) = 0$.

(ii) implies (iii): Let $\mathfrak{r} = \mathfrak{g}^{\perp}$, an ideal. We have a map $\mathfrak{r} \xrightarrow{\mathrm{ad}} \mathfrak{gl}_{\mathfrak{g}}$ and $(x,y)_{\mathrm{ad}} = 0$ for all $x, y \in \mathfrak{r}$, by definition. By Cartan's Criterion $\mathfrak{r}/Z(\mathfrak{r})$ is solvable, and hence $\mathfrak{r}$ is solvable and so $\mathfrak{r} \subseteq R(\mathfrak{g}) = 0$.

(iii), (ii) imply (i): Suppose $(\cdot,\cdot)_{\mathrm{ad}}$ is non-degenerate. Let $\mathfrak{a} \subseteq \mathfrak{g}$ be a non-zero minimal ideal. Then $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}}$ is either zero or non-degenerate. Indeed, the kernel of $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{a}$ is minimal. But if $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}} = 0$ then the Cartan Criterion implies that $\mathfrak{a}$ is solvable. But $R(\mathfrak{g}) = 0$, so there should be no solvable ideals. Therefore $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}}$ is non-degenerate, and we have $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$, a direct sum decomposition as ideals with $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}^{\perp}}$ non-degenerate. $\mathfrak{a}$ is not solvable, so it must be simple (non-Abelian). By induction on $\dim_K \mathfrak{g}$, we may write $\mathfrak{g}$ as a direct sum of simple Lie algebras, so it is s.s.

Finally, if D is a derivation then we get a linear function $\ell : \mathfrak{g} \to K : x \mapsto \operatorname{tr}_{\mathfrak{g}}(D \cdot \operatorname{ad} x)$. Since $\mathfrak{g}$ is s.s., $(\cdot,\cdot)_{\mathrm{ad}}$ is non-degenerate, so $\ell(x) = (y,x)_{\mathrm{ad}}$ for a unique $y \in \mathfrak{g}$. This implies that $\operatorname{tr}_{\mathfrak{g}}((D - \operatorname{ad} y) \cdot \operatorname{ad} x) = 0$ for all $x \in \mathfrak{g}$. Let $E = D - \operatorname{ad} y$, a derivation. We must show $E = 0$. Notice that

$$\operatorname{ad}(Ex)(z) = [Ex,z] = E[x,z] - [x,Ez] = [E,\operatorname{ad} x](z)$$

whence

$$\begin{aligned}(Ex,z)_{\mathrm{ad}} &= \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}(Ex)\cdot \operatorname{ad} z)\\ &= \operatorname{tr}_{\mathfrak{g}}([E,\operatorname{ad} x]\cdot \operatorname{ad} z)\\ &= \operatorname{tr}_{\mathfrak{g}}(E[\operatorname{ad} x,\operatorname{ad} z])\\ &= \operatorname{tr}_{\mathfrak{g}}((D-\operatorname{ad} y)\operatorname{ad}[x,z]) = 0\end{aligned}$$

for all $z \in \mathfrak{g}$, so $Ex = 0$ for all $x \in \mathfrak{g}$. □

3.1.24 Exercises.

1. Prove that $R(\mathfrak{g}) \supseteq \ker(\cdot,\cdot)_{\mathrm{ad}} \supseteq [R(\mathfrak{g}),R(\mathfrak{g})]$.

2. Show that $\mathfrak{g}$ s.s. implies that $\mathfrak{g}$ can be written *uniquely* as a direct sum of its minimal ideals.

3. Show that a nilpotent Lie algebra always has a non-inner derivation.

4. Show that the Lie algebra defined by $\mathfrak{h} = \langle a, b \rangle$, where $[a, b] = b$, has only inner derivations.

Remark. The exact sequence $0 \to R(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/R(\mathfrak{g}) \to 0$ shows that every Lie algebra has a maximal s.s. quotient, and is an extension of this by a maximal solvable ideal. In fact we have the following theorem

3.1.25 Theorem (Levi's Theorem). *If K has characteristic zero then the exact sequence splits, i.e. there is a subalgebra $\mathfrak{h} \hookrightarrow \mathfrak{g}$ such that $\mathfrak{g} \cong \mathfrak{h} \rtimes R(\mathfrak{g})$ (indeed, we may take $\mathfrak{h} = \mathfrak{g}/R(\mathfrak{g})$).*

PROOF: Omitted. □

3.1.26 Exercise. Show $R(\mathfrak{sl}_p(\overline{\mathbb{F}}_p)) = \overline{\mathbb{F}}_p I$, but there does not exist a splitting $\mathfrak{g}/R(\mathfrak{g}) \hookrightarrow \mathfrak{g}$.

3.1.27 Exercises.

1. Let $\mathfrak{g}$ be a simple Lie algebra and $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ be non-degenerate invariant forms. Then there is $\lambda \in K^*$ such that $(\cdot,\cdot)_1 = \lambda(\cdot,\cdot)_2$. (Apply Schur's Lemma.)

2. (essential exercise) Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and put $(A, B) = \operatorname{tr}(AB)$, so $(A, B) = \lambda(A, B)_{\text{ad}}$. Find λ (you may assume $\mathfrak{g}$ is simple).

3.2 Structure Theory

3.2.1 Definition. A *torus* $\mathfrak{t} \subseteq \mathfrak{g}$ is an Abelian subalgebra such that $\operatorname{ad} t : \mathfrak{g} \to \mathfrak{g}$ is semi-simple (i.e. diagonalizable) for all $t \in \mathfrak{t}$. A *maximal torus* is a torus which is maximal with respect to inclusion. A maximal torus is also known as a *Cartan subalgebra*.

3.2.2 Exercises.

1. Let G be an algebraic group, T a subgroup isomorphic to $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$. Then $\mathfrak{t} = T_1 T$ is a torus.

2. (essential exercise) Let $\mathfrak{g} = \mathfrak{sl}_n$, or

$$\mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid JA + A^T J = 0\}$$

or

$$\mathfrak{so}_n = \{A \in \mathfrak{gl}_n \mid MA + A^T M = 0\},$$

where $M = \text{anti-diag}(1, \dots, 1)$. Prove that $\mathfrak{t} = \{\text{diagonal matrices in } \mathfrak{g}\}$ is a maximal torus.

3.2.3 Lemma. *Let $t_1, \dots, t_r : V \to V$, where V is finite dimensional, be pairwise commuting, semi-simple, linear maps. Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r$, and set $V_{\vec{\lambda}} = \{v \in V \mid t_i v = \lambda_i v, i = 1, \dots, r\}$ be the simultaneous eigenspace. Then $V = \bigoplus_{\vec{\lambda} \in \mathbb{C}^r} V_{\vec{\lambda}}$.*

PROOF: By induction on r. The case $r = 1$ is clear since t_1 is assumed to be semi-simple, and hence diagonalizable. If $r > 1$ then $V = \bigoplus_{\mathbb{C}^{r-1}} V_{(\lambda_1,\dots,\lambda_{r-1})}$ by induction. As t_r commutes with $t_1, \dots, t_{r-1}$, $t_r(V_{(\lambda_1,\dots,\lambda_{r-1})}) \subseteq V_{(\lambda_1,\dots,\lambda_{r-1})}$, so decompose $V_{(\lambda_1,\dots,\lambda_{r-1})}$ as a direct sum of eigenspaces. □

We now rephrase the above lemma in the language we have developed so far. Let $\mathfrak{t} := \bigoplus_{i=1}^{r} \mathbb{C}t_i$ be an Abelian Lie algebra isomorphic to $\mathbb{C}^r$. Then V is a representation of $\mathfrak{t}$, by definition. Since a basis of $\mathfrak{t}$ acts on V semi-simply, $V = \bigoplus_{\vec{\lambda}\in\mathbb{C}^r} V_{\vec{\lambda}}$ is the decomposition of V as a representation of $\mathfrak{t}$, and is a direct sum of irreducibles. Hence all $t \in \mathfrak{t}$ act semi-simply (as on each $V_{\vec{\lambda}}$ they act as multiplication by a scalar).

Moreover, it is convenient to think of λ as being in the dual space $\mathfrak{t}^*$. For such a $\lambda \in \mathfrak{t}^*$, the λ-weight space of V is $V_\lambda = \{v \in V \mid tv = \lambda(t)v\}$. (In the above lemma we would take $\lambda(\sum_i x_i t_i) = \sum x_i \lambda_i$.)

3.2.4 Proposition. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\mathfrak{t}$ a maximal torus. Then $\mathfrak{t} \neq 0$, and the collection of elements of $\mathfrak{g}$ which commute with $\mathfrak{t}$ is exactly $\mathfrak{t}$ itself.*

PROOF: Omitted. □

We may write $\mathfrak{g} = \mathfrak{g}_0 + \bigoplus_{\lambda\in\mathfrak{t}^*,\lambda\neq 0} \mathfrak{g}_\lambda$, where for $\lambda \in \mathfrak{t}^*$,

$$\mathfrak{g}_\lambda := \{x \in \mathfrak{g} \mid [t, x] = \lambda(t)x \text{ for all } t \in \mathfrak{t}\}.$$

This is known as the *root-space decomposition* of $\mathfrak{g}$. The proposition implies that when $\mathfrak{g}$ is s.s., $\mathfrak{g}_0 = \mathfrak{t}$.

3.2.5 Definition. If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$ then we say that λ is a *root* of $\mathfrak{g}$. Let $R = R_{\mathfrak{g}} := \{\lambda \in \mathfrak{t}^* \mid \lambda \neq 0, \mathfrak{g}_\lambda \neq 0\}$, the set of roots of $\mathfrak{g}$.

3.2.6 Example. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, so that $\mathfrak{t}$ is the collection of diagonal matrices in $\mathfrak{g}$. Then for $t \in \mathfrak{t}$, $[t, E_{ij}] = (t_i - t_j)E_{ij}$. Let $\varepsilon_i : \mathfrak{t} \to \mathbb{C} : t \mapsto t_{ii}$. Then $[t, E_{ij}] = (\varepsilon_i - \varepsilon_j)(t)E_{ij}$. Note that $\varepsilon_1 + \cdots + \varepsilon_n = 0$ since $\mathfrak{sl}_n$ is the collection of matrices with trace zero. In this case we have that $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$ is one-dimensional, and $R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$. We will see that this holds in general.

3.2.7 Proposition. *$\mathfrak{sl}_n(\mathbb{C})$ is simple (and hence semi-simple).*

The Killing criterion can be used to show that $\mathfrak{sl}_n(\mathbb{C})$ is semi-simple (do so as an exercise).

PROOF: Suppose that $\mathfrak{r} \subseteq \mathfrak{sl}_n$ is a non-zero ideal. Let $r \in \mathfrak{r}$, so by the root-space decomposition, $r = h + \sum_{\alpha\in R} e_\alpha$ for some $h \in \mathfrak{t}$ and $e_\alpha \in \mathfrak{g}_\alpha$. Choose such a non-zero r with a minimal number of non-zero terms.

If $h \neq 0$ then choose a generic h_0 (i.e. choose h_0 so that its diagonal elements of are all different, or equivalently so that $\alpha(h_0) \neq 0$ for all $\alpha \in R$). As $\mathfrak{r}$ is an ideal, $\sum_{\alpha\in R} \alpha(h_0)e_\alpha = [h_0, r] \in \mathfrak{r}$ and if this is a non-zero element then it has fewer non-zero terms than r, contradicting our choice of r. Therefore $[h_0, r] = 0$, and so $r = h$ by 3.2.4. As $h \neq 0$, there are i and j such that $c := (\varepsilon_i - \varepsilon_j)(h) \neq 0$, so $[h, E_{ij}] = cE_{ij} \in \mathfrak{r}$, implying $E_{ij} \in \mathfrak{r}$. From this we will now show that $\mathfrak{r}$ contains a basis for $\mathfrak{sl}_n$ and hence $\mathfrak{r} = \mathfrak{sl}_n$. As $[E_{ij}, E_{jk}] = E_{ik}$ if $i \neq k$ and $[E_{si}, E_{ij}] = E_{ik}$ if

$j \neq s$, we have $E_{ab} \in \mathfrak{r}$ for all $a \neq b$. But now $[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1} \in \mathfrak{r}$ as well. All these matrices together form a basis for the trace zero matrices.

On the other hand, suppose $h = 0$, so $r = \sum_{\alpha \in R} e_\alpha$. If $r = e_\alpha = cE_{ij}$ with $c \neq 0$ then we are done as before. Suppose $r = cE_\alpha + dE_\beta + \sum_{\gamma \in R \setminus \{\alpha,\beta\}} e_\gamma$, where $c, d \neq 0$. Choose $h_0 \in \mathfrak{t}$ such that $\alpha(h_0) \neq \beta(h_0)$, so $[h_0, r] = c\alpha(h_0)E_\alpha + d\beta(h_0)E_\beta + \sum \gamma(h_0)e_\gamma$. Therefore a linear combination of $[h_0, r]$ and r is in $\mathfrak{r}$ and has fewer non-zero terms, contradicting that r was chosen with the minimal number. Therefore $\mathfrak{sl}_n$ is simple. □

3.2.8 Exercises.

1. Show that $A \in \mathfrak{so}_n$ if and only if A is anti-skew-symmetric (i.e. skew-symmetric across the anti-diagonal).
2. Show that $A \in \mathfrak{sp}_{2n}$ if and only if $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with A_2 and A_3 anti-skew-symmetric and $A_4 = -A_1^{aT}$ (anti-transpose).
3. Prove that the maximal tori in $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$ are exactly the diagonal matrices in those algebras.
4. Show that the root spaces for $\mathfrak{so}_n$ are $\mathbb{C}(E_{ij} - E_{n-j+1,n-i+1})$, where $i+j < n+1$ and $i \neq j$. Also show that
$$R = \begin{cases} \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i, j \leq \ell, i \neq j\} & \text{if } n = 2\ell \\ \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid 1 \leq i, j \leq \ell, i \neq j\} & \text{if } n = 2\ell + 1 \end{cases}$$
5. Show that the root spaces for $\mathfrak{sp}_{2n}$ are $\mathbb{C}(E_{ij} - E_{n-j+1,n-i+1})$, where $1 \leq i, j \leq n$ and $i \neq j$ and $\mathbb{C}(E_{ij} + E_{n-j+1,n-i+1})$, where $n < i \leq 2n$ and $j \leq n$ or $i \leq n$ and $n < j \leq 2n$. Also show that
$$R = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \leq i, j \leq n, i \neq j\}.$$
6. Show that $\mathfrak{sp}_{2n}$ is simple, and $\mathfrak{so}_n$ is simple if $n > 4$ or $n = 3$, and show that $\mathfrak{so}_4 = \mathfrak{so}_3 + \mathfrak{so}_3$ and that $\mathfrak{so}_3 = \mathfrak{sl}_2$.

Notice that the root spaces are all one-dimensional.

3.2.9 Theorem (Structure Theorem for S.S. Lie Algebras). *Let $\mathfrak{g}$ be a s.s. Lie algebra, $\mathfrak{t}$ a maximal torus, and let $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\lambda \in R} \mathfrak{g}_\lambda$ be the root-space decomposition. Then*

1. $\mathbb{C}R = \mathfrak{t}^*$*, i.e. the roots of* $\mathfrak{g}$ *span* $\mathfrak{t}^*$*;*
2. $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in R$*;*
3. *For* $\alpha, \beta \in R$*, if* $\alpha + \beta \in R$ *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$*, and if* $\alpha + \beta \notin R$ *and* $\beta \neq -\alpha$ *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$*;*
4. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{t}$ *is one-dimensional, and* $\mathfrak{g}_\alpha + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] + \mathfrak{g}_{-\alpha}$ *is isomorphic to* $\mathfrak{sl}_2$.

PROOF:

1. If R does not span $\mathfrak{t}^*$ then there is $0 \neq h \in \mathfrak{t}$ such that $\alpha(h) = 0$ for all $\alpha \in R$. But then if $x \in \mathfrak{g}_\alpha$ then $[h, x] = \alpha(h)x = 0$, so $[h, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in R$. But $[h, \mathfrak{t}] = 0$ as $\mathfrak{t}$ is Abelian, so $[h, \mathfrak{g}] = 0$, contradicting that $\mathfrak{g}$ is semi-simple and has no center (since it has no Abelian ideals at all).

2,3,4. Observe that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$ as for $t \in \mathfrak{t}$, $x \in \mathfrak{g}_\lambda$, and $y \in \mathfrak{g}_\mu$,

$$[t,[x,y]] = [[t,x],y] + [x,[t,y]] = \lambda(t)[x,y] + \mu(t)[x,y]$$

by the Jacobi identity.

Claim. $(\mathfrak{g}_\lambda, \mathfrak{g}_\mu)_{\mathrm{ad}} = 0$ if $\lambda + \mu \neq 0$. Moreover, if $\mathfrak{g}$ is s.s. then $(\cdot,\cdot)_{\mathrm{ad}}$ restricted to $\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}$ is non-degenerate.

The moreover part follows from the first part of the claim because, as $(\cdot,\cdot)_{\mathrm{ad}}$ is non-degenerate if $\mathfrak{g}$ is s.s., it must be non-degenerate on each $\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}$. To prove the first part, if $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_\mu$ then $(\operatorname{ad} x \operatorname{ad} y)^N \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\alpha+N(\lambda+\mu)}$. But $\mathfrak{g}$ is finite dimensional, so if $\lambda + \mu \neq 0$ then $\mathfrak{g}_{\alpha+N(\lambda+\mu)} = 0$ for N sufficiently large. But this implies that $(\operatorname{ad} x \operatorname{ad} y)$ is nilpotent, and so has zero trace.

Note in particular that $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{t}}$ is non-degenerate. (Warning: $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{t}}$ is not equal to the Killing form of $\mathfrak{t}$, which is obviously zero). Therefore we have an isomorphism

$$\nu : \mathfrak{t} \to \mathfrak{t}^* : t \mapsto \text{“}t' \mapsto (t,t')_{\mathrm{ad}}\text{”}.$$

This induces an inner product on $\mathfrak{t}^*$, denoted $(\cdot,\cdot)$, by $(\nu(t), \nu(t')) := (t,t')_{\mathrm{ad}}$.

Claim. $\alpha \in R$ implies $-\alpha \in R$.

Indeed, $(\cdot,\cdot)_{\mathrm{ad}}$ is non-degenerate on $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$. But $(\mathfrak{g}_\alpha, \mathfrak{g}_\alpha)_{\mathrm{ad}} = 0$ as $2\alpha \neq 0$, so if $\mathfrak{g}_\alpha \neq 0$ then it must be that $\mathfrak{g}_{-\alpha} \neq 0$ also. Moreover, $(\cdot,\cdot)_{\mathrm{ad}}$ defines a perfect pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, namely $\mathfrak{g}_\alpha \cong \mathfrak{g}_{-\alpha}^*$.

Claim. If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ then $[x,y] = (x,y)_{\mathrm{ad}} \nu^{-1}(\alpha)$.

It is clear that $[x,y] \in \mathfrak{t} = \mathfrak{g}_0$, so to check the claim it is enough to check that $(t,[x,y])_{\mathrm{ad}} = (x,y)_{\mathrm{ad}}\alpha(t)$ for all $t \in \mathfrak{t}$, as $(\nu^{-1}(\alpha), t)_{\mathrm{ad}} = \alpha(t)$ by definition. But both sides are equal to $([t,x],y)_{\mathrm{ad}}$ since $x \in \mathfrak{g}_\alpha$.

Let $e_\alpha \in \mathfrak{g}_\alpha$ be non-zero, and pick $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $(e_\alpha, e_{-\alpha})_{\mathrm{ad}} \neq 0$. Then $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})_{\mathrm{ad}} \nu^{-1}(\alpha)$, and $[\nu^{-1}(\alpha), e_\alpha] = \alpha(\nu^{-1}(\alpha)) e_\alpha = (\alpha,\alpha)_{\mathrm{ad}} e_\alpha$. Therefore $\{e_\alpha, e_{-\alpha}, \nu^{-1}(\alpha)\}$ span a Lie subalgebra $\mathfrak{m}$ of dimension three.

3.2.10 Key Lemma. $(\alpha,\alpha)_{\mathrm{ad}} \neq 0$.

Suppose otherwise. If $(\alpha,\alpha)_{\mathrm{ad}} = 0$ then $[\mathfrak{m},\mathfrak{m}] = \mathbb{C}\nu^{-1}(\alpha)$, which implies that $\mathfrak{m}$ is solvable. Lie's theorem implies that $\operatorname{ad}[\mathfrak{m},\mathfrak{m}]$ acts by nilpotent elements, i.e. that $\operatorname{ad}\nu^{-1}(\alpha)$ acts nilpotently. But $\operatorname{ad}\nu^{-1}(\alpha) \in \mathfrak{t}$ and all elements of $\mathfrak{t}$ act semisimply, so these together imply that $\alpha = 0$. This contradiction proves the claim.

Let $h_\alpha = \frac{2\nu^{-1}(\alpha)}{(\alpha,\alpha)_{\mathrm{ad}}}$, and rescale $e_{-\alpha}$ so that $(e_\alpha, e_{-\alpha})_{\mathrm{ad}} = \frac{2}{(\alpha,\alpha)_{\mathrm{ad}}}$. Then $\{e_\alpha, h_\alpha, e_{-\alpha}\}$ span a copy of $\mathfrak{sl}_2$.

Now we use the representation theory of $\mathfrak{sl}_2$ to deduce the structure of R, $\mathfrak{g}$, etc.

Claim. $\dim \mathfrak{g}_{-\alpha} = 1$ for all $\alpha \in R$.

Choose $e_\alpha, h_\alpha, e_{-\alpha}$ as above, so that we have a map $\mathfrak{g}_{-\alpha} \to \mathbb{C}\nu^{-1}(\alpha) = \mathbb{C}h_\alpha$: $x \mapsto [e_\alpha, x]$. If $\dim \mathfrak{g}_{-\alpha} > 1$ then this map has a kernel, i.e. there is a vector v such that $\mathrm{ad}(e_\alpha)v = 0$ and $\mathrm{ad}(h_\alpha)v = -\alpha(h_\alpha)v = -2v$. But then v is a highest weight vector with negative weight, implying that the $\mathfrak{sl}_2$-submodule generated by v is infinite dimensional, contradicting that $\dim \mathfrak{g}$ is finite. □

3.2.11 Proposition. *With the notation from the proof above,*

1. $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ *for all* $\alpha, \beta \in R$*;*
2. *if* $\alpha \in R$ *and* $k\alpha \in R$ *then* $k = \pm 1$*;*
3. $\bigoplus_{k \in Z} \mathfrak{g}_{\beta + k\alpha}$ *is an irreducible* $\mathfrak{sl}_2$*-submodule for all* $\alpha, \beta \in R$*, so the set* $\{\beta + k\alpha \in R \mid k \in \mathbb{Z}\} \cup \{0\}$ *is of the form* $\beta - p\alpha, \dots, \beta, \beta + \alpha, \dots, \beta + q\alpha$*, where* $p - q = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$*, and is called the "*$\alpha$ *string through* β*";*
4. *if* $\alpha, \beta, \alpha + \beta \in R$ *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$*.*

PROOF:

3. Let $q = \max\{k \mid \beta + k\alpha \in R\}$. If $v \in \mathfrak{g}_{\beta+k\alpha}$ is non-zero then $\mathrm{ad}\, e_\alpha v \in \mathfrak{g}_{\beta+(q+1)\alpha} = 0$ by choice of q and $\mathrm{ad}\, h_\alpha v = (\beta + q\alpha)(h)v = (\beta + \alpha)\frac{2\nu^{-1}(\alpha)}{(\alpha,\alpha)}$ so v is a highest weight vector for $\mathfrak{sl}_s = \langle e_\alpha, e_{-\alpha}, h_\alpha \rangle$, so its weight is in $\mathbb{N}$, i.e. $\mathbb{N}$ contains $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2q$, which implies that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

1. It follows from the representation theory of $\mathfrak{sl}_2$ that $\beta + q\alpha, \beta + (q-1)\alpha, \dots, \beta - (q + \frac{2(\alpha,\beta)}{(\alpha,\alpha)})\alpha$ are all roots. We must show that there are no more roots. Let $p = \max\{k \mid \beta - k\alpha \in R\}$ (start at the bottom and squeeze). We now get that $\beta - p\alpha, \beta - (p-1)\alpha, \dots, \beta + (p - \frac{2(\alpha,\beta)}{(\alpha,\alpha)})\alpha$ are roots. Then $q + \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \le p$ by the definition of p, and $p - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \le q$ by the definition of q.

2. If $k\alpha \in R$ then $\frac{2(k\alpha,\alpha)}{(k\alpha,k\alpha)} = \frac{2}{k} \in \mathbb{Z}$ and $\frac{2(\alpha,k\beta)}{(\alpha,\alpha)} = 2k \in \mathbb{Z}$. Hence $k = \pm\frac{1}{2}, \pm 1, \pm 2$, so it is enough to show that $\alpha \in R$ implies that $-2\alpha \notin R$. Suppose $-2\alpha \in R$ and take $v \in \mathfrak{g}_{-2\alpha}$ non-zero. Then $([e_\alpha, v], e_\alpha)_{\mathrm{ad}} = (v, [e_\alpha, e_\alpha])_{\mathrm{ad}} = 0$ by invariance. But $\mathrm{ad}\, e_\alpha v = \lambda e_{-\alpha}$ for some λ, and $\lambda = 0$ since the Killing form is non-degenerate on $\mathfrak{g}_{-\alpha} + \mathfrak{g}_\alpha$. Thus $\mathrm{ad}\, e_\alpha v = 0$, by $h_\alpha v = -4v$, once again a contradiction (negative highest weight).

4. Suppose that $\alpha, \beta, \alpha + \beta$ are roots. Since root spaces are one-dimensional, it is enough to show that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \ne 0$. Now $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$ is an irreducible $\mathfrak{sl}_2$-module, so $\mathrm{ad}\, e_\alpha : \mathfrak{g}_{\beta+k\alpha} \to \mathfrak{g}_{\beta+(k+1)\alpha}$ is an isomorphism if $-p \le k < q$, but $\alpha + \beta \in R$, so $q \ge 1$ and we are done. □

3.2.12 Lemma. $s_\alpha(\beta) := \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ *is a root when* $\alpha, \beta \in R$*.*

PROOF: Let $r = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. If $r \ge 0$ then $p = q + r \ge r$ and if $r \le 0$ then $q = p - r \ge -r$, so $\beta - r\alpha$ is in the string in either case. □

Recall that R spans $\mathfrak{t}^*$.

3.2.13 Proposition.

1. *If $\alpha, \beta \in R$ then $(\alpha, \beta) \in \mathbb{Q}$.*
2. *If we choose a basis $\beta_1, \ldots, \beta_\ell$ of $\mathfrak{t}^*$ with each $\beta_i \in R$ and if $\beta \in R$ then $\beta = \sum_i c_i \beta_i$ where $c_i \in \mathbb{Q}$ for all i.*
3. *$(\cdot, \cdot)$ is positive definite on the $\mathbb{Q}$-span of R.*

PROOF:

1. As $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$, it is enough to show that $(\alpha, \alpha) \in \mathbb{Q}$ for all $\alpha \in R$. Let $t, t' \in \mathfrak{t}$. Then
$$(t, t')_{\text{ad}} = \text{tr}_{\mathfrak{g}}(\text{ad}\, t\, \text{ad}\, t') = \sum_{\alpha \in R} \alpha(t)\alpha(t')$$
as each $\mathfrak{g}_\alpha$ is one-dimensional. So if $\lambda, \mu \in \mathfrak{t}^*$ then
$$\begin{aligned}(\lambda, \mu) &= (\nu^{-1}(\lambda), \nu^{-1}(\mu))_{\text{ad}} \\ &= \sum_{\alpha \in R} \alpha(\nu^{-1}(\lambda))\alpha(\nu^{-1}(\mu)) = \sum_{\alpha \in R} (\alpha, \lambda)(\alpha, \mu).\end{aligned}$$
In particular, $(\beta, \beta) = \sum_{\alpha \in R} (\alpha, \beta)^2$. Dividing by $(\beta, \beta)^2/4$, we get
$$\frac{4}{(\beta, \beta)} = \sum_{\alpha \in R} (\frac{2(\alpha, \beta)}{(\beta, \beta)})^2 \in \mathbb{Z}$$
which implies $(\beta, \beta) \in \mathbb{Q}$.
2. Exercise: if $(\cdot, \cdot)$ is a non-degenerate symmetric bilinear form, $\beta_1, \ldots, \beta_\ell$ are linearly independent vectors then $\det(\beta_i, \beta_j)_{ij} \neq 0$. Set $B = (\beta_i, \beta_j)_{ij}$, the matrix of inner products. Then B is a matrix with rational entries, B^{-1} exists and is rational, so if $\beta = \sum_i c_i \beta_i$ then $(\beta, \beta_j) = \sum_i c_i (\beta_i, \beta_j)$ and $(c_j)_j = B^{-1}(\beta, \beta_j)_j$ is a rational vector.
3. If $\lambda \in \text{span}_{\mathbb{Q}}(R)$ then $(\lambda, \lambda) = \sum_{\alpha \in R} (\lambda, \alpha)^2 \geq 0$ since it is a sum of squares of rational numbers. □

4 Root Systems

4.1 Abstract Root Systems

Let V be a vector space over $\mathbb{R}$ (though the theory could be developed over $\mathbb{Q}$ as well), and let $(\cdot, \cdot)$ be a positive definite symmetric bilinear form. If $\alpha \in V$ is non-zero, define $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$, so $(\alpha, \alpha^\vee) = 2$. Define $s_\alpha : V \to V : v \mapsto v - (v, \alpha^\vee)\alpha$, a linear map.

4.1.1 Lemma. *s_α is the reflection in the hyper-plane perpendicular to α. In particular, all its eigenvalues are 1, except for one which is -1, i.e. $s_\alpha^2 = id$, $(s_\alpha - 1)(s_\alpha + 1) = 0$. Moreover,*
$$s_\alpha \in O(V) = \{\varphi : V \to V \mid (\varphi v, \varphi w) = (v, w) \textit{ for all } v, w \in V\}.$$

PROOF: $V = \mathbb{R}\alpha \oplus \alpha^\perp$, and $s_\alpha v = v$ if $v \in \alpha^\perp$, $s_\alpha \alpha = \alpha - (\alpha, \alpha^\vee)\alpha = -\alpha$. □

4.1.2 Definition. A *root system* R in V is a finite set $R \subseteq V$ such that

1. $\mathbb{R}R = V$ and $0 \notin R$;
2. $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in R$;
3. $s_\alpha R = R$ for all $\alpha \in R$;

(4.) R is *reduced* if $k\alpha \in R$ implies $k = \pm 1$ for all $\alpha \in R$.

Most of the time we will be concerned with reduced root systems.

4.1.3 Example. Let $\mathfrak{g}$ be a s.s. Lie algebra over $\mathbb{C}$ and $\mathfrak{t}$ a maximal torus. Then $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ and $(R, \mathbb{R}R \subseteq \mathfrak{t}^*)$ is a reduced root system (this example is the whole point of looking at root systems).

4.1.4 Definition. Let $W \subseteq GL_V$ be the group generated by the reflections $s_\alpha \in R$ for $\alpha \in R$, the *Weyl group* of the root system R.

4.1.5 Lemma. *The Weyl group W is finite.*

PROOF: By property 3, W permutes R, so $W \to \mathfrak{S}_R$. By property 1 this map is injective. □

4.1.6 Definition. The *rank of a root system* (R, V) is the dimension of V. If (R, V), (R', V') are root systems then the *direct sum root system* is $(R \amalg R', V \oplus V')$. An *isomorphism of root systems* is a bijective linear map (not necessarily an isometry) $\varphi : V \to V'$ such that $\varphi(R) = R'$. A root system which is not isomorphic to a direct sum of root systems is an *irreducible root system*.

4.1.7 Examples.

1. When the rank is 1, $V = \mathbb{R}$ and $(x, y) = xy$. Therefore $A_1 = \{\alpha, -\alpha\}$ $(\alpha \neq 0)$ is the only root system (up to isomorphism). The Weyl group is $\mathbb{Z}/2\mathbb{Z}$.
2. When the rank is 2, $V = \mathbb{R}^2$ and the inner product is the usual inner product. Root systems are $A_1 \times A_1$, A_2, $B_2 = C_2$ and G_2 (insert diagrams).

4.1.8 Lemma. *If (R, V) is a root system then $(R^\vee, V)$ is a root system.*

PROOF: Exercise. □

4.1.9 Definition. A root system is *simply laced* if all the roots have the same length (i.e. (α, α) is the same for all $\alpha \in R$).

4.1.10 Example. A_1, A_2, and $A_1 \times A_1$ are simply laced, B_2 and G_2 are not.

4.1.11 Exercise. If (R, V) is simply laced then $(R, V) \cong (R', V')$, where R' is a root system such that $(\alpha, \alpha) = 2$ for all $\alpha \in R'$, (i.e. $\alpha = \alpha^\vee$ and $|\alpha| = \sqrt{2}$). (Hint: if R is irreducible then just rescale).

4.1.12 Definition. A *lattice* L is a finitely generated free abelian group $\mathbb{Z}^n$, equipped with a bilinear form $(\cdot, \cdot) : L \times L \to \mathbb{Z}$. (We will usually assume that it is positive definite and non-degenerate.)

It follows that $(L\otimes_{\mathbb{Z}}\mathbb{R},(\cdot,\cdot))$ is an inner product space, and it is positive definite if $(\cdot,\cdot)$ is positive definite.

4.1.13 Definition. A *root of a lattice* L is an $\ell\in L$ with $(\ell,\ell)=2$. Write

$$R_L=\{\ell\in L\mid(\ell,\ell)=2\}=\{\ell\in L\mid \ell=\ell^\vee\}.$$

If $\alpha\in R_L$ then $s_\alpha(L)\subseteq L$. (Indeed, $s_\alpha\beta=\beta-(\beta,\alpha^\vee)\alpha\in L$.)

4.1.14 Lemma. *R_L is a root system in $\mathbb{R}R_L$ and it is simply laced.*

PROOF: R_L is finite since it is the intersection of a compact set $\{\ell\in\mathbb{R}L\mid(\ell,\ell)=2\}$ with a discrete set L. The rest of the axioms are obvious (exercise). □

4.1.15 Definition. L is *generated by roots* if $\mathbb{Z}R_L=L$.

If L is generated by roots then L is an *even lattice*, i.e. $(\ell,\ell)\in2\mathbb{Z}$ for all $\ell\in L$.

4.1.16 Example. Let $L=\mathbb{Z}\alpha$ with $(\alpha,\alpha)=\lambda$. Then $R_L=\{\pm\alpha\}$ if $\lambda=2$, and L is generated by roots. If $k^2\lambda\neq2$ for all $k\in\mathbb{Z}$ then $R_L=\varnothing$.

4.1.17 Examples.

A_n: Consider the square lattice $\mathbb{Z}^{n+1}=\bigoplus_{i=1}^{n+1}\mathbb{Z}e_i$, where $(e_i,e_j)=\delta_{ij}$. Let A_n be the sublattice

$$\{\ell\in\mathbb{Z}^{n+1}\mid(\ell,e_1+\cdots+e_{n+1})=0\}=\left\{\sum_{i=1}^{n+1}a_ie_i\mid a_i\in\mathbb{Z},\sum_{i=1}^{n+1}a_i=0\right\}.$$

Then $R_{A_n}=\{e_i-e_j\mid i\neq j\}$ and $|R_{A_n}|=n(n+1)$. If $\alpha=e_i-e_j$ then s_α swaps the i^{th} and j^{th} coordinate, so $W_{A_n}=\langle s_{ij}\mid i\neq j\rangle=\mathfrak{S}_{n+1}$.

D_n: Consider the square lattice $\mathbb{Z}^n=\bigoplus_{i=1}^{n}\mathbb{Z}e_i$, where $(e_i,e_j)=\delta_{ij}$. Take $R_{D_n}=\{\pm e_i\pm e_j\mid i\neq j\}$ and let

$$D_n=\mathbb{Z}R_{D_n}=\left\{\sum_{i=1}^{n}a_ie_i\mid a_i\in\mathbb{Z},\sum_{i=1}^{n}a_i\in2\mathbb{Z}\right\}.$$

Then $|R_{D_n}|=2n(n-1)$. It is seen that $s_{e_i-e_j}$ swaps the i^{th} and j^{th} coordinate and $s_{e_i+e_j}$ changes the sign of the i^{th} and j^{th} coordinate so $W_{D_n}=(\mathbb{Z}/2\mathbb{Z})^{n-1}\rtimes\mathfrak{S}_n$, ($(\mathbb{Z}/2\mathbb{Z})^{n-1}$ is the subgroup of an even number of sign changes). R_{D_n} is irreducible if $n\geq3$, and $R_{D_3}=R_{A_3}$ and $R_{D_2}=R_{A_1\times A_1}$.

E_8: Let $\Gamma_n=\{(k_1,\dots,k_n)\mid$ either $k_i\in\mathbb{Z}$ for all i or $k_i\in\frac12+\mathbb{Z}$ for all i and $\sum_{i=1}^n k_i\in2\mathbb{Z}\}$. Consider $\alpha=(\frac12,\dots,\frac12)$, so that $(\alpha,\alpha)=\frac n4$. Then $\alpha\in\Gamma_n$ and Γ_n is an even lattice only if $8\mid n$. As an exercise, prove that if $n>1$ then the roots of Γ_{8n} form a root system of type D_{8n} (this requires also proving that Γ_{8n} is a lattice). For $n=1$, show that

$$R_{\Gamma_8}=\{\pm e_i\pm e_j,\tfrac12(\pm e_1\pm\cdots\pm e_8)\mid i\neq j,\text{the number of minuses is even}\}.$$

This is the root system of type E_8. Notice that $|R_{E_8}|=\frac{7\cdot8}{2}4+128=240$. Compute W_{E_8} (answer: $|W_{E_8}|=2^{14}3^55^27$).

E_6, E_7: As an exercise, prove that if R is a root system and $\alpha \in R$ then $\alpha^\perp \cap R$ is also a root system. In Γ_8 let $\alpha = (\frac{1}{2}, \dots, \frac{1}{2})$ and $\beta = e_7 + e_8$. Then $\alpha^\perp \cap R_{\Gamma_8}$ is a root system of type E_7 and $(\alpha, \beta)^\perp \cap R_{\Gamma_8}$ is a root system of type E_6. Show that $|R_{E_7}| = 126$ and $|R_{E_6}| = 72$ and describe the cooresponding root lattices.

As an essential exercise, check this whole example and draw the low dimensional systems (e.g. draw $A_n \subseteq \mathbb{Z}^{n+1}$ and R_{A_n} for $n = 1, 2$).

4.1.18 Theorem (ADE classification). *The complete list of simply laced irreducible root systems is* A_n *(*$n \geq 1$*),* D_n *(*$n \geq 4$*),* E_6*,* E_7*, and* E_8*. No two of these root systems are isomorphic.*

4.1.19 Theorem. *The remaining root systems are* $B_2 = C_2$*,* B_n *(*$n \geq 3$*),* C_n *(*$n \geq 3$*),* F_4*, and* G_2*, where* F_4 *is described below,* G_2 *has been drawn, and*

$$R_{B_n} = \{\pm e_i, \pm e_i \pm e_j \mid i \neq j\} \subseteq \mathbb{Z}^n$$

and

$$R_{C_n} = \{\pm 2e_i, \pm e_i \pm e_j \mid i \neq j\} \subseteq \mathbb{Z}^n.$$

(Notice that $R_{B_n} = R^\vee_{C_n}$ *and* $W_{B_n} = W_{C_n} = (\mathbb{Z}/2\mathbb{Z}) \rtimes \mathfrak{S}_n$*.)*

4.1.20 Example. Set $Q = \{(k_1, \dots, k_4) \mid \text{either all } k_i \in \mathbb{Z} \text{ or all } k_i \in \frac{1}{2} + \mathbb{Z}\}$. Then

$$F_4 = \{\alpha \in Q \mid (\alpha, \alpha) = 2 \text{ or } (\alpha, \alpha) = 1\}.$$

We have $R_{F_4} = \{\pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \mid i \neq j\}$.

We want to choose certain "good" bases of V. We make an auxiliary choice of a linear function $f : V \to \mathbb{R}$ such that $f(\alpha) \neq 0$ for all $\alpha \in R$. A root $\alpha \in R$ is said to be "positive" if $f(\alpha) > 0$ and "negative" if $f(\alpha) < 0$. Write R^+ for the collection of positive roots, so that $R = R^+ \amalg (-R^+)$.

4.1.21 Definition. $\alpha \in R^+$ is a *simple root* if it is not the sum of two other positive roots. Write $\Pi = \{\alpha_1, \dots, \alpha_n\}$ for the collection of simple roots.

4.1.22 Proposition.

1. *If* $\alpha, \beta \in \Pi$ *then* $\alpha - \beta \notin R$*;*
2. *If* $\alpha, \beta \in \Pi$ *and* $\alpha \neq \beta$ *then* $(\alpha, \beta^\vee) < 0$*.*
3. *Every* $\alpha \in R^+$ *can be written* $\alpha = \sum_{\alpha_i \in \Pi} k_i \alpha_i$ *with* $k_i \in \mathbb{N}$*.*
4. *Simple roots are linearly independent, so the decomposition above is unique.*
5. *If* $\alpha \in R^+ \setminus \Pi$ *then there is* $\alpha_i \in \Pi$ *such that* $\alpha - \alpha_i \in R^+$*.*
6. Π *is indecomposable (i.e. it cannot be written as a diagonal union of orthogonal sets) if and only if* R *is reducible.*

PROOF: Exercise: check all these properties for the above examples, or find a proof from the definition of root system (this is harder). □

Write $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$, where ℓ is the rank of R, which is $\dim V$. Set $a_{ij} = (\alpha_i, \alpha_j^\vee)$ and $A = (a_{ij})_{\ell \times \ell}$, the *Cartan matrix*.

4.1.23 Proposition.

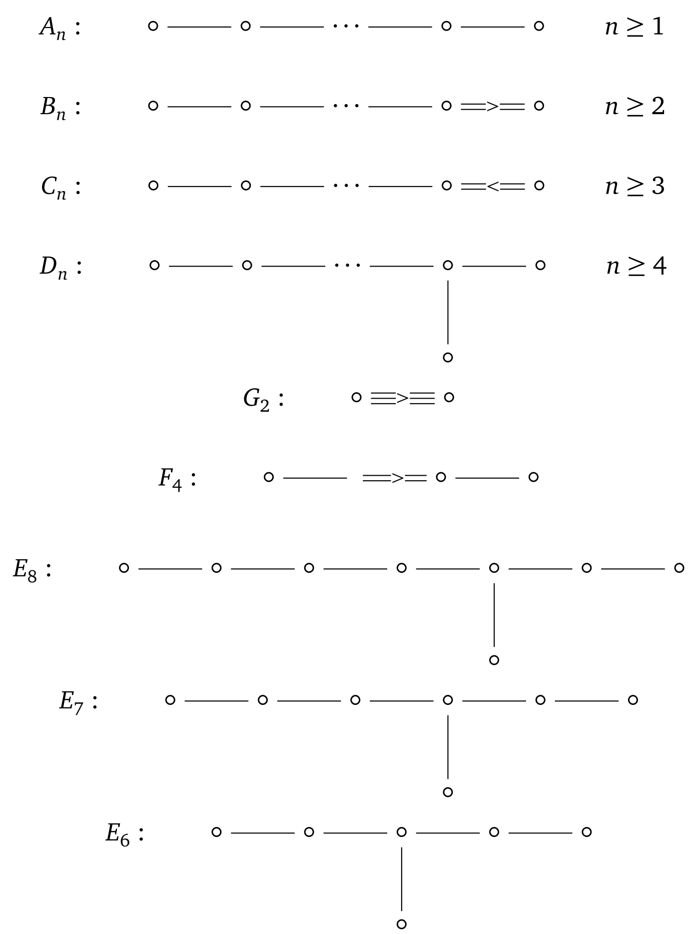
1. *$a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, and $a_{ij} \leq 0$ if $i \neq j$;*
2. *$a_{ij} = 0$ if and only if $a_{ji} = 0$;*
3. $\det A > 0$
4. *All principal subminors have positive determinant.*

PROOF: (written down) □

A matrix which satisfies the conditions of 4.1.23 is an *abstract Cartan matrix*.

4.2 Dynkin Diagrams

Special graphs called *Dynkin diagrams* are used to summarize the information in the Cartan matrix. The vertices are the set of simple roots, and α_i is joined to α_j by $a_{ij}a_{ji}$ lines. But $a_{ij}a_{ji}$ is always 1 if the root system is simply laced, while B_n and C_n have a double edge, and G_2 has a triple edge. If $a_{ij}a_{ji}$ is 2 or 3 then put an arrow in the direction of the shorter root.



4.2.1 Exercises.

1. Show that these really are the Dynkin diagrams of the named root systems.
2. Show that the Dynkin diagram of A^T is that of A, except with any arrows reversed. Also check that A^T is the Cartan matrix of $(A^\vee, V)$.
3. Compute the determinants of the Cartan matrices.
4. For type A_n, compute the inverse of the Cartan matrix. (Hint: A_1 has matrix $\left[\,2\,\right]$, A_2 has matrix $\left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$, so the answer is something of the form $\frac{1}{\det A}\left[\text{ pos. ints }\right]$. What do the positive integers represent?)
5. Redo the above exercise for all other types.

Remark. $SL_n = \{A \in GL_n \mid \det A = 1\}$, and $Z_{SL_n} = \{\lambda I_n \mid \lambda^n = 1\} \cong \mathbb{Z}/n\mathbb{Z}$. Here $n = \det A$. In general, the determinant of the Cartan matrix is $|Z_G|$, where G is a simply connected compact or algebraic group of the given type.

Notice that R is indecomposable if and only if the Dynkin diagram is connected.

4.2.2 Theorem. *An abstract Cartan matrix which is indecomposable is, after reordering simple roots, has Dynkin diagram which is one of the given types.*

PROOF: We will classify connected Dynkin diagrams of the Cartan matrix.

First, it is immediate from 1–4 of the definition of an abstract Cartan matrix (4.1.23) that any subdiagram of a Dynkin diagram is a disjoint union of Dynkin diagrams. All of the rank 2 Cartan matrices are $A = \left[\begin{smallmatrix} 2 & -a \\ -b & 2 \end{smallmatrix}\right]$, where $4 - ab = \det A > 0$, so $(a,b) \in \{(1,2),(2,1),(1,3),(3,1),(1,1)\}$.

Second, Dynkin diagrams contain no cycles. Write $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, and put

$$\alpha = \sum_{i=1}^{n} \frac{\alpha_i}{\sqrt{(\alpha_i, \alpha_i)}}.$$

Then

$$0 < (\alpha, \alpha) = n + \sum_{i=1}^{n} \frac{2(\alpha_i, \alpha_j)}{\sqrt{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}} = n - \sum_{i<j} \sqrt{a_{ij} a_{ji}}$$

as $(\alpha_i, \alpha_j) \le 0$ by definition of a_{ij}. Therefore $\sum_{i<j} \sqrt{a_{ij} a_{ji}} < n$. If there were a cycle on $\alpha_1, \ldots, \alpha_n$ then there would be at least n edges, and if there is an edge from α_i to α_j then $\sqrt{\alpha_{ij} \alpha_{ji}} \ge 1$. A cycle would contradict the identity.

4.2.3 Exercise. If (R, V) is an indecomposable root system and R^+ is a choice of roots, then there exists a unique $\theta \in R^+$ such that $\theta + \alpha_i \notin R$ for all $\alpha_i \in \Pi$. (e.g. for A_n we have $\theta = e_1 - e_{n+1}$ when $\alpha_i = e_i - e_{i+1}$, for B_n, $\theta = e_2 + e_1$, for C_n, $\theta = 2e_1$, and for D_n, $\theta = e_1 + e_2$.) θ is the *highest root*.

Third, the *extended Cartan matrix* $\tilde{A}$ is $(a_{ij})_{i,j=0,\ldots,\ell}$, where $\alpha_0 = -\theta$ (from the exercise). Then $\tilde{A}$ satisfies properties 1 and 2, but $\det \tilde{A} = 0$ since $\alpha_0, \ldots, \alpha_\ell$ are linearly dependent. Property 4 still holds if one takes *proper* subminors, as any subset of the extended simple roots is linearly independent. A Dynkin diagram cannot contain one of the extended Dynkin diagrams as a subdiagram (draw these out).

Fourth, if the diagram is simply laced (i.e. has simple edges) then it is one of A, D or E. Indeed, if it is not A_ℓ then it must have a branch point. But it cannot have $\tilde{D}_4$ as a subdiagram, so it can only have triple branches. Further, it has only one branch point since otherwise it has $\tilde{D}_\ell$ as a subdiagram. Since it does contain any $\tilde{E}$, it must be one of A, D, or E.

Fifth (this part is an exercise), if a Dynkin diagram has G_2 as a subdiagram then it is G_2. (Hint: $\tilde{G}_2$ is prohibited and the rest do not have positive determinant.) If there is a double bond then there is only on double bond (because of $\tilde{C}_n$), and there are no branch points (because of $\tilde{B}_n$). If the double bond is in the middle then it is F_4 (by $\tilde{F}_4$), and if it is on an end then it is B or C series. (Warning: we are often using the fact that the direction of the arrows don't matter from computing the determinant.) □

4.3 Existence and Uniqueness of an Associated Lie Algebra

So far we have accomplished the following:

$$\begin{array}{c} \mathfrak{g} \text{ a s.s. Lie algebra} \\ \big\downarrow \scriptstyle{\text{choosing } \mathfrak{t} \text{ a maximal torus}} \\ R \text{ root system} \\ \big\downarrow \scriptstyle{\text{choosing } f : V \to \mathbb{R} \text{ a linear map}} \\ R^+ \text{ positive roots, } \Pi \text{ simple roots, Cartan matrix } A \end{array}$$

and we have classified all possible Cartan matrices A.

But the choices made are inconsequential.

4.3.1 Theorem. *All maximal tori are conjugate. More precisely, for K algebraically closed and characteristic zero, if $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are maximal tori of $\mathfrak{g}$ then there is*

$$g \in (\operatorname{Aut}\mathfrak{g})^0 = \{g \in GL(\mathfrak{g}) \mid g : \mathfrak{g} \to \mathfrak{g} \textit{ is a Lie algebra homomorphism}\}^0$$

(the "^{0}" means "connected component of the identity") such that $g(\mathfrak{t}_1) = \mathfrak{t}_2$.

4.3.2 Theorem. *All R^+ are conjugate. More precisely, let (R, V) be a root system and $f_1, f_2 : V \to \mathbb{R}$ be two linear maps which define positive roots $R^+_{(1)}$ and $R^+_{(2)}$. Then there is a unique $w \in W_R$ such that $wR^+_{(1)} = R^+_{(2)}$ and we have $w\Pi_{(1)} = \Pi_{(2)}$.*

4.3.3 Corollary. $\mathfrak{g}$ *determines the Cartan matrix, i.e. the Cartan matrix is independent of the choices made.*

There is a one-to-one correspondence between Cartan matrices and s.s. Lie algebras.

4.3.4 Theorem. *Let $\mathfrak{g}_i$ $(i = 1, 2)$ be s.s. Lie algebras over $\mathbb{C}$, and $\mathfrak{t}_i$ maximal tori, R_i, R_i^+, Π_i, A_i be choices of associated objects. If, after reordering the indices, $A_1 = A_2$ then there is a Lie algebra isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\varphi(\mathfrak{t}_1) = \mathfrak{t}_2$, $\varphi(R_1) = R_2$, etc.*

4.3.5 Theorem. *Let A be an abstract Cartan matrix. Then there is a s.s. Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ with Cartan matrix A.*

Remark.

1. We already know this for A_n, B_n, C_n, and D_n (namely $\mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, and $\mathfrak{so}_{2n}$, respectively).

2. It remains to check that E_6, E_7, E_8, F_4, G_2 exist. We can do this case by case, e.g. $G_2 = \mathrm{Aut}(\mathbb{O})$. There are abstract constructions in the literature.

Existence and uniqueness will also follow from the following.

4.3.6 Definition. A *generalized Cartan matrix* is a matrix A such that

1. $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, and $a_{ij} \le 0$ for $i \ne j$; and

2. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Let $A_{n\times n}$ be such a matrix. Define $\tilde{\mathfrak{g}}$ to be a Lie algebra with generators $\{E_i, F_i, H_i \mid i = 1, \dots, n\}$ and relations

1. $[H_i, H_j] = 0$ for all i, j;

2. $[H_i, E_j] = a_{ij} E_j$ for $i \ne j$;

3. $[H_i, F_j] = -a_{ij} F_j$ for $i \ne j$;

4. $[E_i, F_j] = 0$ for $i \ne j$; and

5. $[E_i, F_i] = H_i$ for all i.

Let $\overline{\mathfrak{g}}$ be the quotient of $\tilde{\mathfrak{g}}$ by the additional relations

1. $(\operatorname{ad} E_i)^{1-a_{ij}} E_j = 0$ for $i \ne j$; and

2. $(\operatorname{ad} F_i)^{1-a_{ij}} F_j = 0$ for $i \ne j$.

For example, if $a_{ij} = 0$ then $[E_i, E_j] = 0$, and if $a_{ij} = -1$ then $[E_i, [E_i, E_j]] = 0$. These are known as the *Serre relations*, though they are due to Harish-Chandra and Chevalley.

4.3.7 Theorem.

1. *Let A be indecomposable. Then $\tilde{\mathfrak{g}}$ has a unique maximal ideal and $\overline{\mathfrak{g}}$ is its quotient. Therefore $\overline{\mathfrak{g}}$ is a simple Lie algebra (which need not be finite dimensional).*

2. *Hence if $\mathfrak{g}$ is a finite dimensional s.s. Lie algebra with Cartan matrix A, then the map $\tilde{\mathfrak{g}} \to \mathfrak{g}$ that takes $E_i \mapsto e_{\alpha_i}$, $F_i \mapsto e_{-\alpha_i}$, and $H_i \mapsto \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i,\alpha_i)_{\mathrm{ad}}}$ is surjective, and hence factors as $\tilde{\mathfrak{g}} \to \overline{\mathfrak{g}} \to \mathfrak{g}$.*

3. *$\overline{\mathfrak{g}}$ is finite dimensional if and only if A is a Cartan matrix (i.e. all principal subminors has positive determinant). (This last part is equivalent to the Existence and Uniqueness Theorem.)*

4.3.8 Definition. $\overline{\mathfrak{g}}$ is the *Kac-Moody algebra* associated to the abstract Cartan matrix A.

5 Representation Theory

5.1 Theorem of the Highest Weight

For this section let $\mathfrak{g}$ be a s.s. Lie algebra, $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ be the root space decomposition, R^+ a choice of positive roots, and $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be the corresponding simple roots. Let V be a (finite dimensional) representation of $\mathfrak{g}$. Set, for $\lambda \in \mathfrak{t}^*$, $V_\lambda = \{v \in V \mid tv = \lambda(t)v \text{ for all } t \in \mathfrak{t}\}$, the λ-weight space of V.

Define $\mathfrak{n}^+ = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha}$, so that $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{t} + \mathfrak{n}^-$. Notice that $\mathfrak{n}^+$, $\mathfrak{t}$, and $\mathfrak{n}^-$ are all Lie subalgebras of $\mathfrak{g}$ (and $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent?). Recall also for each $\alpha \in R$, $\mathfrak{g}_\alpha + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] + \mathfrak{g}_{-\alpha} \cong \mathfrak{sl}_2$, and we write $(\mathfrak{sl}_2)_\alpha := \langle e_\alpha, h_\alpha, e_{-\alpha}\rangle \le \mathfrak{g}$.

5.1.1 Proposition. *For $\mathfrak{g}$ s.s. and V f.d.,*

1. *$V = \bigoplus_{\lambda\in\mathfrak{t}^*} V_\lambda$, i.e. each $t \in \mathfrak{t}$ acts semi-simply.*
2. *If $V_\lambda \ne 0$ then $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in R$.*

PROOF: For each $\alpha \in R$, $(\mathfrak{sl}_2)_\alpha$ acts on V, and V is finite dimensional, so h_α acts semi-simply and has eigenvalues in $\mathbb{Z}$ by the representation theory of $\mathfrak{sl}_2$. From this 2 is immediate. As for 1, the h_α span $\mathfrak{t}$ as the roots span $\mathfrak{t}^*$, in fact $\{h_{\alpha_1}, \dots, h_{\alpha_\ell}\}$ is a basis of $\mathfrak{t}$. □

5.1.2 Definition. The *character* of V is $\operatorname{ch}(V) = \sum \dim V_\lambda e^\lambda$, where $e^\lambda \in \mathbb{Z}[\mathfrak{t}^*]$ is a formal symbol intended to remind you of the exponential map.

5.1.3 Examples.
1. Recall the representations $L(n)$ of $\mathfrak{sl}_2$ have characters

$$\operatorname{ch}(L(n)) = z^n + z^{n-2} + \cdots + z^{2-n} + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}},$$

   where $z = e^\alpha$ and $R_{\mathfrak{sl}_2} = \{\pm\alpha\}$.

2. Take $\mathfrak{g} = \mathfrak{sl}_3$ and consider its adjoint representation. Then

$$\operatorname{ch}(\mathfrak{g}) = \dim \mathfrak{t} + \sum_{\alpha\in R} e^\alpha = zw + z + w + 2 + w^{-1} + z^{-1} + z^{-1}w^{-1},$$

   where $w = e^\alpha$ and $z = e^\beta$, recalling $R_{\mathfrak{sl}_3} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$.

5.1.4 Definition. The *lattice of roots* is $Q := \mathbb{Z}R$, and the *lattice of weights* is

$$P := \{\gamma \in \mathbb{Q}R \mid (\gamma, \alpha^\vee) \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

Recall that $(\gamma, \alpha^\vee) = \gamma(h_\alpha)$ by definition of the normalization of h_α, so the second part of 5.1.1 says that if $V_\lambda \ne 0$ then $\lambda \in P$. In particular, $\operatorname{ch}(V) \in \mathbb{Z}[P]$. Notice that if $\alpha, \beta \in R$ then $(\alpha, \beta^\vee) = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, so $Q \subseteq P$ (i.e. each root is a weight).

5.1.5 Examples.
1. In the case of $\mathfrak{sl}_2$, $Q = \mathbb{Z}\alpha$ and $P = \mathbb{Z}(\frac{\alpha}{2})$ since $(\alpha, \alpha) = 2$. Therefore $|P/Q| = 2$.

2. For $\mathfrak{sl}_3$, $Q = \mathbb{Z}\alpha + \mathbb{Z}\beta$ and $P = \mathbb{Z}(\frac{2\alpha+\beta}{3}) + \mathbb{Z}(\frac{\alpha+2\beta}{3})$ so $|P/Q| = 3$.

5.1.6 Exercises.

1. Prove that $|P/Q| = \det A$, the determinant of the Cartan matrix.

2. Show that P is preserved under the action of the Weyl group on $\mathfrak{t}^*$.

5.1.7 Proposition. *If V f.d. then* $\dim V_\lambda = \dim V_{w\lambda}$ *for all* $w \in W$.

PROOF: Since the Weyl group is generated by $\{s_\alpha \mid \alpha \in R\}$ (by definition), it suffices to prove that $\dim V_\lambda = \dim V_{s_\alpha\lambda}$. We give sketches of three possible ways to prove the result.

1. Suppose that G is an algebraic or compact group with Lie algebra $\mathfrak{g}$ and let T be the torus corresponding to $\mathfrak{t}$. Then $W = N(T)/T$. (e.g. in the case of $\mathfrak{g} = \mathfrak{sl}_n$, $G = SL_n$ and T is the collection of diagonal matrices. Then $N(T)$ is the collection of "monomial matrices" and it is seen that $W_{\mathfrak{sl}_n} = \mathfrak{S}_n = N(T)/T$.) Whence for any $w \in T$ there is $\dot{w} \in N(T)$ such that $w = \dot{w}T$, so for any $t \in T$ and $v \in V_\lambda$,

$$t\dot{w}v = \dot{w}\dot{w}^{-1}t\dot{w}v = \dot{w}\lambda(\dot{w}^{-1}t\dot{w})v = (w\lambda)(t)\dot{w}v,$$

   so $\dot{w}(V_\lambda) = V_{w\lambda}$.

2. We can mimic the above construction in $\mathfrak{g}$. Observe that in $\mathfrak{sl}_2$ we may decompose

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \exp(f)\exp(-e)\exp(f).$$

   In general, for each root $\alpha \in R$ define $\dot{s}_\alpha = \exp(e_{-\alpha})\exp(-e_\alpha)\exp(e_{-\alpha})$.

   Exercise:

   a) Each e_α acts nilpotently on V (as V is f.d.) so $\dot{S}_\alpha$ is a finite sum on V and is hence well-defined.

   b) $\dot{s}_\alpha^2 =: \varepsilon_\alpha : V_\lambda \to V_\lambda$, where $\varepsilon_\alpha^2 = 1$ and ε_α is multiplication by a constant on V_λ. (Which constant?)

   c) $\dot{s}_\alpha(V_\lambda) = V_{s\alpha\lambda}$.

3. But we don't really need to do any of this. Consider V as a $\mathfrak{t}\times\mathfrak{sl}_2$-module. By the representation theory of $\mathfrak{sl}_2$, it breaks up as a direct sum of irreducibles, each of which is a string $\mu, \mu - \alpha, \mu - 2\alpha, \ldots, \mu - m\alpha$, where $m = \lambda(h_\alpha)$. These strings are s_α-invariant by the $\mathfrak{sl}_2$-theory. □

It follows that when V is f.d., $\mathrm{ch}(V) \in \mathbb{Z}[P]^W$.

5.1.8 Definition. For $\mu, \lambda \in P$, write $\mu \le \lambda$ if $\lambda - \mu = \sum_{i=1}^{\ell} k_i\alpha_i$ with $k_i \in \mathbb{N}$.

5.1.9 Exercise. Prove that "$\le$" is a partial order on P and show that $\{\mu \in P \mid \mu \le \lambda\}$ is an obtuse cone in P.

5.1.10 Definition. $\mu \in P$ is a *highest weight* if $V_\mu \neq 0$ and $V_\lambda \neq 0$ implies $\lambda \leq \mu$. $v \in V_\mu$ is a *singular vector* if $v \neq 0$ and $e_\alpha v = 0$ for all $\alpha \in R^+$. (Notice that the weight of $e_\alpha v$ is $\mu + \alpha$, so if μ is a highest weight then v is singular.) μ is an *extremal weight* if $w\mu$ is an highest weight vector for some $w \in W$.

It is clear that highest weights exist when V is f.d.

5.1.11 Theorem (Theorem of the Highest Weight).
Let $\mathfrak{g}$ be a s.s. Lie algebra over an algebraically closed field K of characteristic zero.

1. *a) If V is a f.d. representation of $\mathfrak{g}$ then V is a direct sum of irreducible representations.*

   b) Let $P^+ = \{\lambda \in P \mid (\lambda, \alpha^\vee) \geq 0$ for all $\alpha \in R^+\} = \{\lambda \in P \mid (\lambda, \alpha_i) \geq 0$ for all $\alpha_i \in \Pi\}$, the cone of dominant weights. The f.d. irreducible representations of $\mathfrak{g}$ are in bijection with P^+ via $\lambda \mapsto L(\lambda)$.

2. *More precisely, let V be a f.d. representation of $\mathfrak{g}$ and let $v \in V_\lambda$ be a singular vector. Then*

   a) $\dim V_\lambda = 1$, *i.e.* $V_\lambda = Kv$.

   b) If $V_\mu \neq 0$ then $\mu \leq \lambda$, i.e. λ is a highest weight.

   c) $\lambda(h_{\alpha_i}) \in \mathbb{N}$ *for all* $\alpha_i \in \Pi$, *i.e.* $\lambda \in P^+$.

   Moreover, if V' is another f.d. irreducible representation of $\mathfrak{g}$ and $v' \in V'$ is a singular vector of weight λ then there is a unique isomorphism of $\mathfrak{g}$-modules $V \to V'$ sending v to v'. We say that V has highest weight λ and denote this $\mathfrak{g}$-module by $L(\lambda)$.

3. *Given $\lambda \in P^+$ there is a f.d. representation of $\mathfrak{g}$ with highest weight λ.*

5.1.12 Corollary.

1. *For a highest weight λ,*

$$\mathrm{ch}(L(\lambda)) = e^\lambda + \sum_{\mu \leq \lambda} a_{\mu\lambda} e^\mu,$$

   and further

$$\mathrm{ch}(L(\lambda)) = m_\lambda + \sum_{\mu \leq \lambda, \mu \in P^+} \tilde{a}_{\mu\lambda} m_\mu,$$

   where $m_\mu = \sum_{\gamma \in W\mu} e^\gamma$. The $\{m_\mu \mid \mu \in P^+\}$ are a basis of $\mathbb{Z}[P]^W$, and in fact $\{\mathrm{ch}(L(\lambda)) \mid$ irreducible representations $L(\lambda)\}$ are a basis as well.

2. *If V and W are f.d. representations of $\mathfrak{g}$ and $\mathrm{ch}(V) = \mathrm{ch}(W)$ then $V \cong W$.*

We will (slowly) prove (most of) this theorem and corollary.

5.1.13 Definition. Let $\{\omega_1, \dots, \omega_\ell\}$ be a basis of P which is dual to the basis of simple co-roots, i.e. $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for $i, j = 1, \dots, \ell$. These are the *fundamental weights* and we could have defined $P^+ = \bigoplus_i \mathbb{N}\omega_i$.

Yet another way to think of P^+, the parameters of irreducible representations, is as

$$P^+ = \{\lambda \in \mathfrak{t}^* \mid \langle \lambda, h_i \rangle \in \mathbb{N}\}.$$

5.1.14 Examples.

1. Consider the adjoint representation of $\mathfrak{g}$ on itself. If $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ then the character of this representation is $\ell + \sum_{\alpha\in R} e^\alpha$. What is the highest weight? Recall that a highest weight is a weight λ such that $\lambda + \alpha_i \notin R$ for all $\alpha_i \in \Pi$. This implies that $\lambda = \theta$, the highest root, is the highest weight, and $\mathfrak{g} = L(\theta)$.

2. Representations of $\mathfrak{sl}_n$. There is the standard representation over $\mathbb{C}^n$, which has character $e^{\varepsilon_1} + \cdots + e^{\varepsilon_n}$. The highest weight is ε_1 since $\varepsilon_2 + (\varepsilon_1 - \varepsilon_2) = \varepsilon_1$, $\varepsilon_3 + (\varepsilon_2 - \varepsilon_3) = \varepsilon_2$, etc.

3. If V and W are representations then so is $V \otimes W$. Whence $V \otimes V$ is a representation, but $\sigma : V \otimes V \to V \otimes V$ commutes with the $\mathfrak{g}$-action, so S^2V and $\wedge^2 V$ are $\mathfrak{g}$-modules. These need not be irreducible in general, but $\wedge^s \mathbb{C}^n$ and $S^m\mathbb{C}^n$ are irreducible $\mathfrak{sl}_n$-modules.

$\wedge^s\mathbb{C}^n$: for $s \le n-1$ has basis $\{e_{i_1} \wedge \cdots \wedge e_{i_s} \mid i_1 < \cdots < i_s\}$, with associated weights $\varepsilon_{i_1} + \cdots + \varepsilon_{i_s}$. The action of an element $x \in \mathfrak{g}$ on $v_1 \wedge \cdots \wedge v_s$ is $\sum_{i=1}^s v_1 \wedge \cdots \wedge x(v_i) \wedge \cdots \wedge v_s$. Notice $E_i e_j = \delta_{j,i+1} e_{j-1}$, so $E_i(e_{i_1} \wedge \cdots \wedge e_{i_s}) = 0$ for all i if and only if $e_{i_1} \wedge \cdots \wedge e_{i_s} = e_1 \wedge \cdots \wedge e_s$. Whence the highest weight is $\omega_s = \varepsilon_1 + \cdots + \varepsilon_s$, and $\wedge^s\mathbb{C}^n = L(\omega_s)$ for all $1 \le s < n$. (Exercise: compute $\operatorname{ch} L(\omega_s)$.)

$S^m\mathbb{C}^n$: Exercise: Show that $S^m\mathbb{C}^n$ has basis $\{e_{i_1} \wedge \cdots \wedge e_{i_m} \mid i_1 \le \cdots \le i_m\}$, and the unique highest weight vector is e_1^m (so $S^m\mathbb{C}^n = L(m\omega_1)$ is irreducible) and compute its character.

5.1.15 Exercises.

1. Show that $V \otimes V = S^2V + \wedge^2 V$.

2. Show that $V \otimes V^* \to \operatorname{Hom}(V,V) : v \otimes w^* \mapsto (x \mapsto w^*(x)v)$ is an isomorphism of $\mathfrak{g}$-modules, and that $\operatorname{Hom}_{\mathfrak{g}}(V,V)$ is a submodule, which is 1-dimensional if V is irreducible.

3. If $\mathfrak{g} = \mathfrak{sl}_n$ then show that $(\mathbb{C}^n) \otimes (\mathbb{C}^n)^* \xrightarrow{\sim} \mathfrak{sl}_n \oplus \mathbb{C}$ (i.e. the adjoint representation plus the trivial representation).

4. Let algebras $\mathfrak{so}_{2n}$, $\mathfrak{sp}_{2n}$ act on $\mathbb{C}^{2n}$
   a) Compute the highest weight of $\mathbb{C}^{2n}$;
   b) Show that $(\mathbb{C}^{2n}) \cong (\mathbb{C}^{2n})^*$ as $\mathfrak{g}$-modules;
   c) Show that $(\mathbb{C}^{2n}) \otimes (\mathbb{C}^{2n})$ has 3 irreducible components and find them.

5.2 Proof of the Theorem of the Highest Weight

Let $\mathfrak{g}$ be any Lie algebra with a non-degenerate invariant bilinear symmetric form $(\cdot,\cdot)$ (e.g. the Killing form if $\mathfrak{g}$ is s.s.). Let $\{x_1, \ldots, x_N\}$ be a basis of $\mathfrak{g}$, with dual basis $\{x^1, \ldots, x^N\}$ so that $(x_i, x^j) = \delta_{ij}$. Define $\Omega = \sum_{i=1}^N x_i x^i$, the *Casimir* of $\mathfrak{g}$, an operator on any representation of $\mathfrak{g}$. (Exercise: show that Ω is independent of the choice of basis (though it does depend on the form).)

Claim. Ω is central, i.e. $\Omega x = x\Omega$ for all $x \in \mathfrak{g}$.

Indeed,

$$[\Omega, x] = \Big[\sum x_i x^i, x\Big] = \sum x_i [x^i, x] + \sum [x_i, x] x^i,$$

as $[\cdot, x]$ is a derivation. Now write $[x^i, x] = \sum a_{ij} x^j$ and $[x_i, x] = \sum b_{ij} x_j$, so

$$\begin{aligned} a_{ij} &= ([x^i, x], x_j) = -([x, x^i], x_j) = -(x, [x^i, x_j]) \\ &= (x, [x_j, x^i]) = ([x, x_j], x^i) = -([x_j, x], x^i) = -b_{ji}. \end{aligned}$$

Thus $[\Omega, x] = 0$.

Now let $\mathfrak{g}$ be s.s., $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, and $(\cdot, \cdot)$ be the Killing form. Choose a basis $\{u_1, \dots, u_\ell\}$ of $\mathfrak{t}$ and $e_\alpha \in \mathfrak{g}_\alpha$, and a dual basis $\{u^1, \dots, u^\ell\}$ of $\mathfrak{t}^*$ and $e^{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $(e_\alpha, e^{-\alpha}) = 1$. We may normalize the e_α so that $(e_\alpha, e_{-\alpha}) = 1$, and uniqueness of the dual basis implies that $e^{-\alpha} = e_\alpha$, so $[e_\alpha, e_{-\alpha}] = \nu^{-1}(\alpha)$. Then

$$\begin{aligned} \Omega &= \sum_{i=1}^{\ell} u_i u^i + \sum_{\alpha \in R^+} e_\alpha e_{-\alpha} + \sum_{\alpha \in R^+} e_{-\alpha} e_\alpha \\ &= \sum_{i=1}^{\ell} u_i u^i + 2 \sum_{\alpha \in R^+} e_\alpha e_{-\alpha} + \sum_{\alpha \in R^+} \nu^{-1}(\alpha). \end{aligned}$$

Write $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$, so that

$$\Omega = \sum_{i=1}^{\ell} u_i u^i + 2\nu^{-1}(\rho) + 2 \sum_{\alpha \in R^+} e_{-\alpha} e_\alpha.$$

Claim. Let V be a $\mathfrak{g}$-module with singular vector $v \in V$ of weight λ. Then $\Omega v = (|\lambda + \rho|^2 - |\rho|^2) v$.

Indeed, $e_\alpha v = 0$ for all $\alpha \in R^+$ and $tv = \lambda(t) v$ for all $t \in \mathfrak{t}$, so

$$\begin{aligned} \Omega v &= \left[\sum_{i=1}^{\ell} \lambda(u_i) \lambda(u^i) + \lambda\big(2\nu^{-1}(\rho)\big) \right] v \\ &= \big((\lambda, \lambda) + 2(\lambda, \rho)\big) v = \big((\lambda + \rho, \lambda + \rho) - (\rho, \rho)\big) v. \end{aligned}$$

Claim. If V is also irreducible then Ω acts as $(\lambda + \rho, \lambda + \rho) - (\rho, \rho)$ on all of V.

This claim follows from Schur's Lemma.

5.2.1 Definition. Let $\mathfrak{g}$ be any Lie algebra over a field K. The *universal enveloping algebra* $\mathscr{U}\mathfrak{g}$ is the free associative algebra generated by $\mathfrak{g}$ modulo the relations $\{xy - yx - [x, y] \mid x, y \in \mathfrak{g}\}$.

5.2.2 Exercises.

1. An *enveloping algebra* for $\mathfrak{g}$ is an associative algebra A and a map $\iota : \mathfrak{g} \to A$ such that $\iota x \iota y - \iota y \iota x = \iota[x, y]$. Show that $\mathscr{U}\mathfrak{g}$ is the initial object in the category of enveloping algebras. For example, if V is a representation of $\mathfrak{g}$ then $A = \mathrm{End}(V)$ is an enveloping algebra. Naturally, $\Omega \in \mathscr{U}\mathfrak{g}$. Let $(\mathscr{U}\mathfrak{g})_n$ be the collection of sums of products of at most n things from $\mathfrak{g}$, e.g. $(\mathscr{U}\mathfrak{g})_0 = K$, $(\mathscr{U}\mathfrak{g})_1 = K + \mathfrak{g}$, etc.

2. Show that $\mathscr{U}_n\mathscr{U}_m = \mathscr{U}_{n+m}$, i.e. that $\mathscr{U}$ is a filtered algebra. Also show that if $x \in \mathscr{U}_n$ and $y \in \mathscr{U}_m$ then $xy - yx \in \mathscr{U}_{n+m-1}$.

3. Prove that these monomials span $\mathscr{U}\mathfrak{g}$ (i.e. that $S\mathfrak{g} \to \operatorname{gr}\mathscr{U}\mathfrak{g}$ is surjective).

4. Show that the previous exercise gives a well-defined map $S\mathfrak{g} \to \operatorname{gr}\mathscr{U}\mathfrak{g}$.

5.2.3 Theorem (PBW). *$gr\mathscr{U}\mathfrak{g} = \bigoplus_n \mathscr{U}_n/\mathscr{U}_{n-1} \cong S\mathfrak{g}$. The natural map is an isomorphism.*

Equivalently (and in a language I can understand), the PBW theorem states that if $\{x_1, \dots, x_N\}$ is a basis of $\mathfrak{g}$ then $\{x_1^{a_1} \cdots x_N^{a_N} \mid a_1, \dots, a_N \in \mathbb{N}\}$ is a basis of $\mathscr{U}\mathfrak{g}$.

Let V be a representation of $\mathfrak{g}$ and $v \in V_\lambda$ a singular vector of weight λ (so $\mathfrak{n}^+ v = 0$ and $tv = \lambda(t)v$ for all $t \in \mathfrak{t}$). It is clear that the $\mathfrak{g}$-submodule generated by v is $\mathscr{U}\mathfrak{g}v$. If $\mathscr{U}\mathfrak{g}v = V$ then we say that V is a *highest weight module* with highest weight λ.

Claim. If v is a singular vector then $\mathscr{U}\mathfrak{g}v = \mathscr{U}\mathfrak{n}^- v$.

Indeed, since $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{t} + \mathfrak{n}^+$ we have $\mathscr{U}\mathfrak{g} = \mathscr{U}\mathfrak{n}^- \otimes \mathscr{U}\mathfrak{t} \otimes \mathscr{U}\mathfrak{n}^+$, so the easy part of the PBW theorem implies that we can write a basis for $\mathscr{U}\mathfrak{g}$ as a product of bases for $\mathfrak{n}^-$, $\mathfrak{t}$, and $\mathfrak{n}^+$ (in that order). Whence applying v kills the last bit, leaving only the $\mathfrak{n}^-$ part.

Claim. If V is irreducible and f.d. then V is a highest weight module.

Indeed, V f.d. implies a highest weight exists, call it λ. Then each $v \in V_\lambda$ is a singular vector, so $\mathscr{U}\mathfrak{g}v$ is a submodule, whence it is V since V is irreducible.

5.2.4 Proposition. *Let V be a (not necessarily f.d.) highest weight module for $\mathfrak{g}$ and $v \in V$ be a highest weight vector with highest weight Λ. Then*

1. *$\mathfrak{t}$ acts diagonalizably, so $V = \bigoplus_{\lambda \in D(\Lambda)} V_\lambda$, where $D(\Lambda) = \{\mu \mid \mu \le \Lambda\} = \{\mu \in \mathfrak{t}^* \mid \mu = \Lambda - \sum k_i \alpha_i, k_i \in \mathbb{N}\}$;*

2. *$V_\Lambda = Kv$; all weight spaces are f.d.;*

3. *V is irreducible if and only if all singular vectors are in V_Λ;*

4. *$\Omega = |\Lambda + p|^2 - |p|^2$ on V;*

5. *if $v_\lambda \in V$ is a singular vector then $|\lambda + p|^2 = |\Lambda + p|^2$;*

6. *if $\Lambda \in \mathbb{Q}R$, then there are only finitely many λ such that V_λ contains a singular vector;*

7. *V contains a unique maximal proper submodule I, where I is graded with respect to $\mathfrak{t}$, i.e. $I = \bigoplus (I \cap V_\lambda)$ and I is the sum of all proper submodules.*

PROOF:

1. Write $V = \mathscr{U}\mathfrak{n}^- v$ for some v, so it is spanned by

$$\{e_{-\beta_1} e_{-\beta_2} \cdots e_{-\beta_r} v \mid e_\beta \in \mathfrak{g}_\beta, \beta \in R^+ \text{ basis vectors}\}.$$

But for $x = e_{-\beta_1} e_{-\beta_2} \cdots e_{-\beta_r} v$, $t(x) = (\Lambda - \beta_1 - \cdots - \beta_r)(t)x$, whence part 1.

2. There are only finitely many $\beta \in R^+$ which appear in the sum of a given weight $\lambda \in \mathbb{N}R^+$.

3. The third part follows because if $v_\lambda \in V_\lambda$ is a singular vector then $\mathscr{U}\mathfrak{g}v_\lambda = \mathscr{U}\mathfrak{n}^- v_\lambda =: \mathscr{U}$ is a submodule and a highest weight module. By 1, all weights of $\mathscr{U}$ are in $D(\lambda) = \{\lambda - \sum \beta_i \mid \beta_i \in R^+\}$. But $\lambda \neq \Lambda$ implies that $D(\lambda) \subsetneq D(\Lambda)$, so in particular $\Lambda \notin D(\lambda)$. Whence $\mathscr{U}$ is a proper submodule and V is not irreducible if there is a singular vector with weight $\lambda \neq \Lambda$.

   Conversely, if $U \subsetneq V$ is a proper submodule, then $\mathfrak{t}U \subseteq U$, so U is a $\mathfrak{t}$-graded, so its weights are in $D(\Lambda)$ and $U = \bigoplus_{\lambda \in D(\Lambda)} U_\lambda$. Let λ be maximal such that $U_\lambda \neq 0$. If $w \in U_\lambda$ then $e_\beta w \in U_{\lambda+\beta} = 0$ by maximality for $\beta \in R^+$. Whence w is a singular vector. $\lambda \neq \Lambda$ since $U \neq V$.

7. Any proper submodule U is $\mathfrak{t}$-graded, and $U = \bigoplus_{\lambda \in D(\Lambda), \lambda \neq \Lambda} U_\lambda$, and the sum of all such submodules is still of this form, so I is a maximal proper submodule.

4. We know that for any singular vector of weight λ, $\Omega v_\lambda = (|\lambda+\rho|^2 - |\rho|^2)v_\lambda$, so apply to $v \in V_\Lambda$ (recall that Ω is central).

6. If λ is singular then 5 (which is obvious) implies that $|\lambda+\rho|^2 = |\Lambda+\rho|^2$. This defines a sphere in $\mathbb{R}R$, a compact set, but $D(\Lambda)$ is discrete. The singular vectors are the intersection of these, so there are finitely many. □

5.2.5 Lemma. *If λ is a dominant weight (i.e. $\lambda \in P^+$), $\lambda + \mu \in P^+$, and $\lambda - \mu = \sum_i k_i \alpha_i$ with $k_i \geq 0$ for all i then $|\lambda+\rho|^2 = |\mu+\rho|^2$ implies $\lambda = \mu$.*

PROOF: We have

$$\begin{aligned} 0 &= (\lambda+\rho, \lambda+\rho) - (\mu+\rho, \mu+\rho) \\ &= (\lambda-\mu, \lambda+\mu+2\rho) \\ &= \sum_{i=1}^{\ell} k_i(\alpha_i, \lambda\mu+2\rho) \end{aligned}$$

so since $\lambda+\mu+2\rho \in P^+$ and $(\alpha_i, 2\rho) = (\alpha_i, \alpha_i) > 0$, we get that 0 is a positive linear combination of the k_i, so each $k_i = 0$ since they are all non-negative. □

5.2.6 Definition. Let $\Lambda \in \mathfrak{t}^*$. A *Verma module* $M(\Lambda)$ is an initial object in the category of highest weight modules with highest weight Λ and highest weight vector v_Λ, i.e. if V is any highest weight module with highest weight Λ and highest weight vector v then there is a $\mathfrak{g}$-module map $M(\Lambda) \to V : v_\Lambda \mapsto v$.

Such a map is obviously surjective and unique.

5.2.7 Proposition. *Given $\Lambda \in \mathfrak{t}^*$ there exists a unique Verma module $M(\Lambda)$ and there is a unique highest weight module $L(\Lambda)$ which is irreducible.*

PROOF: Uniqueness is clear from the definition of Verma module via universal property. Existence requires the PBW theorem. ($M(\Lambda) = \mathscr{U}\mathfrak{g} \otimes_{\mathscr{U}\mathfrak{b}} k_\Lambda$, where $\mathfrak{b} = \mathfrak{n}^+ + \mathfrak{t}$ and k_Λ is the 1-dimensional $\mathfrak{b}$-module such that $tx = \lambda(t)x$ for $t \in \mathfrak{t}$

and $\mathfrak{n}^+x = 0$ (it is $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} k_\Lambda$). Since $\mathscr{U}\mathfrak{g} = \mathscr{U}\mathfrak{n}^- \otimes \mathscr{U}\mathfrak{t} \otimes \mathscr{U}\mathfrak{n}^+ = \mathscr{U}\mathfrak{n}^- \otimes \mathscr{U}\mathfrak{b}$ as vector spaces (by PBW), so basically $M(\Lambda) = \mathscr{U}\mathfrak{n}^-$ (as K-vector spaces).)

If V is any highest weight module, $V = M(\Lambda)/J$ for some submodule J. If it is an irreducible highest weight module, J must be maximal. But previous proposition implies that the maximal proper submodule of $M(\Lambda)$ is unique, so $L(\Lambda)$ is unique. □

5.2.8 Exercise. Show that $\mathrm{Hom}_{\mathfrak{g}}(\mathscr{U}\mathfrak{g} \otimes_{\mathscr{U}\mathfrak{b}} W, V) = \mathrm{Hom}_{\mathfrak{b}}(W, \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} V)$.

5.2.9 Corollary. *In particular, any irreducible finite dimensional module is $L(\Lambda)$, where $\Lambda(h_i) \in \mathbb{N}$, i.e. $\Lambda \in P^+$.*

We still need to show that if $\Lambda \in P^+$ then $L(\Lambda)$ is f.d. We also need to show that every f.d. module is a direct sum of irreducibles. The proof has been omitted for lack of time, but all the ingredients are in place. (As with $\mathfrak{sl}_2$, the Casimir Ω shows that the different irreducible representations do not interact. The fact that a representation whose composition series consists of a single irreducible is in fact a direct sum follows almost immediately from the fact that $\mathfrak{t}$ acts diagonalizably.)

5.2.10 Exercises.

1. For $\mathfrak{g} = \mathfrak{sl}_2$, there is an $M(\lambda)$ for each $\lambda \in \mathbb{C}$ (where $Hv = \lambda v$). Show that $M(\lambda)$ is irreducible if and only if $\lambda \notin \mathbb{N}$. Moreover, if $\lambda \in \mathbb{N}$ then $M(\lambda)$ contains a unique proper submodule, the span of $F^{\lambda+1}v$, $F^{\lambda+2}v, \ldots$
2. Show that if $\mathfrak{g}$ is an arbitrary s.s. Lie algebra and $\Lambda(h_i) \in \mathbb{N}$ then $F_i^{\Lambda(h_i)+1} v_\Lambda$ is a singular vector and $F_i = e_{-\alpha_i}$ (working in the $\mathfrak{sl}_2$ attached to some simple root α_i).
3. Compute $\mathrm{ch}\, M(\Lambda)$. Note for $\mathfrak{sl}_2$ the characters are $\frac{z^\lambda}{1-z^{-2}}$.

5.3 Weyl Character Formula

Recall that $W := \langle s_\alpha \mid \alpha \in R \rangle$.

5.3.1 Lemma. *If α is a simple root then $s_\alpha(R^+ \setminus \{\alpha\}) = R^+ \setminus \{\alpha\}$. Note that $s_\alpha \alpha = -\alpha \notin R^+$.*

Proof: Write $s_i = s_{\alpha_i}$ for $\alpha_i \in \Pi$. If $\alpha \in R^+$ then $\alpha = \sum_i k_i \alpha_i$ where $k_i \geq 0$, and

$$s_i \alpha = \sum_{j \neq i} k_j \alpha_j + \Big(-k_i + \sum_{j \neq i} (\alpha_j, \alpha_i^\vee) k_j \Big) \alpha_i.$$

If $\alpha \neq \alpha_i$ then some $k_j > 0$, so the coefficient of α_j in $s_i \alpha$ is greater than zero. But $R = R^+ \amalg (-R^+)$, so one positive coefficient implies that all coefficients are positive, so $s_i \alpha \in R^+$. □

If $g \in O(V)$ (the orthogonal group) then $g s_\alpha g^{-1} = s_{g\alpha}$, so in particular $w s_\alpha w^{-1} = s_{w\alpha}$ for all $w \in W$. It is a theorem that all Weyl groups are of the form

$$W = \langle s_1, \ldots, s_\ell \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$$

where the m_{ij} are given by the Cartan matrix according to the following table.

$a_{ij}a_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

5.3.2 Exercise. Check for each root system that the above holds (note that it is enough to check for the rank two root systems (why?)). Compare this to the existence and uniqueness of $\mathfrak{g}$ which was generated by $\{e_i, h_i, f_i\}$, $\alpha_i \in \Pi$.

5.3.3 Lemma. *$\rho(h_i) = 1$ for $i = 1, \ldots, \ell$, i.e. $\rho = \sum_{i=1}^{\ell} \omega_i$.*

PROOF: By 5.3.1,

$$s_i\rho = s_i\left(\frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha\in R^+\setminus\{\alpha_i\}}\alpha\right) = -\frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha\in R^+\setminus\{\alpha_i\}}\alpha = \rho - \alpha_i$$

but also $s_i\rho = \rho - \langle\rho, \alpha_i^\vee\rangle\alpha_i$, so $\langle\rho, \alpha_i^\vee\rangle = 1$ for all i, implying $\rho = \sum_i \omega_i$. □

For $\mathfrak{sl}_n$,

$$\begin{aligned}2\rho &= (\varepsilon_1 - \varepsilon_2) + \cdots + (\varepsilon_{n-1} - \varepsilon_n)\\ &\quad + (\varepsilon_1 - \varepsilon_3) + \cdots + (\varepsilon_{n-2} - \varepsilon_n)\\ &\quad + \cdots + (\varepsilon_1 - \varepsilon_n)\\ &= (n-1)\varepsilon_1 + (n-3)\varepsilon_2 + \cdots + (3-n)\varepsilon_{n-1} + (1-n)\varepsilon_n.\end{aligned}$$

5.3.4 Lemma. *For $\lambda \in \mathfrak{t}^*$ arbitrary, if $M(\lambda)$ is the Verma module with highest weight λ then*

$$\operatorname{ch} M(\lambda) = e^\lambda \prod_{\alpha\in R^+}(1 - e^{-\alpha})^{-1} = e^\lambda \prod_{\alpha\in R^+}(1 + e^{-\alpha} + e^{-2\alpha} + \cdots)$$

PROOF: Write $R^+ = \{\beta_1, \ldots, \beta_s\}$. Then $\{e_{-\beta_1}^{k_1} \cdots e_{-\beta_s}^{k_s} v_\lambda\}$ is a basis of $M(\lambda)$ of weight $\lambda - k_1\beta_1 - \cdots - k_s\beta_s$, so $\dim M(\lambda)_{\lambda-\beta}$ is the number of ways which β can be written as a sum $\sum_i k_i\beta_i$ of roots β_i with non-negative coefficients k_i, which is the coefficient of $e^{-\beta}$ in $\prod_{\alpha\in R^+}(1 - e^{-\alpha})^{-1}$. Indeed, $\operatorname{ch}(V \otimes W) = \operatorname{ch} V \operatorname{ch} W$, and $\operatorname{ch} \mathbb{C}[x] = \frac{1}{1-x}$. □

Let $\Delta = \prod_{\alpha\in R^+}(1 - e^{-\alpha})$, so $\operatorname{ch} M(\lambda) = e^\lambda/\Delta$.

5.3.5 Lemma. *$w(e^\rho\Delta) = \det(w)e^\rho\Delta$ for all $w \in W$. (Notice that $\det(w) \in \{\pm 1\}$ since $\det s_\alpha = -1$ for all $\alpha \in R$.)*

PROOF: It is enough to show that $s_i(e^\rho\Delta) = -e^\rho\Delta$ as W is generated by the s_i. But

$$\begin{aligned}e^\rho\Delta &= e^\rho(1 - e^{-\alpha_i})\prod_{\alpha\in R^+\setminus\{\alpha_i\}}(1 - e^{-\alpha})\\ s_i(e^\rho\Delta) &= e^{\rho-\alpha_i}(1 - e^{\alpha_i})\prod_{\alpha\in R^+\setminus\{\alpha_i\}}(1 - e^{-\alpha})\\ &= e^\rho(e^{-\alpha_i} - 1)\prod_{\alpha\in R^+\setminus\{\alpha_i\}}(1 - e^{-\alpha})\\ &= -e^\rho\Delta.\end{aligned}$$

□

Recall that $P = \{\lambda \in \mathfrak{t}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for all } \alpha_i \in \Pi\}$ and $P^+ = \{\lambda \in \mathfrak{t}^* \mid \lambda(h_i) \in \mathbb{N} \text{ for all } \alpha_i \in \Pi\}$.

5.3.6 Theorem (Weyl Character Formula).
Let $\lambda \in P^+$, so $L(\lambda)$ is finite dimensional. Then

$$\operatorname{ch} L(\lambda) = \frac{1}{\Delta}\left(\sum_{w \in W} \det(w) e^{w(\lambda+\rho)-\rho}\right) = \sum_{w \in W} w\left(\frac{e^\lambda}{\Delta}\right)$$

i.e. $\operatorname{ch} L(\lambda) = \sum_{w \in W} \det(w) \operatorname{ch} M(w(\lambda+\rho)-\rho)$.

Proof: Omitted, but again, all the ingredients are in place. The second equality is immediate from 5.3.5. □

For $\mathfrak{sl}_2$, set $z = e^{-\alpha/2}$, so $\mathbb{C}[P] = \mathbb{C}[z, z^{-1}]$, $\rho = \frac{\alpha}{2}$, $\Delta = (1-z^{-2})^{-1}$, $W = \langle s \mid s^2 = 1\rangle$, and $sz = z^{-1}$. Then

$$\begin{aligned}\operatorname{ch} L(m\tfrac{\alpha}{2}) &= \frac{z^m - z^{-(m+1)-1}}{1-z^{-2}} \\ &= \frac{z^{m+1} - z^{-(m+1)}}{z - z^{-1}} \\ &= z^m + z^{m-2} + \cdots + z^{2-m} + z^{-m}.\end{aligned}$$

5.3.7 Corollary. $\operatorname{ch} L(0) = 1$*, as* $L(0) = \mathbb{C}$*, so*

$$\Delta = \prod_{\alpha \in R^+} (1 - e^{-\alpha}) = \sum_{w \in W} \det(w) e^{w(\rho)-\rho}.$$

This is the *Weyl denominator formula*.

5.3.8 Exercise. For $\mathfrak{g} = \mathfrak{sl}_n$, show that the Weyl denominator formula is just the identity (rewriting allowed)

$$\det\begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & \dots & x_n^{n-1}\end{bmatrix} = \prod_{i<j}(x_j - x_i),$$

i.e. the Vandermond determinant formula, and that $\mathbb{Z}[P] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $W = \mathfrak{S}_n$.

Given $\mu \in P$, define a homomorphism $F_\mu : \mathbb{C}[P] \to \mathbb{C}((q))$ by $e^\lambda \mapsto q^{-(\lambda,\mu)}$. Then $F_0(e^\lambda) = 1$ and $F_0(\operatorname{ch} L(\lambda)) = \dim L(\lambda)$. Let's apply F_μ to the Weyl denominator formula. We get

$$q^{-(\rho,\mu)} \prod_{\alpha \in R^+}(1 - q^{(\alpha,\mu)}) = \sum_{w \in W} \det(w) q^{-(w(\rho),\mu)} = \sum_{w \in W} \det(w) q^{-(\rho, w(\mu))}$$

so

$$F_\mu(\operatorname{ch} L(\lambda)) = \frac{\sum_{w \in W} \det(w) q^{-(w(\lambda+\rho),\mu)}}{q^{-(\rho,\mu)} \prod_{\alpha>0}(1 - q^{(\alpha,\mu)})}$$

if $(\alpha, \mu) \neq 0$ for all $\alpha \in R^+$ (e.g. $\mu = \rho$ satisfies this condition).

5.3.9 Proposition (q-dimension formula).

$$\dim_q L(\lambda) := \sum_\beta \dim L(\lambda)_\beta q^{-(\beta,\rho)} = \frac{q^{-(\rho,\lambda+\rho)}\prod_{\alpha>0}(1-q^{(\alpha,\lambda+\rho)})}{q^{-(\rho,\rho)}\prod_{\alpha>0}(1-q^{(\alpha,\rho)})}$$

PROOF: Immediate from the above discussion. □

5.3.10 Corollary (Weyl dimension formula).

$$\dim L(\lambda) = \prod_{\alpha>0}\frac{(\alpha,\lambda+\rho)}{(\alpha,\rho)}$$

PROOF: Use l'Hopital, letting $q \to 1$. □

5.3.11 Example. For $\mathfrak{sl}_3$ (a.k.a. A_2), $R^+ = \{\alpha,\beta,\alpha+\beta\}$, $\rho = \alpha+\beta = \omega_1+\omega_2$, so if $\lambda = m_1\omega_1 + m_2\omega_2$ ($m_i \in \mathbb{N}$) then

	α	β	$\alpha+\beta$
$(\cdot,\lambda+\rho)$	m_1+1	m_2+1	m_1+m_2+2
$(\cdot,\rho)$	1	1	2

whence $\dim L(\lambda) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$. As an exercise, compute $\dim_q L(\lambda)$, and do this for B_2 and G_2.

5.3.12 Proposition. *The q-dimension $F_\rho(\operatorname{ch} L(\lambda)) = \dim_q L(\lambda)$ is a (symmetric) unimodal polynomial in q.*

PROOF: If V is an representation of $\mathfrak{sl}_2$, then $\operatorname{ch} V$ is a symmetric unimodal polynomial. If we put $H = \nu^{-1}(\rho)$ then $F_\rho(\operatorname{ch} L(\lambda)) = \sum \dim L(\lambda)_n q^n$, where $L(\lambda)_n = \{x \in L(\lambda) \mid Hx = nx\}$, so we need to extend H to an $\mathfrak{sl}_2$. Set $E = e_{\alpha_1} + \cdots + e_{\alpha_\ell}$. Observe that $H = \sum_i c_i h_i$ for some $c_i \in \mathbb{Q}$, as the h_i are a basis of $\mathfrak{t}$. Define $F = \sum_i c_i e_{-\alpha_i}$ (where $\langle e_\alpha, h_\alpha = \alpha^\vee, e_{-\alpha}\rangle$ are the root $\mathfrak{sl}_2$'s). Exercise: $\langle E,H,F\rangle$ is an $\mathfrak{sl}_2$. □

5.3.13 Exercise. Deduce the following polynomials are unimodal.

1. $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \frac{[n]!}{[k]![n-k]!}$, where $[n] = \frac{q^n-q^{-n}}{q-q^{-1}}$ and $[n]! = [n][n-1]\cdots[1]$. (Hint: $\mathfrak{g} = \mathfrak{sl}_n$ and $V = S^k\mathbb{C}^n$ or $\mathfrak{g} = \mathfrak{sl}_{n+k}$ and $V = \wedge^k\mathbb{C}^{n+k}$.)
2. $(1+q)(1+q^2)\cdots(1+q^n)$. (Hint: spin representation of B_n.)

If $\lambda,\mu \in P^+$, then $L(\lambda)\otimes L(\mu) = \sum_{\nu\in P^+} m_{\mu\lambda}^\nu L(\nu)$ for some $m_{\mu\lambda}^\nu \in \mathbb{N}$. Let's determine these coefficients using the Weyl character formula. Define a conjugation on $\mathbb{Z}[P]$ by $\overline{e^\lambda} = e^{-\lambda}$ and $CT : \mathbb{Z}[P] \to \mathbb{Z}$ by $CT(e^\lambda) = 1$ if $\lambda = 0$ and 0 otherwise. Define $(\cdot,\cdot) : \mathbb{Z}[P]\times\mathbb{Z}[P] \to \mathbb{Q}$ by $(f,g) = \frac{1}{|W|}CT(f\overline{g}\Delta\overline{\Delta})$, and let $\chi_\lambda = \operatorname{ch} L(\lambda)$.

5.3.14 Lemma. $(\chi_\lambda,\chi_\mu) = \delta_{\lambda\mu}$ *if $\lambda,\mu \in P^+$.*

PROOF: We have

$$(\chi_\lambda,\chi_\mu) = \frac{1}{|W|}CT\Bigg(\sum_{x,w\in W}\det(x)\det(w)e^{w(\lambda+\rho)-\rho}\overline{e^{x(\mu+\rho)-\rho}}\Bigg),$$

but $\lambda,\mu \in P^+$ so $w(\lambda+\rho) = \mu+\rho$ if and only if $\mu = \lambda$ and $w = 1$ (exercise). Whence $CT(e^{w(\lambda+\rho)-x(\mu+\rho)}) = \delta_{w,x}\delta_{\mu\lambda}$. □

5.3.15 Corollary. $m^{\nu}_{\lambda\mu} = (\chi_\mu \chi_\lambda, \chi_\nu) = \dim \mathrm{Hom}_{\mathfrak{g}}(L(\nu), L(\mu) \otimes L(\lambda))$.

6 Crystals

The following chapter consists mostly of results of Kashiwara.

Let $\mathfrak{g}$ be a s.s. Lie algebra, $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ a choice of simple roots, and P the corresponding weight lattice.

6.0.16 Definition. A *crystal* for $\mathfrak{g}$ is a set B (not containing 0, by convention) with functions $wt : B \to P$ and $\tilde{e}_i : B \to B \amalg \{0\}$ and $\tilde{f}_i : B \to B \amalg \{0\}$ such that

1. if $\tilde{e}_i b \neq 0$ then $wt(\tilde{e}_i b) = wt(b) + \alpha_i$ and if $\tilde{f}_i b \neq 0$ then $wt(\tilde{f}_i b) = wt(b) - \alpha_i$;
2. for all $b, b' \in B$, $\tilde{e}_i b = b'$ if and only if $b = \tilde{f}_i b'$; and
3. $\varphi_i(b) - \varepsilon_i(b) = \langle wt(b), \alpha_i^\vee \rangle$ for all i and $b \in B$, where

$$\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n b \neq 0\} \quad \text{and} \quad \phi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n b \neq 0\}.$$

The coloured, directed graph with vertices B and edge bb' coloured by i if $\tilde{e}_i b' = b$ is the *crystal graph*, i.e. the i-arrows of the crystal graph point along the action of $\tilde{f}_i$.

6.0.17 Example. With $\mathfrak{g} = \mathfrak{sl}_2$, for any n

$$\boxed{n} \longrightarrow \boxed{n-2} \longrightarrow \boxed{n-4} \longrightarrow \cdots \longrightarrow \boxed{2-n} \longrightarrow \boxed{-n}$$

is a crystal, as is

$$\boxed{n} \longrightarrow \boxed{n-2} \longrightarrow \boxed{n-4} \longrightarrow \cdots,$$

where all the edge colours are 1. For $\mathfrak{sl}_2$, if b is of weight $n-2k$ (so $b = (n-2k)\omega_1$, where $\omega_1 = \frac{1}{2}\alpha$), in the finite crystal $\varepsilon_1(b) = k$ and $\varphi_1(b) = n - k$, so $\varphi_1(b) - \varepsilon_1(b) = n - 2k = \langle wt(b), \alpha_1^\vee \rangle$ as required. Also observe that $\varphi_1(b) + \varepsilon_1(b) = n$, the length of the string.

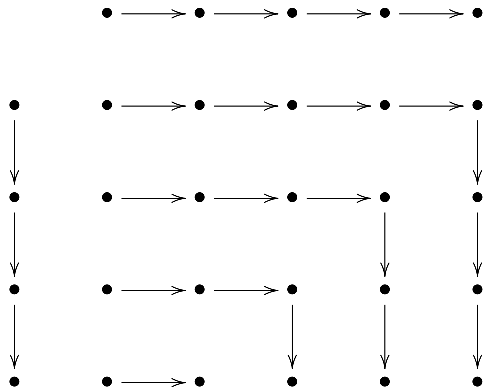
6.0.18 Definition. For crystals B_1 and B_2, define $B_1 \otimes B_2$ as $B_1 \otimes B_2 = B_1 \times B_2$ as sets, $wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$, and

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases},$$

and

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}.$$

Try it and see: in the case of $\mathfrak{sl}_2$ $B_1 \otimes B_2$ breaks up into strings just as $\mathfrak{sl}_2$-representations do.



6.0.19 Exercises.

1. $B_1 \otimes B_2$ is a crystal with these associated functions.

2. Via the obvious map, $(B_1 \otimes B_2) \otimes B_3 \cong B_1 \otimes (B_2 \otimes B_3)$ as crystals.

6.0.20 Definition. $B^\vee$, the *dual crystal* of B, is obtained from B by reversing arrows of the crystal graph. Formally, $wt(b^\vee) = -wt(b)$, $\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee$ and $\tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee$, and $\varepsilon_i(b^\vee) = \varphi_i(b)$ and $\varphi_i(b^\vee) = \varepsilon_i(b)$.

6.0.21 Exercise. $(B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee$ via the obvious map.

6.0.22 Theorem. *Let $L(\lambda)$ be a f.d. irreducible highest weight representation with highest weight λ. Then there is a crystal $B(\lambda)$ in one-to-one correspondence with a basis of $L(\lambda)$.*

1. *It follows that* $\operatorname{ch} L(\lambda) = \sum_{b \in B} e^{wt(b)}$.

2. *Moreover, for each simple root α_i, there is $(\mathfrak{sl}_2)_i \hookrightarrow \mathfrak{g}$, and the decomposition of $L(\lambda)$ as an $\mathfrak{sl}_2$-module is exactly given by the i-strings. In particular, the crystal graph of $B(\lambda)$ is connected (when colours are ignored), and it is generated by the $\tilde{f}_i$'s applied to the unique element of $B(\lambda)_\lambda$.*

3. *The crystal $B(\lambda) \otimes B(\mu)$ is precisely the crystal of $L(\lambda) \otimes L(\mu)$, i.e. it decomposes into connected components, each component a $B(\nu)$ for some $\nu \in P^+$, and we get as many $B(\nu)$'s as the multiplicity with which $L(\nu)$ occurs in $L(\lambda) \otimes L(\mu)$.*

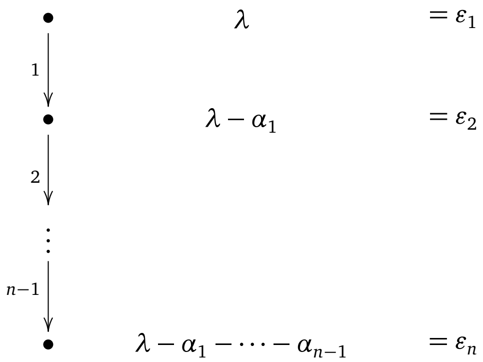
PROOF: Omitted. □

Remark. $B(\lambda)^\vee$ is the crystal of the $\mathfrak{g}$-module $L(\lambda)^* = L(\mu)$, a f.d. irreducible module with lowest weight $-\lambda$. $B \mapsto B^\vee$ corresponds to the automorphism of Lie algebras $\mathfrak{g} \to \mathfrak{g}$ mapping $e_i \mapsto f_i$, $f_i \mapsto e_i$, and $h_i \mapsto -h_i$. (Exercise: compute μ in terms of λ.)

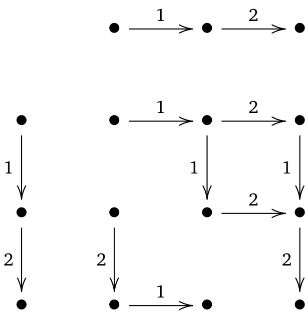
Remark. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra generated by $(\mathfrak{sl}_2)_i$, for $i \in I \subseteq \Pi$, then the crystal for $L(\lambda)|_{\mathfrak{h}}$ is just $B(\lambda)|_{\text{forget stuff } i \text{ for } i \notin I}$.

Notice that $L(\lambda)\otimes L(\mu)\xrightarrow{\sim} L(\mu)\otimes L(\lambda)$, so $B(\lambda)\otimes B(\mu)\xrightarrow{\sim} B(\mu)\otimes B(\lambda)$ by the theorem, but $B\otimes B' \ncong B'\otimes B$ for arbitrary crystals.

6.0.23 Example. For $\mathfrak{g}=\mathfrak{sl}_n$ and $\lambda=\omega_1=\varepsilon_1$, $L(\lambda)=\mathbb{C}^n$ is the standard representation of $\mathfrak{sl}_n$, and it has crystal



Taking the specific example of $n=3$, computing $\mathbb{C}^3\otimes\mathbb{C}^3$ is done by multiplying and decomposing.



From the diagram we see that $\mathbb{C}^3\otimes\mathbb{C}^3\cong S^2\mathbb{C}^3\oplus\wedge^2\mathbb{C}^3$.

6.0.24 Definition. A crystal B is an *integrable crystal* if it is the crystal of a f.d. representation for $\mathfrak{g}$. $b\in B$ is a *highest weight vector* if $\tilde{e}_i b=0$ for all i.

$B(\lambda)$ has a unique highest weight vector. If B is an integrable crystal then its decomposition into connected components is in one-to-one correspondence with highest weight vectors in B. We now apply this to $B_1\otimes B_2$.

6.0.25 Lemma. $\tilde{e}_i(b_1\otimes b_2)=0$ *if and only if* $\tilde{e}_i b_1=0$ *and* $\varepsilon_i(b_2)\le\varphi_i(b_1)$.

PROOF: If $\varepsilon_i(b_2)>\varphi(b_1)$ then $\varepsilon_i(b_2)>0$, so $\tilde{e}_i b_2\neq 0$. The other direction is clear. □

6.0.26 Corollary. $b\otimes b'\in B(\lambda)\otimes B(\mu)$ *is a highest weight vector if and only if* $b\in B(\lambda)_\lambda$ *is a highest weight vector for* $B(\lambda)$ *and* $\varepsilon_i(b')\le\langle\lambda,\alpha_i^\vee\rangle$ *for all* i.

PROOF: $\varepsilon_i(b')\le\varphi_i(b)=\langle\lambda,\alpha_i^\vee\rangle$ (which is the length of the i-string through the highest weight vector b, and by condition 3 since $\tilde{e}_i(b)=0$). □

Whence we have a Clebsch-Gordan type rule

$$L(\lambda)\otimes L(\mu)=\bigoplus_{\substack{b'\in B(\mu)\\ \varepsilon_i(b')\le\langle\lambda,\alpha_i^\vee\rangle}} L(\lambda+wt(b')).$$

Recall that $\wedge^i\mathbb{C}^n$ has highest weight $\omega_i=\varepsilon_1+\cdots+\varepsilon_i$, so if $\lambda\in P^+$ with $\lambda=k_1\omega_1+\cdots+k_{n-1}\omega_{n-1}$, then $L(\lambda)$ is a summand of

$$L(\omega_1)^{\otimes k_1}\otimes\cdots\otimes L(\omega_{n-1})^{\otimes k_{n-1}},$$

as $v_{\omega_i}^{\otimes k_1}\otimes\cdots\otimes v_{\omega_{n-1}}^{\otimes k_{n-1}}$ is a singular vector in this space since v_{ω_i} is a highest weight in each $L(\omega_i)$, and the weight of this vector is $\lambda=k_1\omega_1+\cdots+k_{n-1}\omega_{n-1}$. But in the specific case of $\mathfrak{sl}_n$, $\wedge^i\mathbb{C}^n$ is a summand of $(\mathbb{C}^n)^{\otimes i}$, i.e. $B(\omega_i)$ can be determined from $B(\omega_1)^{\otimes i}$. Therefore the combinatorial rule for product of crystals, together with the standard n-dimensional crystal, already determine the crystal of all $\mathfrak{sl}_n$-modules.

Semi-standard Young tableaux

Write

$$B(\omega_1)=\boxed{1}\xrightarrow{1}\boxed{2}\xrightarrow{2}\cdots\xrightarrow{n-1}\boxed{n},$$

and consider

$$b_i:=\boxed{1}\otimes\boxed{2}\otimes\cdots\otimes\boxed{i}\in B(\omega_1)^{\otimes i},$$

which corresponds to $e_1\wedge e_2\wedge\cdots\wedge e_i\in\wedge^i\mathbb{C}^n\subseteq(\mathbb{C}^n)^{\otimes i}$.

6.0.27 Exercises.

1. Show that b_i is a highest weight vector in $B(\omega_1)^{\otimes i}$, of highest weight $\omega_1=\varepsilon_1+\cdots+\varepsilon_i$. Hence the connected component of $B(\omega_1)^{\otimes i}$ containing b_i is $B(\omega_i)$.

2. Show that this component consists precisely of

$$\{\boxed{a_1}\otimes\cdots\otimes\boxed{a_i}\mid 1\le a_1<a_2<\cdots<a_i\le n\}\subseteq B(\omega_1)^{\otimes i}.$$

Write this vector as $\boxed{\begin{matrix}a_1\\ \vdots\\ a_i\end{matrix}}$, so that the highest weight vector is $\boxed{\begin{matrix}1\\ \vdots\\ i\end{matrix}}$.

Now let $\lambda=\sum_i k_i\omega_i$, and embed

$$B(\lambda)\hookrightarrow B(\omega_1)^{\otimes k_1}\otimes\cdots\otimes B(\omega_{n-1})^{\otimes k_{n-1}}$$

by the highest weight vector $b_\lambda\mapsto b_1^{\otimes k_1}\otimes\cdots\otimes b_{n-1}^{\otimes k_{n-1}}$. Represent an element of the righthand side as (picture of a Young tableaux with no relations for rows and strictly increasing going down each column)

6.0.28 Definition. A *semi-standard tableaux* is a tableaux which strictly increases down columns and decreases (not necessarily strictly) along rows.

6.0.29 Theorem.

1. *The collection of semi-standard tableaux of shape λ is the connected component of $B(\lambda)$.*

2. *$\tilde{e}_i$ and $\tilde{f}_i$ are given by... (determine this as an exercise).*

PROOF: Exercise. □

6.0.30 Example. The crystals of the standard representations of the classical Lie algebras are as follows.

A_n: We have seen that the standard representation of $\mathfrak{sl}_n$ has crystal

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n}.$$

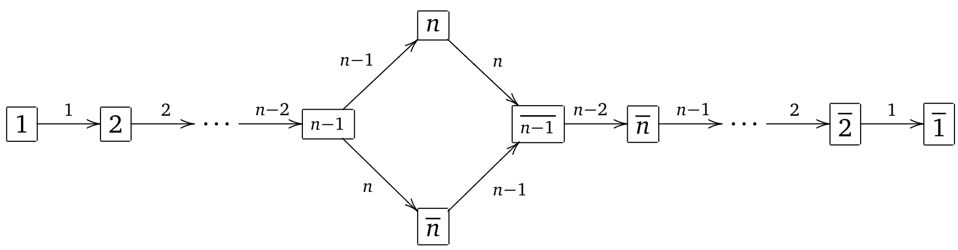
B_n: For $\mathfrak{so}_{2n+1}$,

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\overline{2}} \xrightarrow{1} \boxed{\overline{1}}.$$

C_n: For $\mathfrak{sp}_{2n}$,

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\overline{2}} \xrightarrow{1} \boxed{\overline{1}}.$$

D_n: For $\mathfrak{so}_{2n}$,

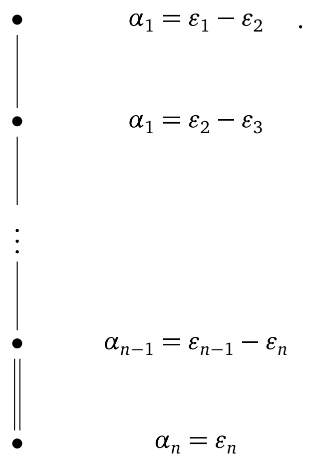


6.0.31 Exercises.

1. Check that these are the crystals of the standard representations.

2. What subcategory of the category of representations of $\mathfrak{g}$ do they generate, i.e. what do you get by taking the tensor product any number of times and decomposing (as we did above for A_n)? (Hint: consider the center of the simply connected group G with Lie algebra $\mathfrak{g}$ and how it acts on the standard representation, i.e. consider the sublattice of P which all summands of tensor powers of the standard representation lie in. The centers are as follows: $\mathbb{Z}/2 \times \mathbb{Z}/2$ for D_{2n}, $\mathbb{Z}/4$ for D_{2n+1}, and $\mathbb{Z}/2$ for B_n and C_n. The center does not act faithfully on the standard representation for B_n, but does for C_n.)

6.0.32 Example (Spin Representation). Recall the a Lie algebra of type B_n has Dynkin diagram of the form



The representation $L(\omega_n)$ is the *spin representation*. As an exercise, show that $\dim L(\omega_n) = 2^n$ (hint: use the Weyl dimension formula). Set $B = \{(i_1, \ldots, i_n) \mid i_j \in \{\pm 1\}\}$ and define $wt(i_1, \ldots, i_n) = \frac{1}{2}\sum_j i_j \varepsilon_j$. Define

$$\tilde{e}_j(i_1, \ldots, i_n) = \begin{cases} (i_1, \ldots, i_{j-1}, 1, -1, i_{j+2}, \ldots, i_n) & \text{if } (i_j, i_{j+1}) = (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{e}_n(i_1, \ldots, i_n) = \begin{cases} (i_1, \ldots, i_{n-1}, 1) & \text{if } i_n = -1 \\ 0 & \text{otherwise} \end{cases}.$$

In the case of D_n, the representations $V^+ := L(\omega_n)$ and $V^- := L(\omega_{n-1})$ are the *half-spin representations*. Set

$$B^{\pm} = \{(i_1, \ldots, i_n) \mid i_j \in \{\pm 1\}, \prod_{j=1}^{n} i_j = \pm 1\},$$

and wt and $e_1, \ldots, e_{n-1}$ as above. Define

$$\tilde{e}_n(i_1, \ldots, i_n) = \begin{cases} (i_1, \ldots, i_{n-2}, 1, 1) & \text{if } (i_{n-1}, i_n) = (-1, -1) \\ 0 & \text{otherwise} \end{cases}.$$

6.1 Littelmann Paths

6.1.1 Definition. Let $P_{\mathbb{R}}$ denote $P \otimes_{\mathbb{Z}} \mathbb{R}$, the $\mathbb{R}$-vector space spanned by P. A *Littelmann path* (or simply *path*) is a piecewise-linear continuous map $\pi : [0,1] \to P_{\mathbb{R}}$.

Paths are considered up to reparameterization, i.e. define $\pi \sim \pi \circ \varphi$, where $\varphi : [0,1] \to [0,1]$ is any piecewise-linear homeomorphism. Let

$$\mathscr{P} = \{\pi \mid \pi(0) = 0, \pi(1) \in P\}/\sim.$$

Define a crystal structure on $\mathscr{P}$ by $wt(\pi) = \pi(1)$ and e_i as follows. Let

$$\begin{aligned} h &= \min\{0, \mathbb{Z} \cap \{\langle \pi(t), \alpha_i^\vee \rangle \mid 0 \le t \le 1\}\} \\ &= \text{smallest integer in } \langle \pi([0,1]), \alpha_i^\vee \rangle \cup \{0\}. \end{aligned}$$

and define $\tilde{e}_i(\pi) = 0$ if $h = 0$. Otherwise, if $h < 0$, let t_1 be the smallest time $t \in [0,1]$ such that $\langle \pi(t), \alpha_i^\vee \rangle = h$ (i.e. the first time that π crosses h) and let t_0 be the largest $t \in [0, t_1]$ such that $\langle \pi(t), \alpha_i^\vee \rangle = h + 1$. Define the path $\tilde{e}_i(\pi)$ by

$$\tilde{e}_i(\pi)(t) = \begin{cases} \pi(t) & 0 \le t < t_0 \\ \pi(t_0) + s_{\alpha_i}(\pi(t) - \pi(t_0)) & t_0 \le t < t_1 \\ \pi(t) + \alpha_i & t \ge t_1 \end{cases}$$

In words, $\tilde{e}_i(\pi)$ is that path that is π up to t_0 (i.e. the last time $h+1$ is hit before the first time hitting h), the reflection of π via s_{α_i} from t_0 to t_1, and then π again from t_1 to 1, but translated for continuity. (The picture from lecture makes this clear.)

6.1.2 Exercise. Show that $\varepsilon_i(\pi) = -h$ (hint: $\langle \alpha_i, \alpha_i^\vee \rangle = 2$).

6.1.3 Example. In $\mathfrak{sl}_2$,

$$\tilde{e}_1\left(-\tfrac{\alpha}{2} \longleftarrow 0 \right) = 0 \longrightarrow \tfrac{\alpha}{2}$$

and applying $\tilde{e}_i$ again gives the zero path. Similarly,

$$\tilde{e}_1\left(-\alpha \longleftarrow 0 \right) = -\tfrac{\alpha}{2} =\!=\!= 0\ ,$$

i.e. the path that goes from 0 to $-\frac{\alpha}{2}$ and back. Applying $\tilde{e}_1$ again gives $0 \to \alpha$, and applying it again give the zero path.

6.1.4 Exercise. What do you get when you hit $-\alpha - \beta \leftarrow 0$ with $\tilde{e}_1$ and $\tilde{e}_2$'s in the case of $\mathfrak{sl}_3$?

Let $\pi^\vee$ denote the reverse of the path π, i.e. define $\pi^\vee$ by $\pi^\vee(t) = \pi(1-t) - \pi(1)$. To complete the definition of the crystal structure let $\tilde{f}_i(\pi) = \tilde{e}_i(\pi^\vee)^\vee$.

6.1.5 Exercise. Prove that $\mathscr{P}$ is a crystal with wt, $\tilde{e}_i$, and $\tilde{f}_i$ defined in this way.

Let $\mathscr{P}^+ = \{\pi \in \mathscr{P} \mid \pi[0,1] \subseteq P_{\mathbb{R}}^+\}$, where $P_{\mathbb{R}}^+ = \{x \in P_{\mathbb{R}} \mid (\forall\ i)\langle x, \alpha_i^\vee \rangle \ge 0\}$. Observe that if $\pi \in \mathscr{P}^+$ then $\tilde{e}_i \pi = 0$ for all i. Let $\mathscr{B}_\pi$ be the subcrystal of $\mathscr{P}$ generated by π, so $\mathscr{B}_\pi = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \pi \mid r \ge 0\}$.

6.1.6 Theorem (Littelmann). *If $\pi, \pi' \in \mathscr{P}^+$ then $\mathscr{B}_\pi \cong \mathscr{B}_{\pi'}$ if and only if $\pi(1) = \pi'(1)$. Furthermore, $\mathscr{B}_\pi$ is isomorphic to the crystal $B(\pi(1))$ (i.e. the crystal associated with the highest wieght module $L(\pi(1))$).*

It follows that we need only consider straight line segements $0 \to \lambda$ and their images under the $\tilde{f}_i$. Littelmann described the paths explicitly in this case.

6.1.7 Definition. For $\pi_1, \pi_2 \in \mathscr{P}$, let $\pi_1 * \pi_2$ be their concatenation, i.e.

$$(\pi_1 * \pi_2)(t) = \begin{cases} \pi_1(2t) & 0 \le t \le \frac{1}{2} \\ \pi_2(2t-1) + \pi_1(1) & \frac{1}{2} \le t \le 1 \end{cases}$$

6.1.8 Exercise. Prove that $* : \mathscr{P} \otimes \mathscr{P} : \pi_1 \otimes \pi_2 \mapsto \pi_1 * \pi_2$ is a well-defined morphism of crystals.

6.1.9 Corollary. *$\mathscr{B}_{\pi_1} \otimes \mathscr{B}_{\pi_2} \cong B_{\pi_1 * \pi_2}$ is an explicit model for the tensor product of crystals.*

6.1.10 Exercises.

1. Describe the tensor product of representations of $\mathfrak{sl}_n$ in terms of Young tableaux. This explicit bit of combinatorics is called the *Littlewood-Richardson rule*.

2. Compute $L_\lambda \otimes \mathfrak{sl}_3$.

Let B be a crystal with $\varepsilon_i(b), \gamma_i(b) < \infty$ for all i. We may define an action of the Weyl group on B by

$$s_i(b) = \begin{cases} \tilde{f}_i^{\langle wt(b), \alpha_i^\vee \rangle} b & \langle\langle wt(b), \alpha_i^\vee \rangle \ge 0 \\ \tilde{e}_i^{-\langle wt(b), \alpha_i^\vee \rangle} b & \langle\langle wt(b), \alpha_i^\vee \rangle \le 0 \end{cases}$$

Then $s_i^2 = 1$ and clearly $wt(s_i b) = s_i(wt(b))$.

$picture$

6.1.11 Proposition (Kashiwara). *If B is an integral crystal then the s_i's satisfy the braid relations, so the above truly gives an action of W on B when extended in the obvious way (since the s_i generate W).*

Proof: It is enough to prove it for rank two Lie algebras, hence for the crystals of all $\mathfrak{sl}_3$, $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{so}_5$, and G_2 modules. Do this as an exercise using Littelmann paths. □

Warning: for $w \in W$, $w\tilde{e}_i \ne \tilde{e}_j w$ in general, even when $w\alpha_i = \alpha_j$.

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