Combinatorial Enumeration
Fall 2004
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Contents

1 Introduction

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1.1 Motivation

Let S be a set of combinatorial objects. Let $\omega : S \to \mathbb{N}$ be a weight function. Find $c_n = |\{s \in S \mid \omega(s) = n\}|$. Equivalently, we want to construct a series $F(x) = \sum_{s\in S} x^{\omega(s)} = \sum_{n\in\mathbb{N}} c_n x^n$. This is called the ordinary generating series for the problem (S, ω). Then $c_n = [x^n]F(x)$. We want to get hold of properties of F. We also wish to discover more algebraic and combinatorial information about S or even F.

Concerning the 4-colour theorem, we deal with planar diagrams (maps, where a map is an embedding of a graph in an orientable surface). Are there any planar algebras? e.g. consider the Tempoley-Lie algebra generated by symbols $1, e_1, \ldots, e_{n-1}$ and the following rules.

1. $e_i^2 = e_i$ for all i
2. $e_i e_j = e_j e_i$ for all $|i - j| > 1$
3. $e_i e_{i\pm1} e_i = e_i$ for all i

See diagrams for interpretation. This gives an algebraic approach to the proof of the 4-colour theorem.

1.2 Ordinary generating series

We have sets $\mathscr{A}$ and $\mathscr{B}$ of combinatorial objects and if we have a bijection $\Omega : \mathscr{A} \xrightarrow{\sim} \mathscr{B} : a \mapsto b$, then $|\mathscr{A}| = |\mathscr{B}|$. The reason we want a bijection is that $\mathscr{B}$ may be easier to count than $\mathscr{A}$. The problem is to find such an Ω, but such constructions are sporatic and not easy find. Sometimes enumerative arguments point to the existence of bijections. We would prefer Ω to be a natural bijection.

A weight function α on $\mathscr{A}$ is any function $\alpha : \mathscr{A} \to \mathbb{N}$. A counting problem: count $\mathscr{A}$ with respect to α. Denoted by $(\mathscr{A}, \alpha)$.

Computational engine for working with formal power series: A formal power series is an element of $\mathbb{C}[[x]]$, where x is an indeterminant. Note that $x^{-1} \notin \mathbb{C}[[x]]$. $\mathbb{C}[[x]]$ is a ring with the obvious sum and product given by

$$\left(\sum_{i\geq0} a_i x^i\right)\left(\sum_{j\geq0} b_j x^j\right) = \sum_{n\geq0}\left(\sum_{i+j=n} a_i b_j\right) x^n$$

Note that this is well defined as each coefficient is just a finite sum of complex numbers.

Given a formal power series f, f^{-1} exists if and only if $f(0) \neq 0$. For example, $(1-x)^{-1} = 1+x+x^2+x^3+\cdots$.

PROOF: (Sketch) Let $f(x) \in \mathbb{C}[[x]]$, $f(x) = \sum a_i x^i$. Suppose that $g(x) \in \mathbb{C}[[x]]$ is such that $g(x)f(x) = 1$, say $g(x) = \sum b_j x^j$. Then by applying the coefficient operator to the equation $(\sum a_i x^i)(\sum b_j x^j) = 1$, we get that the coefficient of x^n is 1 if $n = 0$ and 0 otherwise. In general,

$$\sum_{\substack{i,j\geq0\\ i+j=n}} a_i b_j = \delta_{0,n}$$

In particular, $a_0 b_0 = 1$, so neither of them can be zero. The coefficients of $g(x)$ can be computed from the coefficients of $f(x)$ by working your way up. The general formula is

$$b_n = -a_0^{-1} \sum_{\substack{i>0,b\geq0\\ i+j=n}} a_i b_j$$

As one can easily see, $f(x)$ is invertible in $\mathbb{C}[[x]]$ if and only if $f(0) \neq 0$. □

Sometimes we wish to work in the ring of formal Laurent series, $\mathbb{C}((x))$, where the series may have finitely many terms with a negative exponent on x.

The differential operator is defined formally by $\mathrm{D} : x^n \mapsto nx^{n-1}$ and extended linearly to all of $\mathbb{C}[[x]]$. The sum rule and the product rule for the derivative work as one would expect.

1.1 Lemma. *(Sum Lemma) Let S be a set of combinatorial objects and let $A, B \subseteq S$ with $A \cap B = \emptyset$. Let ω be a wight function on S. Then*

$$[(A \coprod B, \omega)]_o = [(A, \omega)]_o + [(B, \omega)]_o$$

1.2 Lemma. *(Product Lemma) Let A, B be sets of combinatorial objects. Let α be a weight function on A and β be a wight function on B. Let ω be the weight function on $A \times B$ such that $\omega : (a, b) \mapsto \alpha(a) + \beta(b)$. Then*

$$[(A \times B, \omega)]_o = [(A, \alpha)]_o \cdot [(B, \beta)]_o$$

1.3 Types of decompositions

1.3 Example. Say we want to find the generating series for S, given that we already know the generating series for A and B.

1. $\Omega : S \xrightarrow{\sim} A \times B$ is called a direct decomposition for S
2. $\Omega : A \xrightarrow{\sim} S \times B$ is called an indirect decomposition for S. Here S is embedded into a bigger (possibly easier) problem by adjoining B. Under appropriate weight conditions, $S = A \cdot B^{-1}$
3. $S \xrightarrow{\sim} \{\varepsilon\} \coprod S \times S$ is called a recursive decomposition for S. It leads in $\mathbb{C}[[x]]$ under appropriate weight conditions to $S = x + S^2$. This is a functional equation. In more complicated cases we need Lagrange's Theorem for Implicit formal power series to solve this equation

2 Plane planted trees

A plane planted tree is a tree with a unique monovalent root vertex that is embedded in the plane. Let c_n be the number of plane planted trees on n vertices. Find c_n, assuming that trees that are reflections of each other are different trees. We may also wish to count with respect to the number of non-root vertices. Consider the collection of all plane planted trees on 5 vertices (4 non-root vertices). Draw them. There are 5 different trees.

We need a decomposition for the set of all plane planted trees. A simple idea for finding elementary decompositions is to delete the vertex adjacent to the root. This fragments the tree into a single root vertex and a number of trees isomorphic to plane planted trees (i.e. we can identify a monovalent root vertex). This gives us a bijection

$$\Omega : \mathscr{P} \xrightarrow{\sim} \{\varepsilon\} \times \coprod_{m \geq 0} \mathscr{P}^m$$

This is a recursive construction. Let $\omega(t)$ be the number of non-root vertices in $t \in \mathscr{P}$. Is Ω ω-preserving? Yes, and additively! Note that this property is very important in the product lemma.

Let $P = [(\mathscr{P}, \omega)]_o$. Then by the above bijection and the sum and product lemmas, $P(x)$ satisfies the following identity

$$P(x) = x \sum_{m \geq 0} P(x)^m = \frac{x}{1 - P(x)}$$

This uniquely defines P. Can we do anything more with this?

$$\begin{aligned} P &= \frac{x}{1-P} \\ P - P^2 &= x \\ P &= x + P^2 \end{aligned}$$

Now solve the inverse problem, to finds a set $\mathscr{C}$ and a weight function θ on $\mathscr{C}$ such that $C = x + C^2$, where $C = [(\mathscr{C}, \theta)]_o$. Consider

$$\mathscr{C} \xrightarrow{\sim} \{\varepsilon\} \coprod \mathscr{C}^2$$

with $\theta(t)$ the number of non-root monovalent vertices in t.

3 Implicit functions

3.1 Theorem. *(Lagrange's Implicit Function Theorem for $\mathbb{C}[[x]]$) Let $\varphi \in \mathbb{C}[[t]]$ be invertible. Then*

1. *The functional equation $w = t\varphi(w)$ for $w = w(t) \in \mathbb{C}[[t]]$ has a unique solution in $\mathbb{C}[[t]]$. This is called a functional equation of Langrangian type.*
2. *Moreover, if $f(x) \in \mathbb{C}[[x]]$, then*

$$f(w) = f(0) + \sum_{n\geq 1} \frac{x^n}{n} [\lambda^{n-1}](f'(\lambda)\varphi^n(\lambda))$$

3. *If $c_n := [\lambda^n]g(\lambda)\varphi^n(\lambda)$, where $g \in \mathbb{C}[[\lambda]]$, then*

$$\sum_{n\geq 0} c_n t^n = \frac{g(w)}{1 - t\varphi'(w)}$$

where $w = w(t)$ is the unique series defined in (1)

PROOF: See notes for a combinatorial proof.

Prileminaries. We work in $\mathbb{C}((x))$, the ring of formal Laurent series. We consider the operator $[x^{-1}]$. Notice that $[x^n] = [x^{-1}]x^{-n-1}$, so we only need to know how to be able to work out $[x^{-1}]$ and we get the rest for free. For $f \in \mathbb{C}((x))$, $[x^{-1}]f' = 0$ since for any $n \geq 1$ $[x^{-1}]D_x x^n = [x^{-1}]nx^{n-1} = 0$ and $[x^{-1}]$ is linear. For $f, g \in \mathbb{C}((x))$ we have $[x^{-1}]f'g = -[x^{-1}]fg'$ by the product rule. We call $[x^{-1}]$ the formal residue operator. For $f \in \mathbb{C}((x))$ let

$$\mathrm{val}(f) = \begin{cases} k & \text{if } f = x^k g \text{, where } g \in \mathbb{C}[[x]] \text{ and } g(0) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

This is called the valuation of f. For any $f, g \in \mathbb{C}((x))$ we have $\mathrm{val}(fg) = \mathrm{val}(f) + \mathrm{val}(g)$, so val behaves like deg. If $f = x^k g$, where $g \in \mathbb{C}[[x]]$ and $g(0) \neq 0$ then $f^{-1} = x^{-k}g^{-1} \in \mathbb{C}((x))$.

Claim. (Residue Composition Theorem) Let $f, r \in \mathbb{C}((x))$ and let $\mathrm{va}(r) = k > 0$. Then

$$k[x^{-1}]f(x) = [z^{-1}]f(r(z))r'(z)$$

We use the monomial principle to prove this claim. Since $k > 0$ the composition $f(r(z))$ exists. Suppose that $f(x) = x^n$ for some $n \in \mathbb{Z}$. If $n \neq -1$ then

$$[z^{-1}]r^n r' = \frac{1}{n+1} D_z r^{n+1} = 0 = k[x^{-1}]x^n$$

If $n = -1$ then suppose $r = z^k h$ for some $h \in \mathbb{C}[[x]]$ with $h(0) \neq 0$.

$$[z^{-1}]r^{-1}r' = [z^{-1}]z^{-k}h^{-1}(kz^{k-1}h + z^k h') = [z^{-1}](kz^{-1} + \frac{h'}{h}) = k + [z^{-1}]D_z \log h = k = k[x^{-1}]x^{-1}$$

Extending this linearly proves the claim.

Now to prove Langrange's theorem. Let $\Phi(w) = w\varphi^{-1}(w)$. $\operatorname{val}(\Phi) = 1 > 0$ since φ has valuation 0 (it is invertible in $\mathbb{C}[[x]]$). Thus the compositional inverse $\Phi^{[-1]}$ of Φ exists. $\Phi(w) = t$, so $w(t) = \Phi^{[-1]}(t)$ is the unique solution of the functional equation. Let $n \in \mathbb{Z}$, then

$$[t^n]f(w) = [t^{-1}]t^{-(n+1)}f(w) = [t^{-1}]t^{-(n+1)}f(\Phi^{[-1]}(t)) = [w^{-1}]\Phi^{-(n+1)}(w)f(w)\Phi'(w)$$

by the Residue Composition Theorem. If $n \neq 0$ then

$$\begin{aligned}
[t^n]f(w) &= [w^{-1}]\frac{1}{-n}(D_w \Phi^{-n}(w))f(w) \\
&= \frac{1}{n}[w^{-1}]f'(w)\Phi^{-n}(w) && \text{by the Complementation Lemma} \\
&= \frac{1}{n}[w^{-1}]f'(w)\left(\frac{\varphi(w)}{w}\right)^n \\
&= \frac{1}{n}[w^{n-1}]f'(w)\varphi^n(w)
\end{aligned}$$

so the result holds in this case. If $n = 0$ then

$$[t^n]f(w) = f(w)\Big|_{t=0} = f(w)\Big|_{w=0} = f(0)$$

Thus the second part of the theorem is proven.

For the third part, differentiate with respect to t in the functional equation to get

$$\frac{dw}{dt} = \varphi(w) + t\varphi'(w)\frac{dw}{dt}$$

Therefore

$$\frac{f'(w)\varphi(w)}{1 - t\varphi'(w)} = \sum_{n \geq 1} t^{n-1}[\lambda^{n-1}]f'(\lambda)\varphi^n(\lambda) = \sum_{n \geq 0} t^n[\lambda^n]f'(\lambda)\varphi^{n+1}(\lambda)$$

Let $g(\lambda) = f'(\lambda)\varphi(\lambda)$ and the result follows. (Check this.) □

3.2 Example. $P = \frac{x}{1-P}$. Here taking $w = P$, $t = x$, and $\varphi(\lambda) = \frac{1}{1-\lambda}$, we see that this equation is in Lagrangian form, and hence the theorem applies. We want to solve for P, so we should take $f(\lambda) = \lambda$. Thus

$$P(x) = \sum_{n \geq 1} \frac{x^n}{n}[\lambda^{n-1}]\frac{1}{(1-\lambda)^n}$$

by Lagrange's theorem. We still need to actually determine the coefficients.

$$\begin{aligned}\frac{1}{n}[\lambda^{n-1}]\frac{1}{(1-\lambda)^n} &= \frac{1}{n}\binom{-n}{n-1}(-1)^{n-1} \\ &= \frac{1}{n}\binom{n+(n-1)-1}{n-1} \\ \text{so } c_n := [x^n]P(x) &= \frac{1}{n}\binom{2n-2}{n-1}\end{aligned}$$

is the number of plane planted trees on n non-root vertices. Thus

$$P(x) = \sum_{n\geq 1} x^n \frac{1}{n}\binom{2n-2}{n-1}$$

3.3 Example. Find the expansion of $P^6(x)$.

Solution 1: This is a mess, worst in the cosmos.

$$P^6(x) = \left(\sum_{n\geq 1} x^n \frac{1}{n}\binom{2n-2}{n-1}\right)^6$$

Solution 2: Here $f(\lambda) = \lambda^6$. Then

$$\begin{aligned}P^6(x) &= \sum_{n\geq 1}\frac{x^n}{n}[\lambda^{n-1}]\frac{6\lambda^5}{(1-\lambda)^n} \\ &= 6\sum_{n\geq 1}\frac{x^n}{n}[\lambda^{n-6}]\frac{1}{(1-\lambda)^n} \\ &= \sum_{n\geq 1} x^n \frac{6}{n}\binom{2n-7}{n-6}\end{aligned}$$

where the coeffients are determined by doctoring the calculation above. This is the number of plane planted trees on $n+1$ non-root vertices with vertex of degree 7 adjacent to the root vertex.

3.4 Example. Solve $T = xe^T$ in $\mathbb{C}[[x]]$. By Lagrange's theorem

$$T(x) = \sum_{n\geq 1}\frac{x^n}{n}[\lambda^{n-1}]e^{n\lambda} = \sum_{n\geq 1}\frac{x^n}{n}\frac{n^{n-1}}{(n-1)!} = \sum_{n\geq 1} x^n \frac{n^{n-1}}{n!}$$

3.5 Example. Let α, x be indeterminants. Express $e^{\alpha x}$ as a power series in xe^{-x}. (As an aside, if $f \in \mathbb{C}[[x]]$, the valuation of f is defined to be the degree of the term in f of lowest degree. We denote this by $\mathrm{val}(f)$. If $f, g \in \mathbb{C}[[x]]$ then $\mathrm{val}(fg) = \mathrm{val}(f) + \mathrm{val}(g)$. This is an analogue of degree of a polynomial for power series. So why is $\sum_{m\geq 0} c_m (xe^{-x})^m \in \mathbb{C}[[x]]$? It is because the valuation of $(xe^{-x})^m$ is m, and so $[x^k]\sum_{m\geq 0} c_m(xe^{-x})^m$ is a finite sum, for all k.) Let $y = xe^{-x}$. We want to express $e^{\alpha x}$ as a power series in y. Note that $x = ye^x$, and this is a functional equation in Lagrangian form. Then by Lagrange's theorem, this has a unique solution $x = x(y)$. We want $e^{\alpha x}$ rather than $x(y)$.

$$\begin{aligned} e^{\alpha x} &= 1 + \sum_{n\geq 1} \frac{y^n}{n}[\lambda^{n-1}]\alpha e^{\alpha\lambda}e^{n\lambda} \\ &= 1 + \alpha\sum_{n\geq 1}\frac{y^n}{n}[\lambda^{n-1}]\alpha e^{(\alpha+n)\lambda} \\ &= 1+\alpha\sum_{n\geq 1}\frac{y^n}{n}\frac{(\alpha+n)^{n-1}}{(n-1)!} \\ &= 1+\alpha\sum_{n\geq 1}\frac{y^n}{n!}(\alpha+n)^{n-1} \\ &= \alpha\sum_{n\geq 0}\frac{y^n}{n!}(\alpha+n)^{n-1} \end{aligned}$$

by working in $\mathbb{C}[\alpha,\alpha^{-1}][[y]]$. Note that this series lies in $\mathbb{C}[[x]]$.

3.6 Example. Let a, b, x be indeterminants. Then $e^{(a+b)x} = e^{ax}\cdot e^{bx}$, and so

$$(a+b)\sum_{k\geq 0}\frac{x^k}{k!}(a+b+k)^{k-1} = a\sum_{i\geq 0}\frac{x^i}{i!}(a+i)^{i-1} + b\sum_{j\geq 0}\frac{x^j}{j!}(b+j)^{j-1}$$

Applying $[x^n]$ throughout yields

$$\begin{aligned} \frac{(a+b)}{n!}(a+b+n)^{n-1} &= ab\sum_{\substack{i,j\geq 0\\ i+j=n}}\frac{1}{i!}\frac{1}{j!}(a+i)^{i-1}(b+j)^{j-1} \\ (a+b)(a+b+n)^{n-1} &= ab\sum_{i=1}^{n}\binom{n}{i}(a+i)^{i-1}(b+n-i)^{n-i-1} \end{aligned}$$

This is called Abel's extension to the Binomial theorem.

3.7 Theorem. *(Taylor's Theorem for $\mathbb{C}[[x]]$) Let $f(x)\in\mathbb{C}[[x]]$. Then $f(x) = \sum_{k\geq 0}\frac{x^k}{k!}L_0\frac{d^k}{dx^k}f(x)$, where $L_0 : \mathbb{C}[[x]]\to\mathbb{C} : f(x)\to f(0)$.*

PROOF: Consider $[x^n]x^k = \delta_{k,n}$. On the other hand,

$$\frac{1}{k!}L_0\frac{d^k}{dx^k}x^n = \frac{1}{k!}L_0 n(n-1)\cdots(n-k+1)x^{n-k} = \delta_{k,n}$$

since if $n > k$ then $0^{n-k} = 0$, if $n < k$ then $n(n-1)\cdots(n-k+1) = 0$, and finally if $n = k$ then $0^0 = 1$ and $n(n-1)\cdots(n-k+1) = k!$. Thus $[x^n]x^k = \frac{1}{k!}L_0\frac{d^n}{dx^n}x^k$. Extend linearly to the whole of $\mathbb{C}[[x]]$, since $\{1, x, x^2, \ldots\}$ is a basis (?) of $\mathbb{C}[[x]]$. The result follows. □

Examples of the use of Lagrange's Theorem

1. Find the number c_n of plane planted trees with n non-root vertices and such that each vertex has even up degree (except the root). Let $\mathscr{P}$ be the set of all plane planted trees. then by the universal decomposition for plane planted trees (the branch decomposition) we have

$$\Phi : \mathscr{P} \xrightarrow{\sim} \{\varepsilon\}\coprod_{k\geq 0}\mathscr{P}^k$$

where k is the up degree of a generic vertex. Let ω be the obvious weight function on $\mathscr{P}$. Then Φ is additively ω-preserving. Let $\mathscr{E}$ be the desired set of trees. Then $\mathscr{E} \subset \mathscr{P}$. Now consider

$$\Phi|_{\mathscr{E}} : \mathscr{E} \xrightarrow{\sim} \{\varepsilon\} \coprod_{i\geq 0} \mathscr{E}^{2i}$$

Then $\Phi|_{\mathscr{E}}$ is additively ω- preserving. Therefore we use the sum and product lemmas to get

$$E = x(1 + E^2 + E^4 + \cdots) = \frac{x}{1 - E^2}$$

where $E(x) := [(\mathscr{E}, \omega)]_o$. This is a functional equation in Lagrangian form. Solving for E

$$[x^n]E = \frac{1}{n}[\lambda^{n-1}](1+\lambda^2)^{-n} = \frac{1}{n}[\lambda^{n-1}]\sum_{k=0}^{n}\binom{n+k-1}{k}\lambda^{2k}$$

For a non-zero contribution, we need $n-1 = 2k$ for some $k \geq 0$, so n is odd. Let $n = 2m+1$.

$$[x^{2m+1}]E = [\lambda^{2m+1}]\frac{1}{2m+1}\sum_{k\geq 0}\binom{2m+k}{k}\lambda^{2k} = \frac{1}{2m+1}\binom{3m}{m}$$

and is 0 if n is even.

2. Find the average number of non-root vertices of degree one in the set all plane planted trees on n non-root vertices.

$$\Phi : \mathscr{P} \xrightarrow{\sim} \{\varepsilon\} \coprod_{k\geq 0} \mathscr{P}^k$$

with weight function $\omega(t) = (\omega_1(t), \omega_2(t)) \in \mathbb{N}^2$, where $\omega_1(t)$ is the number of non-root monovalent vertices of t and $\omega_2(t)$ is the total number of non-root vertices of t. We call ω the refinement of ω_1 by ω_2 on $\mathscr{P}$. It is also denoted $\omega_1 \otimes \omega_2$. Let

$$P(u,x) = [(\mathscr{P}, \omega)]_o := \sum_{t\in\mathscr{P}} u^{\omega_1(t)} x^{\omega_2(t)} \in \mathbb{C}[u][[x]]$$

Then we have

$$\mathscr{P} \xrightarrow{\sim} \{\varepsilon\} \times (\varepsilon \amalg \mathscr{P} \amalg \mathscr{P}^2 \amalg \cdots)$$

and so

$$P = x(u + P + P^2 + \cdots) = x\left(\frac{1}{1-P} + (u-1)\right)$$

This is a functional equation for P as a series for x in Lagrangian form. Using Lagrange's Theorem in $\mathbb{C}[u][x]]$

$$\begin{aligned}
P &= \sum_{n\geq 0}\frac{x^n}{n}[\lambda^{n-1}]\left(\frac{1}{1-\lambda} + (u-1)\right)^n \\
&= \sum_{n\geq 0}\frac{x^n}{n}[\lambda^{n-1}]\left(\sum_{i=0}^{n}\binom{n}{i}\frac{(u-1)^{n-i}}{(1-\lambda)^i}\right) \\
&= \sum_{n\geq 0}\frac{x^n}{n}\left(\sum_{i=0}^{n}\binom{n}{i}\binom{i+n-2}{n-1}(u-1)^{n-i}\right)
\end{aligned}$$

Now to find the average number of non-root vertices of degree one that a plane planted tree with n non-root vertices has, we must take the average. There are $\frac{1}{n+1}\binom{2n}{n}$ plane planted trees on n non-root vertices. If $c_{n,k} = [x^n u^k]P$, then the number we want is thus

$$\frac{\sum_{k=0}^{n} k c_{n,k}}{\frac{1}{n+1}\binom{2n}{n}}$$

Notice that the numerator is equal to

$$L_{u=1} D_u [x^n] P = \frac{1}{n}\binom{n}{n-1}\binom{2n-3}{n-1}$$

and hence the average number of trees is $\frac{n(n+1)(n-2)(n-3)}{4(2n-1)}$.

4 Partitions of Integers

Let n, k be a positive integers. We say that $(i_1, \dots, i_k)$ is an integer partition of n if

1. $i_1 \geq \cdots \geq i_k \geq 1$
2. $i_1 + \cdots + i_k = n$

Denote that α is a partition of n by $\alpha \vdash n$. The number of partitions of n is denoted by $p(n)$ in most books. It has amazing properties.

4.1 Example. $p(5)$: $(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)$ so $p(5) = 7$.

Is there a closed form expression for $p(n)$? No! There is an asymptotic expression for $p(n)$ such that the integer part is exactly $p(n)$. We are not going to do this.

Can we find a generating series for $\{p(n) \mid n \geq 1\}$? Can we restrict this to other classes of partitions? Can we show that certain subsets of partitions are equicardinal? If so, can we find natural bijections? Can we find non-trivial identities based on these enumerations? The answer to all of these questions is yes!

4.1 Elementary properties

Let $\mathscr{P}$ be the set of all partitions. The universal decomposition is

$$\mathscr{P} \xrightarrow{\sim} \times_{i \geq 1} \{i\}^*$$

Let $\omega(\pi)$ = sum of the parts of π. We want to gain information about $P = [(\mathscr{P}, \omega)]_o$. The bijection is clearly additively ω- preserving, so we can use the sum and product lemmas.

$$P(x) = \prod_{i \geq 1} [(\{i\}^*, \omega)]_o(x) = \prod_{i \geq 1} \frac{1}{1 - x^i}$$

Caveat: This has a barrier of essential singularities on the unit circle.

Let $\mathscr{E} \subseteq \mathscr{P}$ be the set of all partitions with only even parts. Then clearly

$$[(\mathscr{E}, \omega)]_o(x) = \prod_{i \geq 1} \frac{1}{1 - x^{2i}}$$

Let $\mathscr{O}_{\mathscr{D}} \subseteq \mathscr{P}$ be the set of all partitions with distinct odd parts. Then clearly

$$[(\mathscr{O}_{\mathscr{D}}, \omega)]_o = \prod_{i \geq 1} (1 + x^{2i-1})$$

4.2 Haruspication

Conjecture (on flimsy evidence): The number of partitions of n with only odd parts equals the number of partitions of n with distinct parts. It is enough to show that the generating series are equal, no coefficient extraction required.

Let $\mathscr{A}$ be the set of all partitions with only odd parts. Then

$$\begin{aligned} A(x) &= \prod_{i\geq 1} \frac{1}{1-x^{2i-1}} \\ &= \prod_{i\geq 1} \frac{1}{1-x^{2i-1}} \cdot \frac{1-x^{2i}}{1-x^{2i}} \\ &= \frac{\prod_{i\geq 1}(1-x^{2i})}{\prod_{i\geq 1}(1-x^{2i-1})(1-x^{2i})} \\ &= \frac{\prod_{i\geq 1}(1-x^{2i})}{\prod_{i\geq 1}(1-x^{i})} \\ &= \prod_{i\geq 1}(1+x^i) \end{aligned}$$

This is exactly the generating series for $\mathscr{D}$, the set of all partitions with distinct parts. So the conjecture is true! Even though we cannot convienently extract the coefficients, we can still tell many things about the relationships between the numbers of partitions. Since the sets are equicardinal, there is a bijection between them. There should be a natural bijection. Find one.

Consider that $1+x^2+x^3+x^5 = (1+x^2)(1+x^3)$. Using this fact,

$$\begin{aligned} \prod_{i\geq 1}(1+x^{2i}+x^{3i}+x^{5i}) &= \prod_{i\geq 1}(1+x^{2i})(1+x^{3i}) \\ &= \prod_{i\geq 1} \frac{1-x^{4i}}{1-x^{2i}} \frac{1-x^{6i}}{1-x^{3i}} \\ &= \prod_{i\geq 1} \frac{1}{(1-x^{12i-10})(1-x^{12i-9})(1-x^{12i-6})(1-x^{12i-3})(1-x^{12i-2})} \end{aligned}$$

Since

$$\prod_{i\geq 1}(1-x^{6i}) = \prod_{i\geq 1}(1-x^{12i-6})(1-x^{12i})$$

$$\prod_{i\geq 1}(1-x^{4i}) = \prod_{i\geq 1}(1-x^{12i-8})(1-x^{12i-4})(1-x^{12i})$$

$$\prod_{i\geq 1}(1-x^{3i}) = \prod_{i\geq 1}(1-x^{12i-9})(1-x^{12i-6})(1-x^{12i-3})(1-x^{12i})$$

$$\prod_{i\geq 1}(1-x^{2i}) = \prod_{i\geq 1}(1-x^{12i-10})(1-x^{12i-8})(1-x^{12i-6})(1-x^{12i-4})(1-x^{12i-2})(1-x^{12i})$$

We see thusly that the set of partitions in which each part occurs 0, 2, 3, or 5 times is in bijection with the set of partitions in which each part congruent to 2, 3, 6, 9, or 10 modulo 12 only.

4.3 Ferrers Graphs/Diagrams

See the notes for an explanation and some examples.

Let $\alpha \vdash n$. The conjugate of α is the partition of n obtained by reflecting the Ferrers diagram for α about the diagonal, and it is denoted $\tilde{\alpha}$. Note that $(\tilde{\tilde{\alpha}}) = \alpha$ and the number of parts in α is equal to the largest part of $\tilde{\alpha}$. Let $\mathscr{M}_k \subseteq \mathscr{P}$ be the set of all partitions with at most k parts and let $\mathscr{L}_k \subseteq \mathscr{P}$ be the set of all partitions with largest part at most k. Clearly

$$[(\mathscr{L}_k, \omega)]_o = \prod_{i=1}^{k} (1 - x^i)^{-1}$$

To find $[(\mathscr{M}_k, \omega)]_o$ we note that $\mathscr{M}_k \xrightarrow{\sim} \mathscr{L}_k$ by conjugation. Thus $[(\mathscr{M}_k, \omega)]_o = [(\mathscr{L}_k, \omega)]_o$.

Given partition, if we delete the maximal upper left corner square (Durfee square), say it is $k \times k$, then we are left with a partition from $\mathscr{L}_k$ (on the bottom) and a partition from $\mathscr{M}_k$ (to the right). This gives us an additively ω-preserving bijection

$$\Omega : \mathscr{P} \xrightarrow{\sim} \bigcup_{k \geq 0} \{\mathscr{D}_k\} \times \mathscr{M}_k \times \mathscr{L}_k$$

Taking generating series we get

$$\prod_{i \geq 1} \frac{1}{1 - x^i} = \sum_{k \geq 0} \frac{x^{k^2}}{\prod_{i=1}^{k} (1 - x^i)^2}$$

This is due to one of Euler, Gauß, Cauchy.

Can we preserve more combinatorial information? For example, can we preserve the number of parts of α, $\lambda(\alpha)$?

$$[(\mathscr{P}, \omega \otimes \lambda)]_o(x, u) \xrightarrow{\sim} \times_{i \geq 1} \{\varepsilon, i, ii, \dots\}$$

So we have

$$\begin{aligned} P(x,u) &= \prod_{i \geq 1} \sum_{\alpha \in \{\varepsilon, i, \dots\}} x^{\omega(\alpha)} u^{\lambda(\alpha)} \\ &= \prod_{i \geq 1} \sum_{j \geq 0} (x^i u)^j \\ &= \prod_{i \geq 1} \frac{1}{1 - ux^i} \end{aligned}$$

And thus

$$\prod_{i \geq 1} \frac{1}{1 - ux^i} = 1 + \sum_{k \geq 1} \frac{u^k x^{k^2}}{\prod_{i \geq 1}^{k} (1 - x^i) \prod_{i \geq 1}^{k} (1 - ux^i)}$$

which is in $\mathbb{C}[u][[x]]$. Setting $u = -1$, we get

$$\prod_{i \geq 1} \frac{1}{1 + x^i} = 1 + \sum_{k \geq 1} \frac{(-1)^k x^{k^2}}{\prod_{i \geq 1}^{k} (1 - x^{2i})}$$

5 Exponential Generating Series

5.1 A classical problem

The derangement problem is a classical problem. A derangement is a rearrangment of a set of things such that no thing is mapped to itself. Let d_n be the number of derangements of n distinct objects. For examples, when

$n = 3$, the complete set of derangements is $\{(231), (312)\}$, and so $d_3 = 2$. Let $\mathscr{P}_n$ be the set of all permuations of $\{1, 2, \ldots, n\}$. Then a derangement is a permutation with no fixed points. Encoding $\mathscr{P}_3$ in terms of derangements

$$\begin{pmatrix}123\\123\end{pmatrix} \mapsto \binom{123}{123} \mapsto \begin{pmatrix}123\\123\end{pmatrix}_{\{1,2,3\}}$$
$$\begin{pmatrix}123\\132\end{pmatrix} \mapsto \binom{1}{1}\binom{23}{32} \mapsto \begin{pmatrix}1\\1\end{pmatrix}_{\{1\}}\begin{pmatrix}12\\21\end{pmatrix}_{\{2,3\}}$$
$$\begin{pmatrix}123\\213\end{pmatrix} \mapsto \binom{3}{3}\binom{12}{21} \mapsto \begin{pmatrix}1\\1\end{pmatrix}_{\{3\}}\begin{pmatrix}12\\21\end{pmatrix}_{\{1,2\}}$$
$$\begin{pmatrix}123\\231\end{pmatrix} \mapsto \binom{123}{231} \mapsto \begin{pmatrix}123\\231\end{pmatrix}_{\{1,2,3\}}$$
$$\begin{pmatrix}123\\321\end{pmatrix} \mapsto \binom{2}{2}\binom{13}{31} \mapsto \begin{pmatrix}1\\1\end{pmatrix}_{\{2\}}\begin{pmatrix}12\\21\end{pmatrix}_{\{1,3\}}$$
$$\begin{pmatrix}123\\312\end{pmatrix} \mapsto \binom{123}{312} \mapsto \begin{pmatrix}123\\312\end{pmatrix}_{\{1,2,3\}}$$

Clearly $\mathscr{P} := \bigcup_{n=1}^{\infty} \mathscr{P}_n \xrightarrow{\sim} \mathscr{I} * \mathscr{D}$, where $\mathscr{I} = \{\varepsilon, (1), (12), (123), \ldots\}$ is the set of all identity permutations and $\mathscr{D} = \{\varepsilon, (21), (231), (312), \ldots\}$ is the set of all derangements. The $*$ denotes that we take the bijection with all possible labelings of the permutations on symbols $\{1, \ldots, n\}$, where n is the sum of the size of the derangement and size of the identity permutation.

We now have a new combinatorial product of sets, called the $*$-product, defined by

$$\mathscr{A} * \mathscr{B} := \bigcup_{n \geq 1} \bigcup_{\substack{a \in \mathscr{A} \\ b \in \mathscr{B}}} \bigcup_{\substack{\alpha \subseteq \{1,\ldots,n\} \\ \beta = \{1,\ldots,n\} \setminus \alpha}} ((a)_\alpha, (b)_\beta)$$

where the third union is taken over all α, β that partition $\{1, \ldots, n\}$ and $|\alpha| = \#(\text{labels of } a)$, $|\beta| = \#(\text{labels of } b)$. Can we get a product lemma for the $*$-product? If $\mathscr{C} = \mathscr{A} * \mathscr{B}$ then in the generating series for these problems we will have

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$$

since there are $\binom{n}{k}$ ways of choosing the labels for a_k (after which the labels are fixed for b_{n-k}). Thus if we define $C(x) = \sum_{k \geq 0} c_i \frac{x^n}{n!}$ and $A(x)$, $B(x)$ similarily, we have $C(x) = A(x)B(x)$. We call $A(x)$ the exponential generating series for the problem $(\mathscr{A}, \omega)$ and is denoted $[(\mathscr{A}, \omega)]_e$. We have proved

5.1 Lemma. *($*$-Product Lemma)* $[(\mathscr{A} * \mathscr{B}, \omega)]_e(x) = [(\mathscr{A}, \omega)]_e(x)[(\mathscr{B}, \omega)]_e(x)$

Now to solve the problem. $\mathscr{P} \xrightarrow{\sim} \mathscr{I} * \mathscr{D}$, and the generating series for $\mathscr{P}$ is given by

$$[(\mathscr{P}, \omega)]_e(x) = \sum_{\pi \in \mathscr{P}} \frac{x^{\omega(\pi)}}{\omega(\pi)!} = \sum_{k \geq 0} k! \frac{x^k}{k!} = \frac{1}{1-x}$$

The generating series for $\mathscr{I}$ is

$$[(\mathscr{I}, \omega)]_e(x) = \sum_{\pi \in \mathscr{I}} \frac{x^{\omega(\pi)}}{\omega(\pi)!} = \sum_{k \geq 0} \frac{x^k}{k!} = e^x$$

Thus we can derive the generating series for $\mathscr{D}$ implicitly from $P(x) = I(x)D(x)$ to get $D(x) = \frac{e^{-x}}{1-x}$. Thus

$$d_n = \left[\frac{x^n}{n!}\right] D(x) = \left[\frac{x^n}{n!}\right] \frac{e^{-x}}{1-x} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

5.2 An architect's problem

Find the number of $m \times n$ $\{0,1\}$-matrices with

1. no rows of 0's
2. no columns of 0'
3. exactly k 1's

The idea is to bubble row's of 0's to the top of the matrix, taking care of the original row label to the row so that the original matrix can be recovered. Do a similar thing with the columns ("bubble" them to the left). From a general $\{0,1\}$-matrix we get (in the bottom right corner) a smaller $\{0,1\}$-matrix with no rows or columns of 0's. We can decompose this as a zero matrix and a matrix with no rows or columns of 0's, where they are both row and column label disjoint. Let $\mathscr{M}$ be the set of all combinatorial $\{0,1\}$-matrices. Let $\mathscr{A}$ be the set of combinatorial matrices with no rows or columns of 0's and $\mathscr{Z}$ be the set of all combinatorial zero matrices. Then

$$\Omega : \mathscr{M} \xrightarrow{\sim} \mathscr{Z} *_{r,c} \mathscr{A}$$

For indeterminants, take x to be exponential and mark rows, y to be exponential and mark columns, and z to be ordinary and mark the number of 1's. Let

$$\begin{aligned} \omega_1(M) &= \text{ number of rows of } M \\ \omega_2(M) &= \text{ number of columns of } M \\ \lambda(M) &= \text{ number of 1's in } M \end{aligned}$$

Then Ω is additively λ-preserving. Let $\omega = \omega_1 \otimes \omega_2 \otimes \lambda$. Let $A(x,y;u) = \sum_{m,n\geq 0} a_{m,n,k} \frac{x^m}{m!} \frac{y^n}{n!} u^k$. By the $*$-product lemma, we have

$$\begin{aligned} [(\mathscr{M},\omega)](x,y;u) &= [(\mathscr{Z},\omega)](x,y;u)[(\mathscr{A},\omega)](x,y;u) \\ \sum_{m,n\geq 0} (1+u)^{mn} \frac{x^m}{m!} \frac{y^n}{n!} &= A(x,y;u) \sum_{m,n\geq 0} \frac{x^m}{m!} \frac{y^n}{n!} \\ \sum_{m,n\geq 0} (1+u)^{mn} \frac{x^m}{m!} \frac{y^n}{n!} &= A(x,y;u) e^{x+y} \end{aligned}$$

where the series for $\mathscr{M}$ is obtained by labeling the matrix in a boostrophedon manner. Finally,

$$a_{M,N,K} = \left[\frac{x^M}{M!} \frac{y^N}{N!} u^K\right] e^{-x-y} \sum_{m,n\geq 0} (1+u)^{mn} \frac{x^m}{m!} \frac{y^n}{n!}$$

which you can go off and work out yourself.

5.3 Permutations and cycles

Motivation: A derangement is a permutation with no cycles.

$$\pi = \begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10 \\ 3\ 6\ 9\ 1\ 2\ 5\ 8\ 7\ 4\ 10 \end{pmatrix} = \begin{pmatrix} 1\ 3\ 9\ 4 \\ 3\ 9\ 4\ 1 \end{pmatrix} \begin{pmatrix} 2\ 6\ 5 \\ 6\ 5\ 2 \end{pmatrix} \begin{pmatrix} 7\ 8 \\ 8\ 7 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

This is called the disjoint cycle decomposition. We may relabel this as

$$\pi = \begin{pmatrix} 1\ 2\ 3\ 4 \\ 2\ 3\ 4\ 1 \end{pmatrix}_{\{1,3,9,4\}} \begin{pmatrix} 1\ 2\ 3 \\ 2\ 3\ 1 \end{pmatrix}_{\{2,6,5\}} \begin{pmatrix} 1\ 2 \\ 2\ 1 \end{pmatrix}_{\{7,8\}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\{10\}}$$

A permutation is a disjoint union of cycles. Let $\mathscr{U}$ be the set of all canonical sets,

$$\mathscr{U} = \{\varnothing, \{1\}, \{1,2\}, \{1,2,3\} \ldots\}$$

Let $\mathscr{C}$ be the set of all canonical cycles,

$$\mathscr{C} = \{(1), (1\ 2), (1\ 2\ 3), (1\ 3\ 2), \ldots\}$$

We build the set $\mathscr{P}$ of all permutations from these by a new operation $\star$, $*$-composition.

$$\mathscr{P} \xrightarrow{\sim} \mathscr{U} \star \mathscr{C}$$

5.2 Lemma. *($*$-composition)* $[(\mathscr{A} \star \mathscr{B}, \omega_s)]_e = [(\mathscr{A}, \omega_s)]_e \circ [(\mathscr{B}, \omega_s)]_e$ *where $\mathscr{B}$ does not contain the null structure.*

Using this (without proof), letting $C(x) = [(\mathscr{C}, \omega_s)]_e(x)$, we have

$$\begin{aligned} [(\mathscr{P}, \omega_s)]_e(x) &= [(\mathscr{U} \star \mathscr{C}, \omega_s)]_e(x) \\ [(\mathscr{P}, \omega_s)]_e &= [(\mathscr{U}, \omega_s)]_e \circ [(\mathscr{C}, \omega_s)]_e(x) \\ \sum_{n\geq 0} n! \frac{x^n}{n!} &= \sum_{n\geq 0} \frac{\big(C(x)\big)^n}{n!} \\ \frac{1}{1-x} &= e^{C(x)} \\ C(x) &= \log(1-x)^{-1} \\ &:= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \end{aligned}$$

Thus the number of cycles of length n on n symbols is $\left[\frac{x^n}{n!}\right] C(x) = (n-1)!$. For derrangements we have

$$\mathscr{D} \xrightarrow{\sim} \mathscr{U} \star (\mathscr{C} \setminus \{1\})$$

since a derrangement has no fixed points (no one-cycles). This is a direct decomposition for $\mathscr{D}$. Then

$$d_n = \left[\frac{x^n}{n!}\right] D(x) = \left[\frac{x^n}{n!}\right] e^{C(x)-x} = \left[\frac{x^n}{n!}\right] \exp(\log(1-x)^{-1} - x)) = \left[\frac{x^n}{n!}\right] \frac{e^{-x}}{1-x}$$

In a sense we have managed to solve the set equation $\mathscr{P} \xrightarrow{\sim} \mathscr{I} * \mathscr{D}$ for $\mathscr{D}$. We get $\mathscr{D} \xrightarrow{\sim} \mathscr{U} \star (\mathscr{C} \setminus \{1\})$.

5.4 The preimage decomposition

We look as $\{f : \mathbb{N}_n \to \mathbb{N}_m\} = \mathbb{N}_n^{\mathbb{N}_m}$ where $\mathbb{N}_n = \{1, \ldots, n\}$. How many functions/injections/surjections/bijections are there?

Notice that, given a function $f : \mathbb{N}_n \to \mathbb{N}_m$, the set $\{f^{-1}(i) \mid i \in \mathbb{N}_m\}$ of preimages of f partition $\mathbb{N}_n$. Notice that some of these sets may be empty. Given any ordered partition of $\mathbb{N}_n$ into exactly m sets (some of which may be empty) defines a function $f : \mathbb{N}_n \to \mathbb{N}_m$. Let $\mathscr{F}_m = \bigcup_{n \geq 0} \mathbb{N}_m^{\mathbb{N}_n}$. Then we have a bijection

$$\mathscr{F}_m \xrightarrow{\sim} \mathscr{U}^{\star m}$$

Let $\mathscr{A}$ be a set of labeled combinatorial structures. Let $[\mathbb{N}_n]\mathscr{A}$ denote the set of all members of $\mathscr{A}$ with label set $\{1, \ldots, n\}$.

1. How many functions? $[\mathbb{N}_n]\mathscr{F}_m \xrightarrow{\sim} [\mathbb{N}_n]\mathscr{U}^{\star m}$

$$|[\mathbb{N}_n]\mathscr{F}_m| = |[\mathbb{N}_n]\mathscr{U}^{\star m}| = \left[\frac{x^n}{n!}\right][(\mathscr{U}^{\star m}, \omega)]_e(x) = \left[\frac{x^n}{n!}\right][(\mathscr{U}, \omega)]_e^m(x) = \left[\frac{x^n}{n!}\right] e^{mx} = m^n$$

2. How many surjections? $[\mathbb{N}_n]\mathscr{S}_m \xrightarrow{\sim} [\mathbb{N}_n](\mathscr{U} \setminus \{\varnothing\})^{\star m}$

$$|[\mathbb{N}_n]\mathscr{F}_m| = |[\mathbb{N}_n](\mathscr{U} \setminus \{\varnothing\})^{\star m}| = \left[\frac{x^n}{n!}\right][((\mathscr{U} \setminus \{\varnothing\})^{\star m}, \omega)]_e(x) = \left[\frac{x^n}{n!}\right][((\mathscr{U} \setminus \{\varnothing\}), \omega)]_e^m(x)$$

$$= \left[\frac{x^n}{n!}\right](e^x - 1)^m = \left[\frac{x^n}{n!}\right]\sum_{k=0}^{n}\binom{m}{k}(-1)^{m-k}e^{kx} = \sum_{k=0}^{n}\binom{m}{k}(-1)^{m-k}k^n$$

3. How many injections? $[\mathbb{N}_n]\mathscr{I}_m \xrightarrow{\sim} [\mathbb{N}_n]\{\varnothing, \{1\}\}^{\star m}$

$$|[\mathbb{N}_n]\mathscr{F}_m| = |[\mathbb{N}_n]\{\varnothing, \{1\}\}^{\star m}| = \left[\frac{x^n}{n!}\right][(\{\varnothing, \{1\}\}^{\star m}, \omega)]_e(x) = \left[\frac{x^n}{n!}\right][(\{\varnothing, \{1\}\}, \omega)]_e^m(x)$$

$$= \left[\frac{x^n}{n!}\right](1+x)^m = m(m-1)(m-2)\cdots(m-n+1) := (m)_n, \text{ the falling factorial}$$

4. How many bijections? $[\mathbb{N}_n]\mathscr{I}_m \xrightarrow{\sim} [\mathbb{N}_n]\{\{1\}\}^{\star m}$

$$|[\mathbb{N}_n]\mathscr{F}_m| = |[\mathbb{N}_n]\{\{1\}\}^{\star m}| = \left[\frac{x^n}{n!}\right][(\{\{1\}\}^{\star m}, \omega)]_e(x) = \left[\frac{x^n}{n!}\right][(\{\{1\}\}, \omega)]_e^m(x) = \left[\frac{x^n}{n!}\right] x^m = n!\delta_{m,n}$$

5.5 Branch decomposition for labelled rooted trees

Let $\mathscr{T}_R$ be the set of all labelled rooted trees. Then by a variation of the bijection above we have that

$$\mathscr{T}_R \xrightarrow{\sim} \{\varepsilon\} * (\mathscr{U} \star \mathscr{T}_R)$$

Let $T(x) = [(\mathscr{T}_R, \omega_0)]_e(x)$ where $w_0(t)$ is the number of vertices of t. Then $T = xe^T$ and this is a functional equation in Lagrangian form, so

$$\left[\frac{x^n}{n!}\right] T = n!\frac{1}{n}[\lambda^{n-1}]e^{n\lambda} = n!\frac{n^{n-1}}{n(n-1)!} = n^{n-1}$$

5.6 Derivative of Sets

Let $\mathscr{A}$ be a set of labelled combinatorial structures. We refer to the elements that carry lables as s-objects. Let $A(x) = [(\mathscr{A}, \omega_s)]_e(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$. Let $\mathscr{B}$ be the set obtained from $\mathscr{A}$ by deleting a canonical s-ojbect in each $a \in \mathscr{A}$ (for example, the largest label). Assume that every $a \in \mathscr{A}$ has a canonical s-object.

$$B(x) := [(\mathscr{B}, \omega_s)]_e(x) = \sum_{n\geq 0} a_n \frac{x^{n-1}}{(n-1)!} = \frac{d}{dx}A(x)$$

We denote $\mathscr{B}$ by $\frac{\partial}{\partial s}\mathscr{A}$. Then $[(\mathscr{B}, \omega_s)]_e(x) = \frac{d}{dx}[(\mathscr{A}, \omega_s)]_e(x)$ where $\omega_s(a)$ denotes the number of s-objects in $a \in \mathscr{A}$ (i.e. the number of labels used in the labelling of $a \in \mathscr{A}$). For example

$$\frac{\partial}{\partial s}(\mathscr{A} * \mathscr{B}) \xrightarrow{\sim} \left(\frac{\partial \mathscr{A}}{\partial s}\right) \mathscr{B} \coprod \mathscr{A} * \left(\frac{\partial \mathscr{B}}{\partial s}\right)$$

5.3 Example. Alternating permutations are permutations such that the numbers in the even positions are greater the the numbers that they follow and the numbers in the odd positions are less than the numbers that they follow. For example, (2 4 3 5 1) is an alternating permutation on 5 symbols. Let $\mathscr{Q}$ ($\mathscr{P}$) be the set of all alternating permutations of even (odd) length. The minimal element of $\mathscr{P}$ is (1) and the minimal element of $\mathscr{Q}$ is ε. Consider $\frac{\partial}{\partial s}\mathscr{P}$. Let $\pi \in \mathscr{P}$. We delete the greatest label in π, giving us two alternating permutations of odd length (if $\pi \neq (1)$, otherwise it gives us ε). Thus

$$\frac{\partial}{\partial s}(\mathscr{P} \setminus \{(1)\}) \xrightarrow{\sim} \mathscr{P} * \mathscr{P}$$

Similarily, if $\pi \in \mathscr{Q}$ then deleting the greatest label (if $\pi \neq \varepsilon$) gives us an alternating permutation in $\mathscr{P}$ and one in $\mathscr{Q}$. Thus

$$\frac{\partial}{\partial s}(\mathscr{Q} \setminus \{\varepsilon\}) \xrightarrow{\sim} \mathscr{P} * \mathscr{Q}$$

Now, can we solve this simulataneous set of differential equations?

$$\frac{dP}{dx} = 1 + P^2 \text{ and } \frac{dQ}{dx} = PQ$$

This called a pair of Riccati equations. It is also called a matrix Riccati equation. Let $P = \tan(R)$. Then $\frac{dP}{dR} = \sec^2(R)$ and $1 + P^2 = \sec^2(R)$. Thus $dx = \frac{dP}{1+P^2} - \frac{\sec^2(R)}{\sec^2(R)}dR = dR$, and so $R = x + c$, where c is constant. Therefore $P = \tan(x + c)$. But $P(0) = 0$, so $c = 0$ and $P = \tan(x)$. Now $\frac{dQ}{Q} = Pdx = \tan(x)dx$, so (solving) $Q = \sec(x)$.

5.7 Another combinatorial operation

Let $A(x)$ be the exponential generating series for $\mathscr{A}$. Let $A(X) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$. Then $x\frac{d}{dx}A(x) = \sum_{n\geq 0} na_n \frac{x^n}{n!}$. We interpret this replication by distinguishing an s-object in all possible ways. Picture this as “painting a subobject red”. Denote this operation by $\frac{s\partial}{\partial s}$. We have produced a set $\frac{s\partial}{\partial s}\mathscr{A}$. We have

$$[(\frac{s\partial}{\partial s}\mathscr{A}, \omega_s)]_e(x) = \frac{d}{dx}[(\mathscr{A}, \omega_s)]_e(x)$$

Notice that $\frac{s\partial}{\partial s} = s * \frac{\partial}{\partial s}$ as operators.

5.4 Example. Let $\mathscr{F}$ be the set of all functions $f : \mathbb{N}_n \to \mathbb{N}_n$. Let $[\mathbb{N}_n]\mathscr{A}$ denote the set of all $a \in \mathscr{A}$ with n labels $\{1, \ldots, n\}$. Let $\mathscr{T}_R$ be the set of all labelled rooted trees. Then

$$\frac{v\partial}{\partial v}\mathscr{T}_R \xrightarrow{\sim} Path \star \mathscr{T}_R \xrightarrow{\sim} (\mathscr{U} \star \mathscr{C}) \star \mathscr{T}_R \xrightarrow{\sim} \mathscr{U} \star (\mathscr{C} \star \mathscr{T}_R) \xrightarrow{\sim} \coprod_{n\geq 0} \mathbb{N}_n^{\mathbb{N}_n} \xrightarrow{\sim} \mathscr{F}$$

It follows that $[\mathbb{N}_n]\frac{v\partial}{\partial v}\mathscr{T}_R \xrightarrow{\sim} \mathbb{N}_n^{\mathbb{N}_n}$, and so

$$|[\mathbb{N}_n]\frac{v\partial}{\partial v}\mathscr{T}_R| = |\mathbb{N}_n^{\mathbb{N}_n}| \text{ so } nt_n = n^n \text{ and hence } t_n = n^{n-1}$$

PROOF: $\frac{v\partial}{\partial v}\mathscr{T}_R$ is the set of all labelled rooted with exactly 1 vertex in each distinguished by a spot of red paint (for sake of argument). That is, they are bi-rooted trees. Let $t \in \frac{v\partial}{\partial v}\mathscr{T}_R$. Then there is a unique path between the root and the red vertex. Then t can be regarded as a set of trees attached to the path by identifying the rootss of these trees with a vertex of the path. Then

$$\frac{v\partial}{\partial v}\mathscr{T}_R \xrightarrow{\sim} Path \star \mathscr{T}_R$$

Next consider a path. We may write it uniquely as a non-null permutation, thus

$$Path \longleftrightarrow \mathscr{P} \xrightarrow{\sim} \mathscr{U} \star \mathscr{C}$$

by the disjoint cycle decomposition. $\star$ is associative, so

$$\frac{v\partial}{\partial v}\mathscr{T}_R \xrightarrow{\sim} \mathscr{U} \star (\mathscr{C} \star \mathscr{T}_R)$$

What is this as a combinatorial structure? It is some number of cycles with trees hanging off of the vertices of cycle. Direct the tree edges towards a vertex on the cycle (this can be done in a unique way), and notice that the edges of the cycle already have a direction. What do we obtain? A map! Each label in the structure has a unique arrow eminating from it, pointing to another label. This is a function in $\mathbb{N}_n^{\mathbb{N}_n}$. Conversely, given any function from $\mathbb{N}_n$ to $\mathbb{N}_n$ then we may write down a directed graph corresponding to the function. The graph must be in the form of a disjoint union of cycles with trees hanging off, with the arrows in the trees pointing towards the cycle. (Think about it). Thus

$$\frac{v\partial}{\partial v}\mathscr{T}_R \xrightarrow{\sim} \coprod_{n\geq 0} \mathbb{N}_n^{\mathbb{N}_n}$$

and furthermore $[\mathbb{N}_n]\frac{v\partial}{\partial v}\mathscr{T}_R \xrightarrow{\sim} \mathbb{N}_n^{\mathbb{N}_n}$. □

This suggests a combinatorial proof of Lagrange's theorem, which has been given in the course notes. It works by asking how refined a weight function is preserved by the string of bijections.

6 Sign-Reversing Involutions

6.1 Example. Given P_1, P_2, Q_1, Q_2 points in the plane with integer coordinates. We are interested in the number of pairs of paths (π_1, π_2) such that

1. π_1 is a path from P_1 to Q_1 (P_1 is the origin and Q_1 is the terminus) and π_2 is a path from P_2 to Q_2
2. the only steps allowed are $\uparrow: (i,j) \mapsto (i, j+1)$ and $\rightarrow: (i,j) \mapsto (i+1, j)$
3. π_1 and π_2 have no common points

6.2 Definition. We call pair of paths that satisfies (1) and (2) an *ordered 2-path* origins (P_1, p_2) to terminii (Q_1, Q_2). An ordered 2-path satisfying (3) is called *non-intersecting* and one not satisfying (3) is called *intersecting*.

Counting non-intersection 2-paths is hard. However, counting all ordered 2-paths is not: the number of such paths is $\binom{a_1+b_1}{a_1}\binom{a_2+b_2}{a_2}$, where a_i is the difference of the x-coordinates of P_i and Q_i and b_i is the difference of the y-coordinates.

Notation. Let $T = T(P_1, P_2; Q_1, Q_2)$ denote the set of all ordered 2-paths from (P_1, P_2) to (Q_1, Q_2). Let $T_\times$ be the subset of intersecting ordered 2-paths and $T_=$ be the subset of non-intersecting 2-paths.

Let $\widehat{T}(P_1, P_2; Q_1, Q_2) = T(P_1, P_2; Q_1, Q_2) \cup T(P_1, P_2; Q_2, Q_1)$ and define $\widehat{T}_\times$ and $\widehat{T}_=$ similarily. We say that $(P_1, P_2; Q_1, Q_2)$ is *proper* if $T_=(P_1, P_2; Q_2, Q_1) = \varnothing$.

Suppose that $(\pi_1, \pi_2) \in \widehat{T}_\times(P_1, P_2; Q_1, Q_2)$. Let A be the first intersection point encounted upon following π_1 starting at P_1. Let π_1' be the path obtained by following π_1 from P_1 to A and then following π_2, and let π_2' be the path obtained by following π_2 from P_2 to A and then following π_1. If we let

$$\phi : \widehat{T}_\times \to \widehat{T}_\times : (\pi_1, \pi_2) \mapsto (\pi_1', \pi_2')$$

then ϕ is an involution on $\widehat{T}_\times$ (i.e. $\phi^2 = id_{\widehat{T}_\times}$). Notice that ϕ maps elements of $T_\times(P_1, P_2; Q_1, Q_2)$ to elements of $T_\times(P_1, P_2; Q_2, Q_1)$, and *visa versa*. Define $\overline{\omega} : \widehat{T} \to \mathbb{Z}$ by

$$\overline{\omega}(\pi_1, \pi_2) = \begin{cases} 1 & \text{if } (\pi_1, \pi_2) \in T(P_1, P_2; Q_1, Q_2) \\ -1 & \text{otherwise} \end{cases}$$

Then $\overline{\omega}\phi = -\overline{\omega}$. So ϕ is a *sign-reversing involution*. Notice that

$$\begin{aligned} \sum_{\pi \in \widehat{T}_\times} \overline{\omega}(\pi) &= -\sum_{\pi \in \widehat{T}_\times} \overline{\omega}(\phi(\pi)) && \phi \text{ is sign-reversing} \\ &= -\sum_{\phi(\pi) \in \widehat{T}_\times} \overline{\omega}(\phi(\pi)) && \phi \text{ is injective} \\ &= -\sum_{\pi \in \widehat{T}_\times} \overline{\omega}(\pi) && \phi \text{ is surjective} \end{aligned}$$

Thus $\sum_{\pi \in \widehat{T}_\times} \overline{\omega}(\pi) = 0$. Hence, when $(P_1, P_2; Q_1, Q_2)$ is proper,

$$\begin{aligned} |T_=(P_1, P_2; Q_1, Q_2)| &= \sum_{\pi \in T_=} \overline{\omega}(\pi) \\ &= \sum_{\pi \in \widehat{T}_=} \overline{\omega}(\pi) + \sum_{\pi \in \widehat{T}_\times} \overline{\omega}(\pi) \\ &= \sum_{\pi \in \widehat{T}} \overline{\omega}(\pi) \\ &= \sum_{\pi \in T(P_1, P_2; Q_1, Q_2)} \overline{\omega}(\pi) + \sum_{\pi \in T(P_1, P_2; Q_2, Q_1)} \overline{\omega}(\pi) \\ &= |T(P_1, P_2; Q_1, Q_2)| - |T(P_1, P_2; Q_2, Q_1)| \\ &= |T(P_1; Q_1)| \cdot |T(P_2; Q_2)| - |T(P_1; Q_2)| \cdot |T(P_2; Q_1)| \\ &= \begin{vmatrix} |T(P_1; Q_1)| & |T(P_1; Q_2)| \\ |T(P_2; Q_1)| & |T(P_2; Q_2)| \end{vmatrix} \end{aligned}$$

Where $T(P; Q)$ has the obvious meaning.

We can say more. Note that $|T_=| = [(T_=, 0)]_o$ (the zero weight). The same reasoning applies to any weight funtion ω such that

1. $\omega = \psi \otimes \psi$ where ψ is the weight function for a single path
2. $\omega(\pi_1, \pi_2)$ cannot depend on the membership of steps in specific paths

3. $[(T_=, \omega)]_o = \sum_{\pi \in T_=} \overline{\omega}(\pi)$

where

$$\overline{\omega}(\pi_1, \pi_2) := \begin{cases} x^{\omega(\pi_1,\pi_2)} & \text{if } (\pi_1, \pi_2) \in T(P_1, P_2; Q_1, Q_2) \\ -x^{\omega(\pi_1,\pi_2)} & \text{otherwise} \end{cases}$$

Then we get that

$$[(T_=, \omega)]_o = \begin{vmatrix} [(T(P_1; Q_1), \psi)]_o & [(T(P_1; Q_2), \psi)]_o \\ [(T(P_2; Q_1), \psi)]_o & [(T(P_2; Q_2), \psi)]_o \end{vmatrix}$$

6.3 Theorem. *Involution Theorem Let $\mathscr{E} \subseteq \mathscr{S}$ where $\mathscr{S}$ is a set of combinatorial objects. Let $\overline{\omega} : \mathscr{S} \to R$, where R is a commutative ring. If $\phi : \mathscr{S} \setminus \mathscr{E} \to \mathscr{S} \setminus \mathscr{E}$ is a sign-reversing involution (which is to say $\phi^2 = id$ and $\overline{\omega}\phi = -\overline{\omega}$), then*

$$\sum_{\sigma \in \mathscr{E}} \overline{\omega}(\sigma) = \sum_{\sigma \in \mathscr{S}} \overline{\omega}(\sigma)$$

PROOF:

$$\begin{aligned} \sum_{\sigma \in \mathscr{S} \setminus \mathscr{E}} \overline{\omega}(\sigma) &= - \sum_{\sigma \in \mathscr{S} \setminus \mathscr{E}} \overline{\omega}(\phi(\sigma)) && \phi \text{ is sign-reversing} \\ &= - \sum_{\phi(\sigma) \in \mathscr{S} \setminus \mathscr{E}} \overline{\omega}(\phi(\sigma)) && \phi \text{ is one to one} \\ &= - \sum_{\sigma \in \mathscr{S} \setminus \mathscr{E}} \overline{\omega}(\sigma) && \phi \text{ is onto} \end{aligned}$$

Therefore $\sum_{\sigma \in \mathscr{S} \setminus \mathscr{E}} \overline{\omega}(\sigma) = 0$ and the result follows as in the example above. □

Here's how we apply the theorem. Suppose that we have a combinatorial problem $(\mathscr{E}, \omega)$. If we can find

- a superset $\mathscr{S} \supseteq \mathscr{E}$
- a function $\overline{\omega} : \mathscr{S} \to R$
- a function $\mathscr{S} \setminus \mathscr{E} \to \mathscr{S} \setminus \mathscr{E}$ such that

  1. $\phi^2 = id$
  2. $\overline{\omega}\phi = -\overline{\omega}$
  3. $[(\mathscr{E})]_o = \sum_{\sigma \in \mathscr{E}} \overline{\omega}(\sigma)$

Then $[(\mathscr{E})]_o = \sum_{\sigma \in \mathscr{S}} \overline{\omega}(\sigma)$, where hopefully the latter series is easier to compute.

6.4 Example. A *polyomino* is a region in the plane formed by joining unit squares at the edges in such a way the the region formed is simply connected. (Connected in the sense that there is a walk along the integer lattice, no diagonal motions allowed.) We say that two polyominoes are *equivalent* if one can be translated to the other, with no rotations allowed. How many polyonminoes (up to equivalence) are there? The problem of counting general polyominoes is very hard. However, we can count certain subclasses of polyonminoes.

6.5 Definition. A *convex polyomino* is a polyomino for which the intersection with any horizontal or vertical line either has one connected (in the usual sense) component or none at all.

We wish to count the number $c(m)$ of convex polyominoes with perimeter $2m+4$. We will do this by decomposing it into ordered paths.

Given a convex polyomino P, translate it so that the bottom left corner of its smallest bounding rectangle is the origin $O=(0,0)$. Let O' be the opposite corner of the rectangle. Suppose that $O'=(k+1,m-k+1)$, where $k+1$ is the width of P. Pick a and d to be the points on the boundry of P closest to O on the y and x axes, respectively. Pick c and b on the boundry of P closest to O' on the lines parallel to the y and x axes, respectively. Mark the points on the boundry of P closest to a,b,c,d by $\odot$. Let $((\pi_1,\pi_2),(\pi_3,\pi_4))$ be the paths from a to d, b to c, d to c, and a to b, respectively. (Not really, but from the $\odot$ closest to these points in the correct directions.) Notice that π_1 cannot possibly intersect π_3 or π_4, and neither can π_2. Applying our result for non-intersecting paths, if $a=(0,j)$ and $b=(k+1-r,m-k+1)$ and $c=(k+1,m-k+1-s)$ and $d=(i,0)$ then the number of non-intersecting 2-paths (π_1,π_2) is *something* (where $\binom{-2}{-1}:=1$ for today).

$$c(m)=\sum_{k=0}^{m}\sum_{\substack{0\le i,r\le k\\ 0\le j,s\le m-k}} blah$$

which simplifies to

$$c(m)=(2m+7)2^{2m-4}-4(2m-3)\binom{2m-4}{m-2}$$

6.6 Theorem. *(Euler's Pentagonal Number Theorem)*

$$\sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k-1)}{2}}=\prod_{i\ge1}(1-x^i)$$

What is this theorem really saying? Notice that the right hand side looks similar to the generating series for $\mathscr{D}$, the partitions with distinct parts, $\prod_{i\ge1}(1+x^i)$. We define

$$\overline{\omega}:\mathscr{D}\to\mathbb{Z}[[x]]:\alpha\mapsto(-1)^{\ell(\alpha))}x^{|\alpha|}$$

Then we have

$$\sum_{\alpha\in\mathscr{D}}=\prod_{i\ge1}(1-x^i)$$

Now consider the left hand side. The coefficient of x^n is zero for "most" n, and when it is non-zero the coefficient is ±1. So if we believe the theorem, it says that for most n the number of partitions into an even number of distinct parts is equal to the number of partitions of n into an odd number of distinct parts. For $n=\frac{k(3k-1)}{2}$, these numbers differ by 1. Why is this? Define

$$C:\mathscr{D}\to\mathbb{N}:\alpha\mapsto\text{smallest part of }\alpha$$

$$R:\mathscr{D}\to\mathbb{N}:\alpha\mapsto\text{largest }k\text{ such that the sucessive diffs of the first }k\text{ parts are 1}$$

In the Ferrers diagram $\mathcal{F}_\alpha$ for $\alpha\in\mathscr{D}$, replace the $*$'s counted by $C(\alpha)$ with $-$'s. Replace the rightmost $*$ in rows counted by $R(\alpha)$ with $+$'s. Use $\pm$ when necessary. Notice that for any α, there is at most one $\pm$. Let $n_\pm(\alpha)$ be the number of $\pm$'s in the modified Ferrers diagram for α. Define two operations on $\mathscr{D}$:

- Raising (ρ): remove the $-$'s from $\mathcal{F}_\alpha$ and append a $-$ to the end of the first $C(\alpha)$ rows of $\mathcal{F}_\alpha$ (treat $\pm$ as a $-$)
- Lowering (λ): remove the $+$'s from $\mathcal{F}_\alpha$ and add a new row of $R(\alpha)$ $+$'s to the bottom of $\mathcal{F}_\alpha$ (treat $\pm$ as a $+$)

We would like to apply λ and ρ only when the result is in $\mathscr{D}$ and furthermore, we want each $-$ in $\rho\mathcal{F}_\alpha$ to be to the right of a $+$.

ρ can be applied when

- $C(\alpha) < R(\alpha)$
- $C(\alpha) = R(\alpha)$ and $n_\pm(\alpha) = 0$

λ can be applied when

- $C(\alpha) > R(\alpha) + 1$
- $C(\alpha) = R(\alpha) + 1$ and $n_\pm(\alpha) = 0$

Note that at most one of ρ and λ can be applied to any given α. Let $\mathscr{E} \subseteq \mathscr{D}$ the the subset of partitions with distinct parts for which neither ρ nor λ can be applied. Then $\mathscr{E}$ consists of precisely the $\alpha \in \mathscr{D}$ for which either

- $C(\alpha) = R(\alpha)$ and $n_\pm(\alpha) = 1$, or
- $C(\alpha) = R(\alpha) + 1$ and $n_\pm(\alpha) = 1$

But what is $\mathscr{E}$ really? When $C(\alpha) = R(\alpha) = k$, $|\alpha|$ is a pentagonal number, $|\alpha| = \frac{k(3k-1)}{2}$. Whence $\overline{\omega}(\alpha) = (-1)^k x^{\frac{k(3k-1)}{2}}$. Similarily, when $C(\alpha) = R(\alpha) + 1 = k + 1$ we have $|\alpha| = \frac{k(3k+1)}{2}$, so $\overline{\omega}(\alpha) = (-1)^k x^{\frac{k(3k+1)}{2}}$. Thus

$$\sum_{\alpha \in \mathscr{E}} \overline{\omega}(\alpha) = \sum_{k \geq 0} (-1)^k x^{\frac{k(3k-1)}{2}} + \sum_{k \geq 0} (-1)^k x^{\frac{k(3k+1)}{2}} = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k-1)}{2}}$$

Recall the to prove the theorem we need only prove that

$$\sum_{\alpha \in \mathscr{E}} \overline{\omega}(\alpha) = \sum_{\alpha \in \mathscr{D}} \overline{\omega}(\alpha)$$

Define a sign-reversing involution $\phi : \mathscr{D} \setminus \mathscr{E} \to \mathscr{D} \setminus \mathscr{E}$ by

$$\phi(\alpha) = \begin{cases} \rho(\alpha) & \text{if } C(\alpha) < R(\alpha) \\ \rho(\alpha) & \text{if } C(\alpha) = R(\alpha) \text{ and } n_\pm(\alpha) = 0 \\ \lambda(\alpha) & \text{if } C(\alpha) > R(\alpha) + 1 \\ \lambda(\alpha) & \text{if } C(\alpha) = R(\alpha) + 1 \text{ and } n_\pm(\alpha) = 0 \end{cases}$$

Let $\mathscr{P} = \{\alpha \in \mathscr{D} \setminus \mathscr{E} \mid \phi(\alpha) = \rho(\alpha)\}$ and $\mathscr{Q} = \{\alpha \in \mathscr{D} \setminus \mathscr{E} \mid \phi(\alpha) = \lambda(\alpha)\}$. Check that $\rho : \mathscr{P} \to \mathscr{Q}$ and $\lambda : \mathscr{Q} \to \mathscr{P}$, and that $\lambda\rho = id_{\mathscr{Q}}$ and $\rho\lambda = id_{\mathscr{P}}$. It follows from these observations that $\phi^2 = \phi$, so ϕ is an involution. Also,

$$\begin{aligned} \overline{\omega}(\phi(\alpha)) &= (-1)^{\ell(\phi(\alpha))} x^{|\phi(\alpha)|} \\ &= (-1)^{\ell(\alpha) \pm 1} x^{|\alpha|} \\ &= -(-1)^{\ell(\alpha)} x^{|\alpha|} \\ &= -\overline{\omega}(\alpha) \end{aligned}$$

Therefore ϕ is sign-reversing as well. Therefore

$$\sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k-1)}{2}} = \sum_{\alpha \in \mathscr{E}} \overline{\omega}(\alpha) = \sum_{\alpha \in \mathscr{D}} \overline{\omega}(\alpha) = \prod_{i \geq 1} (1 - x^i)$$

Whence the theorem is proved.

7 The Pattern Algebra

This section is under heavy construction. See the notes for a more sane presentation.

7.1 A brief return to $\{0,1\}$-strings

Let a, b be non-commuting indeterminants and let $\mathbb{Q}\langle\langle a, b\rangle\rangle$ be the ring of all formal series in a and b. Encoding $a \longleftrightarrow 0$ and $b \longleftrightarrow 1$, the set of all $\{0,1\}$-stings is generated by

$$(1-(a+b))^{-1} = 1 + (a+b) + (a+b)^2 + \cdots \longleftrightarrow \varepsilon \amalg \{0,1\} \amalg \{0^2, 01, 10, 1^2\} \amalg \cdots$$

We call $(1-(a+b))^{-1}$ the generator for the set $\{0,1\}^*$ What about the well known decomposition $\{0,1\} = 1^*(01^*)$? Can we deduce this algebraically?

$$1-(a+b) = 1-b-a = (1-a(1-b)^{-1})(1-b)$$

Thus

$$(1-(a+b))^{-1} = \left((1-a(1-b)^{-1})(1-b)\right)^{-1} = (1-b)^{-1}(1-a(1-b)^{-1})^{-1}$$

which is exactly what we want. Notice also that

$$1-(a+b) = 1-a-b+ab-ab = (1-a)((1-b)-(1-a)^{-1}ab) = (1-a)(1-(1-a)^{-1}ab(1-b)^{-1})(1-b)$$

Therefore

$$([1-(a+b))^{-1} = (1-b)^{-1}(1-(1-a)^{-1}ab(1-b)^{-1})^{-1}(1-a)^{-1}$$

This gives us the decomposition $\{0,1\}^* = 1^*(0^*010^*)0^*$.

Now let's generalize this to a larger alphabet $\{1,2,3\dots\}$. We look for for complex substructures than just the occurrences of 1's, 2's, etc. We might want:

1. length of maximal strictly increasing substrings
2. length of maximal blocks of symbols
3. restriction to permutations with these conditions (i.e. distinct symbols)

7.2 Patterns

Let $\sigma = 4\ 5\ 9\ 9\ 10\ 7\ 6\ 5\ 5\ 8$. We call $uuddudddu$ the pattern of σ, where $u \longleftrightarrow <$ and $d \longleftrightarrow \geq$. Denote this by $\text{patt}(\sigma) = u^2d^2ud^4u$. Let x_i mark the occurrence of $i \in \{1,2,3\dots\}$. Let f_i mark the occurrence of a maximal $<$-substring. Then

$$\sigma \longleftrightarrow \mathsf{x}_4\mathsf{x}_5\mathsf{x}_9^2\mathsf{x}_8\mathsf{x}_{10}\mathsf{x}_7\mathsf{x}_6\mathsf{x}_5^2\mathsf{x}_8 \text{ and the maximal } <\text{-substrings are } \mathsf{f}_3\mathsf{f}_1\mathsf{f}_2\mathsf{f}_1^3\mathsf{f}_2$$

The contribution to the generating series that counts strings is

$$(\mathsf{f}_3\mathsf{f}_1\mathsf{f}_2\mathsf{f}_1^3\mathsf{f}_2)(\mathsf{x}_4\mathsf{x}_5\mathsf{x}_9^2\mathsf{x}_8\mathsf{x}_{10}\mathsf{x}_7\mathsf{x}_6\mathsf{x}_5^2\mathsf{x}_8) \in \mathbb{Q}\langle \mathsf{f}_1, \mathsf{f}_2, \dots\rangle\langle\langle \mathsf{x}_1, \mathsf{x}_2, \dots\rangle\rangle$$

We may generalize this further to any relation on $\mathbb{N}$ and it's complement. Let $\pi_1 \subseteq \mathbb{N}^2$ and let $\pi_2 = \mathbb{N}^2 \setminus \pi_1$. For example, $<$ and $\geq$, $\leq$ and $>$, $=$ and $\neq$.

7.1 Example. Find all sequences of length 3 and pattern ud over the alphabet $\{1,2,3\}$ where $u \longleftrightarrow <$ and $d \longleftrightarrow \geq$.

$$121,\ 131,\ 132,\ 133,\ 231,\ 232,\ 233$$

These correspond to

$$x_1x_2x_1 + x_1x_3x_1 + x_1x_3x_2 + x_1x_3x_3 + x_2x_3x_1 + x_2x_3x_2 + x_2x_3x_3$$

Now, trivially, the set of all strings with pattern ud is equal to the set of all strings that start with u minus all strings of the form uu. That is, the above expression is equal to

$$(x_1x_2 + x_2x_3 + x_1x_3)(x_1 + x_2 + x_3) - (x_1x_2x_3) =: \gamma_2^{<}\gamma_1^{<} - \gamma_3^{<}$$

7.2 Example. Alternating sequences of odd length have patterm $(ud)^k$ for some k. The generator for such sequences is $1 + ud + (ud)^2 + \cdots = (1-ud)^{-1}$. Let $[(1-ud)^{-1}]_o$ denote the ordinary generating series for sequences with these patterns.

We need an algebraic device to look after the repeated application of the combinatorial construction. Let $\varphi : \mathbb{Q}\langle\langle u,d,w\rangle\rangle/u+d=w \to \mathbb{Q}\langle \mathsf{f}\rangle\langle\langle \mathsf{x}\rangle\rangle$ such that

1. $\varphi(\lambda a + \mu b) = \lambda\varphi(a) + \mu\varphi(b)$ for all linear combinations of patterns a, b and scalars λ, μ (i.e. φ is linear)
2. $\varphi(awb) = \varphi(a)\varphi(b)$ where $w = u+d$ (so $w \longleftrightarrow \pi_1 \amalg \pi_2 = \mathbb{N}^2$, which is to say that w represents no relation)
3. $\varphi(u^{k-1}) = \gamma_k^{\pi_1}(x_1, x_2, \ldots)$, where $\gamma_k^{\pi_1}$ represents the collection of all strings of length k that have pattern u^{k-1}.

Then if a is a linear combination of patterns then $[a]_o = \varphi(a)$.

7.3 Example. Compute $[ud]_o$. We want to calculate $\varphi(ud)$ using the axioms.

$$\begin{aligned}
\varphi(ud) &= \varphi(u(w-u)) \\
&= \varphi(uw - u^2) \\
&= \varphi(uw) - \varphi(u^2) \\
&= \varphi(uw1) - \varphi(u^2) \\
&= \varphi(u)\varphi(1) - \varphi(u^2) \\
&= \gamma_2^{<}\gamma_1^{<} - \gamma_3^{<}
\end{aligned}$$

Now what if we want to restrict this to permutations only? Let $\Delta : \gamma_k^{\pi_1} \mapsto \frac{x^k}{k!}$ and extend as a homomorphism. Then, for example, compute $\left[\frac{x^3}{3!}\right]\Delta[ud]_0$. This gives an an answer of 2 for the above problem, which is correct. Why does it work?

7.4 Example. Compute $[(1-ud)^{-1}]_o =: [h^{-1}]_o$. We need to compute $\varphi(h^{-1})$. Note first that $h = 1 - ud = 1 + u^2 - uw =: q - uw$. But $q^{-1} = q^{-1}hh^{-1} = q^{-1}(q-uw)h^{-1} = h^{-1} - q^{-1}uwh^{-1}$ and thus

$$\begin{aligned}
\varphi(q^{-1}) = \varphi(h^{-1} - q^{-1}uwh^{-1}) &= \varphi(h^{-1}) - \varphi(q^{-1}uwh^{-1}) \\
&= \varphi(h^{-1}) - \varphi(q^{-1}u)\varphi(h^{-1}) \\
&= (1 - \varphi(q^{-1}u))\varphi(h^{-1})
\end{aligned}$$

Whence $\varphi(h^{-1}) = (1 - \varphi(q^{-1}u))^{-1}\varphi(q^{-1})$. But $q^{-1} = (1+u^2)^{-1} = \sum_{k\geq 0}(-1)^k u^{2k}$, so

$$\varphi(q^{-1}) = \varphi(\sum_{k\geq 0}(-1)^k u^{2k}) = \sum_{k\geq 0}(-1)^k\varphi(u^{2k}) = \sum_{k\geq 0}(-1)^k\gamma_{2k+1}^{\pi_1}$$

and

$$1-\varphi(q^{-1}u)=1-\sum_{k\geq 0}(-1)^k\varphi(u^{2k+1})=1-\sum_{k\geq 0}(-1)^k\gamma^{\pi_1}_{2k+2}=\sum_{k\geq 0}(-1)^k\gamma^{\pi_1}_{2k}$$

Thus

$$\varphi(h^{-1})=\left(\sum_{k\geq 0}(-1)^k\gamma^{\pi_1}_{2k}\right)^{-1}\left(\sum_{k\geq 0}(-1)^k\gamma^{\pi_1}_{2k+1}\right)$$

If $\pi_1 =<$ and we wish to find all permutations for which this pattern holds then we must further apply Δ to get

$$\Delta\varphi(h^{-1})=\left(\sum_{k\geq 0}(-1)^k\frac{x^{2k}}{(2k)!}\right)^{-1}\left(\sum_{k\geq 0}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\right)=\frac{\sin x}{\cos x}=\tan x$$

We wish to list all sequences with the order in which the maximal π_1 substrings occur.

7.5 Theorem. *(Maximal Decomposition Theorem) Let f_i mark a maximal π_1-string of length i. Let x_i mark the occurrence of i as a symbol in the sequence. Let*

$$f(x)=1+\mathsf{f}_1x+\mathsf{f}_2x^2+\cdots \text{ and } \gamma^{\pi_1}=(\gamma^{\pi_1}_1(\mathsf{x}_1,\mathsf{x}_2,\ldots),\gamma^{\pi_1}_2,\gamma^{\pi_1}_3,\ldots)$$

Then the enumerator for all such sequences is F, where

$$1+F=(\mathsf{f}^{-1}\circ\gamma^{\pi_1}(\mathsf{x}_1,\mathsf{x}_2,\ldots))^{-1}$$

and $x^k\circ\gamma^{\pi_1}:=\gamma^{\pi_1}_k$ extended linearly.

PROOF: The generator for all patterns is

$$\begin{aligned}(1-w)^{-1}&=(1-(u+d))^{-1}\\&=((1-u)-d)^{-1}\\&=((1-u)(1-(1-u)^{-1}d))^{-1}\\&=(1-(1-u)^{-1}d)^{-1}(1-u)^{-1}\\&=(1-(1+u+u^2+\cdots)d)^{-1}(1+u+u^2+\cdots)\end{aligned}$$

Now mark maximal π_1-substrings of vertex length k by f_k. The generator becomes

$$(1-(\mathsf{f}_1+\mathsf{f}_2u+\mathsf{f}_3u^2+\cdots)d)^{-1}(\mathsf{f}_1+\mathsf{f}_2u+\mathsf{f}_3u^2+\cdots)$$

The generating series F that we want is φ applied to this. Let $g(u)=\mathsf{f}_1+\mathsf{f}_2u+\mathsf{f}_3u^2+\cdots$. Then, noting that

$1 + xg(x) = f(x)$

$$
\begin{aligned}
F &= \varphi\left((1-g(u)d)^{-1}g(u)\right) \\
&= \varphi\left((1-g(u)(w-u))^{-1}g(u)\right) \\
&= \varphi\left((1+ug(u)-g(u)w)^{-1}g(u)\right) \\
&= \varphi\left((1-(1+ug(u))^{-1}g(u)w)^{-1}(1+ug(u))^{-1}g(u)\right) \\
&= \varphi\left((1-f^{-1}(u)g(u)w)^{-1}f^{-1}(u)g(u)\right) \\
1+F &= 1+\varphi\left(\sum_{k\geq 0}(f^{-1}(u)g(u)w)^k f^{-1}(u)g(u)\right) \\
&= 1+\sum_{k\geq 0}\varphi(f^{-1}(u)g(u))^k\varphi(f^{-1}(u)g(u)) \\
&= \sum_{k\geq 0}\varphi(f^{-1}(u)g(u))^k \\
&= \left(1-\varphi(f^{-1}(u)g(u))\right)^{-1}
\end{aligned}
$$

Now $\varphi(u^k) = \gamma^{\pi_1}_{k+1} := x^{k+1}\circ\gamma^{\pi_1} = x(x^k)\circ\gamma^{\pi_1}$, so if $h(x)$ is a power series in x then $\varphi(h(u)) = (xh(x))\circ\gamma^{\pi_1}$. Then

$$1+F = \left(1-(xf^{-1}(x)g(x))\circ\gamma^{\pi_1}\right)^{-1}$$

But $1 - xf^{-1}(x)g(x) = 1 - f^{-1}(x)(f(x)-1) = f^{-1}(x)$, so

$$1+F = \left(f^{-1}(x)\circ\gamma^{\pi_1}\right)^{-1}$$

□

7.6 Example. Find all <-alternating sequences of even length starting with a <. We can characterize the pattern of such sequences by the lengths of maximal <-substrings. These have length 2 only, so set $\mathsf{f}_1 = \mathsf{f}_3 = \mathsf{f}_4 = \cdots = 0$ and $\mathsf{f}_2 = 1$. Then in the maximal decomposition theorem $f(x) = 1 + x^2$, so

$$f^{-1}\circ\gamma^{<} = \left(\sum_{k\geq 0}(-1)^k x^{2k}\right)\gamma^{<} = \sum_{k\geq 0}(-1)^k x^{2k}\circ\gamma^{<} = \sum_{k\geq 0}(-1)^k\gamma^{<}_{2k}$$

So $F = \left(\sum_{k\geq 0}(-1)^k\gamma^{<}_{2k}\right)^{-1}$. For permutations we have $\Delta F = \left(\sum_{k\geq 0}(-1)^k\frac{x^{2k}}{(2k)!}\right)^{-1} = \sec x$.

7.7 Lemma. *(Permutation Lemma) Let Φ be a formal power series in $\gamma^{<}_1, \gamma^{<}_2, \ldots$ and let $\Delta : \gamma^{<}_k \mapsto \frac{x^k}{k!}$ be extended as a homomorphism (we can do this since the $\gamma^{<}_i$'s are algebraically independent). Then*

$$[x_1 \ldots x_n]\chi\Phi = \left[\frac{x^n}{n!}\right]\Delta\Phi$$

where χ denotes the commutative image.

PROOF: See notes. □

7.8 Example. Find the number of permutations of odd length with no maxima or minima in even positions. We first need the pattern generator. The patterns of all sequences of odd length are generated by $\{uu, dd, du, ud\}^*$. The last two are not allowed, so the generator is $(1-(u^2+d^2))^{-1} =: h^{-1}$. We want $\Delta[h^{-1}]_o$. Now

$$h = 1 - u^2 - d^2 = 1 - u^2 - dw + du = 1 - 2u^2 - dw + wu$$

We have put h in the form $h = s + \sum_{j=1}^{k} l_j w r_j = s + LwR^T$ where $L = [l_1, \dots, l_k]$ and $R = [r_1, \dots, r_k]$. Then

$$s^{-1} = s^{-1}hh^{-1} = s^{-1}(s + LwR^T)h^{-1} = h^{-1} + s^{-1}LwR^T h^{-1}$$

Hence

$$R^T s^{-1} = R^T h^{-1} + R^T s^{-1} LwR^T h^{-1}$$

so

$$\varphi(R^T s^{-1}) = \varphi(R^T h^{-1}) + \varphi(R^T s^{-1}L)\varphi(R^T h^{-1}) = (1 + \varphi(R^T s^{-1}L))\varphi(R^T h^{-1})$$

Thus

$$\varphi(R^T h^{-1}) = (1 + \varphi(R^T s^{-1}L))^{-1}\varphi(R^T s^{-1})$$

Now substituting back for our problem yields

$$\left[\begin{bmatrix}1\\u\end{bmatrix} h^{-1}\right]_o = \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \left[\begin{bmatrix}1\\u\end{bmatrix} s^{-1}\begin{bmatrix}-d & 1\end{bmatrix}\right]_o\right)^{-1}\left[\begin{bmatrix}1\\u\end{bmatrix} s^{-1}\right]_o$$

Now take the commutative image. By Cramer's Rule, the generating series we now seek is therefore

$$G = \Delta \frac{\begin{vmatrix}[s^{-1}]_o & [s^{-1}]_o\\ [us^{-1}d]_o & 1 + [us^{-1}]_o\end{vmatrix}}{\begin{vmatrix}1 - [s^{-1}d]_o & [s^{-1}]_o\\ -[us^{-1}d]_o & 1 + [us^{-1}]_o\end{vmatrix}} = \frac{\Delta[s^{-1}]_o}{\begin{vmatrix}1 + \Delta[us^{-1}]_o & \Delta[s^{-1}]_o\\ x + \Delta[u^2s^{-1}]_o & 1 + \Delta[us^{-1}]_o\end{vmatrix}}$$

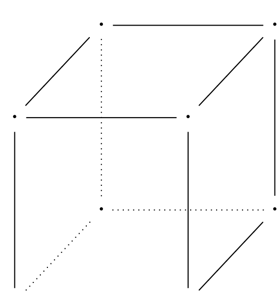
This simplifies as $G = \frac{\sqrt{2}\tanh(\sqrt{2}x)}{1 - \frac{1}{\sqrt{2}}x\tanh(\sqrt{2}x)}$.

8 Pólya Theory

We will now study counting combinatorial structures under the action of a group.

8.1 Colouring Polyhedra

How many different ways are there of painting the faces of a cube with two different colours? $1+1+2+2+2+1+1 = 10$ (What about for more complicated solids?)



The automorphism group of the cube (presented with respect to faces) is a subgroup of $\mathfrak{S}_6$ with 24 elements. The cycle index polynomial of Aut(cube), acting of the faces is

$$\mathscr{Z}_{\text{Aut(cube,faces)}}(x_1, \dots, x_6) = \frac{1}{24}\left(x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^2\right)$$

Suppose we cover the cube with 2 colours white and black. Now replace x_i with $w^i + b^i$. The number of colours of the face cube with i white faces and j black faces is

$$\begin{aligned}[][w^i b^j]\mathscr{Z}_{\mathrm{Aut(cube,faces)}}\langle w+b\rangle &= [w^i b^j]\mathscr{Z}_{\mathrm{Aut(cube,faces)}}(x_1,\dots,x_6)\Big|_{\substack{x_i\mapsto w^i+b^i\\ i=1,\dots,6}} \\ &= [w^i b^j]\left(w^6+w^5b+2w^4b^2+2w^3b^3+2w^2b^4+wb^5+b^6\right)\end{aligned}$$

8.2 Painting the 2×2 board

Number the squares of the board with the numbers $\{1,2,3,4\}$ in a boostrophedon manner. Fix coordinates around the outside of the board. (These are intended to stay fixed as we rotate the board.) Consider the colourings $\begin{bmatrix} W & B \\ B & W \end{bmatrix}$ and $\begin{bmatrix} B & W \\ W & B \end{bmatrix}$. These colourings are the same. We may encode them as functions $\{1,2,3,4\}\to\{W,B\}$, the first being $\binom{1\ 2\ 3\ 4}{WBWB}$ and the second being $\binom{1\ 2\ 3\ 4}{BWBW}$. These functions are different. We need a definition of the equivalence of such functions to reflect the notion of "combinatorially equivalent" colourings.

8.3 Pólya's Theorem

8.1 Definition. Let G be finte group acting on a finite set X. For $g \in G$, let $\tau_i(g)$ be the number of i-cycles in g (presented as an element of a symmetric group). The cycle index polynomial of the permutation group of G on X is

$$\mathscr{Z}_G(x_1,x_2,\dots) := \frac{1}{|G|}\sum_{g\in G} x_1^{\tau_1(g)} x_1^{\tau_1(g)}\cdots$$

8.2 Definition. Let $F(x_1,\dots,x_n)$ and $f(y_1,\dots,y_r)$ be formal power series. The Pólya substitution of f into F is defined to be

$$F\langle f(y_1,\dots,y_r)\rangle := F(x_1,\dots,x_n)\Big|_{\substack{x_i=f(y_1,\dots,y_r)\\ i=1,\dots,n}}$$

8.3 Definition. Recall from Group Theory

1. We say that G has an action on X if
   (a) $gx \in X$ for all $g \in G$ and $x \in X$
   (b) $(hg)x = h(gx)$ for all $g,h \in G$ and $x \in X$
   (c) $1_G x = x$ for all $x \in X$
2. $x, y \in X$ are said to be G-equivalent if there is $g \in G$ such that $gx = y$. We write $x \sim_G y$. $\sim_G$ is an equivalence relation on X.
3. We want to determine $|X/\sim_G|$, the number of equivalence classes in X under G-equivalence.
4. (Counting Lemma) Let $\theta(x)$ be the size of the unique equivalence class in $X/\sim_G$ containing $x \in X$ (i.e. θ is a class function (constant on classes)). Then trivially,

$$|X/\sim_G| = \sum_{x\in X}\frac{1}{\theta(x)}$$

5. H is said to be a subgroup of G if
   (a) H is closed under product in G
   (b) H is closed under taking inverses in G

We write $H < G$

6. For $g \in G$, the R-coset of H in G containing g is a set of the form

$$Hg := \{hg \mid h \in H\}$$

These partition G. We denote the number of distinct R-cosets of H in G by $[G : H]$.

7. (Lagrange's Theorem) Let $H < G$. Then

$$[G : H] = \frac{|G|}{|H|}$$

8. (a) For $x \in X$, the orbit of x with respect to G is

$$Gx := \{gx \mid g \in G\}$$

   (b) For $x \in X$, the stabilizer of x with respect to G is

$$G_x := \{g \in G \mid gx = x\}$$

9. From Lagrange's Theorem, $|G| = |Gx||G_x|$ for any $x \in X$. This is called the Orbit-Stabilizer Theorem.
10. For $g \in G$, the fix of g in X is the set of fixed points of g,

$$\mathrm{Fix}_X(g) := \{x \in X \mid gx = x\}$$

8.4 Lemma. *(Burnside's Lemma) Let G be a finite group acting on a finite set X. Then*

$$|X/\sim_G| = \frac{1}{|G|}\sum_{g\in G}|Fix_X(g)|$$

PROOF:

$$\begin{aligned}
|X/\sim_G| &= \sum_{x\in X}\frac{1}{\theta(x)} \\
&= \sum_{x\in X}\frac{1}{|Gx|} \\
&= \frac{1}{|G|}\sum_{x\in X}|G_x| \\
&= \frac{1}{|G|}\sum_{x\in X}\sum_{\substack{g\in G\\ gx=g}} 1 \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{x\in X\\ gx=g}} 1 \\
&= \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}_X(g)|
\end{aligned}$$

□

8.5 Theorem. *(Pólya's Theorem) Let G be a finite group acting on a finite set $\mathscr{A}$. Let $\mathscr{B} = \{b_1, \ldots, b_n\}$ be a set and let $\mathscr{B}^{\mathscr{A}}$ denote the set of all functions from $\mathscr{A}$ to $\mathscr{B}$.*

1. *We claim that* $g\varphi = \varphi g^{-1}$ *is an action of* G *on* $\mathscr{B}^{\mathscr{A}}$.
2. $\varphi, \psi \in \mathscr{B}^{\mathscr{A}}$ *are said to be (induced)* G*-equivalent if there is* $g \in G$ *such that* $\varphi = g\psi$. *We write* $\varphi \sim' \psi$ *(induced* G*-equivalence). Then*
$$|\mathscr{B}^{\mathscr{A}} / \sim'| =$$
3. *Let* ρ *be a weight function on* $\mathscr{B}$ *that records the number of occurrences of each* b_i.
$$[(\mathscr{B}^{\mathscr{A}} / \sim, \rho)] = \mathscr{Z}_G\langle b_1 + \cdots + b_n\rangle$$

PROOF: 1. Let $h, g \in G$. Two of the conditions for G-action are trivial. The remaining condition is proved by noting
$$(hg)\varphi = \varphi(hg)^{-1} = \varphi g^{-1}h^{-1} = h(\varphi g^{-1}) = h(g\varphi)$$

$$|\mathscr{B}^{\mathscr{A}} / \sim'| = \frac{1}{|G|}\sum_{g\in G}[(\mathrm{Fix}_{\mathscr{B}^{\mathscr{A}}}(g), \rho)]_o \qquad \text{by Burnside's Lemma}$$

<++>

$$[(\psi, \rho)]_o = \prod_{i=1}^{n}[(\{b_i\}, \rho)]_o^{|\psi^{-1}(b_i)|}$$

If $\phi \in \mathrm{Fix}(g)$ then $g\phi = \phi$, so
$$\phi(a) = (g\phi)(a) = \phi(g^{-1}a)$$
by definition of the action of G on ϕ. Thus $\phi(g^k a) = \phi(g^{k-1}a)$, whence $\phi(g^k a) = \phi(a)$ for all $k \geq 1$, so ϕ is constant on cycles of $g \in G$. Thus the theorem is proved? Write this up better. =/ □

For extensions of this theorem, look up deBruijn's Theorem. It can be used for counting self complimenting graphs (a graph Γ is self-complimenting when $\Gamma = \Gamma^C$). Also see the Redfield-Read Superposition Theorem.

8.4 Rooted Trees

Let n be the number of vertices. When $n = 1$ we have a single vertex, when $n = 2$ we have a single stalk. When $n = 3$ there are two rooted trees, one beanpole and one vee. Notice that since these are *not* plane planted trees, reflections of the same tree count as the same tree. Thus there are only four rooted trees on four vertices (whereas there are five plane planted trees). How many trees are there on n vertices for general n?

We use the branch decomposition. Since the order doesn't matter, we must "mod out" by the symmetric group. Hence
$$\mathscr{T}_R \xrightarrow{\sim} \{\odot\} \times \coprod_{k\geq 0} \mathscr{T}_R^{\mathbb{N}_k} / \mathfrak{S}_k$$
so by Pólya's Theorem
$$T(x) = x\sum_{k\geq 0}\mathscr{Z}_{\mathfrak{S}_k}\langle T(x)\rangle$$

The cycle index polynomial of the symmetric group is the generating series for all permutations in $\mathfrak{S}_k$ with respect to cycle type. Now $\mathscr{P} \xrightarrow{\sim} \mathscr{U} \star \mathscr{C}$, so we get
$$\sum_{k\geq 0} u^k \mathscr{Z}_{\mathfrak{S}_k}(x_1, \ldots, x_k) = \exp\left(\sum_{k\geq 1}\frac{u^k x_k}{k}\right)$$

Let $T(x) = \sum_{i\geq 1} t_i x^i$, where t_i is the number of rooted trees with n non-root vertices. Then

$$\begin{aligned}\sum_{i\geq 1} t_i x^i &= x\exp\left(\sum_{k\geq 1}\frac{1}{k}\sum_{i\geq 1} t_i x^{ik}\right)\\ &= x\exp\left(\sum_{i\geq 1} t_i \log(1-x^i)^{-1}\right)\\ &= x\prod_{i\geq 1}\exp\log(1-x^i)^{-t_i}\\ &= x\prod_{i\geq 1}(1-x^i)^{-t_i}\end{aligned}$$

Thus $t_k = [x^{k-1}]\prod_{i\geq 1}(1-x^i)^{-t_i}$. Check a few examples.