Partial Differential Equations II
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Disclaimer: *These notes are* not *the official course notes for this class.* These notes have been transcribed under classroom conditions and as a result accuracy (of the transcription) and correctness (of the mathematics) cannot be guaranteed.

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Contents

1 Sobolev Spaces

1.1 Introduction

This course is concerned with modern methods in the theory of PDE, in particular with linear parabolic and elliptic equations. It will be a technical course more than an applied course, and we will be learning the language required to understand research in PDE.

For the purposes of illustration, consider the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where U is connected and $\overline{U}$ is compact. Multiply by a *test function* $\varphi \in C_c^\infty(U)$ and integrate both sides

$$\int_U -\Delta u \varphi dx = \int_U f \varphi dx.$$

Integrate by parts to get,

$$\int_U \nabla u \cdot \nabla \varphi dx = \int_U f \varphi dx,$$

since $\varphi|_{\partial U} = 0$. Define a bilinear form on some space H, with the property that $u = 0$ on ∂U for $u \in H$, by

$$\langle u_1, u_2 \rangle = \int_U \nabla u_1 \cdot \nabla u_2 dx.$$

For $u \in H$, the norm is $\|u\| = \int_U |\nabla u|^2 dx$, and we hope that H is a Hilbert space and $\varphi \mapsto \int_U f \varphi dx$ is a bounded linear functional. If these hopes hold true then by the Riesz representation theorem there is a unique $u \in H$ such that

$$\int_U \nabla u \cdot \nabla \varphi dx = \langle u, \varphi \rangle = \int_U f \varphi dx.$$

Sounds nice, but does it all work?

1.2 Weak derivatives

1.2.1 Definition. Let $u \in L^1_{loc}(U)$. The function $v \in L^1_{loc}(U)$ is a *weak partial derivative* of u in the i^{th} coordinate if

$$\int_U u \varphi_{x_i} dx = (-1) \int_U v \varphi dx$$

for all $\varphi \in C_c^\infty(U)$. We write $v = u_{x_i}$.

Analogously, v is the α^{th} weak partial derivative for a multiindex α if

$$\int_U u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_U v \varphi \, dx$$

for all $\varphi \in C_c^\infty(U)$, and we write $v = D^\alpha u$.

Of course, to define such notation we need the following.

1.2.2 Proposition. *Weak derivatives are unique if they exist.*

PROOF: Suppose that w_1 and w_2 are α^{th} derivatives of u. Then, for all $\varphi \in C_c^\infty(U)$,

$$(-1)^{|\alpha|} \int_U w_1 \varphi \, dx = (-1)^{|\alpha|} \int_U w_2 \varphi \, dx.$$

So $\int_U (w_1 - w_2)\varphi \, dx = 0$ for all $\varphi \in C_c^\infty(U)$, and it follows that $w_1 = w_2$ a.e.

(Indeed, let η be the standard mollifier, and η_ε be the rescaled mollifier with support $B(0,\varepsilon)$. Then $w_\varepsilon(y) := \int_U w(x)\eta_\varepsilon(y-x)\,dx = 0$ for all ε and all y, and this completes the argument since $w_\varepsilon \to w$ a.e.) $\square$

1.2.3 Examples.

(i) Let $u(x) = |x|$ for $x \in \mathbb{R}$. Then $u_x(x) = 1$ if $x > 0$ and $= -1$ if $x < 0$. Indeed, for $\varphi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} -\int_{\mathbb{R}} u_x \varphi \, dx &= -\int_{-\infty}^0 u_x \varphi \, dx - \int_0^\infty u_x \varphi \, dx \\ &= \int_{-\infty}^0 u \varphi_x \, dx - u(0)\varphi(0) + \int_0^\infty u \varphi_x \, dx + u(0)\varphi(0) \\ &= \int_{\mathbb{R}} |x| \varphi_x \, dx \end{aligned}$$

We will see later that the weak derivative corresponds to the usual generalization of derivative for absolutely continuous functions.

(ii) Let $u(x) = \mathbf{1}_{(0,\infty)}$. Then $u_x(x) = 0$ is the only candidate for the derivative (why?), but

$$\int_{\mathbb{R}} u \varphi_x \, dx = \int_0^\infty \varphi_x \, dx = -\varphi(0)$$

which is not necessarily zero. There is a more general notion (that of a “distribution”) that would give this function a derivative, but for our purposes this function does not have a weak derivative.

1.3 Sobolev spaces

1.3.1 Definition. For $1 \le p < \infty$, let $W^{k,p}(U)$ be the collection of functions $u \in L^1_{loc}(U)$ such that for all α with $|\alpha| \le k$, $D^\alpha u$ exists and is in $L^p(U)$. For $u \in W^{k,p}(U)$,

$$\|u\|_{W^{k,p}(U)} := \left(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p dx \right)^{\frac{1}{p}} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}$$

For $p = \infty$, take instead the essential supremum.

When $p = 2$ we may write $H^k(U) := W^{k,2}(U)$.

1.3.2 Definition. $W_0^{k,p}(U) := \overline{C_c^\infty(U)}^{W^{k,p}(U)}$.

1.3.3 Example. Let $U = B(0,1) \subseteq \mathbb{R}^n$, $\alpha > 0$, and $u(x) = \frac{1}{|x|^\alpha}$. Note that

$$\int_{B(0,\varepsilon)} \frac{1}{|x|^\alpha} dx = C \int_0^\varepsilon \frac{1}{r^\alpha} r^{n-1} dr = \tilde{C} r^{n-\alpha} \Big|_0^\varepsilon,$$

so u is in $L^1_{loc}(U)$ if and only if $\alpha < n$.

For which p and n is u in $W^{1,p}(U)$? u is a.e. differentiable, so

$$u_{x_i} = -\alpha \frac{1}{|x|^{\alpha+1}} \cdot \frac{x_i}{|x|} = -\frac{\alpha x_i}{|x|^{\alpha+2}},$$

and $|Du| = \frac{\alpha}{|x|^{\alpha+1}}$. As above, $u_{x_i} \in L^1_{loc}(U)$ if and only if $\alpha < n-1$. To check that this is truly the weak derivative, let $A_\varepsilon := B(0,1) \setminus \overline{B}(0,\varepsilon)$. By integration by parts,

$$\int_{A_\varepsilon} u \varphi_{x_i} dx = -\int_{A_\varepsilon} u_{x_i} \varphi dx + \int_{\partial B(0,\varepsilon)} u \varphi \nu_i dS.$$

As $\varepsilon \to 0$, $A_\varepsilon \to B(0,1)$ and

$$\left| \int_{\partial B(0,\varepsilon)} \frac{1}{\varepsilon^\alpha} \varphi \nu_i dS \right| \le C \varepsilon^{n-1-\alpha} \|\varphi\|_{L^\infty} \to 0.$$

Finally, $\int_U |Du|^p dx < \infty$ only if

$$\infty > \int_U \frac{1}{|x|^{(\alpha+1)p}} dx = C \int_0^1 r^{n-1-(\alpha+1)p} dx = C r^{n-(\alpha+1)p} \Big|_0^1,$$

which happens when $(\alpha+1)p < n$, or $\alpha < \frac{n}{p} - 1$.

1.3.4 Proposition. *Let $u, v \in W^{k,p}(U)$ and let $|\alpha| \le k$.*

(i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$

(ii) $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$, *for each* β *such that* $|\alpha| + |\beta| \le k$.

(iii) D^α *is a linear operator.*

(iv) *If* $V \subseteq U$ *is open then* $u|_V \in W^{k,p}(V)$.

(v) *If* $\xi \in C_c^\infty(U)$ *then* $\xi u \in W^{k,p}(U)$ *and*

$$D^\alpha(\xi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} u$$

where $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

1.3.5 Theorem. *$W^{k,p}(U)$ is a Banach space.*

Proof: If $\|u\| = 0$ then $\|u\|_p \le \|u\| = 0$, so $u = 0$ a.e. Clearly the norm is homogeneous, so we are left to check the triangle inequality. For $p < \infty$,

$$\begin{aligned}\|u+v\| &= \left(\sum_{\alpha \le k} \|D^\alpha u + D^\alpha v\|_p^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{\alpha \le k} (\|D^\alpha u\|_p + \|D^\alpha v\|_p)^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{\alpha \le k} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}} + \left(\sum_{\alpha \le k} \|D^\alpha v\|_p^p\right)^{\frac{1}{p}} \\ &= \|u\| + \|v\|\end{aligned}$$

by Minkovski's inequality in both cases. The case $p = \infty$ is clear.

Let $\{u_m\}$ be a Cauchy sequence in $W^{k,p}(U)$. Since the Sobolev norm dominates the L^p norm, all of the sequences $\{u_m\}$ and $\{D^\alpha u_m\}$ are Cauchy in L^p, and hence convergent in L^p. Let u and u_α be the limits of these sequences in L^p. Now

$$\begin{aligned}\int_U u D^\alpha \varphi \, dx &= \lim_{m\to\infty} \int_U u_m D^\alpha \varphi \, dx \\ &= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U \varphi D^\alpha u_m \, dx = (-1)^{|\alpha|} \int_U u_\alpha \varphi \, dx\end{aligned}$$

Therefore $u_\alpha = D^\alpha u$, and the sequence converges in $W^{k,p}(U)$. □

1.4 Approximation by smooth functions

Mollifiers

A *mollifier* is a function $\eta \in C_c^\infty(\mathbb{R}^n)$ such that

(i) $\int_{\mathbb{R}^n} \eta = 1$;

(ii) $\eta \geq 0$;

(iii) η is radially symmetric; and

(iv) $\text{supp}(\eta) \subseteq B(0,1)$.

For any $\varepsilon > 0$ define $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$. Then η_ε is also a mollifier for $\varepsilon < 1$. For $f \in L^1_{loc}(\mathbb{R}^n)$, define

$$f^\varepsilon(x) := \eta_\varepsilon * f(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)f(y)dy = \int_{\mathbb{R}^n} \eta_\varepsilon(y)f(x-y)dy.$$

1.4.1 Proposition. *The following properties are true of mollified functions.*

(i) $f^\varepsilon \in C^\infty$ *for every* $\varepsilon > 0$*;*

(ii) $f^\varepsilon \to f$ *a.e.;*

(iii) If $f \in C(U)$*, then* $f^\varepsilon \to f$ *in* $C_{loc}(U)$*.* ***Warning:*** *Mollification does not play well with the* L^∞ *norm.*

(iv) If $1 \leq p < \infty$ *and* $f \in L^p_{loc}(U)$ *then* $f^\varepsilon \to f$ *in* $L^p_{loc}(U)$*.*

1.4.2 Exercise. Is it the case that if $1 \leq p < \infty$ and $f \in L^p(U)$ then $f^\varepsilon \to f$ in $L^p(U)$?

Interior Approximation

1.4.3 Theorem. *Let* $1 \leq p < \infty$ *and* $u \in W^{k,p}(U)$*. For any* $0 < \varepsilon < 1$*, let* $U_\varepsilon := \{x \in U \mid dist(x, \partial U) > \varepsilon\}$*. For* $x \in U_\varepsilon$*, define* $u_\varepsilon(x) := \eta_\varepsilon * u(x)$*. Then*

(i) $u_\varepsilon \in C^\infty(U_\varepsilon)$*; and*

(ii) $u_\varepsilon \to u$ *in* $W^{k,p}_{loc}(U)$ *as* $\varepsilon \to 0$*.*

Notice that if $V \subset\subset U$ then $\text{dist}(\overline{V}, \partial U) > 0$, so u_ε is defined on V for all small enough ε.

PROOF: Let $|\alpha| \leq k$. Then for $x \in U_\varepsilon$, $D^\alpha u_\varepsilon(x) = \eta_\varepsilon * D^\alpha u$. Indeed,

$$\begin{aligned} D^\alpha u_\varepsilon(x) &= D^\alpha_x \int_U \eta_\varepsilon(x-y)u(y)dy \\ &= \int_U D^\alpha_x \eta_\varepsilon(x-y)u(y)dy \\ &= (-1)^{|\alpha|}\int_U D^\alpha_y \eta_\varepsilon(x-y)u(y)dy \\ &= (-1)^{|\alpha|}(-1)^{|\alpha|}\int_U \eta_\varepsilon(x-y)D^\alpha u(y)dy \\ &= \eta_\varepsilon * D^\alpha u \end{aligned}$$

Now $D^\alpha u \in L^p_{loc}(U)$, so $D^\alpha u_\varepsilon = \eta_\varepsilon * D^\alpha u \to D^\alpha u$ in $L^p_{loc}(U)$. It follows that for any $V \subset\subset U$, as $\varepsilon \to 0$,

$$\|u_\varepsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\le k} \|D^\alpha u_\varepsilon - D^\alpha u\|^p_{L^p(V)} \to 0.$$

□

Approximation by smooth functions

1.4.4 Theorem. *Let U be a bounded domain, $1 \le p < \infty$, and $u \in W^{k,p}(U)$. Then there are $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.*

PROOF: Let $U_i := \{x \in U \mid \operatorname{dist}(x, \partial U) > \frac{1}{i}\}$, and let $V_i := U_{i+3} \setminus \overline{U}_{i+1}$ and $V_0 := U_1$. Then $U = \bigcup_{i=0}^\infty V_i$, and each point of U is contained in at most three of the V_i.

Let $\{\xi_i\}$ be a partition of unity subordinate to the cover V_i. That is to say,

(i) $\xi_i \in C^\infty(U, [0,1])$;

(ii) $\operatorname{supp}(\xi_i) \subseteq V_i$; and

(iii) $\sum_i \xi_i = 1$.

Notice that $\operatorname{supp}(\xi_i u) \subseteq V_i$. Fix $\delta > 0$, and pick $\varepsilon > 0$ so small that

(i) $\|\eta_\varepsilon * (\xi_i u) - \xi_i u\|_{W^{k,p}(U)} \le \frac{\delta}{2^{i+1}}$; and

(ii) $\operatorname{supp}(\eta_\varepsilon * (\xi_i u)) \subseteq W_i := U_{i+4} \setminus \overline{U}_i$.

We can do this because, by 1.4.3, $\eta_\varepsilon * (\xi_i u)$ approximates $\xi_i u$ locally, and are they both identically zero off of a compact set.

Let $v := \sum_{i=0}^\infty \eta_\varepsilon * (\xi_i u)$. Then for each $V \subset\subset U$,

$$\|v - u\|_{W^{k,p}(V)} \le \sum_{i=0}^\infty \|\eta_\varepsilon * (\xi_i u) - \xi_i u\|_{W^{k,p}(U)} \le \sum_{i=0}^\infty \frac{\delta}{2^{i+1}} = \delta,$$

so we conclude $\|v - u\|_{W^{k,p}(U)} \le \delta$ by taking the supremum over all $V \subset\subset U$. □

1.4.4 implies that $W^{k,p}$ is the completion (in its norm) of the collection of smooth functions with finite $W^{k,p}$ norm.

Global approximation

In the previous approximation we could only find functions $u_m \in C^\infty(U)$. If the boundary of U is well-behaved then we can do better.

1.4.5 Theorem. *Let U be a bounded domain with C^1 boundary, $1 \le p < \infty$, and $u \in W^{k,p}(U)$. Then there are $u_m \in C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{k,p}(U)$.*

Proof: We write $x = (\hat{x}, x_n)$ for $x \in \mathbb{R}^n$. Fix a point $x_0 \in \partial U$ and write

$$U \cap B(x_0, r) = \{x \in B(0, r) \mid x_n > \gamma(\hat{x})\}$$

for some $r > 0$ and C^1 function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$. Let $V := U \cap B(x_0, \frac{r}{2})$.

For $\lambda \in \mathbb{R}$ and $\varepsilon > 0$, let $x^\varepsilon := x + \lambda\varepsilon e_n$. Choose λ so that

$$B(x^\varepsilon, \varepsilon) \subseteq U \cap B(x_0, r) \quad \text{for all } \varepsilon > 0 \text{ and } x \in V.$$

(Idea: take λ large, depending on $\|D\gamma\|_{L^\infty(\overline{V} \cap \partial U)}$, so that the tip of the cone over x with "slope" λ lies in $B(x_0, r)$.)

Define $u_\varepsilon(x) := u(x^\varepsilon)$ and let $v_\varepsilon := \eta_\varepsilon * u_\varepsilon$. The idea is that we jiggle x slightly so that there is "room" to mollify. Then $v_\varepsilon \in C^\infty(\overline{V})$, and we claim that $v_\varepsilon \to u$ in $W^{k,p}(V)$. Indeed let $|\alpha| \le k$.

$$\|D^\alpha v_\varepsilon - D^\alpha u\|_{L^p(V)} \le \|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p(V)} + \|D^\alpha u_\varepsilon - D^\alpha v_\varepsilon\|_{L^p(V)}$$

The shift operator is continuous on L^p, so the left hand term goes to zero as $\varepsilon \to 0$. Recall $D^\alpha v_\varepsilon = D^\alpha(\eta_\varepsilon * u_\varepsilon) = \eta_\varepsilon * D^\alpha u_\varepsilon$, so

$$\begin{aligned}\int_V |\eta_\varepsilon * D^\alpha u_\varepsilon(x) - D^\alpha u_\varepsilon(x)|^p dx &= \int_{V+\lambda\varepsilon e_n} |\eta_\varepsilon * D^\alpha u(x) - D^\alpha u(x)|^p dx \\ &\le \int_{U \cap B(x_0, r)} |\eta_\varepsilon * g - g|^p dx \to 0\end{aligned}$$

as $\varepsilon \to 0$, where $g(x) = D^\alpha u(x)$ for $x \in U$ and 0 outside. (This redefinition is required so that the convolution is defined over all of U.)

Let $\delta > 0$. ∂U is compact, so choose finitely many boundary points x_i ($1 \le i < N$) such that the corresponding radii r_i and sets $V_i := U \cap B(x_i, \frac{r_i}{2})$ cover ∂U, and choose $v_i \in C^\infty(\overline{V}_i)$ such that

$$\|v_i - u\|_{W^{k,p}(V_i)} \le \frac{\delta}{N}.$$

Choose $V_0 \subset\subset U$ such that $U = \bigcup_{i=0}^{N-1} V_i$ and $v_0 \in C^\infty(\overline{V}_0)$ such that

$$\|v_0 - u\|_{W^{k,p}(V_i)} \le \frac{\delta}{N},$$

which may be done by 1.4.3.

Let $\{\xi_i\}$ be a partition of unity defined on $\overline{U}$ subordinate to the cover $\{V_0, B(x_i, r_i)\}$, and set $v := \sum_{i=0}^{N-1} \xi_i v_i$. Finally,

$$\begin{aligned}\|\xi_i v_i - \xi_i u\|_{W^{k,p}(V_i)} &= \left(\sum_{|\alpha| \le k} \int_{V_i} |D^\alpha(\xi_i v_{\varepsilon_i}) - D^\alpha(\xi_i u)|^p dx \right)^{\frac{1}{p}} \\ &\le C\|v_i - u\|_{W^{k,p}(V_i)}\end{aligned}$$

where C_i is a constant that depends only on ξ_i, $|\alpha|$, n, and p (but not on N or δ). Indeed,

$$\begin{aligned}\int_V |(\xi v)_x - (\xi u)_x|^p dx &= \int_V |\xi(v_x - u_x) + \xi_x(v-u)|^p dx \\ &\leq C\int_V |\xi|^p|v_x - u_x|^p + |\xi_x|^p|v-u|^p dx \\ &\leq C(\|\xi\|_\infty + \|\xi_x\|_\infty)\int_V |v_x - u_x|^p + |v-u|^p dx.\end{aligned}$$

Therefore $\|v-u\|_{W^{k,p}(U)} \leq \sum_{i=0}^{N-1} C_i\|v_i - u\|_{W^{k,p}(V_i)} \leq C\delta$. □

The construction used in this proof also works for Lipschitz domains (i.e. those domains for which the boundary can be smoothed out by a Lipschitz continuous function). The idea is that, for this proof to work, the "corners" of the domain cannot be too sharp.

1.5 Extension

1.5.1 Theorem. *Let U be a bounded domain with C^1 boundary, $1 \leq p < \infty$, and V be open in $\mathbb{R}^n$ and such that $U \subset\subset V$. Then there exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that*

(i) $Eu|_U = u$ *for all* $u \in W^{1,p}(U)$*;*

(ii) $\mathrm{supp}(Eu) \subseteq V$*; and*

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$ *(where C depends only on p, U, and V).*

Proof: Let $u \in C^1(\overline{U})$, and suppose there is $x_0 \in \partial U$ and a small ball B around x_0 such that inside this ball ∂U is the hyperplane $\{x_n = 0\}$. For $x \in B$ let

$$\overline{u}(x) = \begin{cases} u(x) & x_n > 0 \\ -3u(\hat{x}, -x_n) + 4u(\hat{x}, -\frac{x_n}{2}) & x_n < 0 \end{cases}$$

Then $\overline{u}$ is continuous on B, and is in fact C^1 on B. Further, we have the estimate

$$\|\overline{u}\|_{L^p(B^-)} \leq C\|u\|_{L^p(B^+)}$$

since

$$\begin{aligned}\int_{B^-} |-3u(\hat{x}, -x_n) + 4u(\hat{x}, -\tfrac{x_n}{2})|^p dx &\leq C\int_{B^-} |u(\hat{x}, -x_n)|^p + |u(\hat{x}, -\tfrac{x_n}{2})|^p dx \\ &\leq C\int_{B^+} |u|^p dx\end{aligned}$$

by changing variables appropriately. By the same method it can be shown

$$\|\overline{u}\|_{W^{1,p}(B)} \le C\|u\|_{W^{1,p}(B^+)}.$$

Now suppose that ∂U is not necessarily so nice near x_0. There is a C^1 change of coordinates $y = \Phi(x)$ such that $\hat{y} = \hat{x}$ and $y_n = x_n - \gamma(\hat{x})$, and locally the boundary in the image is $\{y_n = 0\}$. Let Ψ denote the inverse mapping, and notice that both $\det(D\Phi) = 1$ and $\det(D\Psi) = 1$. Let $u \in C^1(\overline{U})$ and $v(y) = u(\Psi(y))$. As above, reflect to obtain $\overline{v}$ defined on some ball B around $y_0 = \Phi(x_0)$. For $x \in W = \Psi(B)$, let $\overline{u}(x) = \overline{v}(\Phi(x))$. Then $\overline{u}$ is a C^1 extension of u defined on the neighbourhood W of x_0. Further,

$$\begin{aligned}
\|\overline{u}\|^p_{W^{1,p}(W)} &= \int_W |\overline{u}(x)|^p + \sum_{i=1}^n |\overline{u}_{x_i}(x)|^p dx \\
&= \int_W |\overline{v}(\Phi(x))|^p + \sum_{i=1}^n |\partial_{x_i}\overline{v}(\Phi(x))|^p dx \\
&= \int_B |\overline{v}(y)|^p dy + \sum_{i=1}^n \int_W \left| \sum_{j=1}^n \partial_{y_j}\overline{v}(\Phi(x)) \frac{\partial \Phi_j}{\partial x_i} \right|^p dx \\
&= \int_B |\overline{v}(y)|^p dy + C\sum_{i=1}^n \int_B |\overline{v}_{y_j}(y)|^p dy \\
&\le C\|\overline{v}\|^p_{W^{1,p}(B)} \le C\|v\|^p_{W^{1,p}(B^+)} \le C\|u\|^p_{W^{1,p}(W\cap U)}
\end{aligned}$$

Choose W_0 and $\{W_i \mid 1 \le i < N\}$ such that $U \subseteq \bigcup_{i=0}^{N-1} W_i \subseteq V$ and $\partial U \subseteq \bigcup_{i=0}^{N-1} W_i$. Choose a partition of unity $\{\xi_i\}$ subordinate to this cover and let $\overline{u}_i$ be the reflection obtained above defined on W_i. Define

$$Eu := \xi_0 u + \sum_{i=1}^{N-1} \xi_i \overline{u}_i,$$

so that $\operatorname{supp}(Eu) \subseteq V$ and

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le \sum_{i=1}^{N-1} \|\xi_i\overline{u}_i\|_{W^{1,p}(W_i)} \le \sum_{i=1}^{N-1} C(\xi_i)\|\overline{u}_i\|_{W^{1,p}(W_i)} \le C\|u\|_{W^{1,p}(U)}.$$

For $u \in W^{1,p}(U)$, by 1.4.5 there are $u_m \in C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{1,p}(U)$. E is bounded and linear on a dense subset of $W^{1,p}(U)$, so we can extend it to a continuous linear map by defining $Eu := \lim_{n\to\infty} Eu_m$. □

This theorem also holds true for C^k domains and $W^{k,p}$, but fails for domains with corners.

1.6 Traces

Even for reasonable sets, it is not possible to talk of the value of an element of $L^p(U)$ on the boundary of U. But it is possible for elements of $W^{1,p}(U)$ when U has C^1 boundary.

1.6.1 Theorem. *Let U be a bounded domain with C^1 boundary, and $1 \le p < \infty$. There exists $T : W^{1,p}(U) \to L^p(\partial U)$, linear and bounded such that $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\overline{U})$.*

PROOF: First, we consider $u \in C^1(\overline{U})$. Let $x_0 \in \partial U$ and assume that ∂U is flat near x_0 (i.e. the boundary is given by $\{x_n = 0\}$ in $U \cap B := B(x_0, r)$). Let $\Gamma = (\partial U) \cap B(x_0, \frac{r}{2})$. Let ξ be a cut-off-function, i.e.

(i) $\xi \in C_c^\infty(B, [0,1])$; and

(ii) $\xi = 1$ on $B(x_0, \frac{r}{2})$.

Then

$$\begin{aligned}\int_\Gamma |u|^p dx &\le \int_{B \cap \{x_n = 0\}} \xi |u|^p dx \\ &= -\int_{B^+} (\xi |u|^p)_{x_n} dx \\ &= -\int_{B^+} \xi_{x_n} |u|^p + \xi p |u|^{p-1} \text{sign}(u) u_{x_n} dx \\ &\le C \int_{B^+} |u|^p + |Du|^p dx\end{aligned}$$

by *Young's inequality*, $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ for $a, b \ge 0$, where $\frac{1}{p} + \frac{1}{q} = 1$. (We have $|u_{x_n}||u|^{p-1} \le C(|u_{x_n}|^p + |u|^p)$.) (An equivalent formulation is that $a^\alpha b^\beta \le C(a^{\alpha+\beta} + b^{\alpha+\beta})$.)

Second, if the boundary is not flat then there is a mapping Φ that flattens it out. We have (in the notation of the proof of 1.5.1)

$$\frac{1}{C} \|u\|_{L^p(\Gamma_{x_0}} \le \|v\|_{L^p(\Gamma_{y_0})} \le C \|v\|_{W^{1,p}(B_{y_0})} \le C \|u\|_{W^{1,p}(W)}$$

Therefore for all $x_0 \in \partial U$ there is a relatively open Γ_{x_0} in ∂U with $x_0 \in \Gamma_{x_0}$ such that there is $C_{x_0} = C(\xi_0, U, p)$ with $\|u\|_{L^p(\Gamma_{x_0}} \le C_{x_0} \|u\|_{W^{1,p}(U)}$.

Third, ∂U is compact, so there is a finite cover $\{\Gamma_i\}$, and $\|u\|_{L^p(U)} \le C \|u\|_{W^{1,p}(U)}$ where $C = \max C_i$.

Fourth, consider $u \in W^{1,p}(U)$. There are $u_m \in W^{1,p}(U) \cap C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{1,p}(U)$. We have

$$\|Tu_m - Tu_\ell\|_{L^p(\partial U)} = \|T(u_m - u_\ell)\|_{L^p(\partial U)} \le C \|u_m - u_\ell\|_{W^{1,p}(U)}$$

so $\{Tu_m\}$ is a Cauchy sequence (hence convergent) since $\{u_m\}$ is convergent (hence a Cauchy sequence). Take $Tu := \lim_{m \to \infty} Tu_m$.

We must still check the property for $u \in W^{1,p}(U) \cap C(\overline{U})$. □

1.6.2 Theorem. *Let U be a bounded domain with C^1 boundary, and $1 \le p < \infty$. For $u \in W^{1,p}(U)$, $u \in W_0^{1,p}(U)$ if and only if $Tu = 0$.*

1.7 Sobolev inequalities

Fix $U \subseteq \mathbb{R}^N$. For what N, p, q is there a (continuous) embedding of $W^{1,p}(U)$ into $L^q(U)$?

Gagliardo-Nirenberg-Sobolev inequality

Consider $u \in C_c^\infty(\mathbb{R}^N)$. We consider first a "scale-free" inequality involving only the derivatives. Note first that any inequality that holds must have the powers of the norms balance on each side (e.g. $\|u\| \le C\|Du\|^2$ can not hold for all u, for any norms). This is seen by considering $u_\alpha := \alpha u$.

Suppose that $\|u\|_{L^q(\mathbb{R}^N)} \le C\|Du\|_{L^p(\mathbb{R}^N)}$ holds. Consider $u_\lambda(x) := u(\lambda x)$ to see that

$$\|u_\lambda\|_q = \lambda^{-\frac{N}{q}}\|u\|_q \quad \text{and} \quad \|Du_\lambda\|_p = \lambda^{1-\frac{N}{p}}\|Du\|_p,$$

so

$$\lambda^{-\frac{N}{q}-1+\frac{N}{p}}\|u_\lambda\|_q \le \|Du_\lambda\|_p,$$

and for this to work we must have $\frac{1}{p} + \frac{1}{q} = \frac{1}{N}$ by the reasoning in the paragraph above. The *Sobolev conjugate* is $p^* := q = \frac{Np}{N-p}$.

1.7.1 Theorem (Gagliardo-Nirenberg-Sobolev).
Assume that $1 \le p < N$. There exists $C = C(N,p)$ such that $\|u\|_{p^*} \le C\|Du\|_p$ for all $u \in C_c^1(\mathbb{R}^N)$.

PROOF: Write $u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, x_{i-1}, s_i, x_{i+1}, \dots, x_n) ds_i$, so that by the fundamental theorem of calculus,

$$|u(x)| \le \int_{-\infty}^{\infty} |u_{x_i}(x_1, \dots, x_{i-1}, s_i, x_{i+1}, \dots, x_n)| ds_i.$$

For $p = 1$, $p^* = \frac{N}{N-1}$, and

$$\begin{aligned}
|u(x)|^{\frac{N}{N-1}} &= \prod_{i=1}^N \left(\int_{-\infty}^{\infty} |u_{x_i}| ds_i \right)^{\frac{1}{N-1}} \\
\int_{-\infty}^{\infty} |u(x)|^{\frac{N}{N-1}} dx_1 &= \left(\int_{-\infty}^{\infty} |u_{x_1}| ds_1 \right)^{\frac{1}{N-1}} \int_{-\infty}^{\infty} \prod_{i=2}^N \left(\int_{-\infty}^{\infty} |u_{x_i}| ds_i \right)^{\frac{1}{N-1}} dx_1 \\
&\le \left(\int_{-\infty}^{\infty} |u_{x_1}| ds_1 \right)^{\frac{1}{N-1}} \left(\prod_{i=2}^N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u_{x_i}| ds_i\, dx_1 \right)^{\frac{1}{N-1}}
\end{aligned}$$

Repeat this (see Evans) to get

$$\left(\int_{\mathbb{R}^N} |u(x)|^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} \le \int_{\mathbb{R}^N} |Du(x)| dx.$$

Now for $1 < p < N$, consider $v = |u|^\gamma$. By Hölder's inequality and the result for $p = 1$ above,

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |u(x)|^{\gamma\frac{N}{N-1}}dx\right)^{\frac{N-1}{N}} &= \left(\int_{\mathbb{R}^N} |v(x)|^{\frac{N}{N-1}}dx\right)^{\frac{N-1}{N}} \\ &\le \int_{\mathbb{R}^N} \gamma|u|^{\gamma-1}|Du(x)|dx \\ &\le \gamma\left(\int_{\mathbb{R}^N} |u|^{(\gamma-1)\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N} |Du|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

Take $\gamma = \frac{p(N-1)}{N-p}$, and notice that

$$(\gamma-1)\frac{p}{p-1} = \frac{Np}{N-p} = \gamma\frac{N}{N-1} \quad \text{and} \quad \frac{N-1}{N} - \frac{p-1}{p} = \frac{N-p}{Np},$$

so $\|u\|_{p^*} \le C\|Du\|_p$. □

1.7.2 Theorem. *Let $U \subseteq \mathbb{R}^N$ be a bounded domain with C^1 boundary, and $1 \le p < N$. If $u \in W^{1,p}(U)$ then $u \in L^{p^*}(U)$ and there is a constant $C = C(N,p,U)$ such that $\|u\|_{L^{p^*}(U)} \le C\|u\|_{W^{1,p}(U)}$.*

Proof: Let V be a large ball containing U. By 1.5.1 there is $\overline{u} \in W^{1,p}(\mathbb{R}^N)$ such that $\operatorname{supp}(\overline{u}) \subseteq V$ and there is $C_1 = C_1(N,p,U)$ such that

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^N)} \le C_1\|u\|_{W^{1,p}(U)}.$$

By 1.4.4 there are $u_m \in W^{1,p}(\mathbb{R}^N) \cap C_c^1(\mathbb{R}^N)$ such that $u_m \to \overline{u}$ in $W^{1,p}(\mathbb{R}^N)$. By the Gagliardo-Nirenberg-Sobolev inequality there is $C_2 = C_2(N,p,U)$ such that

$$\|u_m - u_\ell\|_{L^{p^*}(\mathbb{R}^N)} \le C_2\|Du_m - Du_\ell\|_{L^p(\mathbb{R}^N)} \le C_2\|u_m - u_\ell\|_{W^{1,p}(\mathbb{R}^n)}.$$

It follows that u_m is a Cauchy sequence in $L^{p^*}(\mathbb{R}^N)$, and so $u_m \to \overline{u}$ in $L^{p^*}(\mathbb{R}^N)$, and we have

$$\|u\|_{L^{p^*}(U)} \le \|\overline{u}\|_{L^{p^*}(\mathbb{R}^N)} \le C_2\|D\overline{u}\|_{L^p(\mathbb{R}^N)} \le C_1C_2\|u\|_{W^{1,p}(U)}.$$ □

It is not necessarily the case that $W^{1,N}(U)$ embeds in $L^\infty(U)$ (see problem set 1, exercise 13), so the inequality does not hold in the limit $p \nearrow N$.

1.7.3 Theorem (Poincaré Inequality). *Let $U \subseteq \mathbb{R}^N$ be a bounded domain with C^1 boundary, and $1 \le p < N$. If $u \in W_0^{1,p}(U)$ then there is $C = C(N,p)$ such that $\|u\|_{L^{p^*}(U)} \le C\|Du\|_{L^p(U)}$.*

Proof: By definition of $W_0^{1,p}(U)$ there is $u_m \in C_c^1(U)$ such that $u_m \to u$ in $W^{1,p}(U)$.

$$\|u_m - u_\ell\|_{L^{p^*}(U)} \le C\|Du_m - Du_\ell\|_{L^p(U)}$$

so $u_m \to u$ in $L^{p^*}(U)$ and $\|u\|_{L^{p^*}(U)} \le C\|Du\|_{L^p(U)}$. □

If U is bounded and $p > q$ then $L^p(U)$ embeds in $L^q(U)$. Indeed,

$$\int_U |u|^q dx \le \left(\int_U (|u|^q)^{\frac{p}{q}} dx \right)^{\frac{q}{p}} \left(\int_U 1 dx \right)^{\frac{p-q}{p}}$$

so $\|u\|_q \le C\|u\|_p$, where C is a constant depending on the volume of U.

Morrey's inequality

1.7.4 Definition. $u \in C(U)$ is *Hölder continuous* with exponent γ if

$$[u]_\gamma := \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma} < \infty.$$

We write $u \in C^{0,\gamma}(U)$, with norm $\|u\|_{C^{0,\gamma}(U)} := \|u\|_{C(U)} + [u]_\gamma$ Analogously, define $\|u\|_{C^{k,\gamma}(U)} := \|u\|_{C^k(U)} + \sum_{|\alpha|=k} [D^\alpha u]_\gamma$ for $u \in C^k(U)$. (Note that the Hölder semi-norm is only applied to the highest order derivatives.)

1.7.5 Theorem. *Let $N < p \le \infty$. Then there is a constant $C = C(N, p)$ such that for $\gamma = 1 - \frac{N}{p}$ and all $u \in C_c^1(\mathbb{R}^N)$, $\|u\|_{C^{0,\gamma}(\mathbb{R}^N)} \le C\|u\|_{W^{1,p}(\mathbb{R}^N)}$.*

Keep in mind that $u \in W^{1,\infty}(U)$ does not necessarily imply $u \in C^{0,1}(U)$. The function in exercise 13 from problem set 1 (where U is taken to be the unit disc with a slit removed) is not Hölder continuous for any exponent.

Proof: Let $x, y \in \mathbb{R}^n$ and write $y = x + rw$, where w is a unit vector and $r = |x - y|$. Let $\varphi(t) = u(x + tw)$, so that $\varphi'(t) = Du(x + tw) \cdot w$, and by the fundamental theorem of calculus,

$$|u(y) - u(x)| = \left| \int_0^r Du(x + tw) \cdot w \, dt \right| \le \int_0^s |Du(x + tw)| \, dt.$$

Therefore

$$\begin{aligned} \int_{\partial B(0,1)} |u(x + rw) - u(x)| \, dS_w &\le \int_0^r \int_{\partial B(0,1)} |Du(x + tw)| \, dS_w \, dt \\ &= \int_0^r \int_{\partial B(x,t)} |Du(z)| t^{1-N} dS_z \, dt \\ &= \int_{B(x,r)} \frac{|Du(z)|}{|x - z|^{N-1}} dz \end{aligned}$$

taking $z = x + tw$, and by the co-area formula. Now consider

$$\begin{aligned}\int_{B(x,r)} |u(y)-u(x)|dy &= \int_0^r \int_{\partial B(x,t)} |u(y)-u(x)|dS_y dt \\ &= \int_0^r t^{N-1} \int_{\partial B(0,1)} |u(x+tw)-u(x)|dS_w dt \\ &\le \int_0^r t^{N-1} \int_{B(x,t)} \frac{|Du(z)|}{|x-z|^{N-1}} dz\, dt \\ &\le \int_0^r t^{N-1} dt \int_{B(x,r)} \frac{|Du(z)|}{|x-z|^{N-1}} dz \\ &= \frac{r^N}{N} \int_{B(x,r)} \frac{|Du(z)|}{|x-z|^{N-1}} dz\end{aligned}$$

The first inequality below holds since it holds pointwise.

$$\begin{aligned}|u(x)| &\le \fint_{B(x,1)} |u(x)-u(y)|dy + \fint_{B(x,1)} |u(y)|dy \\ &\le C \int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{N-1}} dy + C\|u\|_{L^p(\mathbb{R}^N)} \\ &\le C\left(\int_{B(x,1)} |Du(y)|^p dy\right)^{\frac{1}{p}} \left(\int_{B(x,1)} |x-y|^{(1-N)\frac{p}{p-1}} dy\right)^{\frac{p-1}{p}} + C\|u\|_{L^p(\mathbb{R}^N)} \\ &\le C\|u\|_{W^{1,p}(\mathbb{R}^N)}\end{aligned}$$

by Hölder's inequality, and since $p > N$. Therefore $\|u\|_\infty \le C\|u\|_{W^{1,p}(\mathbb{R}^N)}$.

Let $W = B(x,r) \cap B(y,r)$, and notice that $C_1|B(x,r)| \le |W| \le |B(x,r)|$.

$$\begin{aligned}|u(x)-u(y)| &\le \fint_W |u(x)-u(z)|dz + \fint_W |u(z)-u(y)|dz \\ &\le \frac{1}{C_1}\fint_{B(x,r)} |u(x)-u(z)|dz + \frac{1}{C_1}\fint_{B(y,r)} |u(z)-u(y)|dz\end{aligned}$$

But the two integrals are of comparable size, and

$$\begin{aligned}&\fint_{B(x,r)} |u(x)-u(z)|dz \\ &\quad \le C \int_{B(x,r)} \frac{|Du(z)|}{|x-z|^{N-1}} dz \\ &\quad \le C\left(\int_{B(x,r)} |Du(z)|^p dz\right)^{\frac{1}{p}} \left(\int_{B(x,r)} |x-z|^{(1-N)\frac{p}{p-1}} dz\right)^{\frac{p-1}{p}} \\ &\quad \le \|u\|_{W^{1,p}(\mathbb{R}^N)} s^{1-\frac{N}{p}}.\end{aligned}$$

Therefore, since $r = |x - y|$,

$$\sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^{1-\frac{N}{p}}} \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)}$$

and $\|u\|_{C^{0,\gamma}(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)}$, by combining this with the L^∞ bound obtained above. □

1.7.6 Theorem. *Let $U \subseteq \mathbb{R}^N$ be a bounded domain with C^1 boundary, and $N < p < \infty$. Then $W^{1,p}(U)$ embeds in $C^{0,\gamma}(\overline{U})$, where $\gamma = 1 - \frac{N}{p}$, and the norm of the embedding depends only on N, p, and U.*

Sobolev inequalities

Recall, Gagliardo-Nirenberg-Sobolev says that when $p < N$, $W^{1,p}(U) \hookrightarrow L^q(U)$ for $q \leq p^* = \frac{Np}{N-p}$, and M says that when $p > N$, $W^{1,p}(U) \hookrightarrow C^{0,\gamma}(\overline{U})$ for $\gamma \leq 1 - \frac{N}{p}$.

1.7.7 Theorem (Sobolev Inequalities). *Let $U \subseteq \mathbb{R}^N$ be a bounded domain with C^1 boundary, and $u \in W^{k,p}(U)$.*

(i) If $kp < N$ then $u \in L^q(U)$, where $q = \frac{Np}{N-kp}$ (so $\frac{1}{q} = \frac{1}{p} - \frac{k}{N}$), and the norm of the embedding depends only on N, p, k, and U.

(ii) If $kp > N$ then $u \in C^{k-\lfloor\frac{N}{p}\rfloor-1,\gamma}(\overline{U})$, where γ may be taken $0 \leq \gamma < 1$ if $\frac{N}{p}$ is an integer, and $0 \leq \gamma \leq \lfloor\frac{N}{p}\rfloor + 1 - \frac{N}{p}$ otherwise, and the norm of the embedding depends only on N, p, k, γ, and U.

PROOF: Let $p^{(j)} := \frac{Np}{N-jp}$ (so $p^* = p^{(1)}$, and $\frac{1}{p^{(j)}} = \frac{1}{p} - \frac{j}{N}$). Since, by the properties of the weak derivative, $D^\alpha u \in W^{1,p}(U)$ for all $|\alpha| < k$, the Gagliardo-Nirenberg-Sobolev inequality implies that, for each $j = 1, \dots, k-1$,

$$\|u\|_{W^{1-j,p^{(j)}}(U)} \leq C_j\|u\|_{W^{k-(j-1),p^{(j-1)}}(U)}.$$

One further application implies $\|u\|_{L^q(U)} \leq CC_1 \cdots C_{k-1}\|u\|_{W^{k,p}(U)}$.

Let $\ell := \lfloor\frac{N}{p}\rfloor$. Suppose that ℓ is not an integer and let $r = p^{(\ell)} = \frac{Np}{N-\ell p} > N$. If $u \in W^{k-\ell,r}(U)$ then as above, for all $|\alpha| < k - \ell$ we have $D^\alpha u \in W^{1,r}(U)$, so by Morrey's inequality $D^\alpha u \in C^{0,\gamma}(\overline{U})$, where $\gamma = 1 - \frac{N}{r} = \lfloor\frac{N}{p}\rfloor + 1 - \frac{N}{p}$. □

1.8 Compactness

1.8.1 Definition. Let X and Y be Banach spaces. A linear map $\Phi : X \to Y$ is said to be *sequentially compact* (or *compact*) if Φ is continuous and the image of any bounded sequence in X has a convergent subsequence in Y.

Remark.

(i) Recall the theorem of Arzela-Ascoli: If $\{f_n\}$ is a sequence of functions that is uniformly bounded and equicontinuous then it has a uniformly convergent subsequence.

(ii) $C^{k,\alpha}(U) \subset\subset C^{k,\beta}(U)$ if $1 \geq \alpha > \beta \geq 0$. (Prove this as an exercise.)

(iii) $L^p(U) \hookrightarrow L^q(U)$ when $p > q$, but the embedding is *not* compact.

1.8.2 Theorem (Relich-Kondrachov). *Let $U \subseteq \mathbb{R}^N$ be a bounded domain with C^1 boundary, and $u \in W^{1,p}(U)$. If $p < N$ then for all $1 \leq q < p^* = \frac{Np}{N-p}$, $W^{1,p}(U) \subset\subset L^q(U)$.*

PROOF: Let $\{u_m\}$ be a bounded sequence in $W^{k,p}(U)$. Let V be an open ball containing U. By the extension theorem there are $\overline{u}_m$ such that $\|\overline{u}_m\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|u_m\|_{W^{k,p}(U)}$ and supp$(\overline{u}_m) \subseteq V$. It is enough to show the middle of the following string of inequalities (since the first and last are proved).

$$\|u_m\|_{L^q(U)} \leq \|\overline{u}_m\|_{L^q(\mathbb{R}^n)} \leq \|\overline{u}_m\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|u_m\|_{W^{k,p}(U)}$$

Without loss of generality, we may suppose that $u_m \in W_0^{k,p}(V)$. For $0 < \varepsilon < 1$, let $u_m^\varepsilon := \eta_\varepsilon * u_m$ (supposing that V is large enough to support the convolution).

We claim that

(i) $u_m^\varepsilon \to u_m$ in $L^q(V)$ as $\varepsilon \to 0$, uniformly in m; and

(ii) for fixed ε, $\{u_m^\varepsilon\}$ is uniformly bounded and equicontinuous.

Assuming these claims, for all $\delta > 0$ there is $\varepsilon_\delta > 0$ such that $\|u_m^{\varepsilon_\delta} - u_m\|_{L^q(V)} < \frac{\delta}{3}$. From the second claim and the Arzela-Ascoli theorem, there is a uniformly convergent subsequence $u_{m_k}^{\varepsilon_\delta} \to u^{\varepsilon_\delta}$. Consequently, $u_{m_k}^{\varepsilon_\delta} \to u^{\varepsilon_\delta}$ in $L^q(V)$. In particular, this subsequence is a Cauchy sequence in $L^q(V)$, so there is a k_δ such that for all $k, \ell \geq k_\delta$, by the triangle inequality,

$$\|u_{m_k} - u_{m_\ell}\| < \delta.$$

Note that this does not immediately imply that $\{u_{m_k}\}$ is a Cauchy sequence in $L^q(V)$. Diagonalize to obtain a Cauchy subsequence, which is a convergent subsequence since $L^q(V)$ is complete.

Now for the proof of the first claim. Let $a(t) = x + t(y - x)$, and consider that, by the fundamental theorem of calculus,

$$u_m(y) - u_m(x) = \int_0^1 \frac{d}{dt} u_m(a(t))dt = \int_0^1 Du_m(a(t)) \cdot (y - x)dt$$

so, taking $z := y - x$,

$$u_m^\varepsilon(x) - u_m(x) = \int_{\mathbb{R}^N} \eta_\varepsilon(x-y)(u_m(y) - u_m(x))dy$$
$$= \int_{\mathbb{R}^N} \eta_\varepsilon(-z) \int_0^1 Du_m(x+tz)\cdot z\,dt\,dz$$

$$\begin{aligned}\|u_m^\varepsilon(x) - u_m(x)\|_{L^1(V)} &= \int_V \left| \int_{\mathbb{R}^n} \eta_\varepsilon(-z) \int_0^1 Du_m(x+tz)\cdot z\,dt\,dz \right| dx \\ &\le \int_{\mathbb{R}^n} \int_0^1 \int_V \eta_\varepsilon(-z)|Du_m(x+tz)||z|\,dx\,dt\,dz \\ &\le \varepsilon \int_{\mathbb{R}^n} \int_0^1 \eta_\varepsilon(-z)\|Du_m\|_{L^1(V)} dt\,dz \\ &= \varepsilon\|Du_m\|_{L^1(V)} \le \varepsilon C(V)\|Du_m\|_{L^p(V)} \le C\varepsilon.\end{aligned}$$

Recall the following fact: if $f_n \to f$ in L^1 and $\{f_n\}$ is bounded in L^p then $f_n \to f$ in L^q for all $q \in [1,p)$. This uses the *interpolation inequality*

$$\|f\|_q \le \|f\|_1^\theta \|f\|_p^{1-\theta} \qquad \text{where } \frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p}.$$

Note that $\{u_m^\varepsilon - u_m\}$ is bounded in L^{p^*} since u_m converges in $W^{1,p}(V)$, and by the Sobolev inequalities this space embeds in L^{p^*}.

For the second claim,

$$|u_m^\varepsilon(x)| \le \int_{\mathbb{R}^n} \eta_\varepsilon(z)|u_m(x-z)|dz \le \|\eta_\varepsilon\|_\infty \|u_m\|_{L^1(V)} \le C\|u_m\|_{W^{1,p}(V)}$$

and

$$|Du_m^\varepsilon(x)| \le \int_{\mathbb{R}^n} |D\eta_\varepsilon(z)||u_m(x-z)|dz \le \|D\eta_\varepsilon\|_\infty \|u_m(x-z)\|_{L^1(\mathbb{R}^n)}$$

showing uniform boundedness and uniform Lipschitz continuity. □

Remark. If $X \hookrightarrow Y \hookrightarrow Z$ are Banach spaces and either of the embeddings are compact then the composition $X \hookrightarrow Z$ is compact.

1.9 Poincaré inequalities

We call an inequality giving a bound on u determined entirely by the derivatives of u a *Poincaré inequality*.

1.9.1 Theorem. *Let $U \subseteq \mathbb{R}^N$ be a bounded, connected domain with C^1 boundary. Then $\|u - ⨍_U u\|_p \le C\|Du\|_p$ for all $u \in W^{1,p}(U)$, where C depends on N, p, and U.*

Remark. If U is not connected then a locally constant function on U that is not identically constant on U gives a counterexample to the inequality.

PROOF: Assume that the conclusion of the theorem does not hold. Then for each $k \ge 1$ there is $u_k \in W^{1,p}(U)$ such that $\|u_k - ⨍_U u_k\|_p > k\|Du_k\|_p$. Both sides of the inequality do not change upon adding a constant to u, and the inequality is invariant under multiplying by a scalar. Define $v_k := u_k - ⨍_U u_k$ and normalize v_k so that $\|v_k\|_p = 1$. Then $\|v_k\|_p > k\|Dv_k\|_p$ for all $k \ge 1$. Therefore $\{v_k\}$ is a bounded sequence in $W^{1,p}(U)$. Since $W^{1,p}(U) \subset\subset L^p(U)$ whenever $p < p^*$ (i.e. when $p < N$), and

$$W^{1,p}(U) \subset\subset C^{0,\gamma}(U) \subseteq L^\infty(U) \subseteq L^p(U)$$

whenever $p > N$. (Alternatively, notice that $W^{1,p}(U) \hookrightarrow W^{1,q}(U)$ whenever $p > q$, so since $q^* > q$ and $q^* \to \infty$ as $q \to N$, we get the $p \ge N$ case by taking q only slightly smaller than N, but close enough to N to give $q^* > p$.) Thus there is a subsequence converging in the L^p norm. Without loss of generality we may assume $v_k \to v$ in L^p. Since $\|v_k\|_p = 1$ for all $k \ge 1$, $\|v\|_p = 1$. Further, $⨍_U v = 0$, since $⨍_U \cdot \, dx$ is a continuous linear functional on $L^p(U)$, since U is bounded. We will now show that $Dv \equiv 0$. Indeed, by the dominated convergence theorem,

$$\int_U v\varphi_{x_i} dx = \lim_{k\to\infty} \int_U v_k \varphi_{x_i} dx = -\lim_{k\to\infty} \int_U \partial_{x_i} v_k \varphi \, dx = 0$$

where the last equality is by Hölder's inequality, since $\|Dv_k\|_p \to 0$ as $k \to \infty$. Therefore the weak derivative exist and is zero, and $v \in W^{1,p}(U)$. But this implies $v \equiv 0$, and this contradicts $\|v\|_p = 1$. ☐

Remark. There are other proofs of this inequality that give an estimate on the constant, and yet others which give the optimal constant.

1.10 Difference quotients

1.10.1 Definition. Let $U \subseteq \mathbb{R}^N$ be open, and define

$$U_\delta := \{x \in U \mid \operatorname{dist}(x, \partial U) > \delta\}.$$

For $0 < h < \delta$, define the *difference quotient* in the direction of e_i by

$$D_i^h u(x) := \frac{u(x + he_i) - u(x)}{h}.$$

1.10.2 Theorem. *Let $U \subseteq \mathbb{R}^N$ be open*

(i) *Let $1 \leq p < \infty$ and $u \in W^{1,p}(U)$. For all $V \subset\subset U$ open,*

$$\|D^h u\|_{L^p(V)} \leq C\|Du\|_{L^p(U)},$$

where $C = C(N, p, U, V)$ for all $h \in (0, \frac{1}{2}dist(V, \partial U))$.

(ii) *Let $1 < p < \infty$ and $u \in L^p(U)$, and $V \subset\subset U$ open. Assume there is C such that $\|D^h u\|_{L^p(V)} \leq C$ for all $h \in (0, \frac{1}{2}dist(V, \partial U))$. Then we may conclude that $u \in W^{1,p}(V)$ and $\|Du\|_{L^p(V)} \leq C$.*

PROOF: Without loss of generality, we may assume that u is smooth.

$$\begin{aligned}
\|D_i^h u\|_{L^p(V)} &= \int_V \frac{|u(x + he_i) - u(x)|^p}{|h|^p} dx \\
&\leq \int_V \left| \int_0^1 |Du(x + the_i)| dt \right|^p dx \\
&\leq \int_V \int_0^1 |Du(x + the_i)|^p dt\, dx && \text{by Jensen} \\
&= \int_0^1 \int_V |Du(x + the_i)|^p dx\, dt \\
&\leq \int_0^1 \int_U |Du(y)|^p dy\, dt && y := x + the_i \\
&= \|Du\|_{L^p(U)}.
\end{aligned}$$

For the second part of the theorem, suppose $\varphi \in C_c^\infty(U)$ and $\text{supp}(\varphi) \subseteq V$ and $0 < h < \text{dist}(V, \partial U)$ and $h < \text{dist}(\text{supp}(\varphi), \partial V)$. It can be checked that

$$\int_V u D_i^h \varphi\, dx = -\int_V D_i^{-h} u \varphi\, dx.$$

For lack of a better proof, at this point we must strengthen our assumption on the bound of the difference quotient to all $0 < |h| < \frac{1}{2}\text{dist}(V, \partial U)$. By the Banach-Alouglu theorem, there is a sequence $h_k \to 0$ such that $D_i^{-h_k} u \to v_i$ in $L^p(V)$ for all $i = 1, \ldots, N$.

$$\int_V v_i \varphi\, dx = \lim_{k\to\infty} \int_V D_i^{-h_k} u \varphi\, dx = \lim_{k\to\infty} - \int_V u D_i^{h_k} \varphi\, dx = -\int_V u D_i \varphi\, dx.$$

Therefore $v_i = D_i u$, so u has a weak derivative, and notice that $\|D_i u\|_{L^p(V)} \leq \liminf_k \|D_i^{-h_k} u\|_{L^p(V)}$. □

1.10.3 Definition. $u : U \to \mathbb{R}$ is *differentiable* at $x \in U$ if there is $a \in \mathbb{R}^N$ such that

$$\lim_{y\to x} \frac{|u(y) - u(x) - a \cdot (y - x)|}{|y - x|} = 0$$

and we write $Du = a$.

1.10.4 Theorem. *Assume that $u \in W^{1,p}_{loc}(U)$ for some $N < p \le \infty$. Then u is a.e. differentiable.*

Proof: In the proof of Morrey's inequality we showed that when u is continuous, for $y \in B(x,r)$,

$$|u(y)-u(x)| \le Cr^{1-\frac{N}{p}} \left(\int_{B(x,2r)} |Du|^p \, dx \right)^{\frac{1}{p}}.$$

The Lebesgue Differentiation Theorem states that when $f \in L^p_{loc}(\mathbb{R}^N)$, for $1 \le p < \infty$ and for a.e. $x \in \mathbb{R}^N$,

$$\fint_{B(x,r)} |f(y)-f(x)|^p \, dy \to 0$$

as $r \to 0$. Therefore, for a.e. $x \in U$,

$$\fint_{B(x,r)} |Du(x)-Du(z)|^p \, dz \to 0$$

as $r \to 0$. Let x be such a point and consider

$$v(y) = u(y) - u(x) - Du(x)\cdot(y-x) \in W^{1,p}_{loc}(U).$$

Let $r = |y-x|$, so that

$$\begin{aligned}
|v(y)-v(x)| = |v(y)| &\le Cr^{1-\frac{N}{p}} \left(\int_{B(x,2r)} |D_y v(z) - D_y v(x)|^p \, dx \right)^{\frac{1}{p}} \\
&= Cr\left(r^{-N} \int_{B(x,2r)} |Du(z)-Du(x)|^p \, dx \right)^{\frac{1}{p}} \\
&= \tilde{C}r\left(\fint_{B(x,2r)} |Du(z)-Du(x)|^p \, dx \right)^{\frac{1}{p}} \\
&\to 0 \text{ as } y \to x.
\end{aligned}$$

□

1.11 The dual space H^{-1}

1.11.1 Definition. $H^{-1}(U) := (H^1_0(U))'$, the collection of continuous linear functionals on $H^1_0(U)$.

Remark.

(i) Recall that for $1 \le p < \infty$, $(L^p)' = L^q$ when $\frac{1}{p} + \frac{1}{q} = 1$. More explicitly, for every $\varphi \in (L^p)'$ there is a unique $f \in L^q$ such that $\langle \varphi, g \rangle = \int gf \, dx$ for all $g \in L^p$. We would like a similar "explicit" characterization of H^{-1}.

(ii) If $X \hookrightarrow Y$ then $Y' \hookrightarrow X'$. So $L^2 \hookrightarrow H^{-1}$ since $H_0^1 \hookrightarrow L^2$. The correspondence is, for $f \in L^2$, $\langle f, u\rangle := \int uf\,dx$ for $u \in H^1$.

1.11.2 Theorem. *For every $\varphi \in H^{-1}(U)$ there exist $f^0, f^1, \ldots, f^N \in L^1(U)$ such that*

$$\langle \varphi, u\rangle = \int_U f^0 u\,dx + \sum_{i=1}^N \int_U f^i u_{x_i}\,dx.$$

Moreover, $\|\varphi\|_{H^{-1}(U)} = \inf\{(\int_U \sum_{i=0}^N (f^i)^2 dx)^{\frac{1}{2}}\}$, where the infimum is taken over all representations $(f_0, \ldots, f_N)$ of φ.

Remark.

(i) This is not truly a representation theorem since the representation of φ is not uniquely determined Indeed, $(0,1)$ and $(0,0)$ both represent the zero functional on $H_0^1((0,1))$.

(ii) This theorem holds for $W_0^{1,p}$ (and the analogous dual space) but with L^2 replaced with L^q.

PROOF: $H_0^1(U)$ is a Hilbert space with the inner product

$$(u, v) := \int_U uv + Du \cdot Dv\,dx.$$

Therefore, for $\varphi \in H^{-1}(U)$ there is a unique element $f \in H_0^1(U)$ such that $\langle \varphi, u\rangle = (u, f)$ for any $u \in H_0^1(U)$. Take $f^0 = f$ and $(f^1, \ldots, f^N) = Df$. The infimum in the statement of the theorem is attained for this choice of $(f^0, \ldots, f^N)$, and it is the norm of φ. □

2 Elliptic PDE

2.1 Introduction

For the rest of term we will be studying the following problem

$$(P)\ \begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

This is the problem with *Dirichlet boundary conditions*. We may also consider the *Neumann boundary conditions* $\nabla u = 0$ on ∂U. The *non-divergence form* of the problem is

$$Lu = -\sum_{i,j=1}^N a^{ij} u_{x_i x_j} + \sum_{i=1}^N b^i u_{x_i} + cu$$

where the coefficients are bounded, $a^{ij}, b^i, c \in L^\infty(U)$. The *divergence form* of the problem is

$$Lu = -\sum_{i,j=1}^{N} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{N} b^i u_{x_i} + cu.$$

We will be using the *summation convention* of summing over terms in which there is a are repeated indices, e.g. $Lu = -(a^{ij}u_{x_i})_{x_j} + b^i u_{x_i} + cu$. We can convert between divergence and non-divergence form when the coefficients a^{ij} are in $C^1(\overline{U})$ by only altering the coefficients b^i. We can and will always assume that the matrix $A = (a^{ij})$ is symmetric.

2.1.1 Definition. L is *uniformly elliptic* on U if A is symmetric and positive definite with a uniform lower bound, i.e. if $A(x) \geq \varepsilon I$, i.e. if $\xi^T A(x)\xi \geq \varepsilon |\xi|^2$ for all $\xi \in \mathbb{R}^N$ for all $x \in U$, for some $\varepsilon > 0$ that does not depend on x.

We say that $u \in C^2(U)$ is a *classical solution* if $-(a^{ij}u_{x_i})_{x_j} + b^i u_{x_i} + cu = f$. In this case, for any smooth function $\varphi \in C_c^\infty(U)$,

$$\int_U [-(a^{ij}u_{x_i})_{x_j} + b^i u_{x_i} + cu]\varphi \, dx = \int_U f\varphi \, dx$$

so, by integration by parts,

$$\int_U a^{ij} u_{x_i} \varphi_{x_j} + b^i u_{x_i} \varphi + cu\varphi \, dx = \int_U f\varphi \, dx.$$

But this equation makes sense for any $u \in H^1(U)$, and motivates the definition of a weak solution to (P).

2.1.2 Definition. For $a^{ij}, b^i, c \in L^\infty(U)$, we say that $u \in H_0^1(U)$ is a *weak solution* of (P) for some $f \in H^{-1}(U)$ if for all $v \in H_0^1(U)$,

$$B[u,v] := \int_U a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + cuv \, dx = \langle f, v \rangle.$$

2.2 Existence of solutions and the Fredholm alternative

2.2.1 Example. For $Lu = -\Delta u + u$, $B[u,v] = \int Du \cdot Du + uv \, dx = (u,v)$, the usual inner product on $H_0^1(U)$. Therefore a weak solution to (P) is a u such that $(u,v) = \langle f, v \rangle$ for all $v \in H_0^1(U)$, and such a solution exists for all $f \in H^{-1}(U)$ by the Riesz representation theorem.

2.2.2 Theorem (Lax-Milgram). *Let H be a Hilbert space and $B : H \times H \to \mathbb{R}$ be a bilinear form such that*

(i) B is bounded, i.e. there is α such that $|B[u,v]| \leq \alpha \|u\| \|v\|$ for all $u, v \in H$.

(ii) *B is coercive, i.e. there is $\beta > 0$ such that $\beta\|u\|^2 \le B[u,u]$ for all $u \in H$.*

Then for every $f \in H'$ there is a unique $u \in H$ such that $\langle f, v\rangle = B[u,v]$ for all $v \in H$.

PROOF: Fix $u \in H$ and consider $v \mapsto B[u,v]$, a continuous linear functional on H. By the Riesz representation theorem there is a unique $Au \in H$ such that $B[u,v] = (Au,v)$ for all $v \in H$. We will now show that $A : H \to H$ is invertible.

(i) A is linear by the bilinearity of B.

(ii) A is bounded because $\|Au\|^2 = (Au,Au) = B[u,Au] \le \alpha\|u\|\|Au\|$, so $\|Au\| \le \alpha\|u\|$.

(iii) A is injective because $\|u\|^2 \le \beta B[u,u] = \beta(Au,u) \le \beta\|Au\|\|u\|$, so $\|u\| \le \beta\|Au\|$, and $Au = 0$ if and only if $u = 0$.

(iv) The range of A is closed. Indeed, consider a Cauchy sequence Au_k in the range of A. By coercivity,

$$\frac{1}{\beta}\|u_k - u_\ell\| \le \|A(u_k - u_\ell)\| = \|Au_k - Au_\ell)\| \to 0$$

as $k, \ell \to \infty$. Therefore $u_k \to u \in H$, and $Au_k \to Au$ in the range of A.

(v) A is surjective, because otherwise, for any non-zero $w \in (\text{range}(A))^\perp$, $\beta\|w\|^2 \le B[w,w] = (Aw,w) = 0$, a contradiction.

Given $f \in H'$ there is $z \in H$ such that $\langle f, v\rangle = (z,v)$ for all $v \in H$. Let $u = A^{-1}(z)$, so that $\langle f, v\rangle = (z,v) = (Au,v) = B[u,v]$. Uniqueness follows from coercivity. ☐

2.2.3 Example. Consider $u = \sin x$, so that $u'' = -\sin x$ and u solves

$$\begin{cases} -u'' - u = 0 & \text{in } (0,\pi) \\ u = 0 & \text{at } 0 \text{ and } \pi. \end{cases}$$

But the problem

$$\begin{cases} -u'' - u = 1 & \text{in } (0,\pi) \\ u = 0 & \text{at } 0 \text{ and } \pi. \end{cases}$$

does not have a solution. Indeed, suppose for contradiction that there was a solution. Multiply by $\sin x$ and integrate to see

$$\int_0^\pi (-u'' - u)\sin x \, dx = \int_0^\pi \sin x \, dx = 2$$

but

$$\int_0^\pi u' \cos x - u \sin x \, dx = \int_0^\pi u \sin x - u \sin x \, dx = 0.$$

For one of these problems the solution is not unique, and for the other there are no solutions. We will see that these problems come in pairs.

2.2.4 Theorem (Energy estimates). *There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

(i) $B[u,v] \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$*; and*

(ii) $\gamma \|u\|_{L^2(U)}^2 + B[u,u] \geq \beta \|u\|_{H_0^1(U)}^2$.

PROOF: Recall that $MI \geq A(x) \geq \theta I > 0$ for some θ and M. Indeed,

$$\xi A \xi^T = \xi_i a^{ij} \xi_j \leq \max_{i,j} \|a^{ij}\|_\infty |\xi_i \xi_j| \leq (N \max_{i,j} \|a^{ij}\|_\infty) \|\xi\|^2.$$

Whence,

$$\begin{aligned}
|B[u,v]| &= \left| \int_U Du \cdot A(Dv)^T + (b \cdot Du)v + cuv\, dx \right| \\
&\leq M \int_U |Du||Dv|\, dx + \sum_i \|b^i\|_\infty \int_U |Du||v|\, dx + \|c\|_\infty \int_U |uv|\, dx \\
&\leq M \|Du\|_2 \|Dv\|_2 + \sum_i \|b_i\|_\infty \|Du\|_2 \|v\|_2 + \|c\|_\infty \|u\|_2 \|v\|_2 \\
&\leq \alpha \|u\|_{H_0^1(U)}^2 \|v\|_{H_0^1(U)}^2
\end{aligned}$$

by Cauchy-Schwartz, for some constant $\alpha > 0$.

$$\begin{aligned}
B[u,u] &= \int_U Du \cdot A(Du)^T + (b \cdot Du)u + cu^2 dx \\
&\geq \theta \|Du\|_2^2 - \max_i \|b^i\|_\infty \int_U |Du||u|\, dx + \min_U c \|u\|_2^2 \\
&\geq \theta \|Du\|_2^2 - \max_i \|b^i\|_\infty \left(\varepsilon \int_U |Du|^2 dx + \frac{1}{4\varepsilon} \int_U |u|^2 dx \right) + \min_U c \|u\|_2^2 \\
&\geq (\theta - m_b \varepsilon) \|Du\|_2^2 + \left(m_c - \frac{1}{4\varepsilon} \right) \|u\|_2^2
\end{aligned}$$

by Cauchy's inequality with ε. Set $\varepsilon = \frac{\theta}{2m_b}$ to see that $\gamma = (\frac{\theta}{2} - m_c + \frac{m_b}{2\theta})^+$ will do. □

Remark. Better constants can be found by using the Poincaré inequality.

2.2.5 Theorem (Existence I). *For $a^{ij}, b^i, c \in L^\infty(U)$ there is $\gamma \geq 0$ such that for all $\mu \geq \gamma$ and $f \in H^{-1}(U)$, the problem*

$$(P_\mu) \begin{cases} L_\mu u := Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique weak solution.

PROOF: Note that the linear form associated with L_μ is

$$B_\mu[u,v] := B[u,v] + \mu(u,v),$$

so $B_\mu[u,u] = B[u,u] + \mu\|u\|^2_{L^2(U)}$. That B_μ is bounded for all μ follows from energy estimates, and by energy estimates there is $\gamma \geq 0$ such that for $\mu \geq \gamma$,

$$B_\mu[u,u] = B[u,u] + \mu\|u\|_{L^2(U)} \geq \beta\|u\|^2_{H^1_0(U)} + (\mu - \gamma)\|u\|^2_{L^2(U)} \geq \beta\|u\|^2_{H^1_0(U)}.$$

Lax-Milgram implies the existence of a unique weak solution to (P_μ). □

For $\mu \geq \gamma$, $L_\mu = L + \mu I : H^1_0(U) \to H^{-1}(U)$ is an injective (by uniqueness), surjective (by existence), continuous (by energy estimates) linear mapping. It follows that L_μ is an isomorphism between these spaces. Consider $S = (L_\mu)^{-1} : H^{-1}(U) \to H^1_0(U)$, the *solution mapping*. The composition

$$H^{-1}(U) \xrightarrow{S} H^1_0(U) \hookrightarrow L^2(U) \hookrightarrow H^{-1}(U)$$

is compact since the embedding of $H^1_0(U)$ into $L^2(U)$ is compact. Furthermore, $L^2 \xrightarrow{S} H^1_0 \hookrightarrow L^2$ is compact. In some sense, the solution map is squishing things quite a bit. We will see that the solution map ups the differentiability of the function by two, and functions with two additional derivatives are sparse in some sense.

2.2.6 Definition. For $Lu = -(a^{ij}u_{x_i})_{x_j} + b^i u_{x_i} + cu$, the *formal adjoint* of L is L^*, defined by $L^*v = -(a^{ij}v_{x_i})_{x_j} - b^i v_{x_i} + (c - b^i_{x_i})v$.

The reason for considering L^* is the following.

$$\begin{aligned}(Lu,v) = \int_U (Lu)v\,dx &= \int_U -(a^{ij}u_{x_i})_{x_j}v + b^i u_{x_i}v + cuv\,dx \\ &= \int_U a^{ij}u_{x_i}v_{x_j} + ((b^i u)_{x_i} - b^i_{x_i}u)v + cuv\,dx \\ &= \int_U -(a^{ij}v_{x_i})_{x_j}u - b^i v_{x_i}u + (c - b^i_{x_i})vu\,dx \\ &= \int_U (L^*v)u\,dx = (u, L^*v)\end{aligned}$$

This "formal" adjoint is only helpful if $b \in C^1(\overline{U})$, as only in this case are the coefficients of L^* are in $L^\infty(U)$.

2.2.7 Definition. The *adjoint bilinear form* of B (associated with L) is B^*, defined by $B^*[u,v] = B[v,u]$. We say that $v \in H^1_0(U)$ is a weak solution of

$$(P^*)\ \begin{cases} L^*u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

if $B[u,v] = B^*[v,u] = \langle f, u\rangle$ for all $u \in H^1_0(U)$.

2.2.8 Theorem (Existence II).

(i) *Exactly one of the following two statements is true.*

a) *For all $f \in L^2(U)$ there is a unique weak solution to*

$$(P)\ \begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

b) *There is $u \in H_0^1(U)$, $u \neq 0$, such that*

$$(P_0)\ \begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

(ii) *If (b) holds then let N be the set of solutions to (P_0). Then N is a finite dimensional subspace of $H_0^1(U)$ and $\dim(N) = \dim(N^*)$, where N^* is the set of solutions to (P_0^*). For $f \in L^2(U)$, (P) has a weak solution if and only if $f \in (N^*)^\perp$.*

This theorem is a consequence of the Fredholm alternative from the theory of Banach algebras.

2.2.9 Definition. Let H be a Hilbert space and A be a bounded linear operator on H. The *adjoint* of A is A^*, defined by the condition that $(Au, v) = (u, A^*v)$ for all $u, v \in H$.

It can be show that A^* is a bounded linear operator on H with $\|A^*\| = \|A\|$. Further, if K is a compact linear operator on H then so is K^*.

2.2.10 Fredholm Alternative. *Let K be a compact linear operator on an infinite dimensional Hilbert space H.*

(i) $\ker(I - K)$ *is finite dimensional.*

(ii) *range$(I - K)$ is closed.*

(iii) *range$(I - K) = \ker(I - K^*)^\perp$*

(iv) $\ker(I - K) = \{0\}$ *if and only if range$(I - K) = H$.*

(v) $\dim\ker(I - K) = \dim\ker(I - K^*)$.

Proof:

(i) Assume that $\ker(I - K)$ is not finite dimensional. Let $(u_k)_{k\geq 1}$ be an infinite sequence of pairwise orthogonal unit vectors in $\ker(I - K)$. The sequence $(Ku_k)_{k\geq 1}$ has a convergent subsequence since K is compact, so without loss of generality we may assume that the sequence converges. But $Ku_k = u_k$ for all $k \geq 1$, so distance between any pair of elements is $\sqrt{2}$, and this contradicts that the sequence converges.

(ii) We claim that there is $\beta > 0$ such that $\|(I-K)u\| \geq \beta\|u\|$ for all $u \in \ker(I-K)^\perp$. Assume for contradiction that there is no such β. Let $(u_k)_{k\geq 1}$ be a sequence of unit vectors in $\ker(I-K)^\perp$ such that

$$\|u_k - Ku_k\| < \frac{1}{k} \text{ for all } k \geq 1.$$

By the Banach-Alaoglu theorem, we may assume without loss of generality that $u_i \rightharpoonup u$ weakly. Note that $\|u\| = 1$. Since K is compact, without loss of generality we may assume that $Ku_k \to v$, for some $v \in H$. But linear operators are weakly continuous, so $Ku_k \rightharpoonup Ku$, and it must be the case that $v = Ku$. Therefore $Ku_k \to Ku$. But $u_k - Ku_k \to 0$, so for any $w \in H$,

$$(w, Ku - u) = \lim_{k\to\infty}(w, Ku - Ku_k + Ku_k - u_k + u_k - u) = 0$$

and it follows $u = Ku$. Therefore $u \in \ker(I-K)$, and this is a contradiction since then $(u_k, u) = 0$ and

$$(u,u) = 1 \neq 0 = \lim_{k\to\infty}(u_k, u).$$

To show that $\operatorname{range}(I-K)$ is closed, suppose that $v_k \in \operatorname{range}(I-K)$ is Cauchy. Then there are $u_k \in \ker(I-K)^\perp$ such that $(I-K)u_k = v_k$. By the claim above,

$$\|v_k - v_\ell\| \geq \beta\|u_k - u_\ell\| \text{ for all } k, \ell \geq 1,$$

so $u_k \to u$ for some $u \in H$, and $v_k \to (I-K)u \in \operatorname{range}(I-K)$.

(iii) We claim that for any bounded linear operator A, $\overline{\operatorname{range}}(A) = (\ker A^*)^\perp$. Indeed, if $v \in \overline{\operatorname{range}}(A)$ then there are $u_k \in H$ such that $Au_k \to v$. For any $w \in \ker(A^*)$, $(Au_k, w) = (u_k, A^*w) = 0$, so $(v, w) = 0$.

Conversely, if $v \in \overline{\operatorname{range}}(A)^\perp$ then for all $u \in H$, $0 = (v, Au) = (A^*v, u)$, so $v \in \ker A^*$. Therefore $\operatorname{range}(I-K) = \ker(I-K^*)^\perp$ since the left hand side is closed by part (ii).

(iv) Suppose that $I-K$ is one-to-one, but that $H_1 := \operatorname{range}(I-K) \subsetneq H$. Inductively define $H_{k+1} := (I-K)H_k$, and notice that $H_{k+1} \subsetneq H_k$ for all $k \geq 1$ since $I-K$ is one-to-one. For each $k \geq 1$, choose a unit vector $u_k \in H_k$ orthogonal to H_{k-1}. For $k > \ell \geq 1$,

$$Ku_k - Ku_\ell = \underbrace{-(I-K)u_k + (I-K)u_\ell + u_k}_{=:w \in H_{\ell+1}} - u_\ell$$

so $\|Ku_k - Ku_\ell\|^2 = \|w\|^2 + \|u_\ell\|^2 \geq 1$, which contradicts that K is a compact operator.

Conversely, if $\operatorname{range}(I-K) = H$ then $\ker(I-K^*) = \{0\}$ by part (iii), so $\operatorname{range}(I-K^*) = H$ by the first part, and again by (iii), $\ker(I-K) = \{0\}$.

(v) Apply (iii) and (iv) and induction (exercise). □

PROOF (EXISTENCE II): From Existence I, there is $\gamma \geq 0$ such that

$$(P_\gamma) \begin{cases} Lu + \gamma u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique solution for every $f \in L^2(U)$, with $B_\gamma[u,v] = B[u,v] + \gamma(u,v)$. Consider the solution operator $L_\gamma^{-1} : L^2(U) \to H_0^1(U) \subset\subset L^2(U)$.

$$\beta \|u\|^2_{H_0^1(U)} \leq B_\gamma[u,u] = \int_U fu \leq \|f\|_{L^2(U)}\|u\|_{L^2(U)} \leq \|f\|_{L^2(U)}\|u\|_{H_0^1(U)},$$

so $\|L_\gamma^{-1}\| \leq \frac{1}{\beta}$. Regarding the original equation, it can be checked that u is a solution to (P) if and only if u is a solution to

$$\begin{cases} L_\gamma u = f + \gamma u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

(Indeed, $B[u,v] = \int fv$ for all $v \in H_0^1(U)$ if and only if $B_\gamma[u,v] = \int fv + \gamma \int uv$ for all $v \in H_0^1(U)$.) Therefore any solution satisfies $u = L_\gamma^{-1} f + \gamma L_\gamma^{-1} u$. Yet otherwise stated, u is a solution to (P) if and only if

$$(I - \gamma L_\gamma^{-1})u = L_\gamma^{-1} f.$$

Note that $\gamma L_\gamma^{-1} =: K$ is a one-to-one compact operator $L^2(U) \to L^2(U)$. By the Fredholm alternative for compact operators, either

(i) $(I - K)u = h$ has a unique solution for all $h \in L^2(U)$; or

(ii) $(I - K)u = 0$ has a non-trivial solution.

In the case (i), for $f \in L^2(U)$, $u = (I - \gamma L_\gamma^{-1})^{-1} L_\gamma^{-1} f$ is the unique solution to (P). In the case (ii), $(I - K)u = 0$ has a non-trivial solution, so (P_0) has a non-trivial solution since $L_\gamma^{-1} f = 0$ implies $f = 0$. This establishes the first part of the theorem since (a) and (b) are mutually exclusive.

For the second part, let $N = \ker(I - K)$ be the set of solutions to (P_0). Note that $L_\gamma^* = L^* + \gamma I$, and it can be shown that $(L_\gamma^{-1})^* = (L_\gamma^*)^{-1}$, so the set of solutions to (P_0^*) is exactly $N^* = \ker(I - K^*)$, and $\dim N = \dim N^*$ by the Fredholm alternative. For the solvability condition, $(I - K)u = h$ has a solution if and only if $h \in (N^*)^\perp$. But for all $v \in N^*$,

$$0 = (h, v) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^* v) = \frac{1}{\gamma}(f, v),$$

so (P) has a weak solution if and only if $f \in (N^*)^\perp$. $\square$

2.2.11 Theorem (Existence III).

(i) There exists a set Σ, at most countable and known as the (real) spectrum of L, such that

$$(P_\lambda) \begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique solution for all $f \in L^2(U)$ whenever $\lambda \notin \Sigma$.

(ii) If Σ is infinite then $\Sigma = \{\lambda_k\}_{k \geq 0}$ where $\lambda_k \to \infty$.

PROOF: Notice that $Lu = \lambda u + f$ is equivalent to $L_{-\lambda} u = f$, so by Existence I there is $\gamma \geq 0$ such that (P_λ) has a unique solution for all $f \in L^2(U)$ when $-\lambda \geq \gamma$. For the rest of the proof we consider only $\lambda > -\gamma$.

Existence II shows that "(P_λ) has a unique solution for all $f \in L^2(U)$" is equivalent to "the only solution to $Lu = \lambda u$ is $u \equiv 0$." ($Lu = \lambda u$ is known as the *Helmholtz equation*.) This is trivially equivalent to "the only solution to $L_\gamma u = (\lambda + \gamma) u$ is $u \equiv 0$." Let L_γ^{-1} be the solution operator, and $K := \gamma L_\gamma^{-1}$, a compact operator. Then u is a solution to this last problem if and only if

$$u = L_\gamma^{-1}((\lambda + \gamma) u) = \tfrac{\lambda + \gamma}{\gamma} K u, \text{ or } Ku = \tfrac{\gamma}{\lambda + \gamma} u.$$

By the spectral theory for compact operators (see below, or the appendix in Evans), the spectrum of K is either finite or a sequence converging to zero. If the spectrum of K is finite then there are finitely many λ for which (P_λ) fails to have a unique solution for all $f \in L^2(U)$. Otherwise, if $\mu_k \to 0$ are the eigenvalues of K then $\mu_k = \frac{\gamma}{\lambda_k + \gamma}$ and $\lambda_k = \gamma \frac{1 - \mu_k}{\mu_k} \to \infty$. □

2.2.12 (Real) spectrum of a compact operator.

Let $A : X \to X$ be a bounded linear operator on a Banach space. The spectrum *of A is $\sigma(A) := \mathbb{R} \setminus \rho(A)$, where*

$$\rho(A) = \{\lambda \in \mathbb{R} \mid (\lambda I - A) \text{ is one-to-one and onto}\}$$

is the resolvent set. *The spectrum decomposes into three pieces (defined to be disjoint)*

(i) point spectrum, $\sigma_p(A) := \{\lambda \mid \ker(\lambda I - A) \neq \{0\}\}$

(ii) continuous spectrum, $\sigma_c(A) := \{\lambda \mid \text{range}(\lambda I - A) \text{ is not dense in } X\}$

(iii) residual spectrum, $\sigma_r(A) := \{\lambda \mid \text{range}(\lambda I - A) \text{ is dense in } X \text{ but} \neq X\}$

If K is a compact operator on a Hilbert space then

(i) $0 \in \sigma(K)$

(ii) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$

(iii) Either $\sigma(K)$ is finite or $\sigma(K)$ is a sequence converging to zero.

2.2.13 Theorem. *If $\lambda \notin \Sigma$ then there is a constant $C = C(L, U, \lambda)$ such that $\|u\|_{L^2(U)} \leq C\|f\|_{L^2(U)}$ whenever u solves (P_λ) for $f \in L^2(U)$.*

PROOF: Let $\lambda \notin \Sigma$. Assume for contradiction that for every $k \geq 1$ there is (u_k, f_k) solving (P_λ) such that $\|u_k\|_{L^2(U)} \geq k\|f_k\|_{L^2(U)}$. Without loss of generality $\|u_k\|_{L^2(U)} = 1$. Then $f_k \to 0$ in $L^2(U)$, and by coercivity,

$$\beta\|u_k\|^2_{H^1_0(U)} \leq B_\lambda[u_k, u_k] + \gamma\|u_k\|^2_{L^2(U)} = \int_U f_k u_k \, dx + \gamma \leq 2 + \gamma.$$

Therefore there is a subsequence $u_k \rightharpoonup u$ in $H^1_0(U)$ weakly. It follows that $u_k \to u$ in $L^2(U)$, so $\|u\|_{L^2(U)} = 1$. But in this case $B[u, v] = \int_U 0 \cdot v \, dx = 0$ for all $v \in H^1_0(U)$, so $Lu = \lambda u$, and this is a contradiction since $\lambda \notin \Sigma$. □

2.3 Regularity of solutions

Suppose that $-\Delta u = f$, where $f \in L^2(U)$. Notice that

$$\int_U |D^2 u|^2 dx = \int (\Delta u)^2 dx = \int f \Delta u \, dx \leq \varepsilon \int (\Delta u)^2 dx + \frac{1}{4\varepsilon} \int f^2 dx.$$

so we can get a bound on the second derivative in terms of $\|f\|_{L^2(U)}$, using the Laplacian of u as a "test function." Further,

$$\int |Du|^2 dx = \int u \Delta u \, dx = \int f u \, dx \leq \int u^2 + f^2 \, dx$$

so we can get a bound on the H^2 norm of u in terms of $\|f\|_{L^2(U)}$, and $\|u\|_{L^2(U)}$, using u as a test function.

2.3.1 Theorem (Interior Regularity). *Let $Lu = -(a^{ij} u_{x_i})_{x_j} + b^i u_{x_i} + cu$ be uniformly elliptic and act on $H^1(U)$, where $a^{ij} \in C^1(\overline{U})$ and $b^i, c \in L^\infty(U)$. If $u \in H^1(U)$ is a weak solution of $Lu = f$ in U (i.e. $B[u, v] = \int_U f v \, dx$ for all $v \in H^1_0(U)$), where $f \in L^2(U)$, then*

(i) $u \in H^2_{loc}(U)$; and

(ii) For all $V \subset\subset U$ open there is $C = C(U, V, L)$ such that

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

Remark. If case (i) of Existence II applies and $u \in H^1_0(U)$ then there is $\tilde{C}$ such that $\|u\|_{L^2(U)} \leq \tilde{C}\|f\|_{L^2(U)}$, so we get a bound on the H^2 norm of u in terms of $\|f\|_{L^2(U)}$ alone.

PROOF: Let $V \subset\subset U$ be open. Choose $W \subset\subset U$ open such that $\overline{V} \subseteq W$. Let $\xi : U \to \mathbb{R}$ be a smooth cutoff function that is 1 on V and 0 outside of W. From $B[u,v] = \int_U f v \, dx$ we write

$$\int_U a^{ij} u_{x_i} v_{x_j} dx = \int_U (f - b^i u_{x_i} - cu) v \, dx$$

Let $v := -D_k^{-h}(\xi^2 D_k^h u)$, where

$$D_k^h g(x) = \frac{1}{h}(g(x + he_k) - g(x))$$

is the difference quotient and $h < \mathrm{dist}(\mathrm{supp}(\xi), \partial W)$.

$$\begin{aligned} LHS &= \int_U a^{ij} u_{x_i} (-D_k^{-h}(\xi^2 D_k^h u))_{x_j} dx = -\int_U a^{ij} u_{x_i} D_k^{-h}((\xi^2 D_k^h u)_{x_j}) dx \\ &= \int_U D_k^h (a^{ij} u_{x_i}) (\xi^2 D_k^h u)_{x_j} dx = \int_U \underbrace{a^{ij,h} \xi^2 D_k^h u_{x_i} D_k^h u_{x_j}}_{A_1} \\ &+ \underbrace{(D_k^h a^{ij}) \xi^2 u_{x_i} D_k^h u_{x_j} + 2\xi \xi_{x_j} a^{ij,h} D_k^h u_{x_i} D_k^h u + 2\xi \xi_{x_j} (D_k^h a^{ij}) D_k^h u}_{A_2} \, dx \end{aligned}$$

noting that $D_k^h(fg) = (D_k^h f) g^h + f D_k^h g$, where $g^h(x) = g(x + he_k)$. By uniform ellipticity,

$$A_1 \geq \theta \int_U \xi^2 |D_k^h Du|^2 dx.$$

All of the coefficients involving the (fixed) cutoff function ξ and the a^{ij} are L^∞ with bound independent of h, so

$$\begin{aligned} |A_2| &\leq C \int_W \xi(|Du||D_k^h Du| + |D_k^h u||D_k^h Du| + |Du||D_k^h u|) \, dx \\ &\leq C\varepsilon \int_W \xi |D_k^h Du|^2 dx + \frac{C}{4\varepsilon} \int_W |Du|^2 dx, \end{aligned}$$

recalling the Cauchy inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ for any $\varepsilon > 0$. Whence

$$LHS \geq \frac{\theta}{2} \int_W \xi^2 |D_k^h Du|^2 \, dx - C \int_W |Du|^2 dx$$

choosing ε appropriately, where this C depends on $D\xi$, among other things.

$$RHS = \int_W (f - b^i u_{x_i} - cu) v \, dx \leq \varepsilon \int_W v^2 dx + \frac{C}{\varepsilon} \int_W (f^2 + u^2 + |Du|^2) dx$$

and

$$\begin{aligned}\int_W v^2 dx &\le C \int_W |D(\xi^2 D_k^h u)|^2 dx && \text{by 1.10.2}\\ &\le C \int_W \xi^2(|D_k^h u)|^2 + |D_k^h Du|^2) dx && \text{since } \xi^2 \le 1\\ &\le C \int_W (|Du|^2 + \xi^2 |D_k^h Du|^2) dx.\end{aligned}$$

Choose ε so small that $\varepsilon C = \frac{\theta}{4}$, so that

$$RHS \le \frac{\theta}{4} \int_W \xi^2 |D_k^h Du|^2 dx + C \int_W (f^2 + u^2 + |Du|^2) dx.$$

Therefore, combining the estimates for the LHS and the RHS,

$$\frac{\theta}{4} \int_W \xi^2 |D_k^h Du|^2 \le C \int_W (f^2 + u^2 + |Du|^2) dx.$$

By 1.10.2, $u \in H^2(V)$, and by that theorem we need only show that

$$\int_W |Du|^2 dx \le C \int_U (f^2 + u^2) dx$$

to finish the proof of the theorem. Let $\eta \in C_c^\infty(U)$ be a cut-off function that is 1 on W and zero outside of U. Now let $v := \eta^2 u$.

$$\begin{aligned}LHS &= \int_U (a^{ij} u_{x_i})(\eta^2 u)_{x_j} dx\\ &= \int_U \eta^2 (a^{ij} u_{x_i} u_{x_j}) + 2\eta a^{ij} \eta_{x_j} u \, dx\\ &\ge \theta \int_U \eta^2 |Du|^2 dx - C \int_U \eta |Du||u| \, dx\end{aligned}$$

and

$$RHS = \int_U (f - b^i u_{x_i} - cu)\eta^2 u \, dx \le C \int_U (f^2 + u^2 + \eta |Du||u|) dx.$$

It follows that

$$\begin{aligned}\theta \int_U \eta^2 |Du|^2 dx &\le C \int_U (f^2 + u^2 + |Du||u|) dx\\ &\le \frac{\theta}{2} \int_U \eta^2 |Du|^2 dx + C \int_U u^2 f^2 dx\end{aligned}$$

since

$$\int_U |Du||u|\,dx \leq \varepsilon \int_U \eta^2 |Du|^2 dx + \frac{1}{4\varepsilon} \int_U u^2 dx$$

I'm not so sure about these last steps. □

2.3.2 Theorem (Higher-order Regularity). *Let $m \geq 0$ and assume that $a^{ij} \in C^{m+1}(U)$, $b^i, c \in C^m(U)$, and $f \in H^m(U)$. Assume that u is a weak solution to $Lu = f$ in U. Then*

(i) $u \in H^{m+2}_{loc}(U)$; and

(ii) For all $V \subset\subset U$ open there is $C = C(U, V, L)$ such that

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}).$$

PROOF (IDEA): Consider the simpler problem $-(a^{ij}u_{x_i})_{x_j} = f$, when $m = 1$ and $f \in H^1(U)$. Apply D_k to obtain

$$-(a^{ij}_{x_k}u_{x_i} + a^{ij}u_{x_k x_i})_{x_j} = f_{x_k}.$$

Set $v = u_{x_k}$ and we have

$$-(a^{ij}v_{x_i})_{x_j} = f_{x_k} + (a^{ij}_{x_k}u_{x_i})_{x_j} \in L^2(U).$$

Use the previous theorem to conclude some regularity of v. □

2.3.3 Corollary. *If $a^{ij}, b^i, c, f \in C^\infty(U)$ then any weak solution $u \in H^1(U)$ to $Lu = f$ is in $C^\infty(U)$.*

PROOF: For any k and any $V \subset\subset U$, by Higher-order Regularity and the Sobolev embedding theorems we may conclude that $u \in C^{k,\gamma}(V)$. Therefore $u \in C^\infty(V)$ for any $V \subset\subset U$, so $u \in C^\infty(U)$. □

2.3.4 Theorem (H^2-regularity). *Let U be an open, bounded domain with C^2 boundary, and let $a^{ij} \in C^1(\overline{U})$, $b^i, c \in L^\infty(U)$. Assume that $u \in H^1_0(U)$ is a weak solution to $Lu = f$ in U, where $f \in L^2(U)$. Then*

(i) $u \in H^2(U) \cap H^1_0(U)$; and

(ii) $\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$ for some $C = C(U, L)$.

PROOF: Case 1: the flat case. Suppose that $0 \in \partial U$ assume that

$$B(0,1) \cap U = B(0,1) \cap \{x_N > 0\}.$$

Let $V := B(0, \frac{1}{2}) \cap U$ and let ξ be a cut-off function (defined on $\overline{U}$) that is 1 on V and 0 outside of $B(0,1) \cap U$. Recall $B[u, v] = \int_U f v\,dx$ since u is a weak solution. We write $\int_U a^{ij}u_{x_i}v_{x_j}dx = \int_U \tilde{f} v\,dx$, where $\tilde{f} = f - b^i u_{x_i} - cu$. Fix $k = 1, \ldots, N-1$, and let $v := -D_k^{-h}(\xi^2 D_k^h u)$ for small h. Note that $v \in H_0^1(V)$ since ξ is 0 on the curved boundary and u is zero on the flat boundary. Use estimates similar to the ones used to prove Interior Regularity to show that

$$\int_V |D_k^h Du|^2 dx \le C \int_U (f^2 + u^2 + |Du|^2)\,dx = C(\|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}).$$

From energy estimates the H^1 norm of u is bounded above by constant multiples of the L^2 norms of f and u. By the extension to the theorem on difference quotients, $u_{x_i x_k} \in L^2(V)$, for all $1 \le i \le N$ and $1 \le k < N$. For weak derivatives $u_{x_i x_j} = u_{x_j x_i}$ a.e., and for the last case $j = i = N$ note that $u_{x_N x_N}$ can be written in terms of f, the coefficients, and the other second order partials, all of which are in L^2.

Case 2 is for a general domain. Centred at a point on the boundary, without loss of generality $x_N = \gamma(\hat{x})$. There is a coordinate change $y = \phi(x)$ such that $\hat{y} = \hat{x}$ and $y_N = x_N - \gamma(\hat{x})$. Let $x = \psi(y)$ be the inverse transformation, and consider that $\det D\phi = 1$. Let $\tilde{u} = u(\psi(y))$, so that $u(x) = \tilde{u}(\phi(x))$. Note that $\tilde{u}$ is in H^1 and its trace is zero on the image of the boundary $\{y_N = 0\}$. Our task is to show that $\tilde{u}$ is in H^2. We will do this by applying Case 1, after checking that the properties of the uniformly elliptic operator carry through the coordinate transformation. We claim $\tilde{L}\tilde{u} = \tilde{f}$ on the image $\tilde{B}$, where $\tilde{f}(y) = f(\psi(y))$ and $\tilde{L}$ is given by $\tilde{a}^{k\ell}(y) = a^{ij}(\psi(y))\phi^k_{x_i}(\psi(y))\phi^\ell_{x_j}(\psi(y))$, $\tilde{b}^k(y) = b^i(\psi(y))\phi^k_{x_i}(\psi(y))$, and $\tilde{c}(y) = c(\psi(y))$. □

2.3.5 Theorem (Higher-order Boundary Regularity).
Let U be an open, bounded domain with C^{m+2} boundary, and let $a^{ij} \in C^{m+1}(\overline{U})$, $b^i, c \in C^m(\overline{U})$. Assume that $u \in H_0^1(U)$ is a weak solution to $Lu = f$ in U, where $f \in H^m(U)$. Then

(i) $u \in H^{m+2}(U) \cap H_0^1(U)$*; and*

(ii) $\|u\|_{H^{m+2}(U)} \le C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)})$ *for some* $C = C(U, L, m)$.

PROOF: A mess of induction. □

2.3.6 Corollary. *Let U be an open, bounded domain with smooth boundary. If $a^{ij}, b^i, c, f \in C^\infty(\overline{U})$ then any weak solution $u \in H_0^1(U)$ to $Lu = f$ is in $C^\infty(\overline{U})$.*

Note that if $Lu = -a^{ij}u_{x_i x_j} + b^i u_{x_i} + cu$ is not given in divergence form, then we can rewrite it as $\tilde{L}u = -(a^{ij}u_{x_i})_{x_j} + (a^{ij}_{x_j} + b^i)_{x_i} + cu$. Any classical solution to the second form is also a classical solution to the first form. This completes the theory of linear uniformly elliptic equations.

2.4 Maximum principles

For this section U is a bounded, open domain, $u \in C^2(U) \cap C(\overline{U})$, and

$$Lu := -a^{ij}u_{x_i x_j} + b^i u_{x_i} + cu,$$

where the coefficients are bounded functions and the matrix $A=(a^{ij})$ is (symmetric and) uniformly elliptic.

2.4.1 Theorem (Weak Maximum Principle). *Suppose that $c \equiv 0$ on $\overline{U}$.*

(i) If $Lu \le 0$ in U (i.e. u is a sub-solution) then $\max_{\overline{U}} u = \max_{\partial U} u$.

(ii) If $Lu \ge 0$ in U (i.e. u is a super-solution) then $\min_{\overline{U}} u = \min_{\partial U} u$.

2.4.2 Theorem. *Suppose that $c \ge 0$ on $\overline{U}$.*

(i) If $Lu \le 0$ in U then $\max_{\overline{U}} u \le 0 \vee \max_{\partial U} u$.

(ii) If $Lu \ge 0$ in U then $\min_{\overline{U}} u \ge 0 \wedge \min_{\partial U} u$.

2.4.3 Theorem (Comparison). *Suppose $c \ge 0$ on $\overline{U}$. For $u, v \in C^2(U) \cap C(\overline{U})$, if $Lu \le f$ in U and $f \le Lv$ in U and $u \le v$ on ∂U then $u \le v$ in U.*

PROOF (COMPARISON $\Longrightarrow$ 2.4.1):
Let $v := \max_{\partial U} u$. Then $Lv = 0$ in U and $v \ge u$ on ∂U, so $v \ge u$ in U. Therefore $\max_{\partial U} u \ge \max_{\overline{U}} u$, and the first part follows. For the second part, let $v := u$ and $\tilde{u} = \min_{\partial U} u$ and apply 2.4.3 to v and $\tilde{u}$. □

PROOF (COMPARISON $\Longrightarrow$ 2.4.2):
Let $v := \max_{\partial U} u$ and assume that $v \ge 0$. Then $Lv = cv \ge 0 \ge Lu$. We have $v \ge u$ on ∂U, so $v \ge u$ in U and the conclusion follows. If $\max_{\partial U} u < 0$ then take $v \equiv 0$. Then $Lv = 0 \ge Lu$ and $v \ge u$ on ∂U, so $u \le 0$ in U. Prove the second part as an exercise. □

PROOF (COMPARISON): Assume first that $Lu < f \le Lv$ in U. Assume there is $\hat{x} \in U$ such that $u(\hat{x}) > v(\hat{x})$. Then $\max_{\overline{U}}(u - v) > 0$, and a local maximum is attained at some $x_0 \in U$ (and in particular not on the boundary). We have

$$u(x_0) > v(x_0), \quad Du(x_0) = Dv(x_0), \quad D^2u(x_0) \le D^2v(x_0).$$

It follows that

$$c(x_0)u(x_0) \ge c(x_0)v(x_0), \quad b^i(x_0)u_{x_i}(x_0) = b^i(x_0)v_{x_i}(x_0)$$

and, if $A = I$, that

$$-\Delta u(x_0) \ge -\Delta v(x_0)$$

implying that $Lu(x_0) \ge Lv(x_0)$, a contradiction.

There is an orthogonal matrix O such that $OAO^T = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Let $y = x_0 + O(x - x_0)$ be a change of coordinates. Then $u_{x_i} = u_{y_k} \frac{\partial y_k}{\partial x_i} = u_{y_k} o_{ki}$ and $u_{x_i x_j} = u_{y_k y_\ell} o_{ki} o_{\ell j}$, so $a^{ij} u_{x_i x_j} = o_{ki} a^{ij} o_{\ell j} u_{y_k y_\ell} = \lambda_k u_{y_k y_k}$.

For the second case, note that if $c > 0$ on U then $u^\varepsilon := u - \varepsilon$ satisfies $Lu^\varepsilon = Lu - c\varepsilon < f$, and $u^\varepsilon \le v$ on ∂U. By the first case it follows that $u^\varepsilon \le v$ for all $\varepsilon > 0$. Since this holds for all $\varepsilon > 0$ it follows that $u \le v$.

For the general case, take $u^\varepsilon := u + \varepsilon e^{\lambda x_1} - \delta$, where $\delta := \varepsilon e^{\lambda \max\{|x_1|, x \in \overline{U}\}}$. By the choice of δ, $u^\varepsilon \le u$ for all $\varepsilon > 0$, and $u^\varepsilon \to u$ as $\varepsilon \to 0$.

$$\begin{aligned} Lu^\varepsilon &= Lu + \varepsilon L e^{\lambda x_1} - c\delta \\ &= Lu - \varepsilon\lambda^2 a^{11} e^{\lambda x_1} + \varepsilon b^1 \lambda e^{\lambda x_1} + \varepsilon c e^{\lambda x_1} - c\delta \\ &\le Lu + \varepsilon e^{\lambda x_1}(\underline{-\theta\lambda^2 + \lambda\|b^1\|_\infty + \|c\|_\infty}) - c\delta \end{aligned}$$

Choose λ large enough that the underlined constant is negative. With this choice fixed, $Lu^\varepsilon < f$, and $u^\varepsilon \le v$ on ∂U, so $u^\varepsilon \le v$ in U, and we are done as above. □

2.4.4 Lemma (Hopf). *Suppose $c \equiv 0$. Assume*

(i) $Lu \le 0$ in U;

(ii) there is $x_0 \in \partial U$ such that $u(x_0) > u(x)$ for all $x \in U$; and

(iii) U satisfies the interior ball condition (i.e. for every $x_0 \in \partial U$ there is a closed ball B contained in $\overline{U}$ such that $x_0 \in B$).

Let B be any closed ball in $\overline{U}$ that contains x_0, so that x_0 is on the boundary of B. If ν is the outward normal vector to B at x_0 then $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Further, if instead $c \ge 0$ then the above holds if $u(x_0) \ge 0$.

Proof: Without loss of generality, $B = B(0,r)$. Let $v(x) := e^{-\lambda|x|^2} - e^{-\lambda r^2}$, and note that $v > 0$ on $B(0,r)$. For $x \in A := B(0,r) \setminus \overline{B}(0,\frac{r}{2})$,

$$\begin{aligned} Lv(x) &= (-a^{ij}(-2\lambda\delta_{ij} + 4\lambda^2 x_i x_j) - 2b^i\lambda x_i + c)e^{-\lambda|x|^2} - ce^{-\lambda r^2} \\ &\le (2\lambda \operatorname{tr} A - 4\lambda^2 x^T A x - 2\lambda b^T x)e^{-\lambda|x|^2} \\ &\le \lambda(C_A - \lambda\theta r^2 + C_b r)e^{-\lambda|x|^2} < 0 \end{aligned}$$

if λ is large enough. Let $u^\varepsilon := u + \varepsilon v$, so that $u^\varepsilon = u$ on $\partial B(0,r)$. We have $u(x_0) > \max_{B(0,\frac{r}{2})} u + \delta$ for $\delta > 0$ small enough. Choose $\varepsilon > 0$ so that $\varepsilon \max_A v < \delta$. Then $u^\varepsilon(x) \le u(x_0) = u^\varepsilon(x_0)$ for all $x \in \partial B(0,\frac{r}{2})$. But u^ε is a sub-solution, so by the weak maximum principle, $u^\varepsilon(x) \le u(x_0)$ for all $x \in A$.

Complete the proof by computing $\frac{\partial v}{\partial \nu} = Dv \cdot \frac{x}{|x|} = -2\lambda|x|e^{-\lambda|x|^2} < 0$. □

2.4.5 Theorem (Strong Maximum Principle). *Suppose that U is open, bounded, and connected.*

(i) Suppose $c \equiv 0$.

a) If $Lu \le 0$ in U and u attains its maximum in U then u is constant.

b) If $Lu \le 0$ in U and u attains its minimum in U then u is constant.

(ii) If $c \ge 0$ then the above conclusions hold provided that the maximum and minimum are non-negative and non-positive, respectively.

PROOF: Assume that u has a maximum at $x_0 \in U$. Let

$$A := \{x \in \overline{U} \mid u(x) = u(x_0)\},$$

a closed set. Then $U' := U \setminus A$ is open. If U' is not empty then there is $y \in U'$ such that $\operatorname{dist}(y, A) < \operatorname{dist}(y, \partial U)$. Indeed, let $z \in \partial A \setminus \partial U$, so that $2\varepsilon := \operatorname{dist}(z, \partial U) > 0$. Since $B(z, \varepsilon) \subseteq U$ and $z \in \partial A$, there is $y \in B(z, \varepsilon) \setminus A$, $y \in U$, and y is closer to ∂A than to ∂U.

Let $\overline{y} \in \partial A$ be such that $\operatorname{dist}(y, A) = \|y - \overline{y}\|$. Then $B(y, |y - \overline{y}|) \subseteq U'$. By the Hopf Lemma, applied to u on U', $\frac{\partial u}{\partial \nu}(\overline{y}) > 0$, where $\nu = \frac{\overline{y} - y}{|\overline{y} - y|}$. But then there is $\delta > 0$ such that $u(\overline{y} + \delta' \nu) > u(\overline{y})$ for all $0 < \delta' < \delta$, a contradiction since some of these points are in U. □

2.4.6 Exercise. Suppose that u and $-u$ satisfy the Harnack inequality on $B(0, 1)$. Show that u is α-Hölder continuous on $B(0, \frac{1}{2})$, for some small $\alpha > 0$.

2.5 Eigenvalues and eigenfunctions

In this section we consider the *eigenvalue problem*

$$\text{(EVP)} \quad \begin{cases} Lw = \lambda w & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

where $Lu = -(a^{ij} u_{x_i})_{x_j}$ and $a^{ij} \in C^\infty(\overline{U})$ and $A = (a^{ij})$ is (symmetric and) uniformly elliptic. As usual, let $B[u, v] := \int_U a^{ij} u_{x_i} v_{x_j}\, dx$, and notice that B is symmetric. We say that λ is an *eigenvalue* if the (EVP) has a non-trivial solution for λ, and in this case a non-zero solution is an *eigenfunction*. Note that by the assumptions made on A, all solutions to the (EVP) are smooth.

2.5.1 Theorem. *(i) All eigenvalues λ are real.*

(ii) Let $\Sigma = \{\lambda_k\}$ be the spectrum of L, ordered and with each eigenvalue repeated to its multiplicity. Then $\lambda_2 > \lambda_1 > 0$ and $\lambda_k \to \infty$ as $k \to \infty$.

(iii) The corresponding unit eigenfunctions form an orthonormal basis of $L^2(U)$ and an orthogonal basis of $H_0^1(U)$.

PROOF: Let $S := L^{-1} : L^2 \to H_0^1$ be the solution operator, so that the composition $S : L^2 \to H_0^1 \hookrightarrow L^2$ (which we also call S) is a compact operator. Then S is symmetric since

$$(Sf, g) = \int (Sf)g = B[Sg, Sf] = B[Sf, Sg] = \int f(Sg) = (f, Sg)$$

by the definition of weak solution and the fact that B is symmetric. Further,

$$(Sf, f) = B[Sf, Sf] \geq \beta \|Sf\|_{H^1(U)}$$

since B is coercive (since A is uniformly elliptic). Let

$$m := \inf\{(Su,u) \mid u \in H_0^1(U), \|u\|_{L^2(U)} = 1\}$$

and

$$M := \sup\{(Su,u) \mid u \in H_0^1(U), \|u\|_{L^2(U)} = 1\}.$$

We claim that $\sigma(S) \subseteq [m,M]$ and $m,M \in \sigma(S)$. The first assertion follows because S is symmetric. For the second assertion, we know that $m \geq 0$ since B is coercive, and $0 \in \sigma(S) \subseteq [m,M]$ since S is a compact operator, so $m = 0$. For the second assertion let $\|u_k\|_2 = 1$ be such that $(Su_k, u_k) \to M$. Then there is a subsequence $u_k \rightharpoonup u$ converging weakly to some $u \in L^2(U)$. Since continuous linear operators are continuous for the weak topology, $Su_k \rightharpoonup Su$. Now S is compact, so there is a further subsequence $Su_k \to v \in L^2$. But for any $w \in L^2(U)$,

$$(Su - v, w) = \lim_{k\to\infty}(Su - v, w) = (v - v, w) = 0$$

so $Su_k \to Su$. In particular, $(Su,u) = M$ and $u \neq 0$. In fact $\|u\|_{L^2(U)} = 1$ and $Su = Mu$ can be seen by considering that $(D(u + tw), u + tw) \leq M$.

Let η_k be the eigenvalues of S and H_k be the corresponding eigenspaces. Then the H_k are finite dimensional when $\eta_k \neq 0$ and pairwise orthogonal. Let $H = \operatorname{span}(\bigcup_k H^k)$. To show that $\overline{H} = L^2$ it suffices to show that $H^\perp = \{0\}$. Now $S(H) \subseteq H$ and since S is symmetric, $S(H^\perp) \subseteq H^\perp$. It follows that $(Su,u) = 0$ for all $u \in H^\perp$, so S is zero on $H^\perp$ since it is symmetric. Therefore $H^\perp \subseteq H_0 \subseteq H$, so $H^\perp = \{0\}$.

Next we prove the following *variational principle*

$$\lambda_1 = \min_{\substack{u\in H_0^1(U) \\ \|u\|_{L^2(U)}=1}} B[u,u] = \min_{\substack{u\in H_0^1(U) \\ u\neq 0}} \frac{B[u,u]}{\|u\|_{L^2(U)}^2}.$$

Furthermore, all minimizers are eigenfunctions (corresponding to λ_1) and these eigenfunctions have no zeros in U. Indeed, if $u = \sum_{k=1}^m \alpha_k w_k$ is a unit vector where the w_k are unit eigenfunctions corresponding to λ_k. Then

$$B[u,u] = \sum_{k=1}^m \lambda_k \alpha_k^2 \geq \lambda_1 \sum_{k=1}^m \alpha_k^2 = \lambda_1$$

since λ_1 is the smallest eigenvalue and $\|u\|_{L^2(U)} = 1$. In general u is a limit of functions of the form above, say $u = \sum_{k=1}^\infty \alpha_k w_k =: \lim_{m\to\infty} u_m$. It suffices to prove that $B[u_m, u_m] \to B[u,u]$. This is not trivial because $u_m \to u$ in $L^2(U)$, but evaluating B involves taking some derivatives.

Notice that $B[w_k, w_\ell] = \lambda_k \delta_{k\ell}$, so let's define $v_k := \frac{1}{\sqrt{\lambda_k}} w_k$. Then for $u \in H_0^1(U)$,

$$\beta_k := B[u, v_k] = \frac{1}{\sqrt{\lambda_k}} B[u, w_k] = \frac{1}{\sqrt{\lambda_k}} \lambda_k (u, w_k) = \sqrt{\lambda_k}\alpha_k,$$

Let $u_m = \sum_{i=1}^m \beta_k v_k$.

$$B[u,u] = B[u - u_m + u_m, u - u_m + u_m]$$
$$= B[u - u_m, u - u_m] + 0 + 0 + B[u_m, u_m] \geq B[u_m, u_m]$$

Therefore $\sum_{k=1}^\infty \beta_k^2 < \infty$. Thus the sequence $\{u_m\}$ is Cauchy in $H_0^1(U)$, and its only possible limit is u. Indeed, for all k

$$0 = B[u - \lim_{m\to\infty} u_m, v_k] = B[u - \lim_{m\to\infty} u_m, \frac{1}{\sqrt{\lambda_k}} w_k] = \sqrt{\lambda_k}(u - \lim_{m\to\infty} u_m, w_k)$$

where the last inner product is in L^2.

For the last assertion, suppose that $Lu = \lambda_1 u$. Then from the homework, $u^+, u^- \in H_0^1(U)$, and $Du^+ = Du\mathbf{1}_{u>0}$ and $Du^- = -Du\mathbf{1}_{u<0}$. We have $\int u^+u^- =$ and $B[u^+, u^-] = 0$, so

$$\lambda_1 = B[u,u] = B[u^+,u^+] + B[u^-,u^-] \geq \lambda_1 \|u^+\|_{L^2(U)} + \lambda_1 \|u^-\|_{L^2(U)} = \lambda_1$$

since $\|u\|^2 = \|u^+\|^2 + \|u^-\|^2$. Whence $Lu^+ = \lambda_1 u^+ \geq 0$. By regularity, $u^+ \in C^2(U) \cap C(\overline{U})$, and by strong maximum principle either $u^+ > 0$ in U or $u^+ = 0$. It follows in either case that u has no zeros in U. Note that if we have two positive eigenfunctions u and u' then we can choose σ so that $\int u + \sigma u' = 0$. Such a linear combination is an eigenfunction, so is always positive (or always negative), so it must be the case that u and u' are linearly dependent.

We have the following formula due to Rayleigh

$$\lambda_k = \min_{\substack{u\in H_0^1(U) \\ u\neq 0 \\ u\perp w_i, i<k}} \frac{B[u,u]}{\|u\|^2_{L^2(U)}}$$

□

2.6 Non-symmetric elliptic operators

In finite dimensions there is a classical theorem of Frobenius that says that the smallest real positive eigenvector of a (not necessarily symmetric) real matrix has an eigenvector with all positive coefficients.

2.6.1 Theorem. *Let $Lu = a^{ij}u_{x_ix_j} + b^iu_{x_i} + cu$, where the coefficients are in $C^\infty(U)$ and A is positive but not necessarily symmetric. We consider the problem $Lu = \lambda u$ in U and $u = 0$ on ∂U.*

(i) There is $\lambda_1 \in \mathbb{R}$ that is an eigenvalue of L, the corresponding eigenspace is one-dimensional, and the corresponding unit eigenvector may be chosen to be strictly positive in U.

(ii) $\Re\lambda \geq \lambda_1$ for all eigenvalues λ.

See Evans for the proof.

3 Parabolic PDE

3.1 Bochner Integral

See Appendix E5 of Evans for a list of the results. There is a good reference by Yoshida, *Functional Analysis*.

3.1.1 Definition. Let X be a Banach space.

(i) A function $s : [0,T] \to X$ is said to be a *simple function* if it can be written $s(t) = \sum_{i=1}^m \mathbf{1}_{E_i}(t)u_i$, where $\{E_i\}_{i=1}^m$ is a measurable partition of $[0,T]$ and $u_i \in X$.

(ii) A function $f : [0,T] \to X$ is said to be *strongly measurable* if there is a sequence of simple functions $\{s_k\}$ such that $s_k \to f$ a.e.-$[0,T]$, and *weakly measurable* if for all $u' \in X'$ the function $u'(f(\cdot)) : [0,T] \to \mathbb{R}$ is measurable.

(iii) A function $f : [0,T] \to X$ is said to be *almost separably valued* if there is $N \subseteq [0,T]$ with $|N| = 0$ such that $f([0,T] \setminus N)$ is separable.

Remark. If X is separable then any function is ASV, and if a function is continuous then it is ASV.

3.1.2 Theorem (Pettis). *$f : [0,T] \to X$ is weakly measurable and ASV if and only if f is strongly measurable.*

3.1.3 Definition.

(i) If $s = \sum_{i=1}^m \mathbf{1}_{E_i} u_i$ is simple then

$$\int_0^T s(t)\,dt := \sum_{i=1}^m |E_i| u_i.$$

(ii) We say that $f : [0,T] \to X$ is *summable* (or *integrable*) if there is a sequence $\{s_k\}$ of simple functions such that

$$\int_0^T \|s_k(t) - f(t)\|_X \,dt \to 0 \text{ as } k \to \infty.$$

We then define $\int_0^T f(t)\,dt := \lim_{k\to\infty} \int_0^T s_k(t)\,dt$.

3.1.4 Theorem (Bochner). *A strongly measurable function f is summable if and only if $t \mapsto \|f(t)\|_X$ is summable. In this case*

(i) $\|\int_0^T f(t)\,dt\|_X \le \int_0^T \|f(t)\|_X \,dt$; *and*

(ii) $u'(\int_0^T f(t)\,dt) = \int_0^T u'(f(t))\,dt$ *for all* $u' \in X'$.

3.1.5 Theorem (Fatou). *Let* $\{f_n\}$ *be summable and suppose* $f_n \to f$ *a.e.-*$[0,T]$ *and* $\{\int_0^T \|f_n(t)\|_X\,dt\}$ *is bounded. Then* f *is summable and*

$$\int_0^T \|f(t)\|_X\,dt \le \liminf_n \int_0^T \|f_n(t)\|_X\,dt.$$

3.1.6 Theorem (LDCT). *Let* $\{f_n\}$ *be summable and suppose* $f_n \to f$ *a.e.-*$[0,T]$ *and there is* $g : [0,T] \to \mathbb{R}$ *integrable such that* $\|f_n(s)\|_X \le g(s)$ *a.e.-*$[0,T]$*. Then* f *is summable and*

$$\int_0^T \|f(t) - f_n(t)\|_X\,dt \to 0 \text{ as } n \to \infty.$$

In particular $\int_0^T f_n(t)\,dt \to \int_0^T f(t)\,dt$ *as* $n \to \infty$.

3.2 Spaces involving time

This section follows §5.9 in Evans.

3.2.1 Definition. For $1 \le p < \infty$, $L^p(0,T,X)$ is the set of strongly measurable functions $u : [0,T] \to X$ such that

$$\|u\|_{L^p(0,T,X)} := \left(\int_0^T \|u(t)\|_X^p\,dt \right)^{\frac{1}{p}} < \infty.$$

$L^\infty(0,T,X)$ is defined in the obvious way. $C([0,T],X)$ is the set of continuous functions $[0,T] \to X$.

If X is separable and $p < \infty$ then $L^p(0,T,X)$ is separable.

3.2.2 Theorem (Phillips). *For* $1 < p < \infty$*,* $L^p(0,T,X)' = L^q(0,T,X')$.

The derivative of $f : [0,T] \to X$ is defined in the usual way by

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

provided the limit of the difference quotients exists.

3.2.3 Theorem. *If* $f : [0,T] \to X$ *is summable and* $F(t) := \int_0^t f(s)\,ds$ *then* $F'(t) = f(t)$ *a.e.-*$[0,T]$.

3.2.4 Definition. For $u \in L^1(0,T,X)$ we say that $v \in L^1(0,T,X)$ is the *weak derivative in time* of u and write $v = u'$ if

$$\int_0^T u(t)\varphi'(t)\,dt = -\int_0^T v(t)\varphi(t)\,dt$$

for all $\varphi \in C_c^\infty((0,T),\mathbb{R})$.

Remark. Consider $X = L^1(\mathbb{R})$. If $u \in L^1(0,T,L^1(\mathbb{R}))$ then there is an associated $\tilde{u} \in L^1([0,T]\times\mathbb{R})$, and u' corresponds in a natural way to $(\tilde{u})_t$ (see problem set 5).

3.2.5 Definition. $W^{1,p}(0,T,X)$ is the space of all $u \in L^p(0,T,X)$ such that u' exists and $u' \in L^p(0,T,X)$. In this case define

$$\|u\|_{W^{1,p}(0,T,X)} := \left(\int_0^T \|u(t)\|_X^p + \|u'(t)\|_X^p\,dt\right)^{\frac{1}{p}}.$$

We write $H^1(0,T,X)$ for $W^{1,2}(0,T,X)$.

3.2.6 Theorem (Calculus I). *Let $u \in W^{1,p}(0,T,X)$, $1 \le p \le \infty$. Then*

(i) $u \in C([0,T],X)$;

(ii) $u(t) = u(s) + \int_s^t u'(r)\,dr$ for all $0 \le s \le t \le T$; and

(iii) there is $C = C(T)$ such that

$$\max_{0\le t\le T} \|u(t)\|_X \le C\|u\|_{W^{1,p}(0,T,X)}.$$

Proof: Define $\tilde{u} : \mathbb{R} \to X$ by extending u by 0 outside of $(0,T)$. For $\varepsilon > 0$ define $u_\varepsilon = \eta_\varepsilon * \tilde{u}$. By the same methods as before on mollifiers, it can be checked that $(u_\varepsilon)' = \eta_\varepsilon * (u')\tilde{} = (u')_\varepsilon$, and $u_\varepsilon \to u$ and $u'_\varepsilon \to u'$ in $L^p_{loc}(0,T,X)$ as $\varepsilon \to 0$. For every small $\varepsilon > 0$ we have

$$u_\varepsilon(t) - u_\varepsilon(s) = \int_s^t u'_\varepsilon(r)\,dr.$$

Apply pointwise convergence and convergence in L^1 to conclude the second part, from which the others follow. □

3.2.7 Theorem (Calculus II). *Suppose that $u \in L^2(0,T,H^1_0(U))$ and $u' \in L^2(0,T,H^{-1}(U))$. Then*

(i) $u \in C([0,T],L^2(U))$;

(ii) the mapping $t \mapsto \|u(t)\|_{L^2}$ is absolutely continuous and

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2\langle u'(t),u(t)\rangle;$$

(iii) there is $C = C(T)$ such that

$$\max_{0\le t\le T} \|u(t)\|_{L^2} \le C(\|u\|_{L^2(0,T,H^1_0)} + \|u'\|_{L^2(0,T,H^{-1})}).$$

PROOF: Let $u_\varepsilon := \eta_\varepsilon * u$. For $\varepsilon > 0$ and $\delta > 0$ and appropriate t,

$$\begin{aligned}\frac{d}{dt}\|u_\varepsilon(t)-u_\delta(t)\|^2_{L^2} &= \frac{d}{dt}\int_U (u_\varepsilon(t)-u_\delta(t))^2 dx \\ &= 2\int_U (u'_\varepsilon(t)-u'_\delta(t))(u_\varepsilon(t)-u_\delta(t))\,dx \\ &= 2(u'_\varepsilon(t)-u'_\delta(t), u_\varepsilon(t)-u_\delta(t))_{L^2} \\ &= 2\langle u'_\varepsilon(t)-u'_\delta(t), u_\varepsilon(t)-u_\delta(t)\rangle \\ &\le 2\|u'_\varepsilon(t)-u'_\delta(t)\|_{H^{-1}}\|u_\varepsilon(t)-u_\delta(t)\|_{H^1_0}\end{aligned}$$

It is a property of mollifiers that $u_\varepsilon \to u$ a.e., so there is $s \in (0,T)$ such that $u_\varepsilon(s) \to u(s)$ in $H^1_0(U)$. For all $t \in [0,T]$,

$$\begin{aligned}&\|u_\varepsilon(t)-u_\delta(t)\|^2_{L^2(U)} - \|u_\varepsilon(s)-u_\delta(s)\|^2_{L^2(U)} \\ &\qquad\le 2\int_0^T \|u'_\varepsilon(r)-u'_\delta(r)\|_{H^{-1}}\|u_\varepsilon(r)-u_\delta(r)\|_{H^1_0} dr \\ &\qquad\le \int_0^T \|u'_\varepsilon(r)-u'_\delta(r)\|^2_{H^{-1}} + \|u_\varepsilon(r)-u_\delta(r)\|^2_{H^1_0} dr \\ &\qquad= \|u'_\varepsilon - u'_\delta\|^2_{L^2(0,T,H^{-1})} + \|u_\varepsilon - u_\delta\|^2_{L^2(0,T,H^1_0)}\end{aligned}$$

But three of these terms go to zero as $\varepsilon,\delta \to 0$, so it is seen that $u_\varepsilon \to u$ in $L^2(0,T,L^2(U))$ as $\varepsilon \to 0$, by completeness.

For the second assertion, as above

$$\frac{d}{dt}\|u_\varepsilon(t)\|^2_{L^2(U)} = 2\langle u'_\varepsilon(t), u_\varepsilon(t)\rangle.$$

Then

$$\|u_\varepsilon(t)\|^2_{L^2} = \|u_\varepsilon(s)\|^2_{L^2} + 2\int_s^t \langle u'_\varepsilon(t), u_\varepsilon(t)\rangle\,dt$$

and taking $\varepsilon \to 0$ we get

$$\|u(t)\|^2_{L^2} = \|u(s)\|^2_{L^2} + 2\int_s^t \langle u'(t), u(t)\rangle\,dt$$

□

3.2.8 Theorem (Lions-Aubin). *Let X, Y, and Z be Banach spaces with $X \subset\subset Y \hookrightarrow Z$. Let $1 < p < \infty$, $1 < q < \infty$ and*

$$W = \{u \in L^p(0,T,X), u' \in L^q(0,T,Z)\}$$

Then $W \subset\subset L^p(0,T,Y)$.

In particular $\{u \in L^2(0,T,H^1_0(U)), u' \in L^2(0,T,H^{-1}(U))\} \subset\subset L^2(0,T,L^2(U))$.

3.3 Parabolic equations

Let $U_T := U \times (0,T]$. From now on we identify $L^2(U_T)$ with $L^2(0,T,L^2(U))$ without note.

$$(P)\ \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{0\}, \end{cases}$$

where $Lu := -(a^{ij}u_{x_i})_{x_j} + b^i u_{x_i} + cu$ is uniformly elliptic and in divergence form, $a^{ij}, b^i, c \in L^\infty(U_T)$ are bounded (and in particular may now depend on time) $f \in L^2(U_T)$, and $g \in L^2(U)$.

The associated *time-dependent bilinear form* is

$$B[u,v,t] := \int_U a^{ij}(x,t)u_{x_i}(x)v_{x_j}(x) + b^i(x,t)u_{x_i}(x)v(x) + c(x,t)u(x)v(x)\,dx$$

for $u, v \in H^1(U)$.

3.3.1 Definition. For $u \in L^2(0,T,H_0^1(U))$ with $u' \in L^2(0,T,H^{-1}(U))$, u is a *weak solution* to (P) if, a.e.-$[0,T]$,

$$\langle u'(t), v\rangle + B[u(t),v,t] = \int_U f(x,t)v(x)\,dx$$

for all $v \in H_0^1(U)$, and $u(0) = g$.

The last condition is well-posed because we have seen that such u lie in $C([0,T],L^2(U))$. We will often write $B[u,v,t] := B[u(t),v,t]$ when no confusion could arise.

3.3.2 Theorem (Uniqueness). *There is at most one weak solution of (P).*

PROOF: By linearity it suffices to assume that $f \equiv 0$ and $g \equiv 0$. If u is a weak solution then $\frac{d}{dt}\|u\|_{L^2}^2 = 2\langle u', u\rangle$, so for every $t \in [0,T]$,

$$\begin{aligned} \frac{d}{dt}(\|u\|_{L^2}^2)(t) &= 2\langle u'(t), u(t)\rangle \\ &= -2B[u(t),u(t),t] \\ &\leq -2\beta\|u(t)\|_{H_0^1}^2 + 2\gamma\|u(t)\|_{L^2}^2 \qquad \text{by Energy Estimates} \\ &\leq 2\gamma\|u\|_{L^2}^2(t) \end{aligned}$$

By Gronwall's lemma $\|u(t)\|_{L^2}^2 \leq e^{2\gamma t}\|u(0)\|_{L^2} = 0$. □

3.3.3 Theorem (Energy Esimates). *There is $C = C(L,T,U)$ such that for every weak solution u of (P),*

$$\max_{t\in[0,T]} \|u(t)\|_{L^2} + \|u\|_{L^2(0,T,H_0^1)} + \|u'\|_{L^2(0,T,H^{-1})} \leq C(\|f\|_{L^2(0,T,L^2)} + \|g\|_{L^2}).$$

PROOF: We have $u_t + Lu = f$, so by the definition of weak solution (using $u(t)$ is a test function) and as above,

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|^2_{L^2(U)}\right)(t) + B[u(t), u(t), t] = \int_U f(x,t)u(x,t)\,dx.$$

Whence, again by Energy Estimates,

$$\|u(t)\|^2_{L^2(U)} \le 2\|g\|^2_{L^2(U)} + \|f\|^2_{L^2(U_T)} + (2\gamma + 1)\int_0^t \|u(s)\|^2_{L^2(U)} ds,$$

so by Gronwall's lemma

$$\max_{t\in[0,T]} \|u(t)\|^2_{L^2(U)} \le e^{(2\gamma+1)T}(\|f\|^2_{L^2(U_T)} + \|g\|^2_{L^2(U)}),$$

and it follows that

$$\max_{t\in[0,T]} \|u(t)\|_{L^2(U)} \le C(\|f\|_{L^2(U_T)} + \|g\|_{L^2(U)}).$$

(Fill in the rest of this.) □

3.4 Existence via Galerkin Approximation

Let $\{w_k\}$ be an orthonormal basis for $L^2(U)$ which is also orthogonal in $H^1_0(U)$. (The collection of eigenfunctions corresponding to an elliptic problem has these properties, using the bilinear form associated with the problem as the inner product on H^1_0. For the Laplacian operator the corresponding collection is orthogonal with respect to the usual inner product on H^1_0.) Write

$$u_m(t) = \sum_{k=1}^m d^k_m(t) w_k = d^k(t)_m w_k,$$

where $d^k_m(0) = \int_U g(x)w_k(x)\,dx$ and

$$(u'_m(t), w_k) + B[u_m(t), w_k, t] = (f(t), w_k)$$

for all t and $k = 1, \dots, m$. (i.e., we require that u_m is a weak solution to (P) on the subspace of H^1_0 generated by $\{w_1, \dots, x_m\}$.)

3.4.1 Lemma. *For every $m > 0$ there is a unique u_m of the form above.*

PROOF: Suppose there such a u_m. Since w_k does not depend on time, $u'_m(t) = (d^k_m)'(t) w_k$. By the condition that u_m is a solution

$$(d^m_k)'(t)(w_k, w_j) + d^k_m(t)B[w_k, w_j, t] = (f(t), w_j)$$

for all $j = 1, \dots, m$. Let $e_{kj}(t) = B[w_k, w_j, t]$ and $f_j(t) = (f(t), w_j)$, so that for each j,

$$(d_m^j)'(t) + d_m^k(t) e_{kj}(t) = f_j.$$

The e_{kj} are bounded in time, so by the standard theory from ODE, there is a unique absolutely continuous system of solutions $\{d_m^k, k = 1, \dots, m\}$. This proves existence and uniqueness of u_m. □

3.4.2 Theorem. *There is a constant $C = C(U, L, T)$ (not depending on m) such that*

$$\max_{t \in [0,T]} \|u_m(t)\|_{L^2} + \|u_m\|_{L^2(0,T,H_0^1)} + \|u_m'\|_{L^2(0,T,H^{-1})} \le C(\|f\|_{L^2(U_T)} + \|g\|_{L^2}).$$

PROOF: The proof is very similar to the proof of Energy Estimates above. □

3.4.3 Theorem. *There is a unique weak solution to (P).*

PROOF: Uniqueness has already been proved.

Since $L^2(0, T, H_0^1(U))$ and $L^2(0, T, H^{-1}(U))$ are reflexive there is a subsequence (after relabelling) u_m converging weakly to some $u \in L^2(0, T, H_0^1)$ such that u_m' converges weakly to some $w \in L^2(0, T, H^{-1}(U))$. By the homework $w = u'$. We must show that $\langle u', v\rangle + B[u, v, t] = (f, v)$ for all $v \in H_0^1(U)$. For v of the form $\sum_{i=1}^M \alpha_i w_i$, for $m > M$, we have

$$\int_0^T \langle u_m', v\rangle + B[u_m, v, t]\, dt = \int_0^T (f, v)\, dt.$$

Passing to the limit,

$$\int_0^T \langle u', v\rangle + B[u, v, t]\, dt = \int_0^T (f, v)\, dt$$

for all such v. But such v are dense in $H_0^1(U)$, so this holds for all v. □

3.5 Maximum principles

Let $\Gamma_T = (\partial U \times [0, T]) \cup (U \times \{0\})$. Consider the problem

$$\begin{cases} u_t + Lu + h(u) = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{0\} \end{cases}$$

where $h : \mathbb{R} \to \mathbb{R}$ is non-decreasing, and $Lu = -a^{ij} u_{x_i x_j} + b^i u_{x_i}$ (i.e. $c = 0$).

3.5.1 Definition. We say that $u \in C^{2,1}(U \times [0, T]) \cap C(\overline{U_T})$ is a *sub-solution* if all equalities are replaced by $\le$, and a *super-solution* if all equalities are replaced by $\ge$.

3.5.2 Theorem (Comparison Principle). *Let u be a sub-solution and v be a super-solution. If $u \leq v$ on Γ_T then $u \leq v$ in U_T.*

PROOF: Assume that u is a strict sub-solution, so that $u_t + Lu + h(u) < f$. Assume the conclusion of the theorem is false, and let $(x_0, t_0) \in \overline{U_T}$ be a point where $\max_{\overline{U_T}} u - v > 0$ reaches its maximum. Since $u \leq v$ on Γ_T, we must have $x_0 \in U$ and $t_0 \in (0, T]$. If $t \in (0, T)$ then at (x_0, t_0) we have $Du = Dv$, $u_t = v_t$, $D^2u \leq D^2v$, and $u > v$. Adding all this up, $u_t + Lu + h(u) \geq v_t + Lv + h(v)$ at (x_0, t_0), contradicting $u_t + Lu + h(u) < f \leq v_t + Lv + h(v)$. If $t_0 = T$, then $u_t(x_0, T) \geq v_t(x_0, T)$ and the same contradiction follows.

If u is not a strict sub-solution then consider $u^\varepsilon(x, t) := u(x, t) - \varepsilon t$. Then $u^\varepsilon \leq u$ and u^ε is a strict sub-solution. It follows that $u^\varepsilon \leq v$ on $\overline{U_T}$ for every $\varepsilon > 0$, so $u \leq v$. □

3.5.3 Theorem (Weak maximum principle). *If $u_t + Lu \leq 0$ in U_T then $\max_{\overline{U_T}} u = \max_{\Gamma_T} u$, and the same holds if $(\leq, \max)$ are replaced by $(\geq, \min)$.*

PROOF: Follows from the Comparison Principle. □

Index