From $k+1$ to k heads the descriptive trade-off is non-recursive

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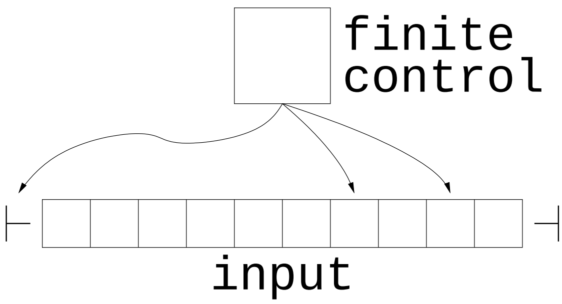
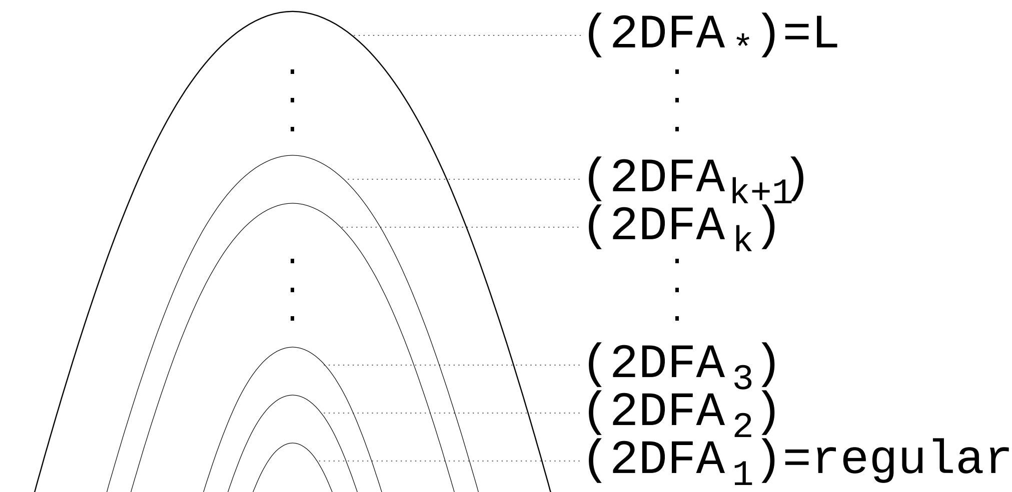
MACHINE MODEL

2DFA$_k$: two-way deterministic finite automaton with k heads

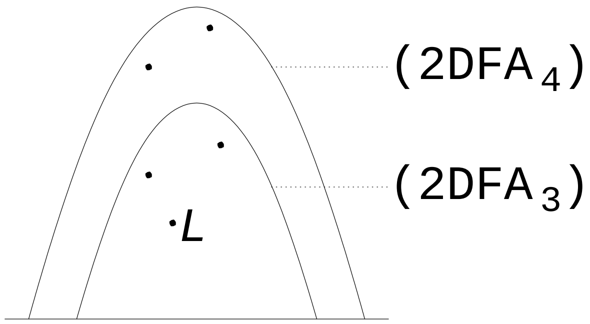
e.g., for $k = 3$:



capabilities:



THE GAP FOR $k = 3$



For $L \in (\text{2DFA}_3)$: $M_3 =$ a smallest 2DFA_3 for L
$M_4 =$ a smallest 2DFA_4 for L

We should have: $|M_4| \leq |M_3|$.

But how larger? $|M_4| \leq |M_3| \leq 2|M_4|$?

$|M_4| \leq |M_3| \leq |M_4|^2$?

$|M_4| \leq |M_3| \leq 2^{|M_4|}$?

QUESTION: If $|M_4| \leq |M_3| \leq f\big(|M_4|\big)$,
how fast does f need to grow?

ANSWER: It must grow so fast that
we cannot even compute it.

IF WE COULD COMPUTE f...

...then we could semidecide the inadequacy of 3 heads:

H_3 = "Given M a $2DFA_4$: is it true that
no $2DFA_3$ solves the same problem as M?"

which is *not* semidecidable.

Need to prove two things:

i. f computable $\Rightarrow$ H_3 semidecidable

ii. H_3 *not* semidecidable

f COMPUTABLE $\Rightarrow$H$_3$ SEMIDECIDABLE

Given M a 2DFA$_4$:

1. Compute $|M| = 53$.
2. Compute $f(|M|) = f(53) = 1013$.
3. Compute the list

$$D_1, D_2, D_3, \ldots, D_{731}$$

   of all 2DFA$_3$'s of size ≤ 1013 and M's alphabet.
4. Check that M is disagrees with every D_i:

   for all inputs x:

   — cross out every D_i such that $D_i(x) \neq M(x)$.

   — if the list of D_i's got empty, *accept*.

M has no equivalent 2DFA$_3$
  $\Rightarrow$ list eventually gets empty
  $\Rightarrow$ we accept

M has an equivalent 2DFA$_3$
  $\Rightarrow$ list contains one [$*$]
  $\Rightarrow$ we loop forever

Hence: if we could compute f, we could semidecide H$_3$

H_3 NOT SEMIDECIDABLE

We prove:

$$\overline{\mathsf{HALT}} \leq \mathsf{E} \leq \mathsf{H}_3$$

$\overline{\mathsf{HALT}}$ = "Given M a TM: is it true that M loops on $\langle M \rangle$?"

E = "Given M a *terminating* $2DFA_2^{u}$ that *obeys a threshold*: is it true that the language of M is empty?"

H_3 = "Given M a $2DFA_4$: is it true that M has no equivalent $2DFA_3$?"

H_3 NOT SEMIDECIDABLE: What's E?

The following problem is *not semidecidable* (known):

E_{TM} = "Given M a TM: is it true that the language of M is empty?"

What if, instead of a TM, the input machine M is:

— just a multihead 2DFA?

— ...with exactly 2 heads?

— ...and unary input alphabet?

— ...and promised to always halt?

— ...and promised to obey a threshold?

for some $l \leq \infty$: $L(M) = \{ \, x \mid l \leq |x| \, \}$

Does the problem get any easier?

E = "Given M a *terminating* $2DFA_2^u$ that *obeys a threshold*: is it true that the language of M is empty?"

E NOT SEMIDECIDABLE: $\overline{\text{HALT}} \leq \text{E}$

M a TM $\longrightarrow$ [?] $\longrightarrow$ M' a 2DFA_2^{u}, *terminating* and *obeys a threshold*

OUTLINE: M a TM

- $\longrightarrow$ A a unary LBA
- $\longrightarrow$ B a 3-counter automaton
- $\longrightarrow$ C a 2-counter automaton
- $\longrightarrow$ M' a 2DFA_2^{u}

Each of A, B, C, M' will be *terminating* and *obey a threshold*.

If M loops on its description:

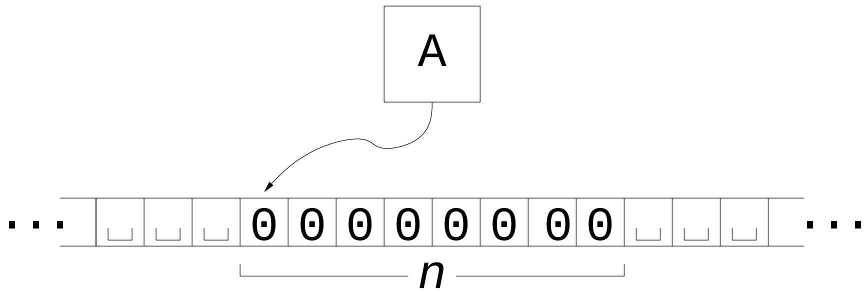
$$L(A) = L(B) = L(C) = L(M') = \emptyset$$

If M halts on its description:

$$L(A),\ L(B),\ L(C),\ L(M') \neq \emptyset$$

E NOT SEMIDECIDABLE: $\boxed{M \to A} \to B \to C \to M'$

M a TM $\longrightarrow$ A a unary LBA,
terminating and
obeys a threshold



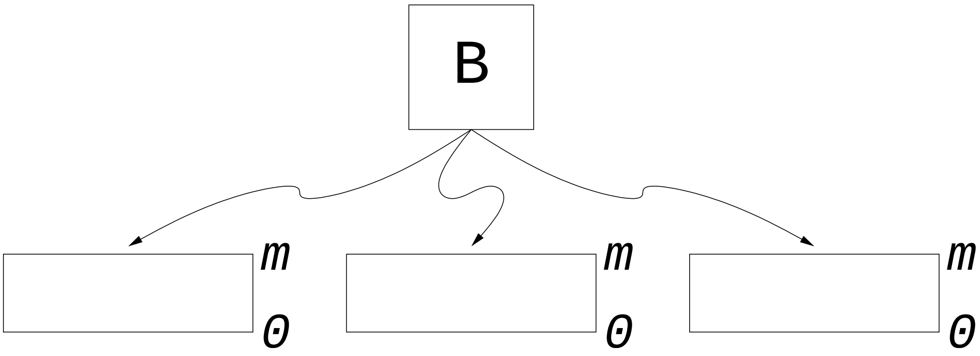
input alphabet$= \{0\}$
tape alphabet$= \{\sqcup, 0, 1, \dot{0}, \dot{1}\}$

$A =$ "On input 0^n:
1. For all $x \in \{0,1\}^n$:
   — if x is an accepting computation history of M on its description, *accept*.
2. *Reject*."

M loops on its description $\Rightarrow$ $L(A) = \emptyset$
M halts on its description $\Rightarrow$ $L(A) \neq \emptyset$

E NOT SEMIDECIDABLE: $M \to \boxed{A \to B} \to C \to M'$

A a unary LBA, *terminating* and *obeys a threshold* $\longrightarrow$ B a DCA$_3$, *terminating* and *obeys a threshold*



input$=m$ (upper bound for the counters)

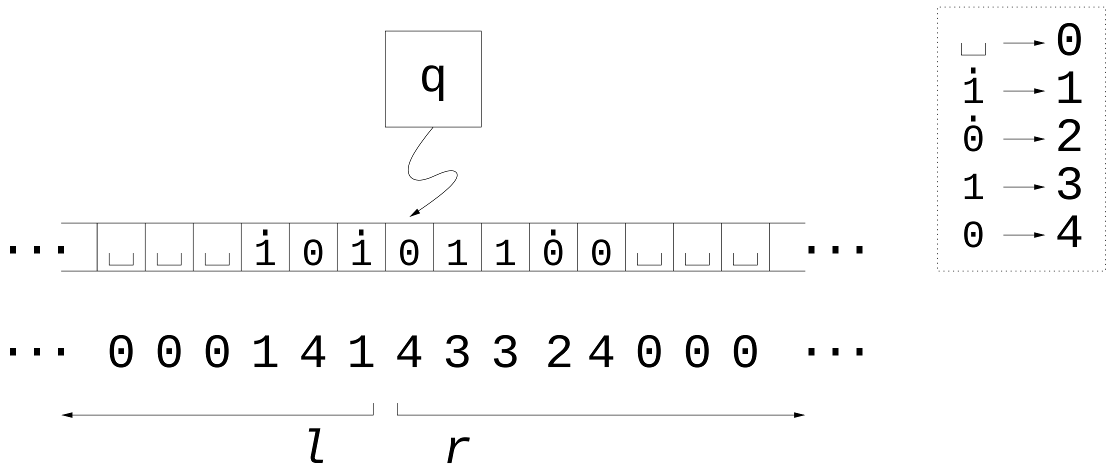
$B =$ "On input m:

— simulate A on input $\lg_5 \lg_{30} m$."

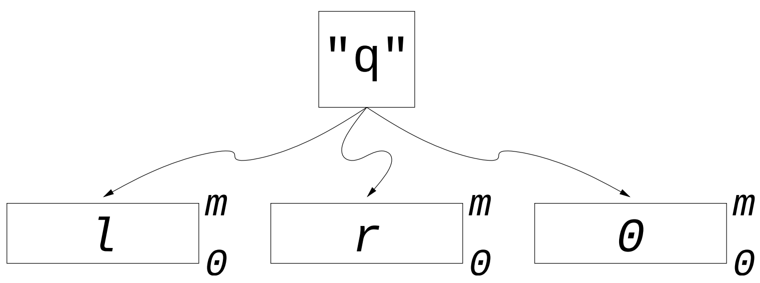
E NOT SEMIDECIDABLE: $M \to \boxed{A \to B} \to C \to M'$

B on input m simulates A on input $\lg_5 \lg_{30} m$

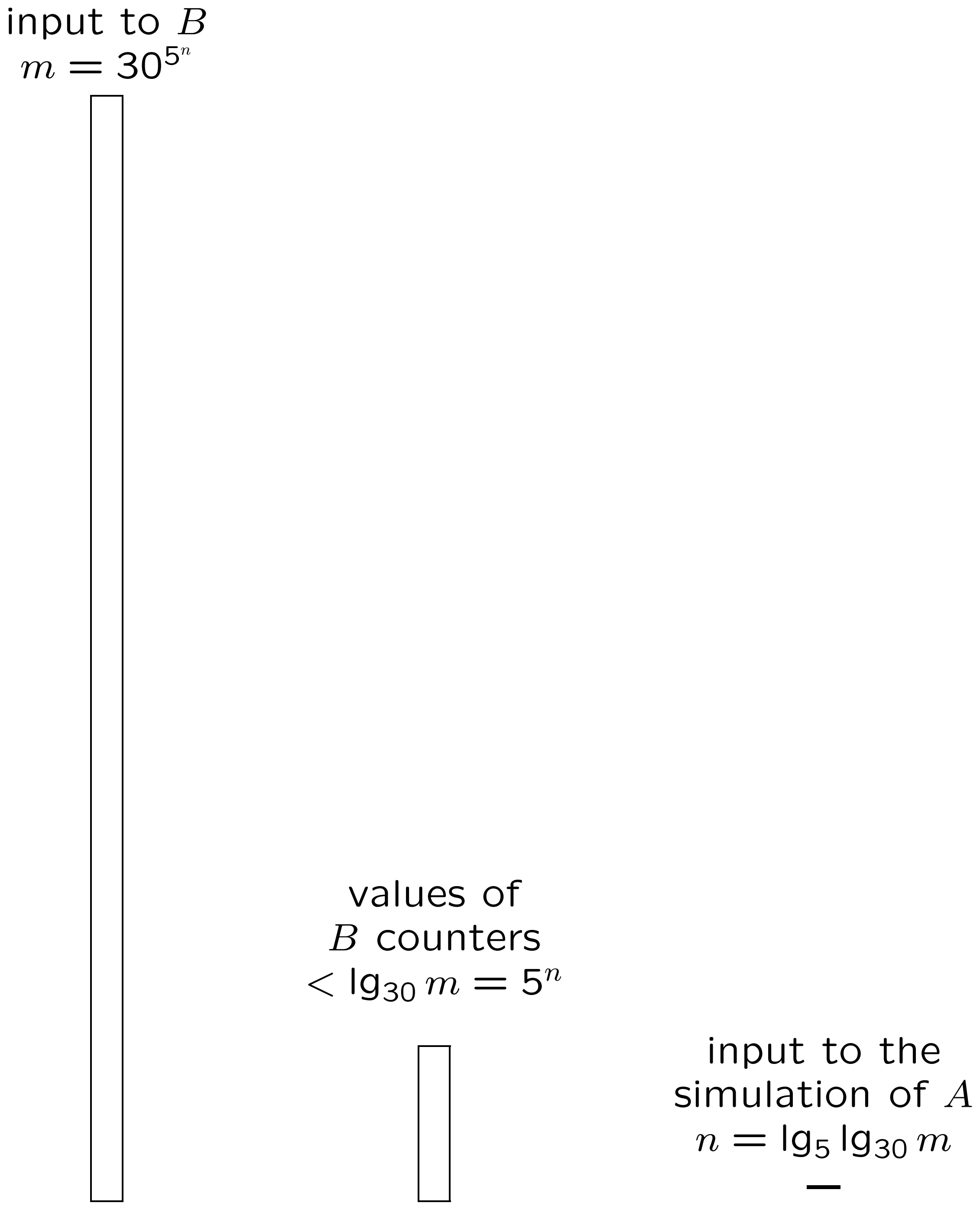
when A on input $n = \lg_5 \lg_{30} m$ is in configuration:



then B on input $m = 30^{5^n}$ is in configuration:

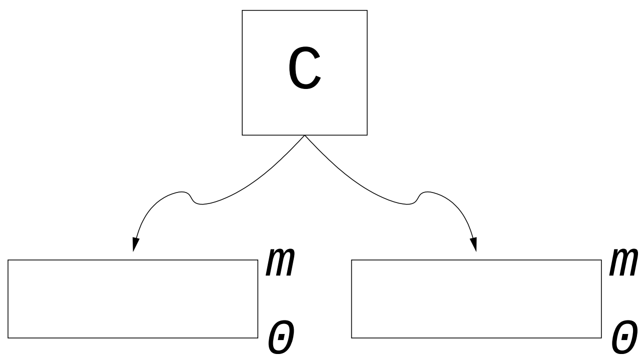


E NOT SEMIDECIDABLE: $M \rightarrow \boxed{A \rightarrow B} \rightarrow C \rightarrow M'$



E NOT SEMIDECIDABLE: $M \to A \to \boxed{B \to C} \to M'$

B a DCA$_3$, *terminating* and *obeys a threshold*	$\longrightarrow$	C a DCA$_2$, *terminating* and *obeys a threshold*



input$=m$ (upper bound for the counters)

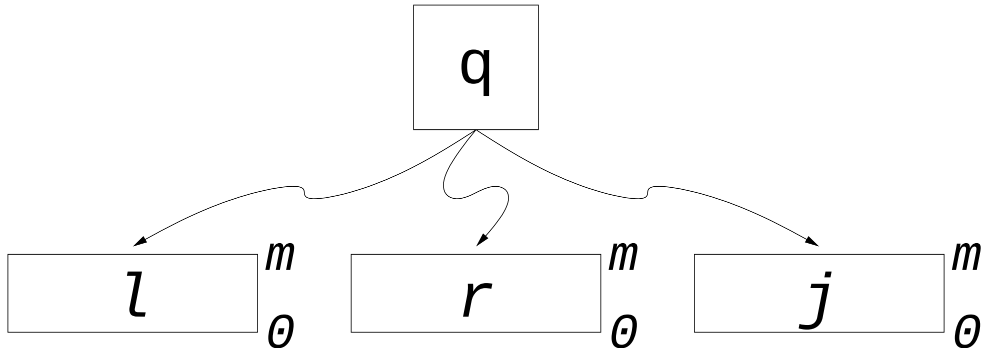
$C =$ "On input m:

— simulate B on input m."

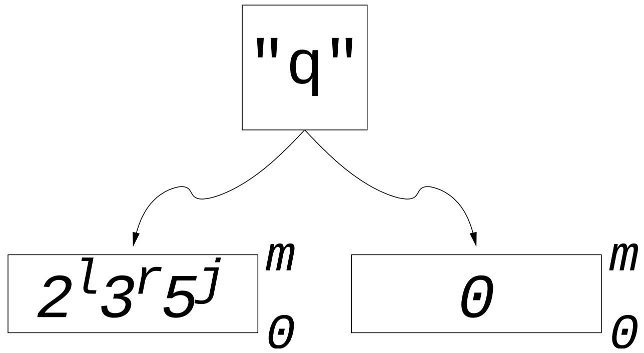
E NOT SEMIDECIDABLE: $M \to A \to \boxed{B \to C} \to M'$

C on input m simulates B on input m

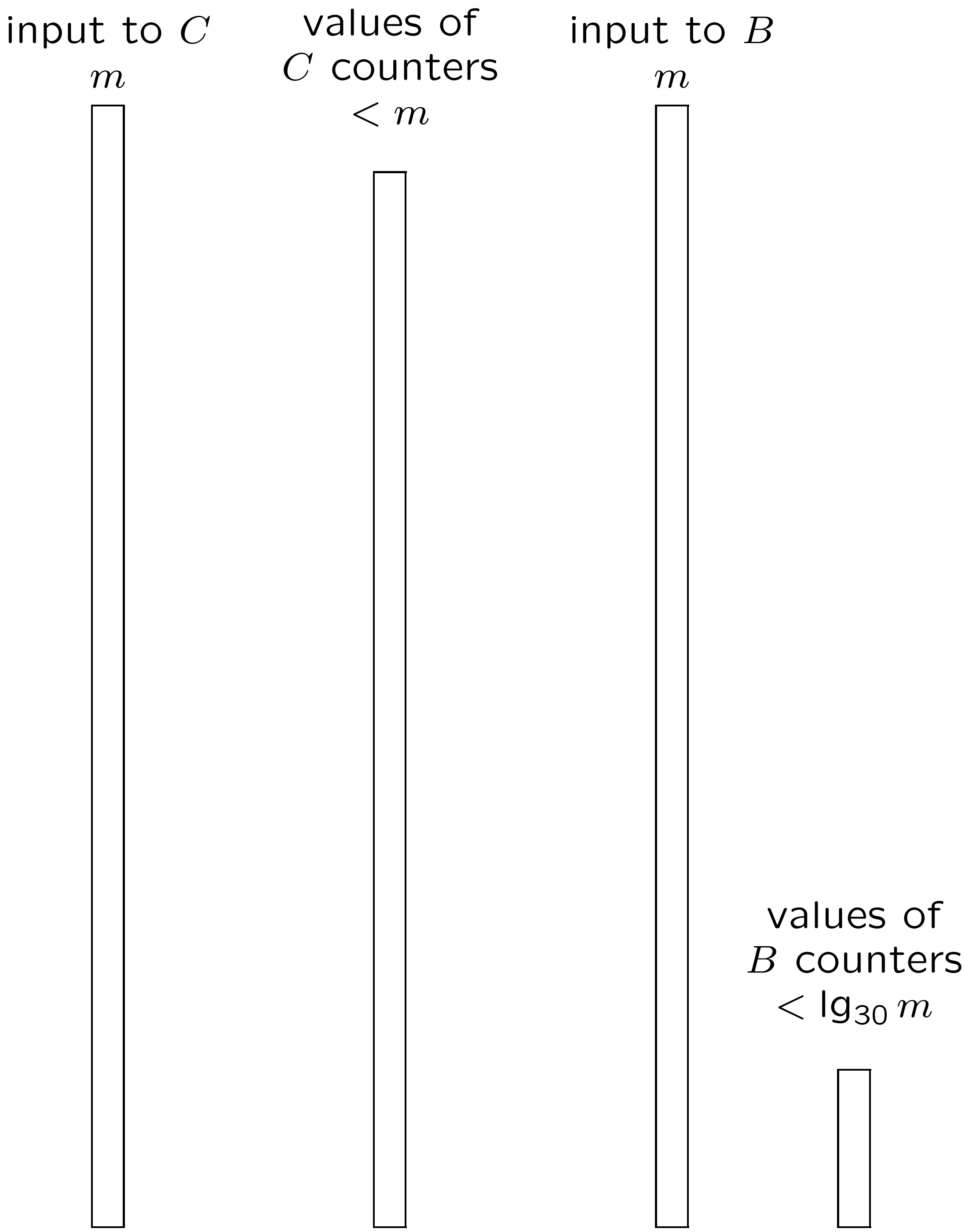
when B on input m is in configuration:



then C on input m is in configuration:



E NOT SEMIDECIDABLE: $M \to A \to \boxed{B \to C} \to M'$

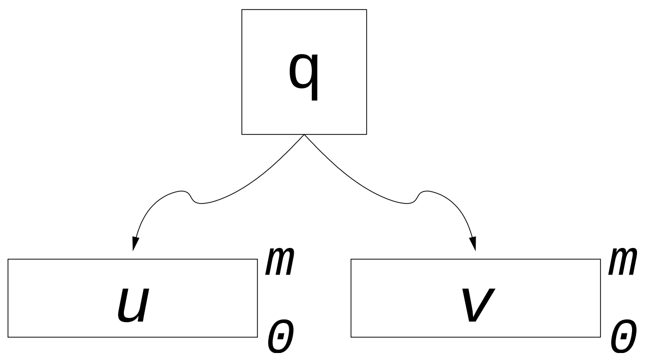


E NOT SEMIDECIDABLE: $M \to A \to B \to \boxed{C \to M'}$

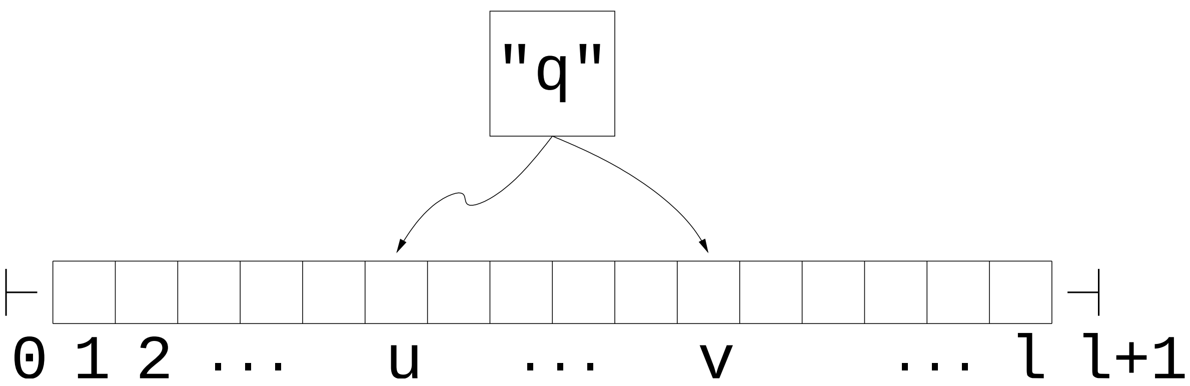
C a DCA_2, *terminating* and *obeys a threshold* $\longrightarrow$ M' a 2DFA_2^{u}, *terminating* and *obeys a threshold*

M' on input l simulates C on input $l+1$

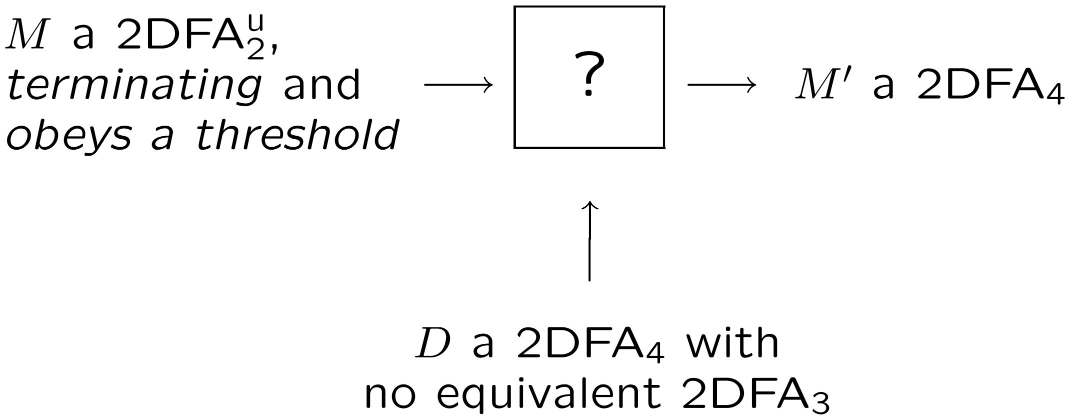
when C on input $m = l+1$ is in configuration:



then M' on input $l = m - 1$ is in configuration:



H_3 NOT SEMIDECIDABLE: $E \leq H_3$



$M' =$ "On input x:
1. Run M. If it accepts, *accept*.
2. Run D."

$L(M) = \emptyset$
$\Rightarrow$ $L(M') = L(D)$
$\Rightarrow$ M' has no equivalent $2DFA_3$

$L(M) \neq \emptyset$
$\Rightarrow$ $L(M) =$ exactly all sufficiently long strings
$\Rightarrow$ $L(M')$ cofinite, hence regular
$\Rightarrow$ M' has equivalent $2DFA_3$

OVERVIEW

Some *computable* f is such that ,

$$|M_4| \leq |M_3| \leq f\big(|M_4|\big)$$

for all $L \in$ (2DFA$_3$)

$\Rightarrow$ H_3 is semidecidable [Hartmanis71]

$\Rightarrow$ E is semidecidable [Kutrib03]

$\Rightarrow$ $\overline{\text{HALT}}$ is semidecidable [because. . .]

$\Rightarrow$ false

Because: Given M a TM

$\longrightarrow$ A a *terminating* unary LBA
that *obeys a threshold*

$\longrightarrow$ B a *terminating* DCA$_3$ [Minsky61]
that *obeys a threshold*

$\longrightarrow$ C a *terminating* DCA$_2$ [Wang57]
that *obeys a threshold*

$\longrightarrow$ M' a *terminating* 2DFA$^{\mathsf{u}}_2$
that *obeys a threshold*

M loops on $\langle M\rangle \;\Rightarrow\; L(M') = \emptyset$
M halts on $\langle M\rangle \;\Rightarrow\; L(M') \neq \emptyset$

CONCLUSION

THEOREM.
Replacing a $2DFA_4$ with an equivalent $2DFA_3$ causes a blow-up in description size that only non-recursive functions can bound.

. . . where nothing is special about
— the 3rd gap,
— determinism, or
— the cardinality of the input alphabet.

THEOREM.
For any $k \geq 1$, and $X \in \{2DFA, 2NFA, 2DFA^u, 2NFA^u\}$: Replacing a X_{k+1} with an equivalent X_k causes a blow-up in description size that only non-recursive functions can bound.

COROLLARY: Same for other types of automata.