Logic in topoi:
Functorial semantics for higher-order logic

Dissertation Abstract

Steven Awodey
The University of Chicago

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This dissertation investigates what may be termed the model theory of higher-order logic using the methods of category theory. Of course, there is no such field of logic as "higher-order model theory," and so our first concern in chapter I will be to specify the basic objects under investigation, viz. higher-order logical theories and their models. This is a fairly straightforward generalization—in two different directions—of the familiar corresponding notions for first-order logic: the notion of a logical theory is generalized from first- to higher-order logic, and the notion of a model is generalized both from first- to higher-order logic, and from the category **Sets** to arbitrary topoi.[1] The remaining four chapters of the dissertation are then devoted to what may be termed 'functorial semantics for higher-order logic,' or 'topos semantics,' by which is meant the study of such higher-order logical theories and their models by functorial methods, i.e. employing the theory of categories and functors, invented by S. Eilenberg and S. Mac Lane (cf. [3]), and in particular the notion of an elementary topos, invented by F. W. Lawvere and M. Tierney (cf. [9], [11]).

The central notion of topos semantics is that of a classifying topos for a logical theory. Our treatment of classifying topoi for higher-order logic is patterned on the now well-developed theory of classifying topoi for geometric

[1]Higher-order theories and some of their connections to topoi have recently been considered by several authors, in particular [2, 8, 1, 4]. Roughly speaking, it is our treatment of models of such theories that is new.

logic, a fragment of first-order logic (cf. [10]). In the geometric case, the relevant topoi are Grothendieck topoi and the relevant morphisms of topoi are geometric morphisms. Indeed, the theory of geometric classifying topoi is as much a tool for studying this category of topoi as for studying geometric logic (as evidenced by [6]). In the case of higher-order logic under consideration here, it is rather elementary topoi and logical morphisms that are most relevant. This is because the logical morphisms of topoi are exactly those that preserve the higher-order logic under consideration here, while the notion of an elementary topos is broad enough to include the classifying topoi that we shall construct, in addition to the topoi like **Sets** in which models are usually taken.

While the theory of geometric classifying topoi provides the starting point for our treatment of topos semantics, it is rather another theory that guides the further development, namely the classical theory of polynomial algebras and algebraic extensions of rings and fields. That this point of view is useful in our case results from the fact that the category at issue of topoi and logical morphisms shares with the category of commutative rings the important property of being "algebraic," in a suitable sense; and this results from the fact that the notion of an elementary topos is itself "algebraic," in a suitable sense,[2] since the definition is given wholly in terms of adjoints. This fact was emphasized by Lawvere in the original presentation of elementary topoi in [9]. Generally, and in rough terms, our goal in developing the theory of classifying topoi for higher-order logic is to take advantage of these facts to replace the traditional logical notion of finitary syntax by the algebraic notions of finite generation and finite presentability.

The theory of classifying topoi for higher-order logic is by no means fully developed in this dissertation; numerous directions remain to be investigated, many of them suggested by the ground covered here. It is hoped, however, that enough of the theory has been carried out for the reader to recognize its outlines, and to share the author's opinion that it is a course worth pursuing. That pursuit is not just an exercise in conceptual unification, however; like the introduction of new methods anywhere, it should also bear fruit. The final chapters of this work are thus intended as sample applications. The method is the same as that used elsewhere: a given logical problem or topic is first translated via our general set-up into an algebraic one (hopefully fa-

[2]More precisely, "essentially algebraic" in the sense of [5]; see §III.3 below for a fuller discussion of these matters.

miliar). The topic is treated by functorial methods, and the solution is then translated back into logical terms. Each of the applications given follows this scheme; the reader will recognize the ideas of algebraic (in)dependence in the treatment of interpolation, descent theory in the treatment of definability, and the sheaf-theoretic methods common in commutative algebra in our treatment of logical completeness.

A summary of the contents and chief results of each chapter now follows.

I. Higher-Order Theories and Models in Topoi

The basic notions under consideration are introduced. A language of higher-order logic is specified, and the notion of a logical theory is defined as a finite list of basic type symbols, basic typed constant symbols, and closed formulas in those parameters. A sequent calculus is given to determine the relation of syntactic entailment between formulas, in the usual way.

The category **Log** of topoi and logical morphisms is then introduced to provide semantics for such logical theories, as follows. For a given theory T and a given topos $\mathcal{E}$, the notion of a model of T in $\mathcal{E}$ is defined, generalizing the usual definition of a model of an elementary theory in a topos (which definition itself generalizes the usual set-valued notion). Morphisms of such models are also specified, determining a category $\mathbf{Mod}_T(\mathcal{E})$ of T-models in $\mathcal{E}$. The relation of semantic entailment between formulas is then specified in terms of models, in much the same way as in the case of first-order logic and set-valued models.

II. Classifying Topoi

The chapter begins with a review of the basic idea of classification, and the definition of a classifying topos for a theory T is given. Given a model $M \in \mathbf{Mod}_T(\mathcal{E})$ of T in a topos $\mathcal{E}$ and any logical morphism of topoi $f : \mathcal{E} \to \mathcal{F}$, the image $f(M)$ of M under f is a T-model in $\mathcal{F}$. In this way, $\mathbf{Mod}_T(-)$ is a functor on the category **Log** of topoi and logical morphisms. A topos $\mathcal{I}[U_T]$ is called a classifying topos for T if $\mathcal{I}[U_T]$ represents this functor $\mathbf{Mod}_T(-)$ (in a suitable 2-categorical sense).

The main result of the chapter is proved in §3:

Theorem (classifying topos). *For any logical theory T there is a classifying topos, denoted $\mathcal{I}[U_T]$; thus for any topos $\mathcal{E}$ there is an equivalence of* *categories, natural in $\mathcal{E}$,*

$$\mathbf{Log}(\mathcal{I}[U_T], \mathcal{E}) \;\simeq\; \mathbf{Mod}_T(\mathcal{E}).$$

Here $\mathbf{Log}(\mathcal{I}[U_T], \mathcal{E})$ is the category of logical morphisms $\mathcal{I}[U_T] \to \mathcal{E}$ and natural isomorphisms between them. It then follows from the theorem that there is a so-called universal T-model U_T in $\mathcal{I}[U_T]$, associated under the above equivalence to the identity logical morphism $\mathcal{I}[U_T] \to \mathcal{I}[U_T]$. The model U_T has the property that any T-model M in any topos $\mathcal{E}$ is the essential image $M \cong M^{\#}(U_T)$ of U_T under a logical morphism $M^{\#} : \mathcal{I}[U_T] \to \mathcal{E}$, determined uniquely up to natural isomorphism. In this sense, $\mathcal{I}[U_T]$ is the free topos on a model of T.

The proof of the classifying topos theorem proceeds in two steps: First, the free topos $\mathcal{I}[X]$ on a single object is constructed syntactically from the basic logical language by identifying closed terms of the form $\{x : \varphi\}$ under provable equality—a Lindenbaum-Tarski style construction already familiar in categorical logic. Next, the general classifying topos $\mathcal{I}[U_T]$ for a theory T is constructed topos-theoretically from $\mathcal{I}[X]$ with the help of a "slice lemma." From the construction of $\mathcal{I}[U_T]$ together with the classifying topos theorem then follows one of the main results of topos semantics:

Theorem (adequacy of topos semantics). *The deductive logical calculus is sound and complete for models in topoi. Specifically, for any logical theory T, a T-sentence is T-provable just if it is true in every T-model.*

The chapter concludes with some examples of classifying topoi and some first applications thereof, e.g. to the completeness of the classical theory of propositional types.

III. The Category Log

In this chapter, some basic facts about the category **Log** of topoi are established. A morphism of theories $T_1 \to T_2$ (called a "translation") is defined to be a logical morphism of classifying topoi $\mathcal{I}[U_{T_1}] \to \mathcal{I}[U_{T_2}]$. By the classifying topos theorem, these correspond uniquely to models of T_1 in the classifying topos $\mathcal{I}[U_{T_2}]$. Such translations can be specified in syntactic terms, generalizing the usual logical notion of a translation (or interpretation) of one theory into another. There results a syntactic notion of equivalence of theories: T_1 and T_2 are equivalent just if they are "intertranslatable" in the sense that there is an equivalence of categories $\mathcal{I}[U_{T_1}] \simeq \mathcal{I}[U_{T_2}]$. Any translation $\tau : T_1 \to T_2$ induces, for each topos $\mathcal{E}$, a "restriction" functor

$$\tau^* : \mathbf{Mod}_{T_2}(\mathcal{E}) \simeq \mathbf{Log}(\mathcal{I}[U_{T_2}], \mathcal{E}) \to \mathbf{Log}(\mathcal{I}[U_{T_1}], \mathcal{E}) \simeq \mathbf{Mod}_{T_1}(\mathcal{E})$$

on the respective model categories. One then has the following result by the classifying topos theorem and the well-known Yoneda lemma:

Theorem. *A translation of theories $\tau : T_1 \to T_2$ is an equivalence just if, for each topos $\mathcal{E}$, the induced restriction functor*

$$\tau^* : \mathbf{Mod}_{T_2}(\mathcal{E}) \to \mathbf{Mod}_{T_1}(\mathcal{E})$$

is an equivalence of model categories.

A topos is called finitary if it is (equivalent to) a slice topos $\mathcal{I}[X]/A$ for some object A in the free topos $\mathcal{I}[X]$ on one object. Any finitary topos is a classifying topos for some logical theory, and every classifying topos is finitary. The category $\mathbf{Log}_f$ of finitary topoi and logical morphisms is thus (equivalent to) the category of logical theories. A notion of finiteness is also specified for logical morphisms, corresponding to the logical notion of a finite extension of a theory. It is shown that any logical morphism $\mathcal{E} \to \mathcal{F}$ between finitary topoi $\mathcal{E}$ and $\mathcal{F}$ is finitary, and that the category $\mathbf{Log}_f$ has all finite colimits. It is further shown that if G is a finite (directed) graph, then the free topos $\mathcal{I}[G]$ on G is finitary; moreover, one has the following characterization of finitary topoi:

Theorem. *For any topos $\mathcal{E}$, the following statements are equivalent:*

(i) $\mathcal{E} \simeq \mathcal{I}[U_T]$ *for a logical theory* T.

(ii) $\mathcal{E} \simeq \mathcal{I}[X]/A$ *for an object A in the free topos $\mathcal{I}[X]$ on one object.*

(iii) $\mathcal{E} \simeq \mathcal{I}[G]/\sigma$ *for a subobject σ of 1 in the free topos $\mathcal{I}[G]$ on a finite graph G.*

(iv) For some finite graphs G, G', $\mathcal{E}$ is a coequalizer of logical morphisms

$$\begin{tikzcd} \mathcal{I}[G'] \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & \mathcal{I}[G] \arrow[r] & \mathcal{E}. \end{tikzcd}$$

Finitary topoi are used to construct colimits of topoi in general, using the fact that every topos is a colimit of finitary topoi, and the colimit can be constructed as the usual one of categories and functors. It follows that so-called "relative classifying topoi" exist, i.e. classifying topoi over an arbitrary base topos.

Finally, a section is devoted to quotient topoi, of which an abstract characterization is given. The kernel of a logical morphism is defined, and a homomorphism theorem for topoi is given, as is a factorization theorem for logical morphisms.

IV. Interpolation and Definability

Results of previous chapters are used to investigate some classical logical topics. Generalizations from first- to higher-order logic of the well-known Craig interpolation and Beth definability theorems are considered. In each case, the theorem is shown (by counter-example) to fail in full generality, i.e. in the case of many-sorted, higher-order theories. A restricted version of each theorem is then obtained by determining a suitable condition; in particular, the single sorted, higher-order case then follows easily. The definability theorem is furthermore shown to hold in some cases where the interpolation theorem fails. Each theorem is stated in both topos-theoretic and syntactical forms.

V. Sheaf Representation and Logical Completeness

The sheaf representation for topoi of Lambek & Moerdijk [7]—an analogue of Grothendieck's sheaf representation for commutative rings—is improved upon, and sharper logical completeness theorems are derived as corollaries. A topos is called hyperlocal (resp. well-pointed) if the terminal object is connected and projective (resp. is a generator). Such topoi have been studied previously; well-pointed topoi as models of so-called weak Zermelo set theory, and hyperlocal topoi as intuitionistic analogues thereof. In logical terms, a classifying topos $\mathcal{I}[U_T]$ is:

- hyperlocal iff $T \vdash p \vee q$ implies $T \vdash p$ or $T \vdash q$, and $T \vdash \exists_x \varphi(x)$ implies $T \vdash \varphi(\tau)$ for some closed term τ;
- boolean iff $T \vdash p \vee \neg p$ (T is "classical");
- well-pointed iff hyperlocal and boolean.

After establishing a result in the theory of stacks (every small stack is equivalent to a sheaf), a recent result in topos theory is applied, and the following theorem is established:

Theorem (sheaf representation). *For any small topos $\mathcal{E}$ there is a topological space $X_{\mathcal{E}}$ with a sheaf of categories $\widetilde{\mathcal{E}}$ such that:*

(i) *$\mathcal{E}$ is equivalent to the category of global sections of $\widetilde{\mathcal{E}}$,*

(ii) *every stalk of $\widetilde{\mathcal{E}}$ is a hyperlocal topos,*

(iii) *$\mathcal{E}$ is boolean if and only if every stalk of $\widetilde{\mathcal{E}}$ is well-pointed.*

Combining this sheaf representation with results of previous chapters, the following completeness theorems for intuitionistic and classical higher-order logic are obtained:

Theorem (strong completeness). *Let T be a (classical) logical theory. A T-sentence is T-provable if it is true in every (well-pointed) hyperlocal model of T.*

By a (well-pointed) hyperlocal model of T is meant here a T-model in a (well-pointed) hyperlocal topos. Thus e.g. if T is a classical theory, then a T-sentence is provable if it is true in every model of T in every model of weak Zermelo set theory. Furthermore, such models give rise to certain of the so-called general models usually considered in this connection, and a sharper version of the classical Henkin completeness theorem for higher-order logic then follows as a corollary to the strong completeness theorem.

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