Quillen model structures on cubical sets

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Acknowledgements

- Parts are joint work with Coquand and Riehl.
- Parts are also joint with Cavallo and Sattler.
- Ideas are also borrowed from Joyal and Orton-Pitts.

Models of HoTT from QMS

The first models of HoTT were built from Quillen model categories.

- A-Warren: general Quillen model structures and weak factorization systems
- van den Berg-Garner: special weak factorization systems on spaces and simplicial sets
- Voevodsky: the Kan-Quillen model structure on simplicial sets

In each case, more specific QMS led to “better” models of type theory, with coherent Id, Σ, Π and eventually univalent U.

QMS from models of HoTT

But one can also start from a model of HoTT and construct a Quillen model structure (cf. Gambino-Garner, Lumsdaine).

Definition (*pace* Orton-Pitts)

A *premodel of HoTT* consists of $(\mathcal{E}, \Phi, \mathbb{I}, \mathsf{V})$ where:

- $\mathcal{E}$ is a topos
- Φ is a representable class of monos $\Phi \rightarrowtail \Omega$ that form a *dominance* and ...
- $\mathbb{I}$ is an interval $1 \rightrightarrows \mathbb{I}$ in $\mathcal{E}$ that is *tiny* $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$ and ...
- $\dot{\mathsf{V}} \to \mathsf{V}$ is a universe of *small families*, closed under Σ, Π and ...

A model of HoTT is then constructed internally using the *extensional* type theory of $\mathcal{E}$ (see Orton-Pitts).

QMS from models of HoTT

Our goal here is to show that from such a set-up for modelling HoTT one can also construct a QMS:

Construction

From a premodel $(\mathcal{E}, \Phi, \mathbb{I}, \mathsf{V})$ *one can construct a QMS on* $\mathcal{E}$*.*

The resulting QMS is right proper and has descent, so it also admits a model of HoTT in the pre-Orton-Pitts sense.

QMS from models of HoTT

The construction of a QMS $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ from a premodel $(\mathcal{E}, \Phi, \mathbb{I}, \mathsf{V})$ is general, but the details depend on the specifics of the premodel.

We consider three special cases of *cubical sets*.

$$\mathcal{E} = \mathsf{Set}^{\mathbb{C}^{op}}$$

1. Cartesian cubical sets
2. Cartesian cubical sets with equivariance
3. Dedekind cubical sets

QMS from models of HoTT

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$$\mathcal{E} = \mathsf{Set}^{\mathbb{C}^{op}}$$

1. Cartesian cubical sets (new)
2. Cartesian cubical sets with equivariance (new jww/CCRS)
3. Dedekind cubical sets (Sattler)

Outline of the construction

Let $(\mathcal{E}, \Phi, \mathbb{I}, \mathsf{V})$ be a premodel of HoTT where $\mathcal{E} = \mathsf{cSet}$.

We construct a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{E}$ in 3 steps:

1. use Φ to determine an awfs $(\mathcal{C}, \mathsf{TFib})$,
2. use $\mathbb{I}$ to determine another awfs $(\mathsf{TCof}, \mathcal{F})$,
3. let $\mathcal{W} = \mathsf{TFib} \circ \mathsf{TCof}$ and prove 3-for-2 from FEP (done!)

To prove the Fibration Extension Property:

4. show that $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ satisfies the EEP,
5. use V and $\mathbb{I}$ to construct a universe U of fibrations,
6. use EEP to show that U is fibrant, which implies FEP.

NB: (5) seems to be a detour; maybe one can prove FEP directly?

1. The cofibration awfs $(\mathcal{C}, \mathsf{TFib})$

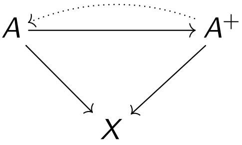
The monos classified by $\Phi \rightarrowtail \Omega$ are called *cofibrations*.
The generic one $1 \rightarrowtail \Phi$ determines a polynomial endofunctor,

$$X^+ := \sum_{\varphi:\Phi} X^\varphi \,,$$

which is a (fibered) monad,

$$+ : \mathsf{cSet}/\cdot \;\longrightarrow\; \mathsf{cSet}/\cdot$$

Algebras for the pointed endofunctor of this monad,



form the right class of an awfs – they are the *trivial fibrations*.

2. The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

For any $c : A \to B$ in $\mathrm{cSet}^{\mathbb{2}}$, the *Leibniz adjunction*

$$(-)\otimes c \ \dashv \ c \Rightarrow (-)$$

relates the pushout-product with c and the pullback-hom with c. These operations satisfy

$$(f \otimes c) \boxslash g \ \Leftrightarrow \ f \boxslash (c \Rightarrow g)$$

with respect to the diagonal filling relation $f \boxslash g$.

Definition

A map $f : Y \to X$ is a *biased fibration* if $\delta_\varepsilon \Rightarrow f$ is a $+$-algebra for both endpoints $\delta_0, \delta_1 : 1 \to \mathbb{I}$. Equivalently, $f \in \mathcal{F}$ if for all cofibrations $c \in \mathcal{C}$ and $\varepsilon = 0, 1$,

$$c \otimes \delta_\varepsilon \boxslash f\ .$$

This notion of fibration is used for the *Dedekind cubes*.

2. The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

For the *Cartesian cubes*, we pass to the slice category $\mathrm{cSet}/\mathbb{I}$, where there is a *generic point* $\delta : 1 \to \mathbb{I}$.

Definition

A map $f : Y \to X$ is an *(unbiased) fibration* if $\delta \Rightarrow f$ is a $+$-algebra. Equivalently, $f \in \mathcal{F}$ if $c \otimes \delta \boxslash f$ for all $c \in \mathcal{C}$.

Proposition

There is an awfs $(\mathsf{TCof}, \mathcal{F})$ *with these fibrations as* $\mathcal{F}$*.*

Remark

There is also an *equivariant* version of this awfs, in which the fibration structure respects the symmetries of the cubes $\mathbb{I}^n$ (this is explained in Emily's talk).

3. The weak equivalences $\mathcal{W}$

Now define

$$\mathcal{W} = \mathsf{TFib} \circ \mathsf{TCof}$$

thus a map is a weak equivalence if it factors as a trivial cofibration followed by a trivial fibration.

It is easy to show that

$$\mathsf{TCof} = \mathcal{W} \cap \mathcal{C}$$
$$\mathsf{TFib} = \mathcal{W} \cap \mathcal{F}$$

so we just need the 3-for-2 property for $\mathcal{W}$.

We will compare $\mathcal{W}$ with the following, which does satisfy 3-for-2.

Definition

A map $f : Y \to X$ is a *weak homotopy equivalence* if the map

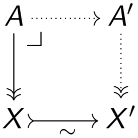
$$K^f : K^X \longrightarrow K^Y$$

is a bijection on connected components for all fibrant objects K.

The QMS $(\mathcal{C}, \mathcal{W}, \mathcal{F})$

Definition (FEP)

The *Fibration Extension Property* says that fibrations extend along trivial cofibrations:



Lemma

If the FEP holds, then a map $f : Y \to X$ is a weak equivalence iff it is a weak homotopy equivalence.

Corollary

If the FEP holds, then $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a QMS.

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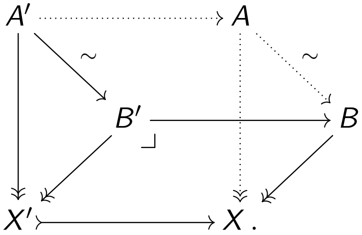
4. The equivalence extension property

Definition (EEP)

The EEP says that weak equivalences extend along any cofibration $X' \rightarrowtail X$: given a fibration $B \twoheadrightarrow X$, and a weak equivalence $A' \simeq B'$ over X', where $A' \twoheadrightarrow X'$ and $B' = X' \times_X B$,



there is a fibration $A \twoheadrightarrow X$, and a weak equivalence $A \simeq B$ over X that pulls back to $A' \simeq B'$.

This is was shown by Voevodsky for modelling univalence in Kan simplicial sets. A related proof by Sattler works in our setting.

5. The universe U of fibrations

There is a *universal (small) fibration* $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$.
Every small fibration $A \twoheadrightarrow X$ is a pullback of $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ along a canonical classifying map $X \to \mathsf{U}$.

$$\begin{array}{ccc} A & \longrightarrow & \dot{\mathsf{U}} \\ \downarrow\!\!\downarrow & \lrcorner & \downarrow\!\!\downarrow \\ X & \longrightarrow & \mathsf{U} \end{array}$$

Take $\mathsf{U} \to \mathsf{V}$ to be the *object of fibration structures on* $\dot{\mathsf{V}} \to \mathsf{V}$.

$$\mathsf{U} = \mathsf{Fib}(\dot{\mathsf{V}})$$

Then define $\dot{\mathsf{U}} \to \mathsf{U}$ by pulling back the universal small family.

$$\begin{array}{ccc} \dot{\mathsf{U}} & \longrightarrow & \dot{\mathsf{V}} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{V} \end{array}$$

5. The universe U of fibrations

We said $\mathsf{U} = \mathsf{Fib}(\dot{\mathsf{V}})$, and we defined $\dot{\mathsf{U}} \to \mathsf{U}$ by:

$$
\begin{CD}
\dot{\mathsf{U}} @>>> \dot{\mathsf{V}} \\
@VVV @VVV \\
\mathsf{U} @>>> \mathsf{V}
\end{CD}
$$

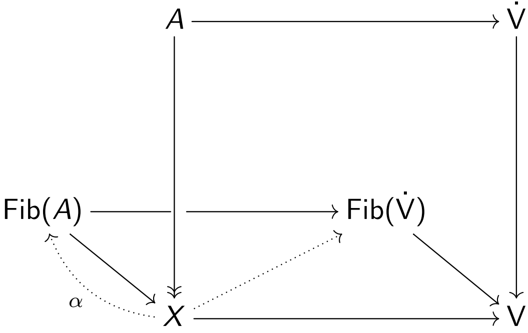
But $\mathsf{Fib}(-)$ is stable under pullback, so there is a section

$$
\begin{CD}
\dot{\mathsf{U}} @>>> \dot{\mathsf{V}} \\
@VVV @VVV \\
\mathsf{U} @>>> \mathsf{V} \\
@AAA @AAA \\
\mathsf{Fib}(\dot{\mathsf{U}}) @>>> \mathsf{Fib}(\dot{\mathsf{V}})
\end{CD}
$$

Thus $\dot{\mathsf{U}} \longrightarrow \mathsf{U}$ is a fibration.

5. The universe U of fibrations

A fibration structure α on a family $A \to X$ therefore gives rise to a factorization of the classifying map to $\dot{\mathsf{V}} \to \mathsf{V}$.



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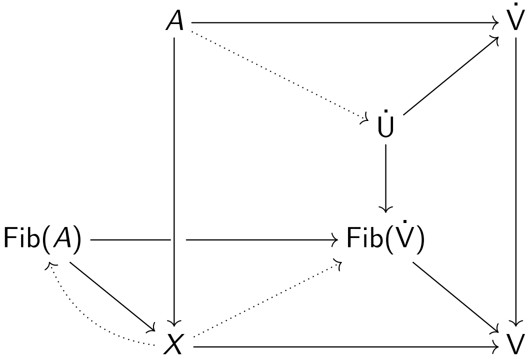
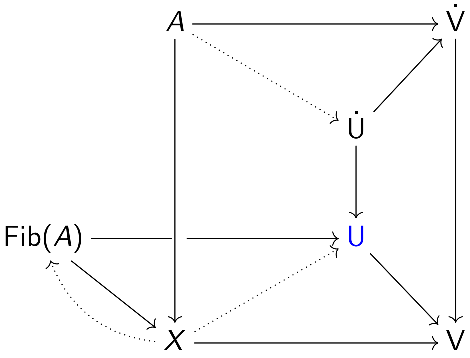


5. The universe U of fibrations

A fibration structure α on a family $A \to X$ therefore gives rise to a factorization of the classifying map to $\dot{\mathsf{V}} \to \mathsf{V}$ through the fibration classifier $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$.

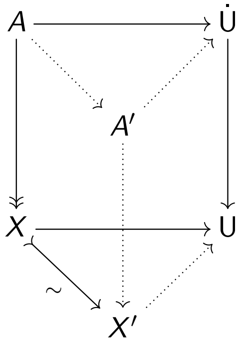


The construction of Fib uses the *root functor* $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

FEP and EEP in terms of U

Given a universe U, the EEP and FEP take on new meaning.

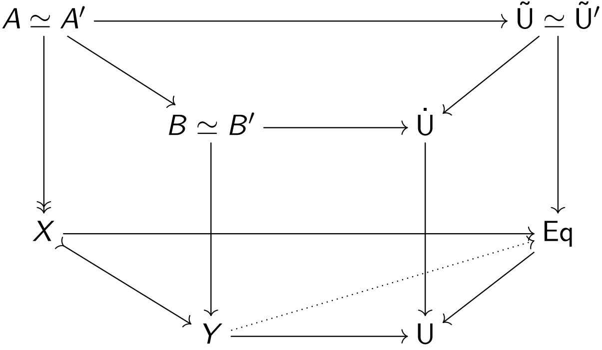
The FEP says just that U is fibrant:



Voevodsky proved this for Kan simplicial sets.

FEP and EEP in terms of U

The EEP says that $\mathsf{Eq} \longrightarrow \mathsf{U}$ is a TFib:



Shulman gave a neat proof of FEP from EEP, but it uses 3-for-2.

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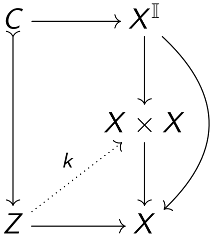
6. FEP from EEP

Coquand gave a proof of FEP from EEP using *Kan composition*.

Definition

An object X has *(biased) composition* if for every cofibration $C \rightarrowtail Z$ and commutative rectangle as on the outside below,



there is an arrow $k : Z \longrightarrow X \times X$ making the diagram commute.

Lemma

If X has composition, then X is fibrant. □

6. FEP from EEP

We can now show:

Proposition

The universe U *is fibrant.*

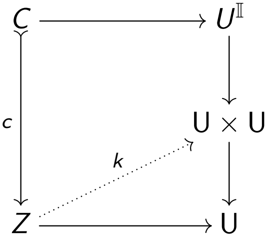
By the previous lemma it suffices to show:

Lemma

The universe U *has composition.*

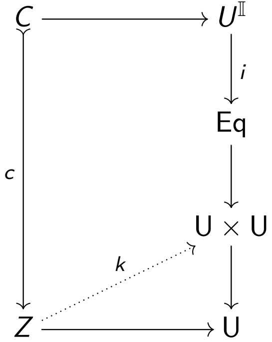
Proof.

Consider a composition problem



6. FEP from EEP

The canonical map $\mathsf{U}^{\mathbb{I}} \longrightarrow \mathsf{U} \times \mathsf{U}$ factors (over $\mathsf{U} \times \mathsf{U}$) through the object Eq of equivalences via $i = \mathsf{IdtoEq}$,



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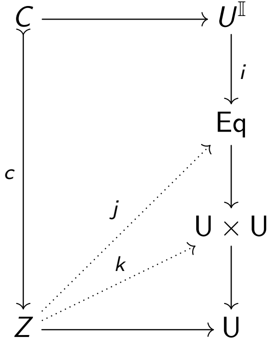


But the projection $\mathsf{Eq} \longrightarrow \mathsf{U}$ is a trivial fibration by EEP, so there is a diagonal filler j. Composing gives the required k. □

Done!

But is our QMS *right proper*?

Postscript: Frobenius

Definition (Frobenius)

The *Frobenius Property* says that trivial cofibrations pull back along fibrations,

$$\begin{array}{ccc} A' & \longrightarrow & X' \\ {\scriptstyle\sim}\downarrow & \lrcorner & \downarrow{\scriptstyle\sim} \\ A & \twoheadrightarrow & X\,. \end{array}$$

It is equivalent to the condition that fibrations "push forward" along fibrations,

$$\begin{array}{ccc} B & & \Pi_A B \\ \downarrow & & \downarrow \\ A & \twoheadrightarrow & X\,. \end{array}$$

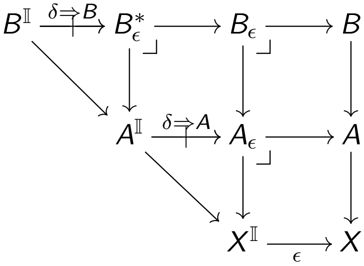
This is related to the existence of Π-types. It implies that our QMS is *right proper*.

Frobenius

Proposition

The Frobenius property holds for (TCof, $\mathcal{F}$).

Proof.

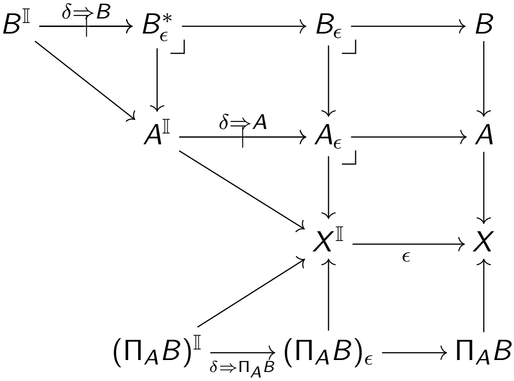


Frobenius

Proposition

The Frobenius property holds for $(\mathsf{TCof}, \mathcal{F})$*.*

Proof.

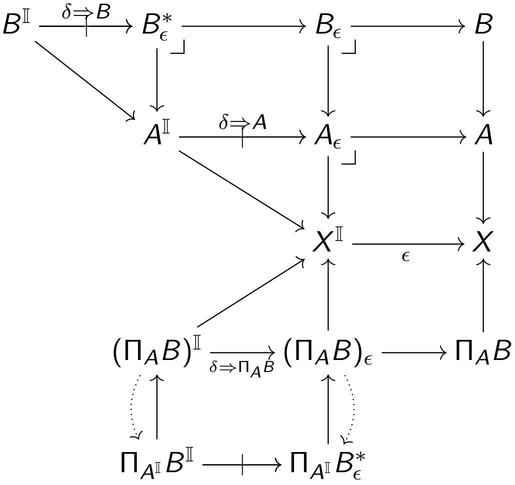


Frobenius

Proposition

The Frobenius property holds for $(\mathsf{TCof}, \mathcal{F})$.

Proof.



□

That's all Folks!