Univalence in ∞-Topoi

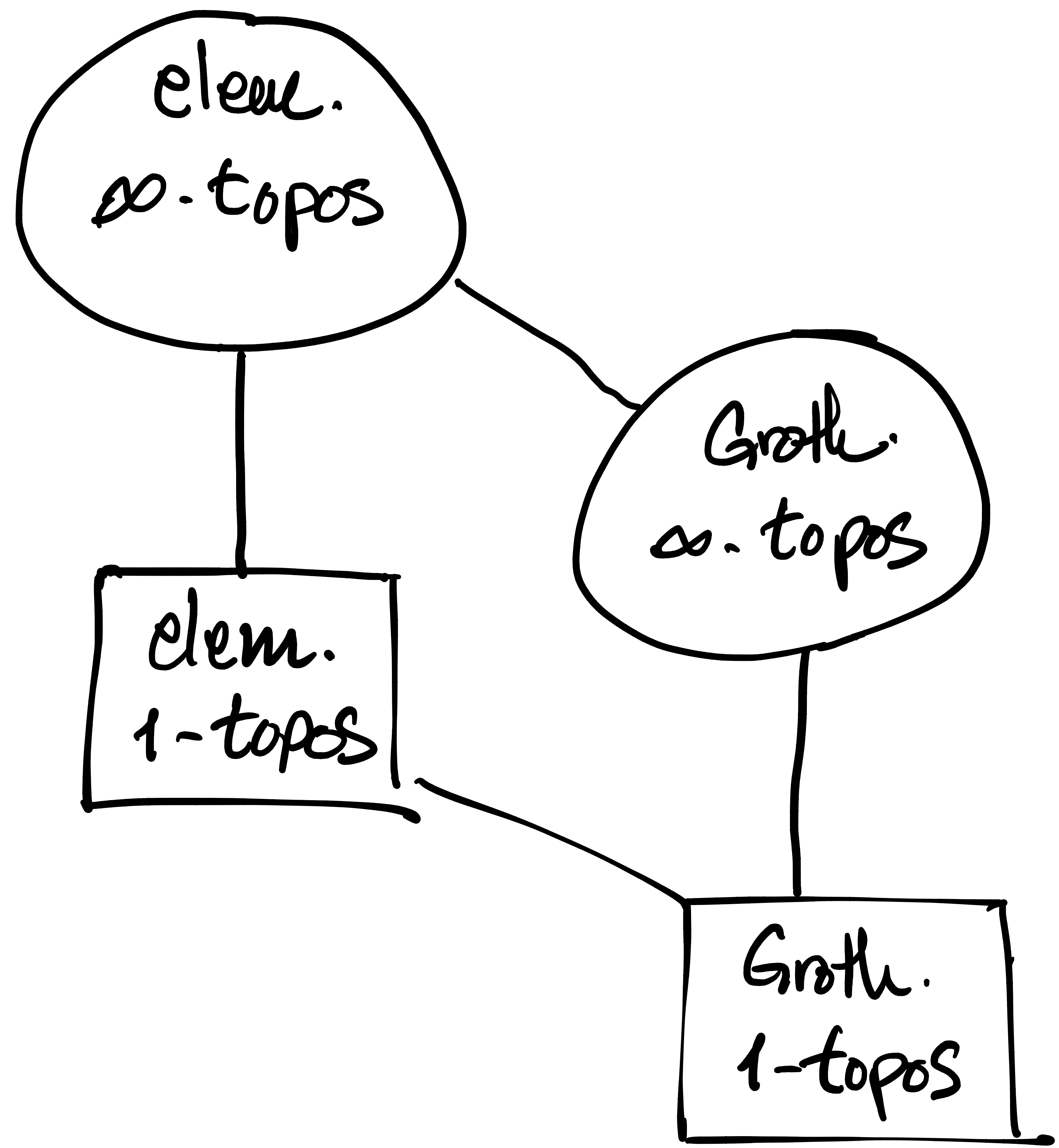
(toward an elementary ∞-topos)

Steve Awodey

CT 2020 → 21

GENOA

What is an elementary ∞-topos?



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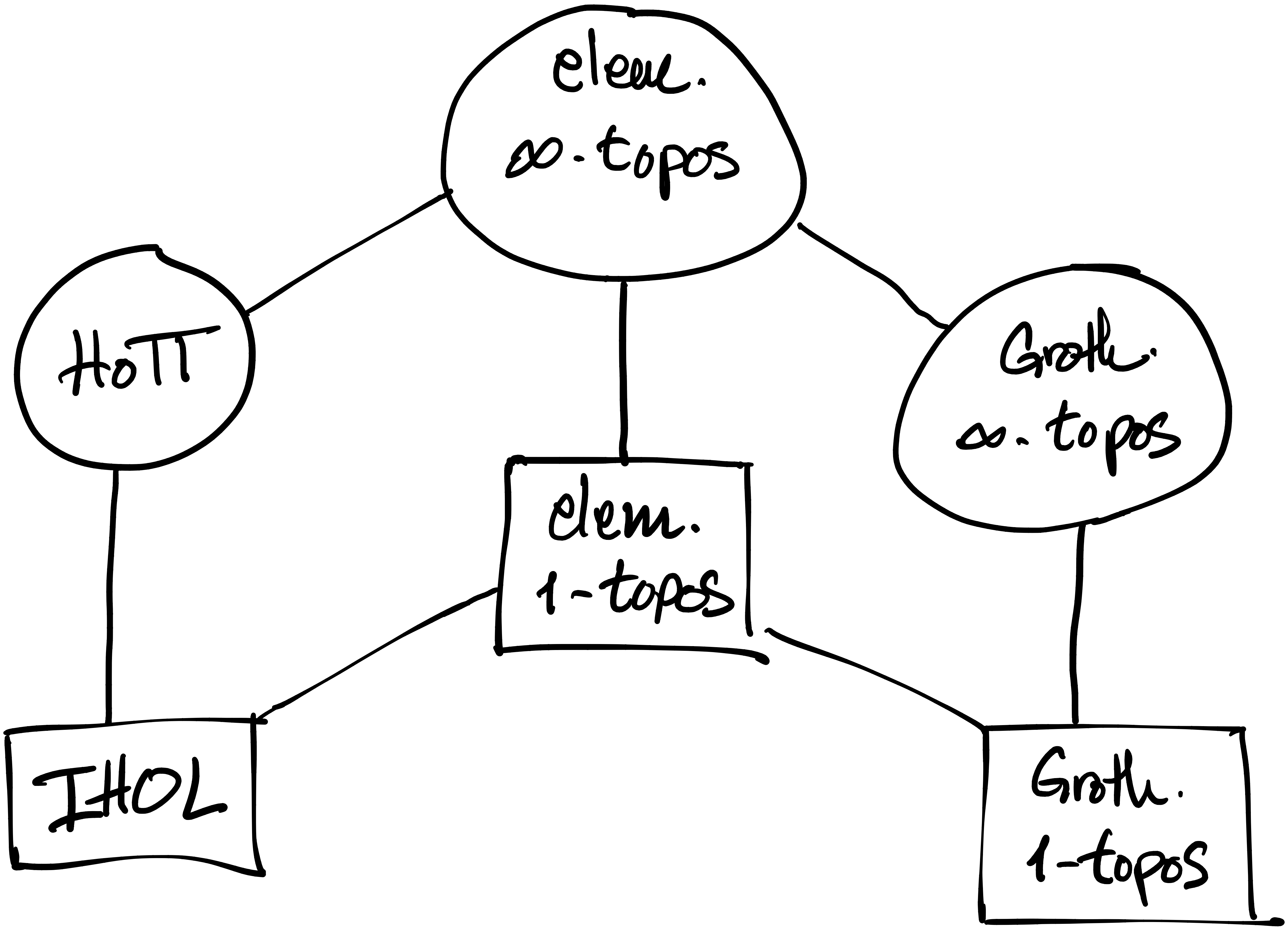

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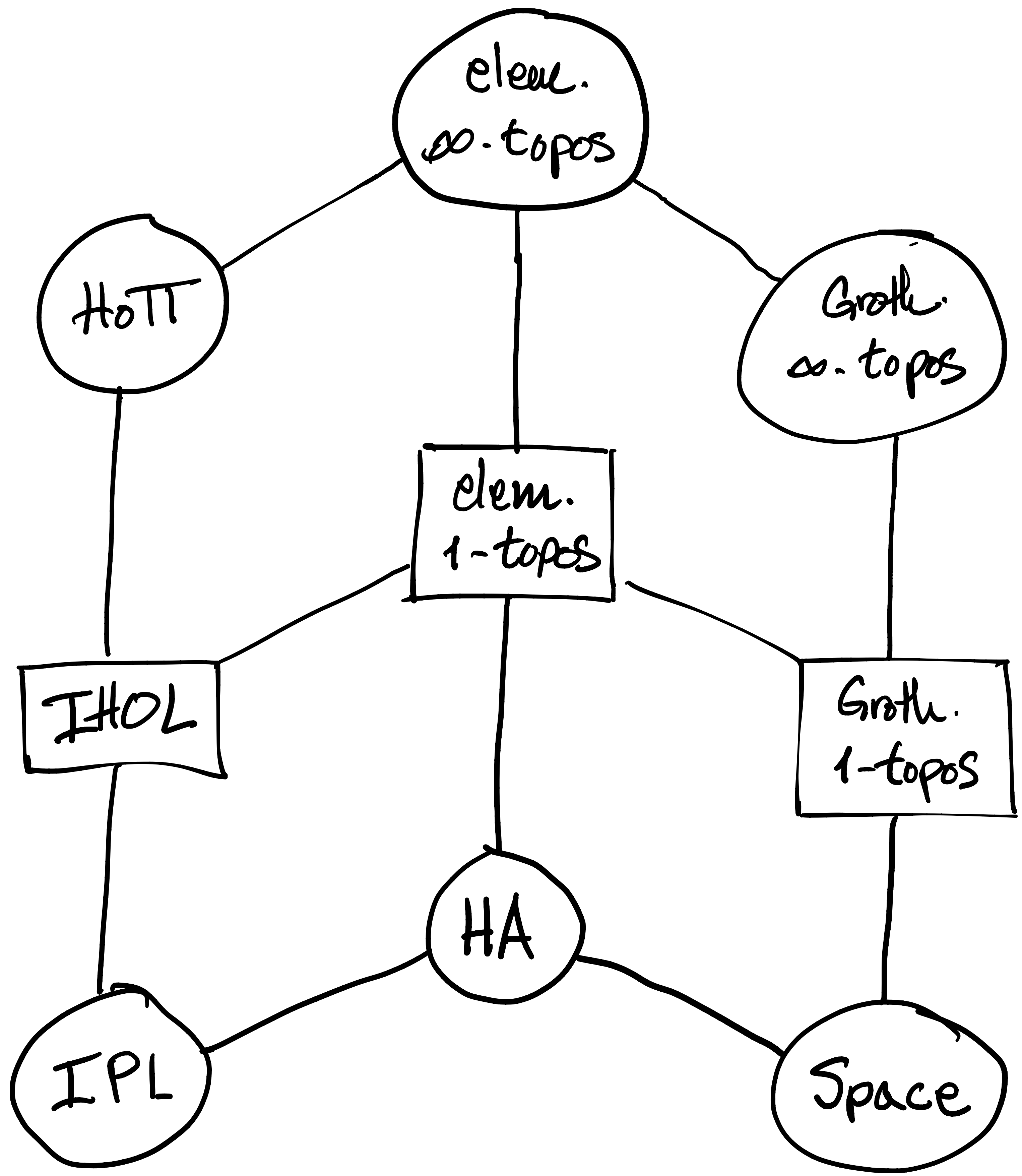

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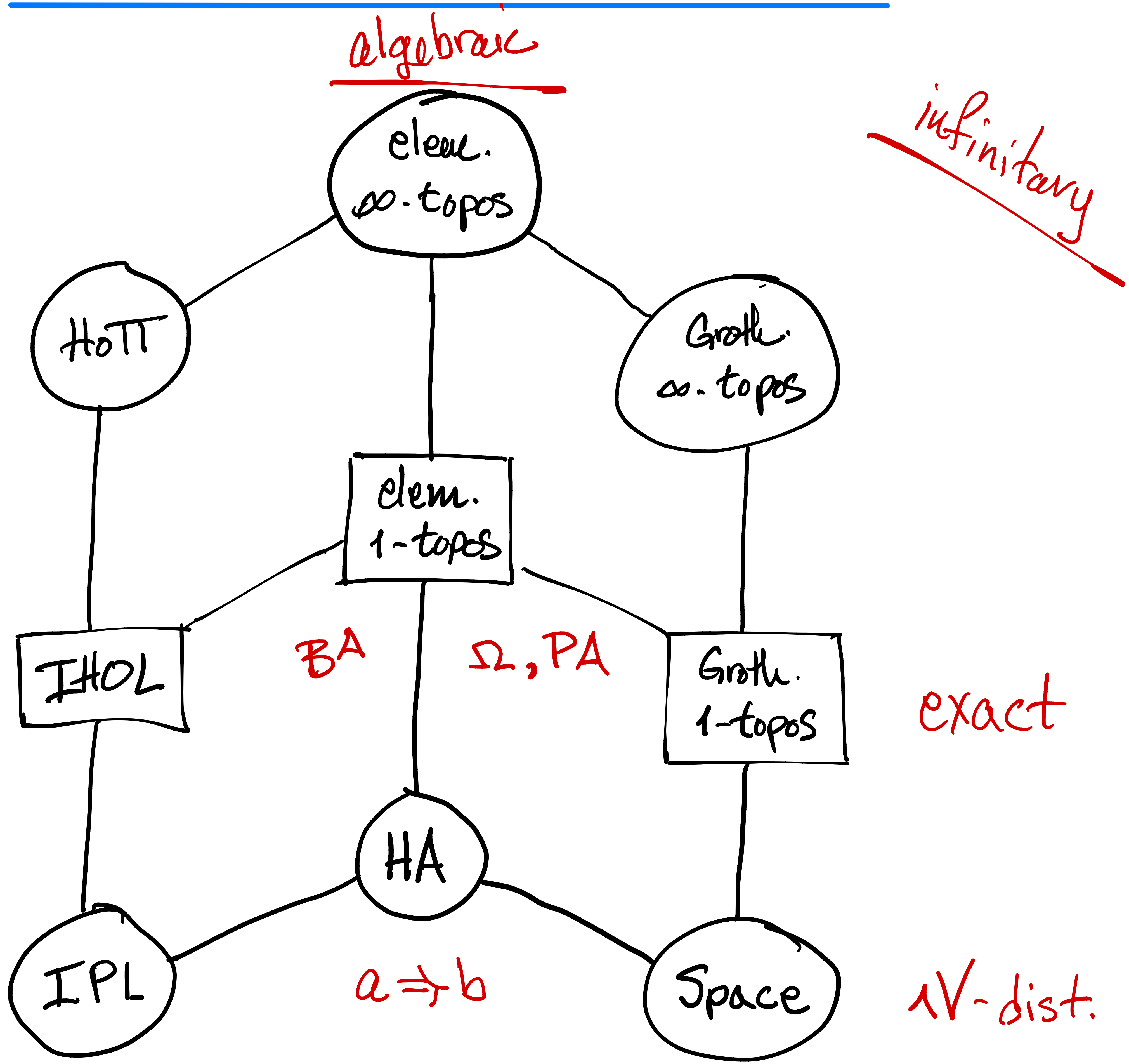

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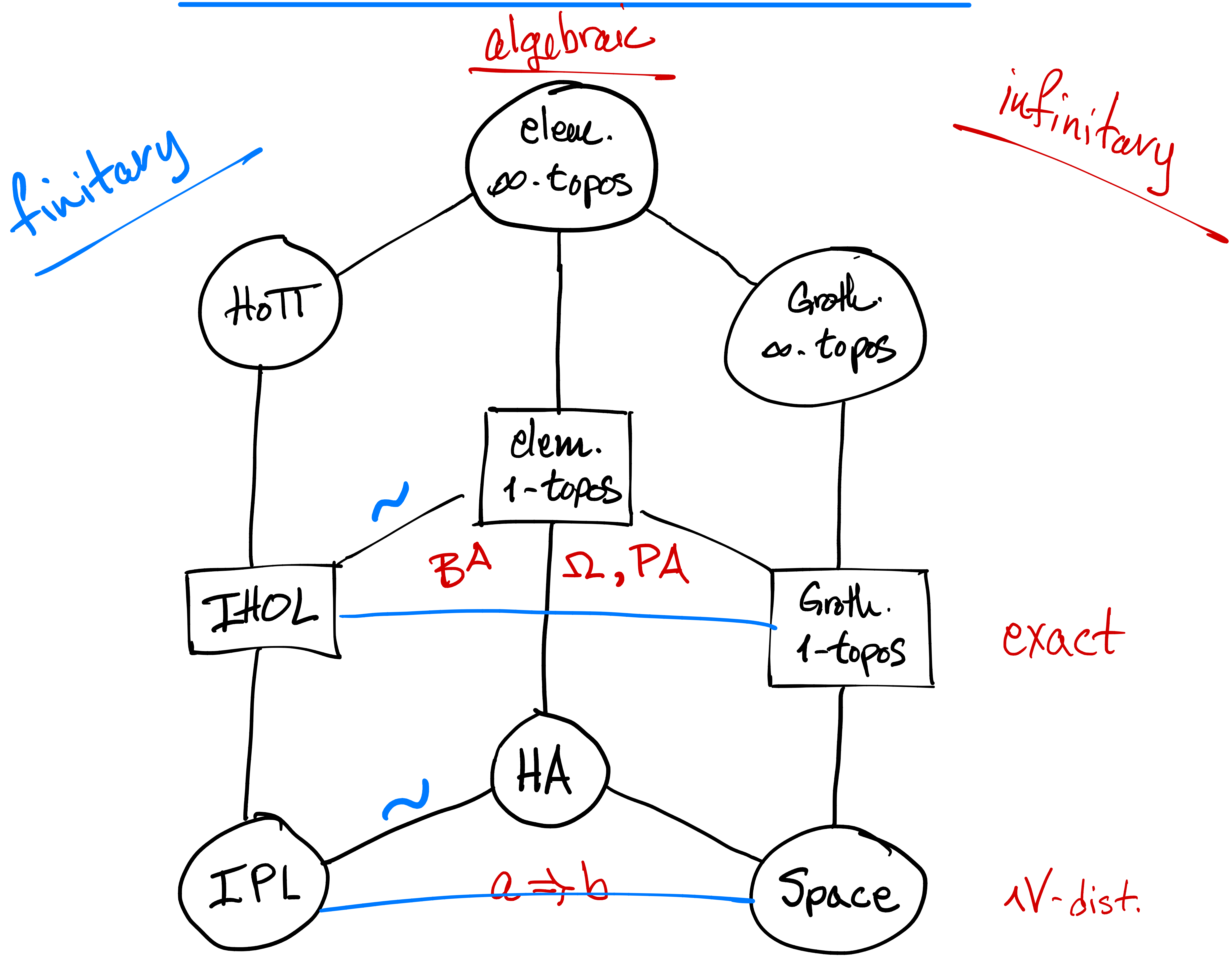

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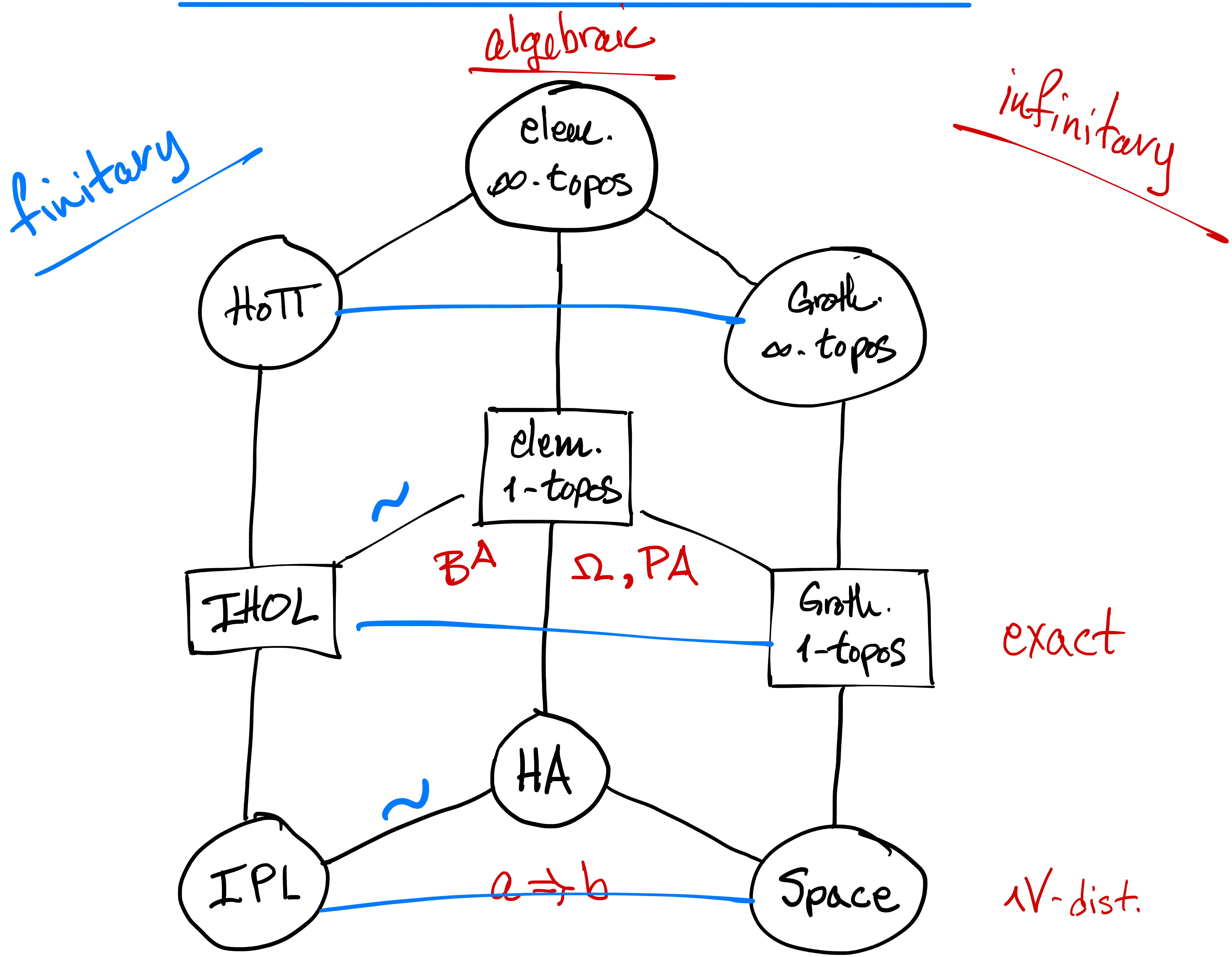

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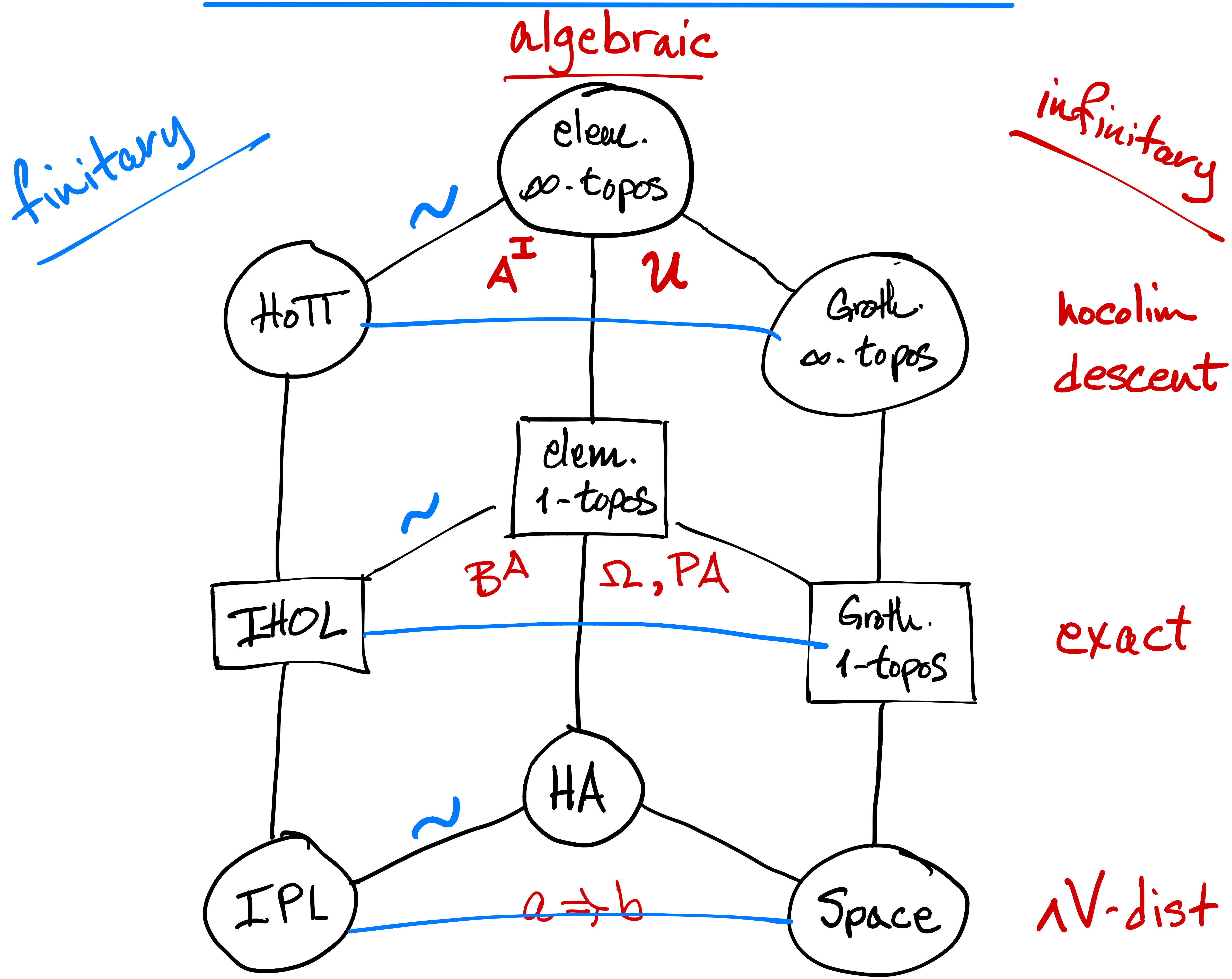


Background

HoTT	∞-Topos
DTT	∞-Cat
Σ, Π, subst.	LCCC
Prop	SOC
$\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$	Object classifiers
Id	QMS/WFS
Univalence:	Descent:
$(A = B) \simeq (A \simeq B)$	$\mathcal{E}/\varinjlim X_i \simeq \varprojlim \mathcal{E}/X_i$

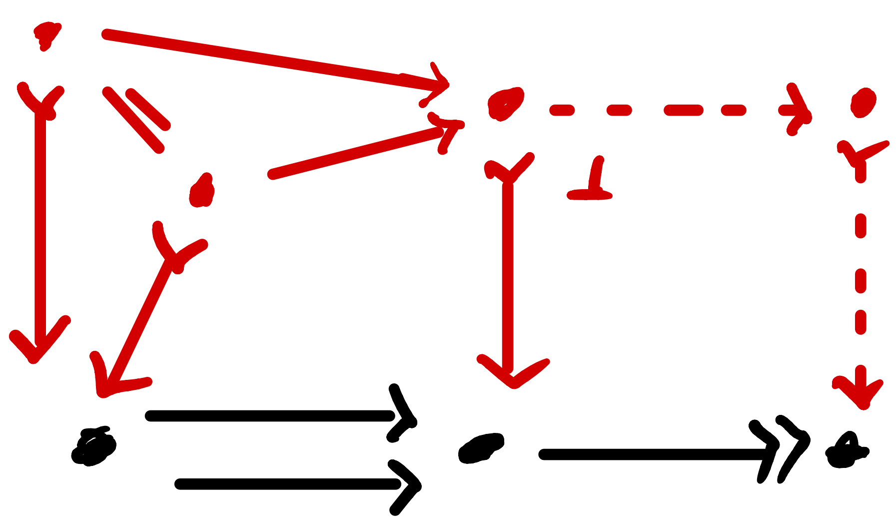
SOC in 1-Topos

$$1 \longrightarrow \Omega$$

- $\mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega)$
- so $\mathrm{Sub}(-)$ is continuous:

$$\mathrm{Sub}(\varinjlim_i X_i) \cong \varprojlim_i \mathrm{Sub}(X_i)$$

- descent for subobjects:



OC in ∞-Topos

$$\tilde{\mathcal{V}} \longrightarrow \mathcal{V}$$

- weakly classifies (small) maps:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & \tilde{\mathcal{V}} \arrow[d] \\ X \arrow[r, "\alpha"'] & \mathcal{V} \end{tikzcd}$$

- but α is **not** unique
- so we do **not** have

$$\mathcal{E}/X \cong \mathrm{Hom}(X, \mathcal{V})$$

(even for small maps).

Object Classifiers in $\mathrm{Psh}(\mathbb{C})$

- In $\mathbf{Set}$ we have: $\quad \mathbf{Set}_2 \hookrightarrow \mathbf{Set}_\omega \hookrightarrow \mathbf{Set}_\kappa \hookrightarrow \cdots$

- In $\mathbf{Set}^{\mathbb{C}^{op}}$ we get: $\quad \mathcal{V}_2 \hookrightarrow \mathcal{V}_\omega \hookrightarrow \mathcal{V}_\kappa \hookrightarrow \cdots$

  where: $\quad \mathcal{V}_\kappa(c) := \mathrm{Hom}_{\mathbf{Cat}}(\mathbb{C}/c^{op}, \mathbf{Set}_\kappa)$

- Note that then: $\quad \mathcal{V}_2 = \Omega$

- And $\mathcal{V}_\kappa$ classifies "families of objects of size κ"

$$
\begin{tikzcd}
A_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{V}}_\kappa \arrow[d] \\
y^c \arrow[r, "x"'] & X \arrow[r, "\alpha"'] & \mathcal{V}_\kappa
\end{tikzcd}
\qquad
\begin{array}{c} \Gamma(y^c, A_x) \\ < \kappa \end{array}
$$

Object Classifiers in $Psh(\mathbb{C})$

Prop $\tilde{V}_\kappa \longrightarrow V_\kappa$ determines an internal category $\mathbb{V}_\kappa$ s.th.

$$Psh(\mathbb{C})/^{\kappa}_{X} \simeq \underline{\mathrm{Hom}}(X, \mathbb{V}_\kappa).$$

So in the classifying pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & \tilde{V} \arrow[d] \\ X \arrow[r, "\alpha"'] & V \end{tikzcd}$$

α is unique up to natural iso (in $Cat(Psh(\mathbb{C}))$
$= Cat^{\mathbb{C}^{op}}$).

<u>Corollary</u> For $\mathcal{E} = \mathrm{Psh}(\mathbb{C})$, the pseudofunctor

$$\mathcal{E}/^k_- : \mathcal{E}^{op} \longrightarrow \mathrm{Cat}$$

is representable & therefore continuous,

$$\mathcal{E}/^k_{\varinjlim_i X_i} \simeq \varprojlim_i \mathcal{E}/^k X_i .$$

In particular, $\mathcal{E}/^k_-$ is a stack (for e.g. the coherent topology).

Generalizing to Higher Dimensions

We have the internal poset Ω & 1-cat $\mathcal{V}$ with:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{V}} \arrow[d] \\ \Omega \arrow[r] & \mathcal{V} \end{tikzcd}$$

$$\underline{\mathrm{Hom}}(X, \Omega) \cong \mathrm{Sub}(X),$$

$$\underline{\mathrm{Hom}}(X, \mathcal{V}) \simeq \mathcal{E}/X.$$

Want to generalize to higher cats & their $\simeq$.

For ∞-cats, we can use an idea from HoTT.

Univalence

In HoTT, a universe $\tilde{\mathcal{U}} \to \mathcal{U}$ is <u>univalent</u> iff there is an equivalence:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\simeq"] & & \tilde{\mathcal{U}} \arrow[dd] \\
& B \arrow[ur] \arrow[dl] & \\
X \arrow[rr, shift left=1.5, "\alpha"] \arrow[rr, shift right=1.5, "\beta"'] \arrow[rr, phantom, "\sim"] & & \mathcal{U}
\end{tikzcd}$$

$$\frac{A \simeq B}{\alpha \sim \beta}$$

where $A \simeq B$ is an <u>equivalence of types</u> over X, and $\alpha \sim \beta$ is a homotopy consisting of <u>paths</u> in $\mathcal{U}$.

Univalence in HoTT

More precisely, there is a classifying type for $\simeq$,

$$\begin{tikzcd}
A \simeq B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Equ} \arrow[d] \\
X \times X \arrow[r, "\alpha \times \beta"'] & \mathcal{U} \times \mathcal{U}
\end{tikzcd}$$

and a factorization

$$\begin{tikzcd}
\mathcal{U}^I \arrow[rr, "*"] \arrow[dr] & & \mathrm{Equ} \arrow[dl] \\
& \mathcal{U} \times \mathcal{U} &
\end{tikzcd},$$

so that the equivalence

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\simeq"] & & \tilde{\mathcal{U}} \arrow[dd] \\
& B \arrow[ur] \arrow[dl] & \\
X \arrow[rr, shift left=2, "\alpha"] \arrow[rr, shift right=2, "\beta"'] \arrow[rr, phantom, "\simeq"] & & \mathcal{U}
\end{tikzcd}$$

is the univalence axiom:

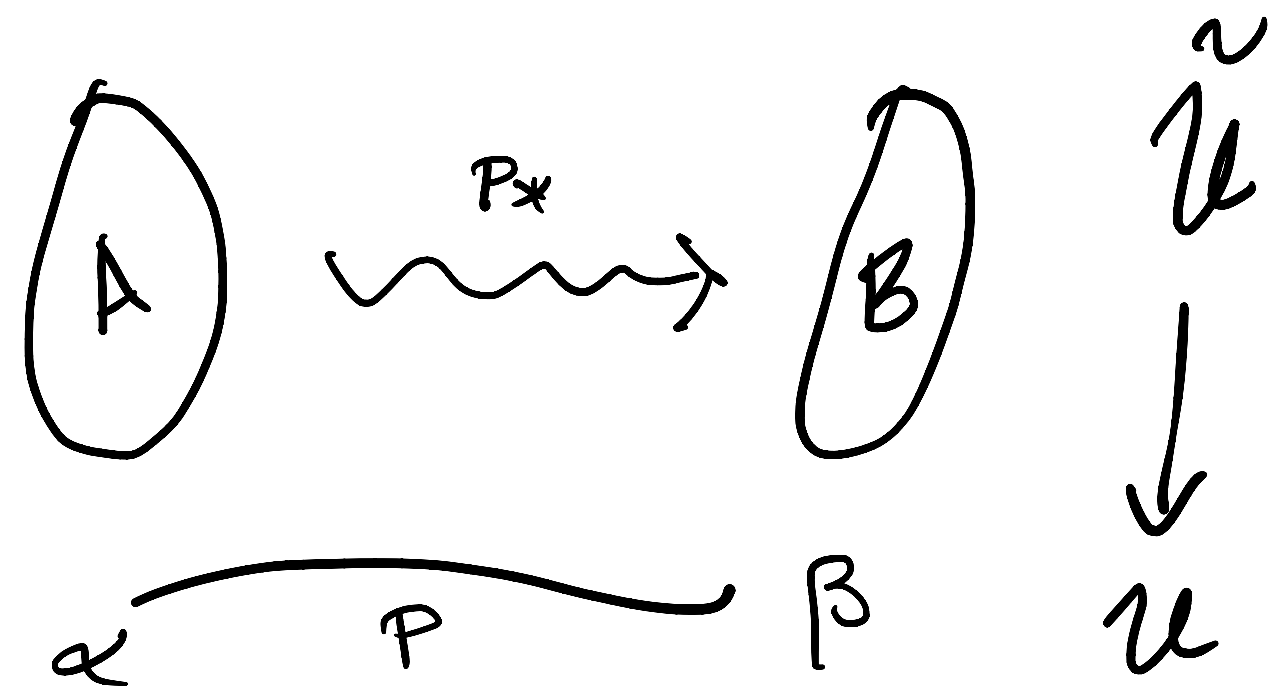
$$(\alpha \simeq \beta) \underset{*}{\simeq} (A \simeq B)$$

Univalence in a QMC

Def. Let $\mathcal{E} = \mathrm{Psh}(\mathbb{C})$ w/ a Cisinski model str.

An object classifier $\tilde{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is <u>univalent</u> if the canonical transport map

$$\begin{tikzcd} \mathcal{U}^I \arrow[rr, "*"] \arrow[dr] & & \mathrm{Eq}\,\mathcal{U} \arrow[dl] \\ & \mathcal{U} \times \mathcal{U} & \end{tikzcd}$$



is itself a weak equivalence.

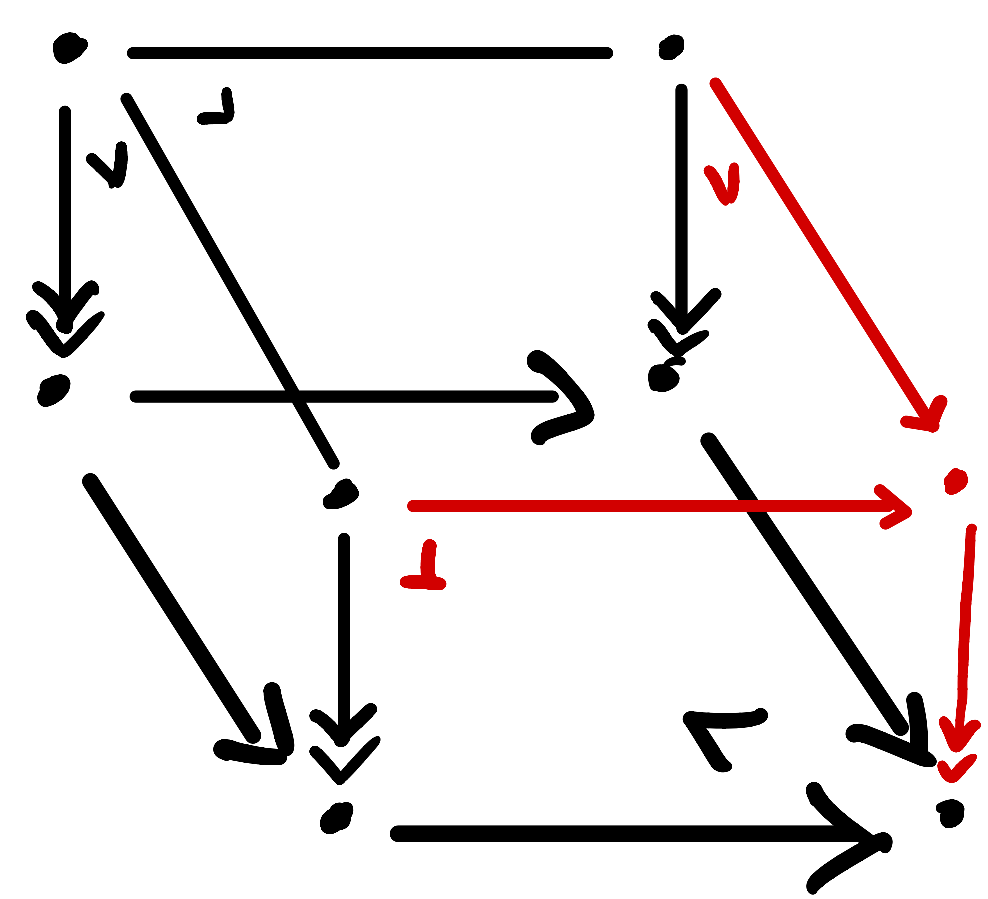
Univalence in an ∞-Topos

Prop. For a univalent universe $\mathcal{U}$ of (small) families, there's an equivalence:

$$\mathcal{E}/^{s}_{X} \simeq \underline{\mathrm{Hom}}(X, \mathcal{U}) .$$

Cor. (descent)

$$\mathcal{E}/^{s}_{\varinjlim_{i} X_i} \simeq \varprojlim_{i} \mathcal{E}/^{s}_{X_i}$$



Models of Univalence

Thm The following QMCs have univalent $\mathcal{U}$:

(i) $(sSet, Kan)$ VV 2011

(ii) $(sSet^{\mathbb{C}}, Reedy)$ Shulman 2014

(iii) $(cSet, TT)$ Coquand-Sattler 2016

(iv) $(Set^{\mathbb{C}}, \text{tiny } I)$ A. 2020

Examples of (iv) include realizability and various (symmetric) cubical sets.