First-Order Logical Duality

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A dissertation submitted in partial fulfillment of the
requirements for the degree of
PhD in Logic, Computation and Methodology

Carnegie Mellon University
2008

Abstract

Generalizing Stone duality for Boolean algebras, an adjunction between Boolean coherent categories—representing first-order syntax—and certain topological groupoids—representing semantics—is constructed. The embedding of a Boolean algebra into a frame of open sets of a space of 2-valued models is replaced by an embedding of a Boolean coherent category, $\mathcal{B}$, into a topos of equivariant sheaves on a topological groupoid of set-valued models and isomorphisms between them. The latter is a groupoid representation of the topos of coherent sheaves on $\mathcal{B}$, analogously to how the Stone space of a Boolean algebra is a spatial representation of the ideal completion of the algebra, and the category $\mathcal{B}$ can then be recovered from its semantical groupoid, up to pretopos completion. By equipping the groupoid of sets and bijections with a particular topology, one obtains a particular topological groupoid which plays a role analogous to that of the discrete space 2, in being the dual of the object classifier and the object one 'homs into' to recover a Boolean coherent category from its semantical groupoid. Both parts of the adjunction, then, consist of 'homming into sets', similarly to how both parts of the equivalence between Boolean algebras and Stone spaces consist of 'homming into 2'.

By slicing over this groupoid (modified to display an alternative setup), Chapter 3 shows how the adjunction specializes to the case of first-order single sorted logic to yield an adjunction between such theories and an independently characterized slice category of topological groupoids such that the counit component at a theory is an isomorphism.

Acknowledgements

I would like, first and foremost, to thank my supervisor Steve Awodey.
I would like to thank the members of the committee: Jeremy Avigad, Lars Birkedal, James Cummings, and Dana Scott.
I would like to thank the faculty and staff of the department of philosophy at Carnegie Mellon University. I would like to thank Michael Warren, Kohei Kishida, and my fellow students at the departments of philosophy and mathematics at Carnegie Mellon University.
I would like to thank Elise for her patience and support.

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Chapter 1

Introduction

1.1 Algebra, Geometry, and Logic

In this thesis, I present an extension of Stone Duality for Boolean Algebras from classical propositional logic to classical first-order logic. The leading idea is, in broad strokes, to take the traditional logical distinction between syntax and semantics and analyze it in terms of the classical mathematical distinction between algebra and geometry, with syntax corresponding to algebra and semantics to geometry. Insights from category theory allows us glean a certain duality between the two notions of algebra and geometry. We see a first glimpse of this in Stone's duality theorem for Boolean algebras, the categorical formulation of which states that a category of 'algebraic' objects (Boolean algebras) is the categorical dual of a category of 'geometrical' objects (Stone spaces). "Categorically dual" means that the one category is opposite to the other, in that it can be obtained from the other by formally reversing the morphisms. In a more far reaching manner, this form of algebra-geometry duality is exhibited in modern algebraic geometry as reformulated in the language of schemes in the Grothendieck school. E.g. in the duality between the categories of commutative rings and the category of affine schemes.

On the other hand, we are informed by the area of category theory known as categorical logic that algebra is closely connected with logic, in the sense that logical theories can be seen as categories and suitable categories can be seen as logical theories. For instance, Boolean algebras correspond to classical propositional theories, equational theories correspond to categories with finite products, Boolean coherent categories correspond to classical first-order logic, and topoi—e.g. of sheaves on a space—correspond to higher-order intuitionistic logic. Thus the study of these algebraic objects has logical interpretation and, vice versa, reasoning in or about logical theories has application in their corresponding algebraic objects. With the connection between algebra and logic in hand, instances of the algebra-geometry duality can be seen to manifest a syntax-semantics duality between an algebra of syntax and a geometry of semantics.

Stone duality, in its logical interpretation, manifests a syntax-semantics duality for propositional logic. As the category of Boolean algebras can be considered as the category of propositional theories modulo 'algebraic' equivalence, the category of Stone spaces can be seen as the category of spaces of corresponding two-valued models. We obtain the set of models corresponding to a Boolean algebra by taking morphisms in the category of Boolean algebras from the given algebra into the two-element Boolean algebra, 2,

$$\mathrm{Mod}_{\mathcal{B}} \cong \mathrm{Hom}_{\mathbf{BA}}(\mathcal{B}, 2) \tag{1.1}$$

And with suitable topologies in place, we can retrieve the Boolean algebra by taking morphisms in the category of Stone spaces from that space into the two-element Stone space, 2,

$$\mathcal{B} \cong \mathrm{Hom}_{\mathbf{Stone}}(\mathrm{Mod}_{\mathcal{B}}, 2)$$

Here, the two-element set, 2, is in a sense living a 'dual' life, and 'homming into 2' forms an adjunction between (the opposite of) the 'syntactical' category of Boolean algebras and the category of topological spaces, which becomes an equivalence once we restrict to the 'semantical' subcategory of Stone spaces.

For equational (or algebraic) theories, i.e. those formulated in languages without relation symbols and with all axioms equations, an example of syntax-semantics duality occurred already in F.W. Lawvere's thesis ([16]). Such a theory $\mathbb{A}$ can be considered, up to 'algebraic' or categorical equivalence, as a particular category $\mathcal{C}_{\mathbb{A}}$ with finite products, and the category of set-valued models of the theory $\mathbb{A}$ is then the category of finite product preserving functors $\mathcal{C}_{\mathbb{A}} \longrightarrow \mathbf{Sets}$ from $\mathcal{C}_{\mathbb{A}}$ into the category of sets and functions,

$$\mathrm{Mod}_{\mathbb{A}} \simeq \mathrm{Hom}_{\mathcal{FP}}(\mathcal{C}_{\mathbb{A}}, \mathbf{Sets}) \tag{1.2}$$

Then $\mathcal{C}_{\mathbb{A}}$ can be recovered from the category $\mathrm{Mod}_{\mathbb{A}}$ of models as the category $\mathrm{Hom}_{\mathcal{G}}(\mathrm{Mod}_{\mathbb{A}}, \mathbf{Sets})$ of those functors $F : \mathrm{Mod}_{\mathbb{A}} \longrightarrow \mathbf{Sets}$ that preserve all limits, filtered colimits, and regular epimorphisms,

$$\mathcal{C}_{\mathbb{A}} \simeq \mathrm{Hom}_{\mathcal{G}}(\mathrm{Mod}_{\mathbb{A}}, \mathbf{Sets})$$

We think of those as the 'continuous' maps in this context. In a wider perspective, 'homming into the dual object **Sets**' creates an adjunction between (the opposite of) the category of categories with finite products—in which the theories live—and category of cocomplete categories—in which the corresponding model categories live. An adjunction which can be restricted to an equivalence between the 'syntactical' and 'semantical' subcategories of theories and models.

Full first-order theories, too, form a category when considered up to algebraic equivalence, namely the category of Boolean coherent categories and coherent functors between them. This category contains the category of sets and functions, **Sets**, homming into which produces the usual set-valued models that often are of particular interest for first-order theories. We consider these models together with model isomorphisms between them, so as to form a 'semantical' groupoid. We then show how these semantical groupoids can be 'geometrized' (i.e. topologized) so that we can retrieve, up to 'Morita' equivalence, the theories that produced them by homming into the geometrized groupoid of sets and bijections. This then forms the basis for a syntax-semantics adjunction between the category of theories and a category of geometrized groupoids. In so doing, we produce an alternative to the Stone type adjunction constructed by M. Makkai. In [19], Makkai considers the collection of models of a theory as a groupoid equipped with additional structure involving ultraproducts, limit ultrapowers, ultramorphisms, and certain relations on hom-sets, and it is shown that a theory can, then, be recovered up to equivalence as the structure preserving morphisms from this object into **Sets**. Our approach equips groupoids of models instead with topological structure and employs the theory of Grothendieck topoi to study the relationship between 'logical' or 'syntactical' categories and topological groupoids. We equip the semantical groupoid of a theory with a 'logical' topology—generalizing the 'logical' topology used to topologize the set of models of a propositional theory to obtain a Stone space—and we consider (equivariant) sheaves on the resulting topological groupoid in order to recover the theory. Thus we have a comparatively more geometrical setup which equips the semantical side with topological structure rather than structure based on ultraproducts. The result is a setup which uses a natural generalization of the structure which yields the classical Stone duality in the propositional case, and a first-order syntax-semantics duality which specializes in the propositional case to Stone's classical result.

1.2 Logical Dualities

The first-order logical duality presented in this thesis is a generalization of classical Stone duality both in the overall formal structure of the proof that we present and in the sense of yielding the classical result as a specialization. We now present the most relevant aspects of the classical Stone duality and its logical interpretation, as well as briefly recalling two other examples of syntax-semantics duality in Lawvere's duality for equational theories and Makkai's duality for first-order logic. We devote particular attention to these due to their logical interpretations. However, logical interpretation aside, it should also be emphasized that Stone duality stands as a classical and important result of mathematics, with points in common with other classical mathematical algebra-geometry or discrete-compact dualities, such as Pontryagin duality or Gelfand duality. See e.g. [9].

1.2.1 Propositional Logic and Stone Duality

Any propositional theory has an associated Boolean algebra: Identify a propositional theory $\mathbb{S}$ in language $\mathcal{L}_{\mathbb{S}}$ with the set S consisting of $\mathbb{S}$-provable equivalence classes of formulas in $\mathcal{L}_{\mathbb{S}}$. That is to say, formulas ϕ, ψ are equivalent if and only if $\mathbb{S} \vdash \phi \leftrightarrow \psi$. Then observe that the operations $[\phi] \wedge [\psi] = [\phi \wedge \psi]$, $[\phi] \vee [\psi] = [\phi \vee \psi]$, $\neg[\phi] = [\neg\phi]$, and the order $[\phi] \leq [\psi] \Leftrightarrow \mathbb{S} \vdash \phi \rightarrow \psi$ respects the equivalence relation and makes S a Boolean algebra. We refer to this as the *Tarski-Lindenbaum algebra* of $\mathbb{S}$, and denote it $B_{\mathbb{S}}$. Conversely, we can associate a Boolean algebra with a particular propositional theory. For Boolean algebra B, build language $\mathcal{L}_B$ by having a propositional constant P_b for each $b \in B$. Construct propositional theory $\mathbb{S}_B$ in $\mathcal{L}_B$ by adding for each pair $b, c \in B$ an axiom $P_b \rightarrow P_c$ if $b \leq c$, and axioms $P_b \wedge P_c \leftrightarrow P_{b \wedge c}$, $P_{\neg b} \leftrightarrow \neg P_b$. Call $\mathbb{S}_B$ for the *theory of* B. Now, a Boolean algebra is isomorphic to the Tarski-Lindenbaum algebra of its theory. But a propositional theory can not in general be said to be the same as the theory of its Tarski-Lindenbaum algebra, as it will usually be in a different language. Call two propositional theories algebraically or categorically equivalent if their Tarski-Lindenbaum algebras are isomorphic. Then a propositional theory and the theory of its Tarski-Lindenbaum algebra are algebraically equivalent. Now that we know to think of propositional theories and Boolean algebras as essentially the same, we recall the classical Stone representation and duality for Boolean algebras before we return to its logical interpretation and use.

Stone Duality

We sketch a presentation of Stone duality in a form that we proceed to generalize to first-order logic in Chapter 2. Recall the Stone representation theorem:

Theorem 1.2.1.1 (Stone Representation) *Any Boolean algebra is isomorphic to a field of sets.*

PROOF Given Boolean algebra B, let X^u_B be the set of ultrafilters on B. Define a map from B to the (complete) lattice of subsets of X^u_B by:

$$\begin{array}{ccc} B & \xrightarrow{\quad M_d \quad} & \mathcal{P}(X^u_B) \\ & & \\ b & \mapsto & \{U \in X^u_B \mid b \in U\} \end{array}$$

Then verify that M_d ('M discrete') is an injective morphism of lattices. ⊣

We identify ultrafilters on B with Boolean algebra (lattice) morphisms

$$B \longrightarrow 2$$

and write

$$X_B := \mathrm{Hom}_{\mathbf{BA}}(B, 2) \cong X^u_B$$

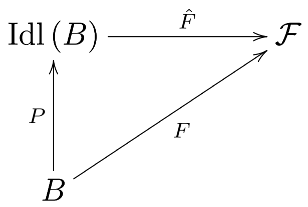
One can verify that the morphism $M_d : B \longrightarrow \mathcal{P}(X^u_B) \cong \mathcal{P}(X_B)$ is *cover reflecting* in the sense that for any set of elements $\{b_i \mid i \in I\}$ and b in B, if $M_d(b) \subseteq \bigcup_{i\in I} M_d(b_i)$ then there exists $b_{i_1}, \ldots, b_{i_n}$ such that $b \leq b_{i_1} \vee \ldots \vee b_{i_n}$.

By equipping X_B with a topology, one can characterize the sets in the image of M_d in terms of that topology. In fact, one can do somewhat better. Recall that a *frame* is a lattice with infinite joins satisfying the infinitary distributive law

$$a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I} a \wedge b_i$$

A frame is necessarily a complete lattice, but a morphism of frames is (a lattice morphism which is) only required to preserve infinite joins, and not infinite meets. The open sets of a topological space X form a frame $\mathcal{O}(X)$, and a continuous map $f : X \to Y$ induces a morphism of frames $f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ by inverse image. For each Boolean algebra B, there is an associated *ideal completion*, $\mathrm{Idl}\,(B)$, consisting of ideals of B, that is nonempty sets of elements of B that are closed downwards and under finite joins.

Proposition 1.2.1.2 *The ideal completion* $\mathrm{Idl}\,(B)$ *is the free frame on B, in the sense that for any frame* $\mathcal{F}$ *and any lattice morphism* $F : B \longrightarrow \mathcal{F}$ *there is a unique morphism of frames* $\hat{F} : \mathrm{Idl}\,(B) \longrightarrow \mathcal{F}$ *such that the triangle*



commutes, where P is the principal ideal embedding $b \mapsto \downarrow b$.

Proof Straightforward. The joins in $\mathrm{Idl}\,(B)$ are given by closing unions under finite joins. ⊣

B can, thus, be identified with the sublattice of principal ideals in $\mathrm{Idl}\,(B)$. The principal ideals can also be characterized as the *compact* elements of $\mathrm{Idl}\,(B)$, that is, the elements A such that for any covering of A by a family of elements A_i,

$$A \leq \bigvee_{i \in I} A_i$$

there exists $A_{i_1}, \ldots, A_{i_n}$ such that

$$A \leq A_{i_1} \vee \ldots \vee A_{i_n}$$

We display this fact:

Lemma 1.2.1.3 *B can be recovered from* $\mathrm{Idl}\,(B)$ *as the sublattice of compact elements.*

Theorem 1.2.1.4 *There exists a topology on X_B so that*

$$\mathcal{O}(X_B) \cong \mathrm{Idl}\,(B)$$

PROOF Define a topology on X_B by taking as a basic all sets of the form

$$U_b = M_d(b) = \{G : B \to 2 \mid G(b) = 1\}$$

for $b \in B$. Then M_d factors through $\mathcal{O}(X_B)$,

$$B \xrightarrow{M} \mathcal{O}(X_B) \hookrightarrow \mathcal{P}(X_B)$$

and with $M : B \longrightarrow \mathcal{O}(X_B)$ a lattice morphism, there is a unique morphism of frames $\hat{M}$ such that the triangle

$$\begin{array}{ccc} \mathrm{Idl}\,(B) & \xrightarrow{\hat{M}} & \mathcal{O}(X_B) \\ {\scriptstyle P}\uparrow & \nearrow_{M} & \\ B & & \end{array}$$

commutes. Then, from the fact that M is injective and cover reflecting and that the objects in the image of M generate $\mathcal{O}(X_B)$ (i.e. they form a basis for the topology), conclude that $\hat{M} : B \longrightarrow \mathcal{O}(X_B)$ is an isomorphism of frames. $\dashv$

Corollary 1.2.1.5 *B can be recovered from $\mathcal{O}(X_B)$ as the sublattice of compact elements, i.e. B can be recovered from X_B as the compact open sets.*

Moreover, since B is a Boolean algebra, the complement of the (compact) open set $M(b) = \{G : B \to 2 \mid G(b) = 1\}$ is the (compact) open set $M(\neg b) = \{G : B \to 2 \mid G(\neg b) = 1\}$. So $X_B = M(\top)$ is compact with a basis of clopen sets. And if $G, H : B \to 2$ are two distinct lattice morphisms, there exists $b \in B$ so that $G \in M(b)$ and $H \notin M(b)$, so X_B is Hausdorff. A compact Hausdorff space with a basis of clopen sets is a *Stone space.*

In summary, then, any Boolean algebra B has an associated Stone space X_B from which it can be recovered as the compact open sets, or equivalently, the clopen sets. Now, the assignment of a Boolean algebra to its Stone space is functorial, in the sense of being the object part of a contravariant functor from the category of Boolean algebras to the category of topological spaces

$$\mathrm{Hom}_{\mathbf{BA}}\,(-, 2) : \mathbf{BA}^{\mathrm{op}} \longrightarrow \mathbf{Top}$$

which sends a Boolean algebra B to the Stone space X_B and a Boolean algebra morphism $F : \mathcal{B} \longrightarrow \mathcal{D}$ to the continuous function $\mathrm{Hom}_{\mathbf{BA}}(F, 2)$ which is defined by precomposition:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{BA}}(F, 2) : X_{\mathcal{D}} &\longrightarrow X_B \\ G : \mathcal{D} \to 2 &\longmapsto F \circ G : B \to 2 \end{aligned}$$

This functor has a left adjoint,

$$\mathrm{Hom}_{\mathbf{Top}}(-, 2) : \mathbf{Top} \longrightarrow \mathbf{BA}^{\mathrm{op}}$$

which sends a topological space Y to its Boolean algebra of clopen sets, also describable as the set of continuous functions from Y to the Stone space 2,

$$\mathrm{Clopen}(Y) \cong \mathrm{Hom}_{\mathbf{Top}}(Y, 2)$$

and a continuous function $f : Y \longrightarrow Z$ to the Boolean algebra morphism $\mathrm{Hom}_{\mathbf{Top}}(f, 2)$ which is defined by precomposition:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Top}}(f, 2) : \mathrm{Hom}_{\mathbf{Top}}(Z, 2) &\longrightarrow \mathrm{Hom}_{\mathbf{Top}}(Y, 2) \\ g : Z \to 2 &\longmapsto f \circ g : Y \to 2 \end{aligned}$$

From Corollary 1.2.1.5, we deduce that the lattice morphism

$$M : B \longrightarrow \mathcal{O}(X_B)$$

factors as an isomorphism followed by an embedding:

$$\mathcal{B} \cong \mathrm{Clopen}(X_B) \hookrightarrow \mathcal{O}(X_B)$$

That isomorphism is the counit component at B of the adjunction

$$\mathbf{BA}^{\mathrm{op}} \underset{\mathrm{Hom}_{\mathbf{BA}}(-,2)}{\overset{\mathrm{Hom}_{\mathbf{Top}}(-,2)}{\rightleftarrows}} \mathbf{Top} \quad (\bot)$$

whence the adjunction restricts to an equivalence on the image of the (full and faithful) right adjoint, which is the full subcategory **Stone** of Stone spaces,

$$\mathbf{BA}^{\mathrm{op}} \underset{\mathrm{Hom}_{\mathbf{BA}}(-,2)}{\overset{\mathrm{Hom}_{\mathbf{Stone}}(-,2)}{\overset{\longleftarrow}{\underset{\longrightarrow}{\simeq}}}} \mathbf{Stone}$$

and we have:

Theorem 1.2.1.6 (Stone Duality) *There is a contravariant equivalence of categories between the category* **BA** *of Boolean algebras and Boolean algebra homomorphisms and the category* **Stone** *of Stone spaces and continous functions.*

$$\mathbf{BA}^{\mathrm{op}} \simeq \mathbf{Stone}$$

Logical Interpretation and Significance

Consider now a Boolean algebra as a Lindenbaum-Tarski algebra $B_{\mathbb{S}}$ for a propositional theory $\mathbb{S}$ in language $\mathcal{L}_{\mathbb{S}}$. A *two-valued structure* for $\mathcal{L}_{\mathbb{S}}$ is a map from the set of propositional constants to the set $\{0,1\} =: 2$, which we inductively extend in the usual manner to a map from $\mathcal{L}_{\mathbb{S}}$ to 2. A (two-valued) *model* is a structure where all formulas provable in $\mathbb{S}$ receive the value 1 under this extension. Equivalently, a model is a Boolean algebra homomorphism from $B_{\mathbb{S}}$ to the Boolean algebra 2, so the set

$$X_{B_{\mathbb{S}}} = \mathrm{Hom}_{\mathbf{BA}}\left(B_{\mathbb{S}}, 2\right)$$

can be considered to be the set of two-valued models of $\mathbb{S}$. Furthermore, the topology of Theorem 1.2.1.4 on the set of models is a 'logical' topology, in that the basic open sets are given in terms of sentences ϕ of $\mathcal{L}_{\mathbb{S}}$ as the sets of models in which ϕ is true:

$$U_{[\phi]} = \left\{\mathbf{M} \in X_{B_{\mathbb{S}}} \;\middle|\; \mathbf{M} \vDash \phi\right\}, \quad [\phi] \in X_{B_{\mathbb{S}}}$$

Thus, if a Boolean algebra is considered as a propositional theory, then its corresponding Stone space can be considered as its space of models equipped with a logical topology. This endows Stone duality with both conceptual and technical significance for logic. Conceptually, for instance, we obtain a geometrical perspective on the classical completeness and compactness theorems: First, that a Boolean algebra $B_{\mathbb{S}}$ embeds, by the Stone Representation Theorem (1.2.1.1), homomorphically into the Boolean algebra of subsets of $\mathbb{S}$-models by sending a sentence to the set of models where it is true, now tells us that the provable sentences are exactly those that are true in all models. Second, if we are given a set Γ of sentences of $\mathbb{S}$ such that any finite subset has a model, then that is saying that the family of clopen sets $\left\{U_{[\phi]} \;\middle|\; \phi \in \Gamma\right\}$ has the finite intersection property, and since $X_{B_{\mathbb{S}}}$ is compact, that means that

$$\bigcap_{\phi\in\Gamma} U_{[\phi]} \neq \emptyset$$

so Γ has a model. Whence the compactness theorem for propositional logic. On the more technical side, an important application of Stone duality in modern model theory is found in the *Stone space of types*. For a first-order theory $\mathbb{T}$, a complete n-type is a set $p(\vec{x})$ of formulas with all free variables in $x_1, \ldots, x_n$ such that for any formula $\phi(\vec{x})$, either $\phi(\vec{x})$ or $\neg\phi(\vec{x})$ is in $p(\vec{x})$ and $\mathbb{T} \cup p(\vec{x})$ is (finitely) satisfiable ([20]). Accordingly, a complete n-type $p(\vec{x})$ can be seen as an ultrafilter in the Boolean algebra of $\mathbb{T}$-provably equivalent formulas with free variables in $x_1, \ldots, x_n$ ordered by provability, and the corresponding Stone space (denoted $S_n(\mathbb{T})$ in e.g. [20]) is known as the Stone space of types. As an example of its use, we have e.g. the following ([20, Theorem 4.2.10]):

Theorem 1.2.1.7 *Let $\mathcal{L}$ be a countable language and let $\mathbb{T}$ be a complete $\mathcal{L}$-theory with infinite models. Then, the following are equivalent:*

1. *$\mathbb{T}$ has a prime model;*
2. *$\mathbb{T}$ has an atomic model;*
3. *the isolated points in $S_n(\mathbb{T})$ are dense for all n.*

For a propositional theory $\mathbb{S}$, there are of course only complete 0-types, and the Stone space of 0-types is just the Stone space $X_{B_{\mathbb{S}}}$,

$$S_0(\mathbb{S}) \cong X^u_{B_{\mathbb{S}}} \cong B_{\mathbb{S}}$$

As a final example, we give an alternative proof of the Beth Definability Theorem for propositional logic in terms of the connection between syntax and semantics that Stone duality provides. Let a propositional theory $\mathbb{S}(P)$ in language $\mathcal{L}(P)$ be given, where P is a propositional constant in the language. Let $\mathcal{L}$ be the same language with P removed, and let $\mathbb{S}$ be the theory $\mathbb{S}(P) \cap \mathcal{L}$. Say that $\mathbb{S}(P)$ *implicitly defines* P if any $\mathbb{S}$-model extends to at most one $\mathbb{S}(P)$-model. Say that $\mathbb{S}(P)$ *explicitly defines* P if there exists an $\mathcal{L}$-sentence ϕ such that $\mathbb{S}(P) \vdash \phi \leftrightarrow P$.

Theorem 1.2.1.8 (Beth Definability: Propositional Logic) *$\mathbb{S}(P)$ implicitly defines P if and only if $\mathbb{S}(P)$ explicitly defines P.*

PROOF The right to left case is immediate. Let

$$F : B_{\mathbb{S}} \longrightarrow B_{\mathbb{S}(P)}$$

be the Boolean algebra morphism defined by $[\phi] \mapsto [\phi]$, for ϕ in $\mathcal{L}$. Applying Stone duality, the dual of F is then the continuous function

$$\mathrm{Hom}_{\mathbf{BA}}(F) = f : X_{B_{\mathbb{S}(P)}} \longrightarrow X_{B_{\mathbb{S}}}$$

which sends a $\mathbb{S}(P)$-model to its underlying $\mathbb{S}$-model. Suppose that $\mathbb{S}(P)$ implicitly defines P. Then f is injective. Since $X_{B_{\mathbb{S}(P)}}$ is compact and $X_{B_{\mathbb{S}}}$ is Hausdorff, f is a closed map. Hence f is a (closed) subspace embedding, and therefore a regular monomorphism in **Top**. It is straightforward to check that it is therefore also a regular monomorphism in **Stone**, and by the duality $\mathbf{BA}^{\mathrm{op}} \simeq \mathbf{Stone}$, we have that $F : B_{\mathbb{S}} \longrightarrow B_{\mathbb{S}(P)}$ is a regular epimorphism in **BA**. Since the category **BA** is algebraic (see e.g. [9]), regular epimorphisms are surjections, and so there exist a sentence ϕ in $\mathcal{L}$ so that $F([\phi]) = [P]$, whence $\mathbb{S}(P) \vdash \phi \leftrightarrow P$. ⊣

1.2.2 Lawvere's Duality for Equational Logic

An *equational* theory is a theory in a language with no relation symbols (except equality), quantifiers, or connectives and all axioms equations. See eg [8] for a formal presentation. Such theories are often referred to as *algebraic*. Each equational theory has an associated category. Consider an algebraic theory $\mathbb{A}$ in language $\mathcal{L}_{\mathbb{A}}$. The associated *syntactic* category $\mathcal{C}_{\mathbb{A}}$ consists of denumerable many objects $T_0, T_1, \ldots, T_n, \ldots$ and it has as morphisms finite lists of $\mathbb{A}$-provable equivalence classes of terms, e.g.:

$$[f_1, \ldots, f_m] : T_n \longrightarrow T_m$$

where f_i has arity n. T_0 is then the terminal object of $\mathcal{C}_{\mathbb{A}}$, and the product of T_n and T_m is T_{n+m}. Conversely, any *equational* category, that is any category with denumerably many objects $T_0, T_1, \ldots, T_n, \ldots$ where T_n is the nth power of T_1, gives rise to an equational theory by taking the morphisms of the category as function symbols and letting the equalities between morphisms of the category determine the axioms of the theory (see [2] or [8]). Equational theories are *categorically* equivalent if their syntactic categories are equivalent. The theory of a syntactic category $\mathcal{C}_{\mathbb{A}}$ is categorically equivalent to $\mathbb{A}$.

An equational category is, in particular, a category with finite products. We can take a morphism between equational categories $F : \mathcal{T} \longrightarrow \mathcal{R}$ to be a finite product preserving functor such that $F(T_n) = R_n$ for all n. The category $\mathcal{E}$ of equational categories is then a subcategory (not full) of the category $\mathcal{FP}$ of categories with finite products and finite product preserving functors between them. In $\mathcal{FP}$ we also find the category **Sets** as an object. The functor $\mathrm{Hom}_{\mathcal{FP}}(-, \mathbf{Sets})$ is contravariant from $\mathcal{FP}$ into the category $\mathcal{G}$ of categories with all limits and colimits, with morphisms being limit, filtered colimit, and regular epimorphism preserving functors. In $\mathcal{G}$ we find, again, the category **Sets** as an object. For each $\mathcal{M} \in \mathcal{G}$, the functor η_M, or 'evaluate at (-)':

$$\eta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathrm{Hom}_{\mathcal{FP}}(\mathrm{Hom}_{\mathcal{G}}(\mathcal{M}, \mathbf{Sets}), \mathbf{Sets})$$

$$M \longmapsto (-)(M)$$

is universal from $\mathcal{M}$ to the functor $\mathrm{Hom}_{\mathcal{FP}}(-, \mathbf{Sets}) : \mathcal{FP} \longrightarrow \mathcal{G}$, and so it is the unit of an adjunction:

$$\mathcal{FP}^{\mathrm{op}} \underset{\mathrm{Hom}_{\mathcal{FP}}(-,\mathbf{Sets})}{\overset{\mathrm{Hom}_{\mathcal{G}}(-,\mathbf{Sets})}{\rightleftarrows}} \mathcal{G} \qquad \bot \tag{1.3}$$

Returning now to the subcategory $\mathcal{E} \hookrightarrow \mathcal{FP}$ of equational categories, we recall that the syntax-semantics duality for equational theories is obtained by restricting the adjunction to $\mathcal{E}$, as follows:

A set-valued model of an algebraic theory $\mathbb{A}$ is precisely a finite product preserving functor from $\mathcal{C}_{\mathbb{A}}$ to **Sets**. And a morphism of models is a natural transformation of such functors. Thus we can identify the category of set-valued models of $\mathbb{A}$, $\mathrm{Mod}_{\mathbb{A}}$, with the full subcategory of finite product preserving functors in the functor category $\mathbf{Sets}^{\mathcal{C}_{\mathbb{A}}}$:

$$\mathrm{Mod}_{\mathbb{A}} \cong \mathrm{Hom}_{\mathcal{FP}}\left(\mathcal{C}_{\mathbb{A}}, \mathbf{Sets}\right) \hookrightarrow \mathbf{Sets}^{\mathcal{C}_{\mathbb{A}}}$$

Now, considering algebraic theories up to equivalence—i.e. as objects of $\mathcal{E}$—we can thus consider the restriction of the functor $\mathrm{Hom}_{\mathcal{FP}}\left(-, \mathbf{Sets}\right) : \mathcal{FP}^{\mathrm{op}} \longrightarrow \mathcal{G}$ to the category $\mathcal{E}$ as a semantic functor, taking a theory to its models:

$$\mathrm{Mod}_{-} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{MOD}$$

$$\mathcal{T} \longmapsto \mathrm{Hom}_{\mathcal{FP}}\left(\mathcal{T}, \mathbf{Sets}\right)$$

where $\mathcal{MOD}$ is the image of the functor $\mathrm{Hom}_{\mathcal{FP}}\left(-, \mathbf{Sets}\right) : \mathcal{FP}^{\mathrm{op}} \longrightarrow \mathcal{G}$ restricted to $\mathcal{E}^{\mathrm{op}}$.

The category $\mathcal{MOD}$ can be independently characterized. For $\mathcal{T}$ with objects $T_0, T_1, \ldots, T_n, \ldots$ the object $\mathrm{Hom}_{\mathcal{T}}\left(T_1, -\right) \in \mathrm{Mod}_{\mathcal{T}}$ represents the forgetful functor

$$U : \mathrm{Mod}_{\mathcal{T}} \longrightarrow \mathbf{Sets}$$

which send a model to its underlying set,

$$U(\mathbf{M}) = M \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{T}}}\left(\mathrm{Hom}_{\mathcal{T}}\left(T_1, -\right), \widetilde{M}\right)$$

with $\mathbf{M}$ a model, M its underlying set, and $\widetilde{M}$ the functor $\mathcal{T} \longrightarrow \mathbf{Sets}$ corresponding to the model. Notice that all functors of the form $\mathrm{Hom}_{\mathcal{T}}\left(T_n, -\right) : \mathcal{T} \longrightarrow \mathbf{Sets}$ preserves limits, so they correspond to $\mathcal{T}$-models. Now, U preserves regular epimorphisms and reflects isomorphisms, and it has a left adjoint F,

$$\mathrm{Mod}_{\mathcal{T}} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Sets} \qquad (F \dashv U)$$

such that UF preserves filtered colimits. F sends a set X to the free $\mathcal{T}$-model on X. For a finite set n, we have that $F(n) \cong \mathrm{Hom}_{\mathcal{T}}\left(T_n, -\right)$, and the full subcategory of $\mathrm{Mod}_{\mathcal{T}}$ with objects $F(n)$, $n \in \mathbb{N}$, is equivalent to the opposite of the equational category $\mathcal{T}$. Thus, writing $\mathcal{F}_{\mathcal{T}}$ for the category of finitely generated free models of $\mathcal{T}$,

$$\mathcal{T}^{\mathrm{op}} \simeq \mathcal{F}_{\mathcal{T}}$$

Conversely, for any category $\mathcal{C}$ and functor $U : \mathcal{C} \longrightarrow \mathbf{Sets}$ such that $\mathcal{C}$ has coequalizers and kernel pairs and U preserves regular epimorphisms, reflects isomorphisms, and has a left adjoint F such that UF preserves filtered colimits, the full subcategory $\mathcal{R}$ of $\mathcal{C}$ generated by the objects $F(n)$, $n \in \mathbb{N}$, has finite coproducts, and $\mathcal{C}$ is equivalent to the category of models of $\mathcal{R}^{\mathrm{op}}$:

$$\mathcal{C} \simeq \mathrm{Mod}_{\mathcal{R}^{\mathrm{op}}}$$

We can recover a theory $\mathcal{T}$ from $\mathrm{Mod}_{\mathcal{T}}$ by homming into **Sets**. Any limit preserving functor $H : \mathrm{Mod}_{\mathcal{T}} \longrightarrow \mathbf{Sets}$ is representable, and if it in addition preserves filtered colimits, then it is represented by a finitely presentable model. If H furthermore preserves regular epimorphisms, then it is represented by a finitely generated free model, so that

$$\mathcal{F}_{\mathcal{T}}{}^{\mathrm{op}} \simeq \mathcal{T} \simeq \mathrm{Hom}_{\mathcal{G}}\left(\mathrm{Mod}_{\mathcal{T}}, \mathbf{Sets}\right)$$

where $\mathrm{Hom}_{\mathcal{G}}\left(\mathrm{Mod}_{\mathcal{T}}, \mathbf{Sets}\right)$ is the category of limit, filtered colimit, and regular epimorphism preserving functors. Hence the functor $\mathrm{Hom}_{\mathcal{G}}\left(-, \mathbf{Sets}\right) : \mathcal{G} \longrightarrow \mathcal{FP}^{\mathrm{op}}$ when restricted to the subcategory $\mathcal{MOD}$ takes values in $\mathcal{E}^{\mathrm{op}}$, and we have that the adjunction (1.3) between $\mathcal{FP}^{\mathrm{op}}$ and $\mathcal{G}$ restricts to an equivalence

$$\mathcal{E}^{\mathrm{op}} \simeq \mathcal{MOD}$$

so that, for $\mathcal{T} \in \mathcal{E}$ and $\mathcal{M} \in \mathcal{MOD}$ we have the following equivalences:

$$\mathcal{T} \simeq \mathrm{Hom}_{\mathcal{G}}\left(\mathrm{Hom}_{\mathcal{FP}}\left(\mathcal{T}, \mathbf{Sets}\right), \mathbf{Sets}\right)$$
$$\mathcal{M} \simeq \mathrm{Hom}_{\mathcal{FP}}\left(\mathrm{Hom}_{\mathcal{G}}\left(\mathcal{M}, \mathbf{Sets}\right), \mathbf{Sets}\right)$$

(which display a compelling analogy to the previous case of propositional logic).

Theorem 1.2.2.1 *There is an adjunction between the category of equational categories and the category of models*

$$\mathcal{E}^{\mathrm{op}} \underset{\mathrm{Hom}_{\mathcal{E}}(-,\mathbf{Sets})}{\overset{\mathrm{Hom}_{\mathcal{MOD}}(-,\mathbf{Sets})}{\rightleftarrows}} \mathcal{MOD} \quad (\perp) \tag{1.4}$$

such that unit and counit components are all equivalences.

1.2.3 Makkai's First-Order Logical Duality

A (classical, many-sorted) first-order theory, $\mathbb{T}$, has an associated *syntactic category*, $\mathcal{C}_{\mathbb{T}}$. The objects of $\mathcal{C}_{\mathbb{T}}$ consists of alpha-equivalence classes of formulas-in-context, $[\vec{x} \,|\, \phi]$ of $\mathbb{T}$. An arrow between objects

$$[\vec{x} \,|\, \phi] \longrightarrow [\vec{y} \,|\, \psi]$$

consists of a $\mathbb{T}$-provable equivalence class of formulas-in-context $[\vec{x}, \vec{y} \,|\, \sigma]$ such that σ is a $\mathbb{T}$-provable functional relation between ϕ and ψ. That is, the sequents

- $\sigma \vdash_{\vec{x},\vec{y}} \phi \wedge \psi$
- $\phi \vdash_{\vec{x}} \exists \vec{y}.\, \sigma$
- $\sigma \wedge \sigma[\vec{z}/\vec{y}] \vdash_{\vec{x},\vec{y},\vec{z}} \vec{y} = \vec{z}$

are provable in $\mathbb{T}$ (this definition also makes sense for the fragment of first-order logic, called *coherent logic*, which lack the connectives $\rightarrow$, $\neg$, and $\forall$). See e.g. [10, D1] for a full introduction to syntactic categories. It is also possible to factor objects by $\mathbb{T}$-provable equivalence, and we shall find it convenient to do so in Chapters 2 and 3. The syntactic theory of a classical first-order theory is a Boolean coherent category: that is, a category with finite limits; coequalizers of kernel pairs; stable (under pullback) finite joins or *unions* (coherent); and complements for all subobjects (Boolean). (The syntactic category of a coherent theory is coherent, but not necessarily Boolean.) We call two theories *categorically equivalent* if their syntactic categories are equivalent. Conversely, any Boolean coherent category is, up to equivalence, the syntactic category of a theory, obtained e.g. by having a sort for each object, a function symbol for each arrow, and appropriate axioms. Accordingly, we consider the category of first-order theories to be the category of Boolean coherent categories with coherent functors between them. That is, functors that preserve finite limits, coequalizers of kernel pairs, and unions (it follows that complements are preserved). The coherent functors from a Boolean coherent category to the (Boolean coherent) category of **Sets** are considered as its models, with invertible natural transformations the model isomorphisms. If a Boolean coherent category has, in addition, stable, disjoint finite coproducts and coequalizers of equivalence relations, it is a *Boolean pretopos*. Any first-order theory can be conservatively extended so that its syntactical category forms a Boolean pretopos.

The Makkai Stone-type adjunction between Boolean pretopoi and so called *ultra-groupoids* proceeds from the fact that the models of a theory (Boolean pretopos) comes equipped with ultraproduct structure inherited from **Sets** and the idea that ultraproducts can be used as a basis for equipping the category of models of a pretopos with sufficient structure so that the pretopos can be recovered as the structure preserving functors from the structure enriched groupoid of models and isomorphisms into the (similarly equipped) groupoid of sets and isomorphisms. Again, we can only give the coarsest of outlines, the full details can be found in [17], [18], and [19].

An ultrafilter (I, U)—that is, U is an ultrafilter on $\mathcal{P}(I)$—gives rise to a ultraproduct functor $\mathbf{Sets}^I \longrightarrow \mathbf{Sets}$. Since **Sets** has all small colimits, one can also define the notion of a limit ultrapower, and a limit ultrapower functor $\mathbf{Sets} \longrightarrow \mathbf{Sets}$, for an ω-sequence of ultrafilters. The same structure is induced on the groupoid of models, $\underline{\mathrm{Hom}}^*(\mathcal{B}, \mathbf{Sets})$. An ultra-groupoid is a groupoid with ultraproducts and limit ultrapowers, as well as additional ultraproduct and limit ultrapower related structure, including certain relations on arrows and a notion of ultra-morphism, see [19]. Together with structure preserving (up to specified transition isomorphisms) groupoid functors (*ultra-functors*) and certain invertible natural transformations between them (*ultra-transformations*), these form the groupoid-enriched category of ultra groupoids, $\mathcal{UG}$. Let $\mathcal{BP}^*$ be the groupoid-enriched category of Boolean pretopoi, i.e. the 2-category of Boolean pretopoi, coherent functors between the pretopoi, and invertible natural transformation between functors. Sending a Boolean pretopos $\mathcal{B}$ to its ultra-groupoid $\underline{\mathrm{Hom}}^*(\mathcal{B}, \mathbf{Sets})$ of models and isomorphism defines a functor $\mathcal{BP}^{*\mathrm{op}} \longrightarrow \mathcal{UG}$. In the other direction, the category of functors from any groupoid to **Sets** is a Boolean coherent category, and it is shown that for any ultra-groupoid $\mathbf{K}$, the category consisting of ultra-functors into the ultra-groupoid of sets and lax (i.e. not necessarily invertible) ultra-transformations between them, which we can denote $\underline{\mathrm{Hom}}^l(-, \mathbf{Sets})$, also forms a Boolean coherent category. Thus an adjunction

$$\mathcal{BP}^{*\mathrm{op}} \underset{\underline{\mathrm{Hom}}^*(-,\mathbf{Sets})}{\overset{\underline{\mathrm{Hom}}^l(-,\mathbf{Sets})}{\overset{\longleftarrow}{\underset{\longrightarrow}{\perp}}}} \mathrm{UG}$$

is constructed, with evaluation functors forming unit and counit components. With the ultra-product and limit ultra-power related structure appropriately specified, the counit components

$$\epsilon_T : T \longrightarrow \underline{Hom}^l(\underline{Hom}^*(T, \mathbf{Sets}), \mathbf{Sets})$$

form equivalences of pretopoi, so that a pretopos is recovered (up to equivalence) from its groupoid of models and isomorphisms.

Theorem 1.2.3.1 *There is an adjunction between the category of Boolean pretopoi and the category of ultra-groupoids*

$$\mathcal{BP}^{*\mathrm{op}} \underset{\underline{\mathrm{Hom}}^*(-,\mathbf{Sets})}{\overset{\underline{\mathrm{Hom}}^l(-,\mathbf{Sets})}{\rightleftarrows}} \mathrm{UG} \quad (\bot)$$

such that counit components at small pretopoi are equivalences.

1.3 A Sheaf-Theoretical Approach

We present a 'syntax-semantics' duality for (classical) first order logic. As in Stone duality, as well as in Makkai's duality, the heart of the matter is the representation of the syntax of a theory in terms of its semantics. That is to say, the equipping of the models of a theory with additional, e.g. geometric, structure such that the theory can be recovered from the resulting object, at least up to some reasonable form of equivalence. The 'syntax-semantics' duality is then obtained by showing that these equivalences form the counit components of an adjunction between the category of theories and the category of semantical objects.

Analogously with the Makkai duality, we consider the groupoid of models and isomorphisms of a theory. Analogously with Stone duality, we use topological structure to equip the models and model isomorphisms of a theory with sufficient structure to recover the theory from them. The result is a first order logical duality which, in comparison to Makkai's, is more geometrical, in that it uses topology and sheaves on spaces and topological groupoids rather than ultraproducts, and that moreover specializes to the traditional Stone duality.

The 'syntax-semantics' adjunction has Boolean coherent categories representing first-order theories on the syntactical side. In fact, we can also consider coherent categories that are not Boolean, as long as they are *decidable* in the sense that each diagonal is complemented. Such categories represent coherent theories with a predicate $\neq$ for each sort satisfying axioms of inequality. We pass to the 'semantical' side by considering their models and model isomorphisms, i.e. by homming into the category **Sets** of sets and functions. Thus, as Makkai, we consider the groupoid of coherent functors into **Sets** together with invertible natural transformations. Generalizing the Stone setup for Boolean algebras, we then equip these semantical objects with a 'logical' topological structure. The Boolean or decidable coherent category with which we started can then be recovered, up to a form of equivalence, from the topological groupoid of its models and isomorphisms. In the Boolean algebra case, this was done simply by considering the frame of open sets of the space of models. In the case of a topological groupoid, however, the natural thing to consider is a 'generalized space' in the form of the topos of (equivariant) sheaves on the groupoid of models.

A topos is a locally cartesian closed category with a subobject classifier, but we shall only concern ourselves with those topoi that are *Grothendieck* topoi, that is, those that arise as sheaves on a site ([10, C2]). Accordingly, whenever we say "topos", we mean "Grothendieck topos". A geometric morphism $f : \mathcal{E} \longrightarrow \mathcal{F}$ between topoi consists of an adjoint pair,

$$\mathcal{E} \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{F}$$

where the left adjoint—also known as the *inverse image* part, while the right adjoint is known as the *direct image* part—is cartesian. In particular, a topos is a coherent category and an inverse image functor is a coherent functor. A geometric transformation between a parallel pair of geometric morphisms is a natural transformation between their inverse image parts.

We will be mainly interested in three kinds of topoi. First, for any coherent category $\mathcal{C}$, equipping $\mathcal{C}$ with the so called *coherent coverage*, J—that is, the coverage generated by finite epimorphic families—produces a topos $\mathrm{Sh}(\mathcal{C}, J)$ of sheaves with the property that there is an equivalence of categories,

$$\underline{\mathrm{Hom}}_{\mathbf{Coh}}(\mathcal{C}, \mathcal{E}) \simeq \underline{\mathrm{Hom}}_{\mathcal{TOP}}(\mathcal{E}, \mathrm{Sh}(\mathcal{C}, J))$$

between the category of coherent functors $\mathcal{C} \longrightarrow \mathcal{E}$ and natural transformations and the category of geometric morphisms $\mathcal{E} \longrightarrow \mathrm{Sh}(\mathcal{C}, J)$ and geometric transformations. In particular, for a coherent functor $F : \mathcal{C} \longrightarrow \mathcal{E}$ there exists a geometric morphism $f : \mathcal{E} \longrightarrow \mathrm{Sh}\,(\mathcal{C}, J)$ such that the triangle

$$\begin{array}{ccc} & & \mathcal{E} \\ & \overset{F}{\nearrow} & \uparrow{\scriptstyle f^*} \\ \mathcal{C} & \xrightarrow[y]{} & \mathrm{Sh}\,(\mathcal{C}, J) \end{array}$$

commutes, with y the (coherent) Yoneda embedding. We refer to $\mathrm{Sh}\,(\mathcal{C}, J)$ as the *topos of coherent sheaves* on $\mathcal{C}$, and whenever $\mathcal{C}$ is a coherent category, we omit J and just write $\mathrm{Sh}(\mathcal{C})$ for $\mathrm{Sh}\,(\mathcal{C}, J)$ (that is, if the coverage on a coherent category is not mentioned, it is assumed to be the coherent coverage). In extension, if $\mathcal{C}_{\mathbb{T}}$ is the (coherent) syntactical category of a first-order or decidable coherent theory, we refer to $\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ as the topos of coherent sheaves on $\mathbb{T}$. We say that two coherent categories (or theories) are *Morita equivalent* if their topoi of coherent sheaves are equivalent.

Second, for a topological space X, we take the topos of sheaves on X, denoted $\mathrm{Sh}\,(X)$, to be the category consisting of local homeomorphisms $a : A \to X$ over X, with continuous maps over X between them,

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & B \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & X & \end{array}$$

A continuous function of spaces $f : X \to Y$ induces a geometric morphism, which we will denote by the same name, $f : \mathrm{Sh}\,(X) \longrightarrow \mathrm{Sh}\,(Y)$, such that the inverse image $f^* : \mathrm{Sh}\,(Y) \longrightarrow \mathrm{Sh}\,(X)$ acts by pullback along f,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

to send a sheaf on Y to a sheaf X.

Finally, a topological groupoid, $\mathbb{G}$, is a groupoid in the category of topological spaces, i.e. a space G_0 of objects, a space G_1 of arrows, and continuous morphisms

$$G_1 \times_{G_0} G_1 \xrightarrow{\ c\ } G_1 \overset{i}{\circlearrowleft} \ \underset{t}{\overset{s}{\underset{\xleftarrow{\ e\ }}{\rightrightarrows}}}\ G_0$$

of *source* (or domain); *target* (or codomain); insertion of identities (*Einheit*); *inverse*; and *composition*, satisfying the usual axioms (stating that it is a category with all arrows invertible). We write $e(x) = 1_x$. The topos of equivariant sheaves on $\mathbb{G}$, which we write as $\mathrm{Sh}_{G_1}(G_0)$ or just as $\mathrm{Sh}\,(\mathbb{G})$, has as objects pairs $\langle a : A \to G_0, \alpha \rangle$ where $a : A \to G_0$ is a sheaf on G_0 and α is a continuous action,

$$\alpha : G_1 \times_{G_0} A \longrightarrow A$$

where the pullback is along the source map $s : G_1 \to G_0$. That is to say, α is a continuous function such that $a(\alpha(g, x)) = t(g)$, satisfying the axioms of an action. An arrow $g : \langle a : A \to G_0, \alpha \rangle \longrightarrow \langle b : B \to G_0, \alpha \rangle$ is an arrow $g : A \to B$ of $\mathrm{Sh}\,(G_0)$ which commutes with the actions, i.e. such that

$$\begin{array}{ccc} G_1 \times_{G_0} A & \xrightarrow{\ \alpha\ } & A \\ {\scriptstyle 1_{G_1} \times g} \downarrow & & \downarrow {\scriptstyle g} \\ G_1 \times_{G_0} B & \xrightarrow[\ \beta\ ]{} & B \end{array}$$

commutes. A morphism of topological groupoids $f : \mathbb{G} \longrightarrow \mathbb{H}$ consist of continuous functions $f_1 : G_1 \to H_1$ and $f_0 : G_0 \to H_0$ commuting with source, target, insertion of identities, and composition maps (i.e. so as to form a functor of categories). Such a morphism induces a geometric morphism, which we shall also, as a rule, denote by the same name, $f : \mathrm{Sh}\,(\mathbb{G}) \longrightarrow \mathrm{Sh}\,(\mathbb{H})$, where the inverse image functor $f^* : \mathrm{Sh}\,(\mathbb{H}) \longrightarrow \mathrm{Sh}\,(\mathbb{G})$ takes an equivariant sheaf $\langle b : B \to H_0, \beta \rangle$ on $\mathbb{H}$ to the equivariant sheaf $\langle a : A \to H_0, \alpha \rangle$ on $\mathbb{G}$, the sheaf component of which is obtained by pullback,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

and the action component of which is defined by $\alpha(g,x) \mapsto \beta(f_1(g),x)$ (considering the fiber $a^{-1}(y)$ to be the fiber $b^{-1}(f_0(y))$).

For any Boolean or decidable coherent category, the collection of coherent functors into **Sets** forms a proper class which we need to restrict to a set, thereby also forcing us to restrict the coherent categories we consider. We restrict the category of **Sets** to those sets that are hereditarily of less than κ size for some uncountable cardinal κ. Call the resulting (Boolean coherent) category $\mathbf{Sets}_\kappa$. We then restrict to those Boolean or decidable coherent categories for which $\mathbf{Sets}_\kappa$ is still 'large enough' for our purposes, which are the categories, $\mathcal{B}$, such that the set of coherent functors into $\mathbf{Sets}_\kappa$ is 'saturated', in the sense that the coherent functors into $\mathbf{Sets}_\kappa$ jointly reflect covers when $\mathcal{B}$ is equipped with the coherent coverage and $\mathbf{Sets}_\kappa$ is equipped with its canonical coverage. Any Boolean or decidable coherent category which is strictly smaller than κ has a saturated set of coherent functors into $\mathbf{Sets}_\kappa$. The groupoid $\underline{\mathrm{Hom}}^*_{\mathbf{Coh}}(\mathcal{B}, \mathbf{Sets}_\kappa)$ of models and isomorphisms obtained by homming into $\mathbf{Sets}_\kappa$ can then be equipped with a 'logical' topology to form a topological groupoid, $\mathbb{G}_\mathcal{B}$. We show that $\mathrm{Sh}(\mathbb{G}_\mathcal{B}) \simeq \mathrm{Sh}(\mathcal{B})$, and so we can recover $\mathcal{B}$ from $\mathbb{G}_\mathcal{B}$ up to equivalence, if it is a pretopos, and up to Morita equivalence otherwise. Sending a coherent category $\mathcal{B}$ to its semantical groupoid $\mathbb{G}_\mathcal{B}$, and a coherent functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ to the resulting composition morphism

$$\underline{\mathrm{Hom}}^*_{\mathbf{Coh}}(\mathcal{D}, \mathbf{Sets}_\kappa) \xrightarrow{-\circ F} \underline{\mathrm{Hom}}^*_{\mathbf{Coh}}(\mathcal{B}, \mathbf{Sets}_\kappa)$$

defines a contravariant functor into the category of topological groupoids. We show that factored through a suitable subcategory of topological groupoids, this functor has an adjoint which extracts a Boolean or decidable (depending on the groupoid) coherent category from a topological groupoid $\mathbb{G}$ by considering the decidable *compact* objects in $\mathrm{Sh}(\mathbb{G})$, which is then the analogy with extracting a Boolean algebra from a Stone space by considering the compact open sets in the lattice of open subsets (we expand on this analogy in Section 2.1). Still in analogy with the Stone duality between Boolean algebras and topological spaces, the objects thus extracted from $\mathrm{Sh}(\mathbb{G})$ correspond to morphisms in our restricted category of groupoids from $\mathbb{G}$ to a topological groupoid of sets and isomorphisms, which is the dual (up to equivalence) of the object classifier in decidable coherent categories. The counit components of the adjunction are equivalences at pretopoi, and Morita equivalences otherwise. Restricted to the semantical groupoids, the unit components are also Morita equivalences. In (the self-contained) Chapter 3 we show that if we apply the general result to the special case of Boolean coherent categories that are syntactic categories of single-sorted first-order theories, then the existence of the single-sort allows us to consider a slice category of groupoids on the semantical side, which we can characterize without reference to induced sheaf topoi, and which results in an adjunction where the counit components are isomorphisms, as are the unit components at semantical groupoids.

The first-order logical duality presented in this thesis is intended to provide fresh conceptual and technical perspectives on issues concerning the study of first-order theories and the relationship between first-order syntax and semantics, in line with those provided by Stone duality for propositional logic (as indicated in Section 1.2.1). The algebraic representation of syntax and the manipulation of semantics in terms of sheaves on spaces and groupoids of models provide alternative perspectives on traditional issues already in the setup. For instance, it may be of interest to note that the type spaces $S_n(\mathbb{T})$ for a theory $\mathbb{T}$ mentioned in Section 1.2.1 occur centrally in the representation of a theory $\mathbb{T}$ in terms of its groupoid of models and isomorphisms, as the issue to a significant extent revolves around using model isomorphisms together with topological structure in order to be able to identify $\mathbb{T}$-definable classes of subsets of models which correspond to open subsets of the spaces $S_n(\mathbb{T})$, and which then form certain equivariant sheaves on the semantical groupoid of $\mathbb{T}$. As for the results of the thesis, the general purpose of the representation and duality theorems is to allow for a going back and forth between syntactical and semantical characterizations, so as e.g. to study questions concerning syntactic relations between theories in terms of relations between their corresponding groupoids of models. Some examples of this is presented in Section 3.5, where we indicate a few first steps towards studying theory extensions in terms of continuous morphisms of topological groupoids. Using only the most immediate consequences of the duality, and bringing but very little of the available machinery of Grothendieck topoi and topological groupoids to bear, a weaker version of the Beth Definability Theorem is nevertheless quickly arrived at. We also sketch a potentially interesting Galois connection between sub-theories of a theory $\mathbb{T}$ and certain 'intermediate' groupoids of its corresponding semantical groupoid. Finally, in addition to the further development of those issues, it falls within the category of future work to compare more fully the setup of this thesis with that of Makkai ([17], [19]) and Zawadowski ([27]) and determine the extent to which the results presented there—in particular the Descent Theorem for Boolean Pretopoi ([19])—can be developed within the current framework.

1.4 Representing Topoi by Groupoids

The idea of the current approach is thus that in the first-order logical case, topoi of sheaves can play the role that is played by frames of open sets in the propositional case. This idea springs from the view of topoi as generalized spaces and the representation theorem of Joyal and Tierney ([11]) to the effect that any topos can be represented as equivariant sheaves on a localic groupoid. Briefly (see [10, C5] for a concise presentation), proceeding from the theorem that for every Grothendieck topos $\mathcal{E}$, there exists a locale $\mathcal{L}$ and an open surjective geometric morphism $f : \mathrm{Sh}(\mathcal{L}) \longrightarrow \mathcal{E}$, one forms a truncated simplicial topos, $\mathrm{Sh}(\mathcal{L})_\bullet$,by pullbacks of topoi:

$$\mathrm{Sh}(\mathcal{L}) \times_{\mathcal{E}} \mathrm{Sh}(\mathcal{L}) \times_{\mathcal{E}} \mathrm{Sh}(\mathcal{L}) \underset{\pi_{13}}{\overset{\pi_{12}}{\substack{\longrightarrow \\ \xrightarrow{\pi_{13}} \\ \longrightarrow}}} \mathrm{Sh}(\mathcal{L}) \times_{\mathcal{E}} \mathrm{Sh}(\mathcal{L}) \underset{\pi_2}{\overset{\pi_1}{\substack{\longrightarrow \\ \xleftarrow{\Delta} \\ \longrightarrow}}} \mathrm{Sh}(\mathcal{L})$$

Since localic morphisms pull back, the pullbacks are again localic, whence the whole simplicial topos can be seen as arising from a localic groupoid, $\mathbb{G}$, i.e. a groupoid in the category of locales. Therefore, the topos, $\mathbf{Desc}(\mathrm{Sh}(\mathcal{L})_\bullet)$ of sheaves on $\mathcal{L}$ equipped with descent data can be taken to be the topos $\mathrm{Sh}(\mathbb{G})$ of equivariant sheaves on the localic groupoid $\mathbb{G}$. And since open surjections are descent, this means that $\mathcal{E}$ is equivalent to sheaves on a localic groupoid, $\mathcal{E} \simeq \mathrm{Sh}(\mathbb{G})$. In [5], Butz and Moerdijk show that if the topos $\mathcal{E}$ has enough points, in the sense that the collection of inverse image functors for geometric morphisms $\mathbf{Sets} \longrightarrow \mathcal{E}$ are jointly faithful, then $\mathcal{E}$ has a representation in terms of a *topological* groupoid, i.e. there is a topological groupoid $\mathbb{G}$ such that $\mathcal{E} \simeq \mathrm{Sh}(\mathbb{G})$. Furthermore, $\mathbb{G}$ can be constructed as a space of (equivalence classes of) points and enumerations, with (equivalence classes of) natural isomorphisms between them. As coherent topoi—that is, topoi of the form $\mathrm{Sh}(\mathcal{C})$ for a coherent category $\mathcal{C}$—have enough points, by Deligne's theorem (see e.g. [14, IX,11]), and the points of a coherent topos $\mathrm{Sh}(\mathcal{C})$ correspond to coherent functors (models) $\mathcal{C} \longrightarrow \mathbf{Sets}$, it is a short step to try and modify this approach to the special case of Boolean coherent topoi, for the sake of constructing a duality between Boolean coherent categories and topological groupoids of models and isomorphisms. (The debt to Butz and Moerdijk is particularly visible in the semantical groupoid construction of Chapter 3.) Once we know that the topos of coherent sheaves $\mathrm{Sh}(\mathcal{B})$ on a Boolean coherent category $\mathcal{B}$ is equivalent to the topos of equivariant sheaves on a topological groupoid of models and isomorphisms, we know how to recover $\mathcal{B}$ from its groupoid of models. The adjoint functor from groupoids to

Boolean coherent categories is then obtained by generalizing the method by which we extract $\mathcal{B}$ from its semantical groupoid.

Chapter 2

The Syntax-Semantics Adjunction

2.1 Overview

The duality between the category of Boolean algebras and Stone spaces can be seen as a restriction of an adjunction between Boolean algebras and the category of topological spaces. There is also another sense in which Stone duality is a restriction of a more inclusive adjunction. Let L be a distributive lattice. The set of lattice morphisms into the two-point distributive lattice, $\mathrm{Hom}_{\mathbf{DLat}}(L, 2)$, can be equipped with the topology generated by basic opens of the form $U_c = \{f : L \longrightarrow 2 \mid f(c) = 1\}$ for $c \in L$. In the resulting space, call it X_L for short, the distributive lattice L embeds, by $c \mapsto U_c$, as the compact open sets to make X_L a *coherent space*. That is, a compact space such that the compact open subsets are closed under intersection and form a basis for the space (unlike [9], we do not require the space to be sober). In particular, the compact open sets in a coherent space form a distributive sublattice of the lattice of open sets. A lattice homomorphism $F : L \longrightarrow K$ yields by composition a function $f : X_K \to X_L$ which is a *coherent continuous map* in the sense that the inverse image of a compact open set is again compact. Thus we have a contravariant functor $\mathbf{DLat} \longrightarrow \mathbf{Top}$ from the category of distributive lattices to the category of topological spaces. This functor is sometimes called Spec, and the space X_L is the *spectrum* of the lattice, that is, it can be considered as the space of prime ideals. Now, define a *stably compact* open subset C of an arbitrary space X to be a compact open such that for any compact open set $D \subseteq X$, the intersection $C \cap D$ is again compact. Notice that if X is a compact space, then the stably compact open subsets form a distributive lattice.

Definition 2.1.0.2 Let **ctTop** be the subcategory of **Top** consisting of compact spaces and those continuous functions between them such that the inverse image of a stably compact open is again stably compact.

We can call the arrows in **ctTop** *stably compact* functions. Notice that any coherent function between coherent spaces is automatically stably compact, as any compact open is stably compact in a coherent space. Then the functor, $\mathcal{O}_{sc}$, which sends a space to its lattice of stably compact opens and a stably compact function in **ctTop** to the lattice morphism obtained by inverse image, is (left) adjoint to Spec,

$$\mathbf{DLat}^{\mathrm{op}} \underset{\mathrm{Spec}}{\overset{\mathcal{O}_{sc}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{ctTop}$$

with the counit components in **DLat** being isomorphisms, meaning that we recover a distributive lattice L up to isomorphism from X_L as the (stably) compact opens. Notice that taking the compact opens of a space X in **ctTop** can be described as taking the stably compact functions from X into the Sierpiński space 2 (where $\{1\}$ is open and $\{0\}$ is not),

$$\mathcal{O}_{sc}(X) \cong \mathrm{Hom}_{\mathbf{ctTop}}(X, 2)$$

and we can describe the action of $\mathcal{O}_{sc}$ on coherent functions accordingly as composition. Notice also that the Sierpiński space is the dual (model space) of the 'object classifier' in **DLat**—i.e. the free distributive lattice on one generator, which is the three-element distributive lattice—just as the discrete space 2 is the dual of the 'object classifier' in **BA**—i.e. the four element Boolean algebra. Now, if we take **clTop** to be the category of compact spaces such that the compact open sets are closed under intersection—i.e. such that the compact open sets form a distributive lattice—and coherent functions between them, then **clTop** is a full subcategory of **ctTop**. For reference:

Definition 2.1.0.3 The category **clTop** consists of those topological spaces such that the intersection of any two compact open sets is again compact, with those continuous maps between them that pull an open compact set back to a compact (open) set. Denote by **CohTop** the full subcategory thereof consisting of coherent spaces, i.e. spaces such that the compact open sets both form a distributive lattice and generate the topology.

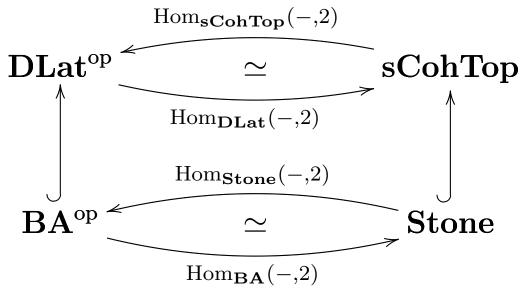
Continuing to restrict, then, the category **CohTop** of coherent topological spaces and coherent functions between them is a full subcategory of **clTop**, and the category **sCohTop** of sober coherent spaces and coherent functors is a full subcategory of **CohTop**,

$$\mathbf{sCohTop} \hookrightarrow \mathbf{CohTop} \hookrightarrow \mathbf{clTop} \hookrightarrow \mathbf{ctTop}$$

Since Spec factors through **sCohTop**, and hence through **CohTop** and **clTop** as well, the adjunction restricts to all of these subcategories, and in the foremost case it yields an equivalence:

$$\mathbf{DLat}^{\mathrm{op}} \simeq \mathbf{sCohTop}$$

(see [9, II]), between the category of distributive lattices and sober coherent spaces (also called *spectral* spaces). Finally, this equivalence restricts to the full subcategory of Boolean algebras, $\mathbf{BA} \hookrightarrow \mathbf{DLat}$ on the 'algebraic' side, and to the full subcategory of Stone spaces and continuous functions $\mathbf{Stone} \hookrightarrow \mathbf{sCohTop}$ on the 'geometric' side, to yield the familiar Stone duality.



We shall present an approach to duality for first-order logic which is analogous to, and indeed a generalization of, the propositional setup in the following way and sense. A Boolean algebra is a Boolean coherent category, and the category of Boolean algebras can be seen as the poset part of the category of Boolean coherent categories, which we see as representing propositional and full first-order classical logic, respectively. Similarly, a distributive lattice is a decidable (in the sense that every object is decidable) coherent category, and the category of distributive lattices can be seen as the poset part of the category of decidable coherent categories, which represent coherent decidable logic, i.e. the fragment of first order logic with only the connectives $\top$, $\bot$, $\wedge$, $\vee$, and $\exists$, and with a predicate symbol $\neq$ (for each sort) obeying axioms of inequality:

$$x \neq y \wedge x = y \vdash_{x,y} \bot$$
$$\top \vdash_{x,y} x \neq y \vee x = y$$

In the propositional case (distributive lattices, Boolean algebras), a poset P is assigned to a space X_P by homming into the two element set, 2, considered as a poset. Equipped with topology, this yields a space such that the poset occurs as the compact elements in the frame of open subsets $\mathcal{O}(X_P)$, where by a frame we mean a lattice with infinite joins satisfying the infinitary distributive law. In fact, the frame of open sets is isomorphic to the frame, $\mathrm{Idl}(P)$, of ideals of P, where an ideal of P is a nonempty subset of P closed downward and under finite joins. The compact elements of $\mathrm{Idl}(P)$ are the principal ideals, and thus correspond to elements of P. By restricting to spaces, X, such that the compact elements in the frame of open sets $\mathcal{O}(X)$ form a poset of the right kind (distributive lattice or Boolean algebra), and by restricting to continuous functions $f : X \to Y$ such that the frame morphisms $f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ induced by f preserves compact elements, we obtain an adjoint from spaces to posets by extracting the compact elements. This adjoint can be seen as homming into a particular space, namely the 'schizophrenic' object 2 equipped with a topology.

Now, in the first-order case, where decidable coherent and Boolean coherent categories replace distributive lattices and Boolean algebras respectively, we have the role of spaces played by topological groupoids, and the role of frames of open sets and ideals is played by topoi of sheaves (which can also be considered as a form of generalized spaces). The role of the 'schizophrenic' object 2 is played by the object of sets, where we restrict the hereditary size of the sets involved to get a small category of sets and functions on the one side, and a small groupoid of sets and bijections on the other. We restrict the decidable coherent categories involved accordingly, to ensure that they have enough models—i.e. coherent functors into **Sets**—of the appropriate size for our purposes. Now, as a distributive lattice generates a frame of ideals, a decidable coherent category, $\mathcal{C}$, generates a topos $\mathrm{Sh}(\mathcal{C})$ of coherent sheaves—that is, sheaves for the coherent coverage—on $\mathcal{C}$ from which it can be recovered, up to Morita equivalence, as the compact decidable objects. And as a space has an associated frame of open subsets, a topological groupoid, $\mathbb{G}$, has an associated topos, $\mathrm{Sh}(\mathbb{G})$, of equivariant sheaves on $\mathbb{G}$. By homming into the decidable (and Boolean) coherent category **Sets** of sets and functions we obtain a topological groupoid $\mathbb{G}_{\mathcal{C}}$ of models and model isomorphisms of $\mathcal{C}$, for which it can be shown that (Theorem 2.4.1.3):

$$\mathrm{Sh}(\mathcal{C}) \simeq \mathrm{Sh}(\mathbb{G}_{\mathcal{C}})$$

By restricting e.g. to those topological groupoids that have associated topoi of equivariant sheaves such that the decidable compact objects form a coherent category (as in the poset case, one can, and we will, formulate more or less inclusive requirements), and by restricting to those morphisms of topological grupoids that induce inverse image functors of topoi that preserve such objects, we obtain an adjoint from groupoids to categories by extracting the decidable compact elements (Theorem 2.4.4.3). This adjoint, too, can be seen as homming into a particular topological groupoid, namely the 'schizophrenic object' of sets, now considered as a groupoid $\mathbb{S}$ of sets and bijections equipped with a topology. This groupoid $\mathbb{S}$, then, plays the role played by the Sierpiński space in the propositional case. The adjunction can then be restricted to those groupoids such that their associated topoi of sheaves are generated by the compact decidable objects (in analogy to coherent spaces) and further to the subcategory of Boolean coherent categories, on the one hand, and the groupoids such that the compact decidable objects, in addition, form a Boolean coherent category (in analogy to Stone spaces). For the latter adjunction, recovering a Boolean coherent category can also be seen as 'homming' into a 'Stone' groupoid of sets and bijections—with a finer topology, then, than the mentioned 'Sierpiński' groupoid—playing a role analogous to that played by the (discrete) Stone space 2. The properties of this groupoid are presented in Section 2.5. Finally, we should emphasize that our 'first-order' case is not a mere analogy with the posetal case, but a generalization: the concepts, and even to some extent the proofs of the classical case occur as the special 'posetal' case of our generalization (see Section 2.4.5).

Note that there are two points where our generalization has to deal with issues which do not arise in the propositional case. First, we cannot consider all models of a decidable coherent category, for this is a proper class. We restrict therefore to a small (in the sense of being a set) subcategory of the category of sets and functions: the hereditarily less than κ large sets, for some uncountable cardinal κ. This necessitates some extra care throughout the construction. For instance, we quite often make sure to have on-the-nose definitions of certain functors often otherwise left specified only up to natural isomorphism, since our subcategory of hereditarily less than κ sets is not closed under isomorphism. We consider this a technicality which could perhaps be dealt with by other means as well. Second, while a distributive lattice can be identified, up to isomorphism, from its frame of ideals as the compact elements, we cannot expect such precision when recovering a coherent category, $\mathcal{C}$, from its topos of sheaves, $\mathrm{Sh}(\mathcal{C})$ (even up to equivalence). In general, one can only expect to recover a category up to pretopos completion. This can be remedied by only considering categories that are pretoposes, which allows us to recover them up to equivalence. Indeed, the counit component at a pretopos for the adjunction we present is an equivalence. However, in Chapter 3 we apply our setup to the special case of a single-sorted first-order theory, where the peculiarity of that situation can be exploited to yield an adjunction between the category of such theories and an independently characterized category of 'Stone fibrations' over the groupoid of sets and isomorphisms, which has the property that the original theory is recovered up to isomorphism.

2.2 The Semantical Groupoid of a Decidable Coherent Category

Sections 2.2–2.4.1 now following present the main representation result. Section 2.2.1 introduces the topological groupoid of models and isomorphisms for a decidable coherent category. Section 2.2.2 introduces a 'strictification' in terms of syntactic structure, so as to functorially equip decidable coherent categories with canonical coherent structure. Section 2.3 then presents the representation result (Theorem 2.3.4.14) for strictified categories. Section 2.4.1 then reconnects up with arbitrary decidable coherent categories and presents the general representation result (Theorem 2.4.1.3). The material may equally well be read in the order 2.3–2.2–2.4.1, especially if the reader is primarily interested in the logical interpretation, i.e. the perspective according to which we are representing a theory in terms of its groupoid of models.

2.2.1 Spaces of Models and Isomorphisms

We introduce the 'semantical' groupoid of a decidable coherent category: Let $\mathcal{D}$ be a (small) decidable coherent category, that is, a category with finite limits, images, stable covers, finite unions of subobjects, and complemented diagonals ([10, A1.4]).

Definition 2.2.1.1 For an uncountable cardinal κ, we say that $\mathcal{D}$ *has a saturated set of* $< \kappa$*-models* if the set of coherent functors $\mathcal{D} \longrightarrow H(\kappa) := \mathbf{Sets}_\kappa$ (the hereditarily $< \kappa$ sets, see e.g. [12, IV,6]) jointly reflect covers, where $\mathcal{D}$ is considered to have the coherent coverage and $\mathbf{Sets}_\kappa$ has its canonical coverage.

Explicitly, this means that for any family of arrows $f_i : C_i \to C$ in $\mathcal{D}$, if for all $M : \mathcal{D} \longrightarrow \mathbf{Sets}_\kappa$ in $X_\mathcal{D}$

$$\bigcup_{i \in I} \mathrm{Im}\,(M(f_i)) = M(C)$$

then there exists $f_{i_1}, \ldots, f_{i_n}$ such that $\mathrm{Im}\,(f_{i_1}) \vee \ldots \vee \mathrm{Im}\,(f_{i_n}) = C$.

It is sufficient that (the set of objects and arrows of) a coherent category $\mathcal{D}$ is of cardinality $< \kappa$ for it to have a saturated set of $< \kappa$ models, see Lemma 2.3.1.2 below.

Definition 2.2.1.2 Let $\mathbf{DC}$ be the category of small decidable coherent categories with coherent functors between them, and let $\mathbf{DC}_\kappa$ be the full subcategory of those categories with a saturated set of $< \kappa$-models.

Assume that $\mathcal{D}$ is in $\mathbf{DC}_\kappa$.

Definition 2.2.1.3 Let $X_\mathcal{D}$ be the set of coherent functors from $\mathcal{D}$ to $\mathbf{Sets}_\kappa$,

$$X_\mathcal{D} = \mathrm{Hom}_{\mathbf{DC}}\left(\mathcal{D}, \mathbf{Sets}_\kappa\right).$$

Let $G_\mathcal{D}$ be the set of invertible natural transformations between functors in $X_\mathcal{D}$, with s and t the source and target, or domain and codomain, maps,

$$s, t : G_\mathcal{D} \rightrightarrows X_\mathcal{D}$$

thus forming the (discrete) groupoid:

$$\mathbb{G}_\mathcal{D} = \underline{\mathrm{Hom}}^*_{\mathbf{DC}}(\mathcal{D}, \mathbf{Sets}_\kappa).$$

The *coherent topology* on $X_{\mathcal{D}}$ is given by taking as a subbasis the collection of sets of the form,

$$\begin{aligned} U_{(\vec{f},\vec{a})} &= (\langle f_1 : A \to B_1, \ldots, f_n : A \to B_n \rangle, \langle a_1, \ldots, a_n \rangle) \\ &= \{ M \in X_{\mathcal{D}} \mid \exists x \in M(A).\, M(f_1)(x) = a_1 \wedge \ldots \wedge M(f_n)(x) = a_n \} \end{aligned}$$

for a finite span of arrows

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{f_1} & \downarrow^{f_i} & \searrow^{f_n} & \\ B_1 & \ldots & B_i & \ldots & B_n \end{array}$$

in $\mathcal{D}$ and $a_1, \ldots, a_n \in \mathbf{Sets}_\kappa$. Let the *coherent topology* on $G_{\mathcal{D}}$ be the coarsest topology such that $s, t : G_{\mathcal{D}} \rightrightarrows X_{\mathcal{D}}$ are both continuous and all sets of the form

$$V_{(A,a,b)} = (A, a \mapsto b) = \{ f : M \Rightarrow N \mid a \in M(A) \wedge f_A(a) = b \}$$

are open, for A an object of $\mathcal{D}$ and $a, b \in \mathbf{Sets}_\kappa$.

For any coherent category, $\mathcal{C}$, the topos of sheaves on $\mathcal{C}$ equipped with the coherent coverage is denoted $\mathrm{Sh}(\mathcal{C})$. It has the property that for any (Grothendieck) topos $\mathcal{E}$ there is an equivalence of categories, $\underline{\mathrm{Hom}}_{\mathbf{Coh}}(\mathcal{C})(\mathcal{E}) \simeq \underline{\mathrm{Hom}}_{\mathcal{TOP}}(\mathcal{E})(\mathrm{Sh}(\mathcal{C}))$, between the category of coherent functors from $\mathcal{C}$ to $\mathcal{E}$ and natural transformations between them, and the category of geometric morphisms from $\mathcal{E}$ to $\mathrm{Sh}(\mathcal{C})$ and geometric transformations between them. We refer to $\mathrm{Sh}(\mathcal{C})$ as the *topos of coherent sheaves* on $\mathcal{C}$. The main results of the current chapter (Theorem 2.4.1.3 and Theorem 2.4.4.3, respectively) are as follows:

(Theorem 2.4.1.3) $\mathbb{G}_{\mathcal{D}} = G_{\mathcal{D}} \rightrightarrows X_{\mathcal{D}}$ is a topological groupoid, and the topos of coherent sheaves on $\mathcal{D}$ is equivalent to the topos of equivariant sheaves on $\mathbb{G}_{\mathcal{D}}$,

$$\mathrm{Sh}(\mathcal{D}) \simeq \mathrm{Sh}\left(\mathbb{G}_{\mathcal{D}}\right).$$

(Theorem 2.4.4.3) The assignment $\mathcal{D} \mapsto \mathbb{G}_{\mathcal{D}}$ is functorial, and defines a contravariant functor from decidable coherent categories into a subcategory, **ctGpd** of the category **Gpd** of topological groupoids,

$$\mathcal{G} : \mathbf{DC}_\kappa \longrightarrow \mathbf{ctGpd} \hookrightarrow \mathbf{Gpd}.$$

Moreover, $\mathcal{G}$ is one half of an adjunction—if $\mathbf{DC}_\kappa$ is op'ed, $\mathcal{G}$ is the right adjoint—and the counit (in that case) at $\mathcal{D}$ is an equivalence of categories whenever $\mathcal{D}$ is a pretopos (and a Morita equivalence otherwise).

2.2.2 Strictification: Equipping Coherent Categories with Canonical Coherent Structure

For a decidable coherent category $\mathcal{D}$ in $\mathbf{DC}_\kappa$, the coherent topologies on $G_\mathcal{D}$ and $X_\mathcal{D}$ are presented in terms of subbases which can be somewhat cumbersome to work with. Exploiting the intended interpretation, so to speak, we can, however, give homeomorphic spaces in which the topology has a more convenient presentation. A presentation which, moreover, displays the sense in which the coherent topology is a logical topology. The close connection between categories and logical theories, through the lens of which the algebra-geometry adjunction we construct can be seen as a syntax-semantics adjunction, can also provide an inessential but convenient technical tool for the presentation, in providing an instrument for 'strictification' of coherent categories: For each $\mathcal{D}$ in $\mathbf{DC}_\kappa$ we specify (functorially) an equivalent category $T\mathcal{D}$ which has a canonical choice of finite products and subobject representatives. For the category **Sets**, we assume that it comes equipped with a fixed choice of finite products, with reference to which we use the usual tuple notation, e.g. $\langle a, b, c\rangle$, unambiguously. The fixed terminal object (empty product) we denote $1 = \{\star\}$ ($\star$ may be taken to be the empty set, $\star = \emptyset$, as usual). Since the usual subset inclusions provide **Sets** with a canonical choice of subobject representatives, this means that the category **Sets** already has a distinguished choice of coherent structure. For a general decidable coherent category, which may not come equipped with such a choice, a straightforward way to equip it with one is, then, to exploit the correspondence between categories and logical theories.

Let a decidable coherent category $\mathcal{D}$ be given. The coherent language of $\mathcal{D}$, $\mathcal{L}_\mathcal{D}$, has a sort A for each object A of $\mathcal{D}$. For each arrow $f : A \longrightarrow B$ in $\mathcal{D}$, there is a function symbol f with type A, B. The relation symbols of $\mathcal{L}_\mathcal{D}$ are, for each sort A, the identity $=_A$ and a relation, $\neq_A$, both of the type A, A. The subscript shall usually be left implicit. $\mathcal{L}_\mathcal{D}$ is then generated in the usual way with the coherent logical connectives $\bot$, $\top$, $\wedge$, $\vee$, and $\exists_A$. The theory $\mathbb{T}_\mathcal{D}$ of $\mathcal{D}$ consists of those coherent sequents which are true in $\mathcal{D}$ under the canonical interpretation of $\mathcal{L}_{\mathcal{D}}$ in $\mathcal{D}$, where each sort and function symbol is interpreted as the object or arrow whence it came, the identity is interpreted as the diagonal, and $\neq$ is interpreted as the complement of the diagonal. Here, the reader is welcome to assume that any coherent category comes equipped with a canonical choice of structure, which then determines the canonical interpretation. Otherwise, the interpretation depends on a choice of finite limits and image factorizations, but the theory $\mathbb{T}_{\mathcal{D}}$ is nevertheless well defined, since it does not depend on this choice.

For a coherent theory, the syntactic category, $\mathcal{C}_{\mathbb{T}}$, is defined in the usual way, see e.g. [10, D1.4], except for the following: the objects of $\mathcal{C}_{\mathbb{T}}$ are equivalence classes of alpha-equivalence classes of formulas-in-context

$$[x_1 : A_1, \ldots, x_n : A_n \mid \phi]$$

in $\mathcal{L}_{\mathbb{T}}$, where two formulas in the same context (up to alpha-equivalence) $[x_1 : A_1, \ldots, x_n : A_n \mid \phi]$ and $[x_1 : A_1, \ldots, x_n : A_n \mid \psi]$ are equivalent iff the sequents

$$\phi \vdash_{x_1:A_1,\ldots,x_n:A_n} \psi \qquad \psi \vdash_{x_1:A_1,\ldots,x_n:A_n} \phi$$

are both in $\mathbb{T}$. We denote such an object by a representative,

$$[x_1 : A_1, \ldots, x_n : A_n \mid \phi]$$

leaving it understood that it is an equivalence class. An arrow between objects $[x_1 : A_1, \ldots, x_n : A_n \mid \phi]$ and $[y_1 : B_1, \ldots, y_n : B_m \mid \psi]$ consists as usual of, again, a $\mathbb{T}$-provable equivalence class of formulas-in-context

$$[x_1 : A_1, \ldots, x_n : A_n, y_1 : B_1, \ldots, y_n : B_m \mid \sigma]$$

such that the following sequents,

- $\sigma \vdash_{\vec{x},\vec{y}} \phi \wedge \psi$
- $\phi \vdash_{\vec{x}} \exists \vec{y}.\, \sigma$
- $\sigma \wedge \sigma[\vec{z}/\vec{y}] \vdash_{\vec{x},\vec{y},\vec{z}} \vec{y} = \vec{z}$

are in $\mathbb{T}$. Although different from the definition of [10, D1.4] which does not identify provably equivalent formulas in the same context, this definition of the syntactic category of a coherent theory clearly produces an equivalent category to the one defined there (see also [10, D1.4.3]). The reason for identifying provably equivalent formulas in the same context is that we will mostly be considering models in **Sets**, with respect to which this is convenient. Although we often somewhat loosely refer to any coherent functor from a coherent category into **Sets** as a model, whenever we are talking about a theory $\mathbb{T}$, a $\mathbb{T}$-*model* always means a standard set-model of $\mathbb{T}$. That is to say, an assignment of sorts to sets, function symbols to functions, and relation symbols to subsets of appropriate products, so as to inductively define an interpretation of each formula-in-context of $\mathbb{T}$ in the usual manner. Thus, a $\mathbb{T}$-model can be seen as a coherent functor $\mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$, but an arbitrary coherent functor $\mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$, although naturally isomorphic to a $\mathbb{T}$-model, need not be a $\mathbb{T}$-model in this strict sense.

Now, any syntactical category $\mathcal{C}_{\mathbb{T}}$ comes equipped with canonical coherent structure, which is given as follows. We consider the objects $[x : A \mid \top]$ representing the sorts of $\mathbb{T}$ to be distinguished. These objects have distinguished finite products of the form $[x : A, y : B, z : C \mid \top]$, with the distinguished terminal object (empty product) being $[\mid \top]$. Furthermore, a monomorphism between objects in the same context (modulo alpha equivalence), $[x_1 : A_1, \ldots, x_n : A_n \mid \phi]$ and $[y_1 : A_1, \ldots, y_n : A_n \mid \psi]$, which is represented by a formula of the form

$$\left[\vec{x} : \vec{A}, \vec{y} : \vec{A} \,\middle|\, \phi \wedge \psi \wedge \vec{x} = \vec{y}\right]$$

may be considered an distinguished monomorphism, or *inclusion*. A syntactic category then has (and can be characterized up to isomorphism by) the following property of 'strictness':

Definition 2.2.2.1 We say that a decidable coherent category is *strict* if it has the following structure:

- a set $\mathfrak{S}$ of distinguished objects, with distinguished finite products (including a distinguished terminal object);
- a *system of inclusions*, that is, a set $\mathfrak{I}$ of distinguished monomorphisms which is closed under composition and identities; such that for every object R there is a unique tuple $\vec{A} \in \bigcup_{\mathbb{N}} \mathfrak{S}^n$ such that there is an inclusion $R \hookrightarrow A_1 \times \ldots \times A_n$ into the corresponding (distinguished) finite product; and such that every subobject, considered as a set of monomorphisms, of an object contains a unique inclusion.

A coherent functor between two such categories is *strict* if it preserves inclusions, distinguished objects, and distinguished finite products of distinguished objects on the nose.

A coherent category which is strict has, as a consequence, distinguished finite products for all objects, so that such a category has canonical coherent structure given in terms of the distinguished terminal object, the distinguished finite products, and inclusions as distinguished subobject representatives. For a decidable coherent theory $\mathbb{T}$, a $\mathbb{T}$-model, in the usual sense, is then a coherent functor

$$\mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$$

which sends finite distinguished products of distinguished objects to (the fixed) finite cartesian products and inclusions to subset inclusions. We call such set-valued functors *strict* as well, and use bold faced upper case letters, $\mathbf{M}$, $\mathbf{N}$, to denote them, while arbitrary coherent functors are denoted by ordinary upper case letters M, N. As usual, for a coherent theory $\mathbb{T}$, a $\mathbb{T}$-isomorphism between two $\mathbb{T}$-models, $f : \mathbf{M} \Rightarrow \mathbf{N}$ is given by a sort-indexed family of bijections $f_A : [\![x : A \,|\, \top]\!]^{\mathbf{M}} \to [\![x : A \,|\, \top]\!]^{\mathbf{N}}$ such that for any formula-in-context $[x_1 : A_1, \ldots, x_n : A_n \,|\, \phi]$ there is a pullback square

$$\begin{array}{ccc} [\![x_1 : A_1, \ldots, x_n : A_n \mid \phi]\!]^{\mathbf{M}} & \longrightarrow & [\![x_1 : A_1, \ldots, x_n : A_n \mid \phi]\!]^{\mathbf{N}} \\ \downarrow & \lrcorner & \downarrow \\ [\![x_1 : A_1, \ldots, x_n : A_n \mid \top]\!]^{\mathbf{M}} & \xrightarrow[f_{A_1} \times \ldots f_{A_n}]{} & [\![x_1 : A_1, \ldots, x_n : A_n \mid \top]\!]^{\mathbf{N}} \end{array}$$

As such, $\mathbb{T}$-isomorphisms between $\mathbf{M}$ and $\mathbf{N}$ are invertible natural transformations of the strict functors $\mathbf{M}$ and $\mathbf{N}$.

Consider now a decidable coherent category $\mathcal{D}$ in $\mathbf{DC}_\kappa$, with its associated theory $\mathbb{T}_{\mathcal{D}}$. We refer to the syntactic category $\mathcal{C}_{\mathbb{T}_{\mathcal{D}}}$ also by the name $T\mathcal{D}$ in order to avoid proliferation of subscripts. Now, the functor which sends an object, A, in $\mathcal{D}$ to the object $[x : A \,|\, \top]$ in $T\mathcal{D}$ and an arrow $f : A \longrightarrow B$ in $\mathcal{D}$ to the arrow $[x : A, y : B \,|\, f(x) = y]$ in $T\mathcal{D}$ defines an equivalence of categories $\iota_{\mathcal{D}} : \mathcal{D} \longrightarrow T\mathcal{D}$.

Definition 2.2.2.2 Let $\mathbf{T}$ be the category consisting of syntactical categories, $\mathcal{C}_{\mathbb{T}}$, of decidable coherent theories, $\mathbb{T}$, with a saturated set of $< \kappa$ models, and strict functors between them, i.e. coherent functors that preserve distinguished objects (the sorts), distinguished finite products (including the distinguished terminal object), and inclusions.

Notice that any coherent functor $F : \mathcal{A} \longrightarrow \mathcal{D}$ lifts to a morphism $\hat{F} : T\mathcal{A} \longrightarrow T\mathcal{D}$ in $\mathbf{T}$ to make the following square commute:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle \iota_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \iota_{\mathcal{D}}} \\ T\mathcal{A} & \xrightarrow[\hat{F}]{} & T\mathcal{D} \end{array} \tag{2.1}$$

so that we have a functor $T : \mathbf{DC}_\kappa \longrightarrow \mathbf{T}$. Now, a $\mathbb{T}_\mathcal{D}$-model in $\mathbf{Sets}_\kappa$, seen as a strict functor $\mathbf{M} : T\mathcal{D} \longrightarrow \mathbf{Sets}_\kappa$, composes with $\iota_\mathcal{D}$ to yield a coherent functor $\mathcal{D} \longrightarrow \mathbf{Sets}_\kappa$.

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{M} & \mathbf{Sets}_\kappa \\ {\scriptstyle \iota_{\mathcal{D}}}\downarrow & \nearrow{\scriptstyle \mathbf{M}} & \\ T\mathcal{D} & & \end{array}$$

Conversely, a coherent functor $M : \mathcal{D} \longrightarrow \mathbf{Sets}_\kappa$, being an assignment of objects of $\mathcal{D}$ to sets and arrows to functions, determines a $\mathbb{T}_\mathcal{D}$-structure, which is a $\mathbb{T}_\mathcal{D}$-model since M is coherent. Thus, the set $X_\mathcal{D}$ is in bijective correspondence with the set of $\mathbb{T}_\mathcal{D}$-models in $\mathbf{Sets}_\kappa$. And similarly, the set $G_\mathcal{D}$ is in bijective correspondence with the set of $\mathbb{T}_\mathcal{D}$-isomorphisms. With $X_\mathcal{D}$ and $G_\mathcal{D}$ being equipped with the coherent topologies, these bijections induce topologies to become homeomorphisms, and thus we can choose between $\mathbb{G}_\mathcal{D}$ or the isomorphic topological groupoid of $\mathbb{T}_\mathcal{D}$-models and isomorphisms to work with, depending on which is the more convenient, and in accordance with the guiding idea that what is an algebra-geometry duality from one perspective is a syntax-semantics duality from another. The following section (2.3) presents the proof of the promised representation theorem—Theorem 2.4.1.3—in terms of the latter. Beginning with a theory, we equip its groupoid of models and isomorphisms with a 'logical' topology. Section 2.4 then reconnects this construction with un-strictified decidable coherent categories and their groupoids of set-valued functors equipped with the coherent topology. We summarize this section in the following 'strictification' result:

Lemma 2.2.2.3 *The functor $T : \mathbf{DC}_\kappa \longrightarrow \mathbf{T}$ sends a decidable coherent category $\mathcal{D}$ to an equivalent category $T\mathcal{D}$ which is strict, i.e. such that $T\mathcal{D}$ is equipped with a set of distinguished objects, distinguished finite products, and a system of inclusions. Moreover, the (discrete) groupoid of coherent functors $\mathcal{D} \longrightarrow \mathbf{Sets}_\kappa$ and invertible natural transformations is isomorphic to the groupoid of strict functors $T\mathcal{D} \longrightarrow \mathbf{Sets}_\kappa$ and invertible natural transformations.*

We now proceed to consider strict decidable coherent categories in the form, then, of syntactic categories. The material is presented in terms of the syntax of first-order theories rather than in terms of distinguished objects and inclusions, not in the least because this makes for a much more readable presentation, but can also be presented in terms of the properties of Definition 2.2.2.1. We introduce a 'logical' topology on the groupoid of $\mathbf{Sets}_\kappa$-valued strict functors and invertible natural transformations (i.e. the groupoid of models and isomorphisms) of a syntactic category. In Section 2.4.1, we continue where Lemma 2.2.2.3 leaves off by comparing the logical topology with that of Definition 2.2.1.3, and showing that the groupoid of coherent set-valued functors and invertible natural transformations equipped with the coherent topology for a category $\mathcal{D}$ in $\mathbf{DC}_\kappa$ is isomorphic to the groupoid of strict set-valued functors and invertible natural transformations equipped with the logical topology for the strictified category $T\mathcal{D}$.

2.3 Representing Decidable Coherent Topoi by Groupoids of Models

Recall that we call a coherent theory *decidable* if for each sort there is a predicate $\neq$ satisfying axioms of inequality (see p. 28). The goal of the current section is the representation of such a theory $\mathbb{T}$ in terms of its semantical groupoid $G_\mathbb{T} \rightrightarrows X_\mathbb{T}$ of models and isomorphisms in Theorem 2.3.4.14, stating that the topos of sheaves for the coherent coverage on $\mathcal{C}_\mathbb{T}$ is equivalent to the topos of equivariant sheaves on $G_\mathbb{T} \rightrightarrows X_\mathbb{T}$,

$$\mathrm{Sh}\,(\mathcal{C}_\mathbb{T}) \simeq \mathrm{Sh}_{G_\mathbb{T}}(X_\mathbb{T})$$

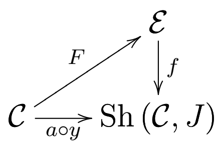
The overall form of the argument resembles that sketched in Section 1.2.1 for the representation of a Boolean algebra in terms of its space of models: Generalizing the Stone Representation Theorem, we embed $\mathcal{C}_\mathbb{T}$ in the topos of sets over the set of models, $\mathbf{Sets}/X_{\mathbb{T}}$, in Lemma 2.3.2.1. Proceeding to introduce a logical topology on the set $X_{\mathbb{T}}$ of models and then to introduce $\mathbb{T}$-model isomorphisms for additional structure, we show how the embedding of $\mathcal{C}_{\mathbb{T}}$ into $\mathbf{Sets}/X_{\mathbb{T}}$ factors through, first, the topos $\mathrm{Sh}\,(X_{\mathbb{T}})$ of sheaves on the space $X_{\mathbb{T}}$ (Proposition 2.3.2.7) and, finally, the topos $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ of equivariant sheaves on the topological groupoid $G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}}$ (Lemma 2.3.4.7). Showing that the image of the embedding generates $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ in Lemma 2.3.4.13, we are then in a position to conclude that the embedding lifts to an equivalence $\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}}) \simeq \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ in Theorem 2.3.4.14.

2.3.1 Cover-Reflecting, Set-Valued Functors

We use the condition *cover reflecting* in the following sense:

Definition 2.3.1.1 A morphism of sites $F : (\mathcal{C}, \mathrm{J}) \longrightarrow (\mathcal{D}, \mathrm{K})$ *reflects covers* if for any sieve S on $C \in \mathcal{C}$, if the image of S in $\mathcal{D}$ generates a covering sieve on $F(C)$, then S is a covering sieve on C.

Now, suppose that we are given a cartesian site $(\mathcal{C}, J)$, a cocomplete topos $\mathcal{E}$ (the canonical coverage of which is all epimorphic families) and a cartesian, cover preserving functor $F : \mathcal{C} \longrightarrow \mathcal{E}$,



with f the induced classifying geometric morphism. Then f is a surjection iff F reflects covers in the sense of Definition 2.3.1.1. For consider a parallel pair $g, h : A \rightrightarrows B$ of arrows in $\mathrm{Sh}(\mathcal{C})$. If $c : ayC \longrightarrow A$ is a morphism with representable domain, then by composing with g and h and taking the equalizer,

$$E \overset{e}{\rightarrowtail} ayC \overset{c}{\longrightarrow} A \overset{g}{\underset{h}{\rightrightarrows}} B$$

we obtain a subobject of ayC which corresponds to a (closed) sieve S on C, with $\varinjlim ay(S) \cong E$, such that $gc = hc$ iff e is iso iff S covers C. Now, assume that $F : \mathcal{C} \longrightarrow \mathcal{E}$ reflects covers, and suppose we are given a parallel pair of distinct morphisms, $g, h : A \rightrightarrows B$. Choose $c : ayC \longrightarrow A$ such that $g \circ c \neq h \circ c$, take the equalizer, and apply the equalizer and colimit preserving inverse image f^* to obtain the following diagram:

$$\varinjlim F(S) \cong f^*(E) \xrightarrow{f^*(e)} F(C) \xrightarrow{c} f^*(A) \underset{f^*(h)}{\overset{f^*(g)}{\rightrightarrows}} f^*(B)$$

where we have taken advantage of the fact that $F \cong f^* \circ ay$. Then $f^*(g)$ and $f^*(h)$ must also be distinct. For otherwise $f^*(e)$ would be iso, and since F is cover reflecting, that would imply that S would be a covering sieve, which would mean that $e : \varinjlim ay(S) \cong E \longrightarrow ay(C)$ was iso, contradicting that $gc \neq hc$. Therefore, F cover reflecting implies f^* faithful, that is, f is a surjection. Conversely, if f is a surjection, then f^* reflects isomorphisms. So for any sieve S on $C \in \mathcal{C}$, if $\varinjlim F(S) \cong F(C)$ in $\mathcal{E}$, then $\varinjlim ay(S) \cong ay(C)$ in $\mathrm{Sh}(\mathcal{C}, J)$, and so S covers C. Therefore f surjective implies that F is cover reflecting. Now, suppose $\mathcal{E} = \mathbf{Sets}$ and we are giving a set X of cover preserving and *jointly* cover reflecting functors $x : \mathcal{C} \longrightarrow \mathbf{Sets}$. Then we can put these functors together to form a functor

$$F : \mathcal{C} \longrightarrow \prod_{x \in X} \mathbf{Sets}_x \cong \mathbf{Sets}/X$$

which is cartesian, cover preserving, and cover reflecting, since finite limits and covers are computed fibrewise in $\mathbf{Sets}/X$. Whence it induces a surjective geometric morphism

$$f : \mathbf{Sets}/X \twoheadrightarrow \mathrm{Sh}(\mathcal{C}, J)\,.$$

In what follows, we exploit this to construct a cover of the topos of coherent sheaves $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ on a theory $\mathbb{T}$. Notice that the following lemma depends on Deligne's Theorem and hence on the axiom of choice.

Lemma 2.3.1.2 *Let $\mathbb{T}$ be a decidable coherent theory of cardinality $< \kappa$. Then the set X of coherent functors $\mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$ which take values in $\mathbf{Sets}_\kappa$ jointly reflect covers.*

Proof Since we can factor out images, we can consider coverages given in terms of subobjects. Suppose we are given a family

$$\mathcal{F} = \{[\vec{x} \mid \psi_i] \hookrightarrow [\vec{x} \mid \phi] \mid i \in I\}$$

of subobjects of $[\vec{x} \mid \phi]$ in $\mathcal{C}_{\mathbb{T}}$. Add sequents $\vdash \phi(\vec{c})$ and $\psi_i(\vec{c}) \vdash \bot$, for $\vec{c}$ fresh constants and $i \in I$, to $\mathbb{T}$ to obtain $\mathbb{T}^+$. If $\mathcal{F}$ does not cover $[\vec{x} \mid \phi]$, then $\mathbb{T}^+$ is consistent by Deligne's theorem (see e.g. [14, IX.11]), and by the classical completeness theorem for coherent logic (see e.g. [10, D1.5]), $\mathbb{T}^+$ has a model in $\mathbf{Sets}_\kappa$ which restricts to a $\mathbb{T}$-model $M : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}_\kappa$ such that $M(\mathcal{F})$ does not cover $M([\vec{x} \mid \phi])$. $\dashv$

Note that the category of (small, i.e. not proper class) distributive lattices is a full subcategory of $\mathbf{DC}_\kappa$.

2.3.2 The Semantical Groupoid of a Theory $\mathbb{T}$

Fix a decidable coherent theory $\mathbb{T}$ and assume that $\mathbb{T}$ has a saturated set of models in $\mathbf{Sets}_\kappa$, in the sense that the coherent functors $\mathbf{M} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}_\kappa$ jointly reflect covers, where $\mathcal{C}_{\mathbb{T}}$ is equipped with the coherent coverage and $\mathbf{Sets}_\kappa$ is equipped with its canonical coverage. We construct the topological groupoid $\mathbb{G}_{\mathbb{T}}$ of $\mathbb{T}$-models and isomorphisms equipped with a logical topology, and show that the topos of coherent sheaves $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ on $\mathbb{T}$ is equivalent to the topos of equivariant sheaves on $\mathbb{G}_{\mathbb{T}}$. Thereafter (Section 2.4), we verify that the groupoid $\mathbb{G}_{\mathcal{D}}$ of coherent set-valued functors and natural transformations on a decidable coherent category $\mathcal{D}$ equipped with the coherent topology is the same as the topological groupoid of $\mathbb{T}_{\mathcal{D}}$-models, and conclude that we have a groupoid representation of the topos of coherent sheaves on $\mathcal{D}$ in terms of $\mathbb{G}_{\mathcal{D}}$.

Set $X_{\mathbb{T}}$ to be the set of $\mathbb{T}$-models in $\mathbf{Sets}_\kappa$ (i.e. inclusion-preserving coherent functors $\mathbf{M} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}_\kappa$ that sends a (distinguished) n-product of distinguished objects, $A \times \ldots \times B$, to a cartesian n-product $\mathbf{M}(A) \times \ldots \times \mathbf{M}(B)$). Set $G_{\mathbb{T}}$ to be the set of $\mathbb{T}$-isomorphisms (natural transformations) between the models in $X_{\mathbb{T}}$. We use double-arrow notation for $\mathbb{T}$-isomorphisms, e.g. $f : \mathbf{M} \Rightarrow \mathbf{N}$, and write $f_A : [\![A]\!]^{\mathbf{M}} \to [\![A]\!]^{\mathbf{N}}$ for the component at the sort A of $\mathbb{T}$ (or, at the distinguished object $[x : A \mid \top]$ in $\mathcal{C}_{\mathbb{T}}$). We have, then, a (discrete) groupoid:

$$G_{\mathbb{T}} \times_{X_{\mathbb{T}}} G_{\mathbb{T}} \xrightarrow{\ \mathrm{c}\ } G_{\mathbb{T}} \overset{\mathrm{i}}{\circlearrowleft} \; \underset{\mathrm{t}}{\overset{\mathrm{s}}{\underset{\longleftarrow\,\mathrm{e}\,}{\rightrightarrows}}} \; X_{\mathbb{T}} \tag{2.2}$$

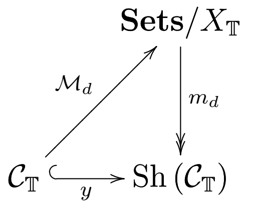
Lemma 2.3.2.1 *The assignment*

$$\left[\vec{x} : \vec{B} \,\middle|\, \phi\right] \mapsto E_{[\vec{x} \mid \phi]} = \left\{ \langle \mathbf{M}, \vec{b} \rangle \;\middle|\; \mathbf{M} \in X_{\mathbb{T}}, \vec{b} \in [\![\vec{x} : \vec{B} \mid \phi]\!]^{\mathbf{M}} \right\} \xrightarrow{\ \pi_1\ } X_{\mathbb{T}}$$

where π_1 *projects to the model, defines a coherent functor*

$$\mathcal{M}_d : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}/X_{\mathbb{T}}$$

which induces a surjective geometric morphism $m_d : \mathbf{Sets}/X_{\mathbb{T}} \twoheadrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$*:*



PROOF Ignoring the 'indexing' left component of the pairs, we may thus regard $\mathcal{M}_d$ as the functor that sends an object A in $\mathcal{C}_{\mathbb{T}}$ to the set E_A over $X_{\mathbb{T}}$ the fiber, $(E_A)_{\mathbf{M}}$, over a model $\mathbf{M} \in X_{\mathbb{T}}$ of which is $\mathbf{M}(A)$.

$$E_A = \{\langle \mathbf{M}, a\rangle \mid \mathbf{M} \in X_{\mathbb{T}} \wedge a \in \mathbf{M}(A)\}$$

An arrow $f : A \longrightarrow B$ of $\mathcal{C}_{\mathbb{T}}$ is, accordingly, sent to the function $E_f : E_A \to E_B$ over $X_{\mathbb{T}}$ the restriction of which to the fibers over a model $\mathbf{M}$ is the function $\mathbf{M}(f) : \mathbf{M}(A) \to \mathbf{M}(B)$. All models $\mathbf{M} \in X_{\mathbb{T}}$ being coherent functors, $\mathcal{M}_d$ is coherent since the coherent structure in $\mathbf{Sets}/X_{\mathbb{T}}$ is computed fiberwise, and for an element $\mathbf{K} \in X_{\mathbb{T}}$, if $k : \mathbf{Sets} \longrightarrow \mathbf{Sets}/X_{\mathbb{T}}$ is the corresponding point, then

$$\mathbf{K} = k^* \circ \mathcal{M}_d : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}/X_{\mathbb{T}} \longrightarrow \mathbf{Sets}.$$

Finally, $m_d : \mathbf{Sets}/X_{\mathbb{T}} \longrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ is a surjection since the functor $\mathcal{M}_d : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}/X_{\mathbb{T}}$ reflects covers. $\dashv$

For an arrow $f : A \longrightarrow B$ in $\mathcal{C}_{\mathbb{T}}$, we denote the underlying sets and functions of $\mathcal{M}_d(f) : \mathcal{M}_d(A) \longrightarrow \mathcal{M}_d(B)$ as $E_f : E_A \to E_B$. The *logical topology* on $X_{\mathbb{T}}$ is defined by taking as basic open sets those of the form

$$U_{[\vec{x}:\vec{B} \mid \phi],\vec{b}} = \left\{ \mathbf{M} \in X_{\mathbb{T}} \;\middle|\; \vec{b} \in [\![\vec{x} : \vec{B} \mid \phi]\!]^{\mathbf{M}} \right\}$$

for $b \in \mathbf{Sets}_\kappa$, with $\vec{b}$ the same length as $\vec{x}$. (Of course, the definition makes sense also for arbitrary sets a replacing $\vec{b}$, but unless a is a tuple of the same length as $\vec{x}$ we automatically obtain the empty set, so we usually assume without further comment that basic open sets are presented in terms of appropriate tuples.) Now, this is a basis, since it obviously covers $X_{\mathbb{T}}$, and

$$U_{[\vec{x}:\vec{B} \mid \phi],\vec{b}} \cap U_{[\vec{y}:\vec{C} \mid \psi],\vec{c}} = U_{[\vec{x}:\vec{B},\vec{y}:\vec{C} \mid \phi\wedge\psi],\vec{b}*\vec{c}}$$

(where we use $*$ to indicate concatenation of tuples. We shall also sometimes use a comma to the same effect.) The *logical topology* on $G_{\mathbb{T}}$ is defined by taking as sub-basic open sets those of the form

- $s^{-1}(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}}) = \left\{ f \in G_{\mathbb{T}} \;\middle|\; \vec{a} \in [\![\vec{x} : \vec{A} \mid \phi]\!]^{s(f)} \right\}$
- $V_{B:b\mapsto c} = \left\{ f \in G_{\mathbb{T}} \;\middle|\; b \in [\![x : B \mid \top]\!]^{s(f)} \wedge f_B(b) = c \right\}$
- $t^{-1}(U_{[\vec{x}:\vec{D}\,|\,\psi],\vec{d}}) = \left\{ f \in G_{\mathbb{T}} \;\middle|\; \vec{d} \in [\![\vec{x} : \vec{D} \mid \psi]\!]^{t(f)} \right\}$

We make the notation for a basic open set of $G_{\mathbb{T}}$ an array displaying the three sets of relevant data, e.g.:

$$V = \begin{pmatrix} \left[\vec{x} : \vec{A} \middle| \phi\right], \vec{a} \\ \vec{B} : \vec{b} \mapsto \vec{c} \\ \left[\vec{y} : \vec{D} \middle| \psi\right], \vec{d} \end{pmatrix} \tag{2.3}$$

which, written out, is the set

$$\begin{aligned} &V = \{f : \mathbf{M} \Rightarrow \mathbf{N} | \\ &\quad \vec{a} \in [\![\vec{x} : \vec{A} \mid \phi]\!]^{\mathbf{M}} \wedge \vec{b} \in [\![\vec{x} : \vec{B} \mid \top]\!]^{\mathbf{M}} \wedge f_{\vec{B}}(\vec{b}) = \vec{c} \wedge \vec{d} \in [\![\vec{y} : \vec{D} \mid \psi]\!]^{\mathbf{N}}\} \end{aligned}$$

These sets cover $G_{\mathbb{T}}$ and are closed under finite intersection.

Lemma 2.3.2.2 *Equipped with the logical topologies, the groupoid*

$$G_{\mathbb{T}} \times_{X_{\mathbb{T}}} G_{\mathbb{T}} \xrightarrow{\ \mathrm{c}\ } G_{\mathbb{T}} \overset{\mathrm{i}}{\circlearrowleft} \quad G_{\mathbb{T}} \underset{\mathrm{t}}{\overset{\mathrm{s}}{\rightrightarrows}} X_{\mathbb{T}}, \quad X_{\mathbb{T}} \xrightarrow{\ \mathrm{e}\ } G_{\mathbb{T}}$$

is a topological groupoid.

Proof We need to verify that the functions s, t, c, i, and e are continuous. s and t are continuous by the definition of the topology on $G_{\mathbb{T}}$.
(e): Given a basic open set V of $G_{\mathbb{T}}$ of the form (2.3). Say the length of $\vec{b}$ is n. If for some $1 \leq i \leq n$, $b_i \neq c_i$, then $\mathrm{Id}^{-1}(V) = \emptyset$. Otherwise, $\mathrm{e}^{-1}(V) = U_{[\vec{x}:\vec{A},\vec{y}:\vec{B},\vec{z}:\vec{D}\,|\,\phi\wedge\psi]\vec{a}*\vec{b}*\vec{d}}$. Thus e is continuous.

(i): Given a basic open set V of $G_{\mathbb{T}}$ of the form (2.3). Then

$$i^{-1}(V) = \left(\begin{array}{c} \left[\vec{y} : \vec{D} \,\middle|\, \psi\right], \vec{d} \\ \vec{B} : \vec{c} \mapsto \vec{b} \\ \left[\vec{x} : \vec{A} \,\middle|\, \phi\right], \vec{a} \end{array} \right)$$

thus i is continuous. Note that i, being self-inverse is therefore a homeomorphism.

(c): Given a basic open set V (as in (2.3)) and a pair of composable isomorphisms $\langle g, f\rangle$ such that $c(\langle g, f\rangle) = g \circ f \in V$, the following box is seen to contain $\langle g, f\rangle$ and to map into V by c:

$$\left(\begin{array}{c} - \\ \vec{B} : f_{\vec{B}}(\vec{b}) \mapsto \vec{c} \\ \left[\vec{y} : \vec{D} \,\middle|\, \psi\right], \vec{d} \end{array} \right) \times_{X_{\mathbb{T}}} \left(\begin{array}{c} \left[\vec{x} : \vec{A} \,\middle|\, \phi\right], \vec{a} \\ \vec{B} : \vec{b} \mapsto f_{\vec{B}}(\vec{b}) \\ - \end{array} \right)$$

thus c is continuous. ⊣

For an object $\left[\vec{x} : \vec{A} \,\middle|\, \phi\right]$ of $\mathcal{C}_{\mathbb{T}}$, the *logical topology* on the set

$$E_{\left[\vec{x}:\vec{A} \,|\, \phi\right]} = \left\{ \langle \mathbf{M}, \vec{a}\rangle \;\middle|\; \mathbf{M} \in X_{\mathbb{T}}, \vec{a} \in [\![\vec{x} : \vec{A} \mid \phi]\!]^{\mathbf{M}} \right\}$$

is given by basic opens of the form

$$V_{\left[\vec{x}:\vec{A},\vec{y}:\vec{B} \,|\, \psi\right],\vec{b}} = \left\{ \langle \mathbf{M}, \vec{a}\rangle \;\middle|\; \vec{a} * \vec{b} \in [\![\vec{x} : \vec{A}, \vec{y} : \vec{B} \mid \phi \wedge \psi]\!]^{\mathbf{M}} \right\}$$

(where the x's are distinct from the y', but not necessarily of distinct type, and $\vec{b}$ is of the same length as $\vec{y}$) which clearly covers and is closed under intersections, since

$$V_{\left[\vec{x}:\vec{A},\vec{y}:\vec{B} \,|\, \psi\right],\vec{b}} \cap V_{\left[\vec{x}:\vec{A},\vec{y'}:\vec{D} \,|\, \xi\right],\vec{d}} = V_{\left[\vec{x}:\vec{A},\vec{y}:\vec{B},\vec{y'}:\vec{D} \,|\, \psi\wedge\xi\right],\vec{b}*\vec{d}}$$

For an object $\left[\vec{x} : \vec{A} \,\middle|\, \phi\right]$ in $\mathcal{C}_{\mathbb{T}}$, we now have the following:

Lemma 2.3.2.3 *The projection*

$$\pi_1 : E_{\left[\vec{x}:\vec{A} \,|\, \phi\right]} \to X_{\mathbb{T}}$$

defined by $\langle \mathbf{M}, \vec{a}\rangle \mapsto \mathbf{M}$ *is continuous.*

PROOF Let a basic open $U_{[\vec{y}:\vec{B}\,|\,\psi],\vec{b}} \subseteq X_{\mathbb{T}}$ be given. Then

$$\pi_1^{-1}(U_{[\vec{y}:\vec{B}\,|\,\psi],\vec{b}}) = V_{[\vec{x}:\vec{A},\vec{y}:\vec{B}\,|\,\psi],\vec{b}} \qquad \dashv$$

Lemma 2.3.2.4 *The projection $\pi_1 : E_{[\vec{x}:\vec{A}\,|\,\phi]} \to X_{\mathbb{T}}$ is open.*

PROOF Given a basic open $V_{[\vec{x}:\vec{A},\vec{y}:\vec{B}\,|\,\psi],\vec{b}} \subseteq E_{[\vec{x}:\vec{A}\,|\,\phi]}$ Then

$$\pi_1(V_{[\vec{x}:\vec{A},\vec{y}:\vec{B}\,|\,\psi],\vec{b}}) = U_{[\vec{y}:\vec{B}\,|\,\exists\vec{x}:\vec{A}.\phi\wedge\psi],\vec{b}} \qquad \dashv$$

Lemma 2.3.2.5 *The projection $\pi_1 : E_{[\vec{x}:\vec{A}\,|\,\phi]} \to X_{\mathbb{T}}$ is a local homeomorphism.*

PROOF Let $\langle \mathbf{M}, \vec{a}\rangle \in E_{[\vec{x}:\vec{A}\,|\,\phi]}$ be given. Then

$$\langle \mathbf{M}, \vec{a}\rangle \in V := V_{[\vec{x}:\vec{A},\vec{y}:\vec{A}\,|\,\vec{x}=\vec{y}],\vec{a}} \subseteq E_{[\vec{x}:\vec{A}\,|\,\phi]}$$

and $\langle \mathbf{N}, \vec{a'}\rangle \in V$ if and only if $\vec{a} = \vec{a'}$. Thus $\pi_1 \restriction_V$ is injective. By Lemma 2.3.2.3 and Lemma 2.3.2.4, $\pi_1 \restriction_V : V \to \pi_1(V)$ is continuous, open, and bijective, and therefore a homeomorphism. $\dashv$

Lemma 2.3.2.6 *Given an arrow*

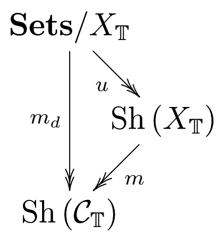
$$\left[\vec{x}:\vec{A},\vec{y}:\vec{B}\,\middle|\,\sigma\right] : \left[\vec{x}:\vec{A}\,\middle|\,\phi\right] \longrightarrow \left[\vec{y}:\vec{B}\,\middle|\,\psi\right]$$

in $\mathcal{C}_{\mathbb{T}}$, the corresponding function $E_\sigma : E_{[\vec{x}:\vec{A}\,|\,\phi]} \to E_{[\vec{y}:\vec{B}\,|\,\psi]}$ is continuous.

PROOF Given a basic open $V_{[\vec{y}:\vec{B},\vec{z}:\vec{C}\,|\,\xi],\vec{c}} \subseteq E_{[\vec{y}:\vec{B}\,|\,\psi]}$

$$E_\sigma^{-1}(V_{[\vec{y}:\vec{B},\vec{z}:\vec{C}\,|\,\xi],\vec{c}}) = V_{[\vec{x}:\vec{A},\vec{z}:\vec{C}\,|\,\exists\vec{y}:\vec{B}.\sigma\wedge\xi],\vec{c}} \qquad \dashv$$

Proposition 2.3.2.7 *The geometric morphism $m_d : \mathbf{Sets}/X_{\mathbb{T}} \twoheadrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ factors through $\mathrm{Sh}(X_{\mathbb{T}})$ as*



where $u^* : \mathrm{Sh}(X_{\mathbb{T}}) \longrightarrow \mathbf{Sets}/X_{\mathbb{T}}$ is the forgetful functor. Moreover, the second component, m, of the factorization is a surjection.

Proof By Lemma 2.3.2.5 and Lemma 2.3.2.6, the functor

$$\mathcal{M}_d : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}/X_{\mathbb{T}}$$

factors through $\mathrm{Sh}\,(X_{\mathbb{T}})$,

$$\begin{array}{ccc} \mathbf{Sets}/X_{\mathbb{T}} & & \\ {\scriptstyle u^*}\uparrow & \nwarrow^{\mathcal{M}_d} & \\ \mathrm{Sh}\,(X_{\mathbb{T}}) & \xleftarrow[\mathcal{M}]{} & \mathcal{C}_{\mathbb{T}} \end{array}$$

Since $\mathcal{M}_d$ is coherent and the forgetful functor u^* reflects coherent structure, $\mathcal{M}$ is coherent, and so it induces a geometric morphism

$$m : \mathrm{Sh}\,(X_{\mathbb{T}}) \longrightarrow \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$$

through which m_d factors. Finally, m is a surjection because m_d is. $\dashv$

2.3.3 Topological Groupoids and Equivariant Sheaves

Recall that if $\mathbb{H}$ is an arbitrary topological groupoid, which we also write as $H_1 \rightrightarrows H_0$, the topos of *equivariant sheaves* (or the topos of *continuous actions*) on $\mathbb{H}$, written $\mathrm{Sh}\,(\mathbb{H})$ or $\mathrm{Sh}_{H_1}(H_0)$, consists of the following[1] [10, B3.4.14(b)], [21], [23]:

- An object of $\mathrm{Sh}\,(\mathbb{H})$ is a pair $\langle a : A \to H_0, \alpha \rangle$, where a is a local homeomorphism (that is, an object of $\mathrm{Sh}\,(H_0)$) and $\alpha : H_1 \times_{H_0} A \to A$ is a continuous function from the pullback (in **Top**) of a along the source map $s : H_1 \to H_0$ to A such that

  $$a(\alpha(f, x)) = t(f)$$

  and satisfying the axioms for an action:

  (i) $\alpha(1_h, x) = x$ for $h \in H_0$.

  (ii) $\alpha(g, \alpha(f, x)) = \alpha(g \circ f, x)$.

  For illustration, it follows that for $f \in H_1$, $\alpha(f, -)$ is a bijective function from the fiber over $s(f)$ to the fiber over $t(f)$.

[1] [21] denotes the topos of equivariant sheaves on $\mathbb{G}$ by $B(\mathbb{G})$. [10] uses the notation $\mathrm{Cont}\,(\mathbb{H})$.

- An arrow $h : \langle a : A \to H_0, \alpha\rangle \longrightarrow \langle b : B \to H_0, \beta\rangle$ is an arrow of $\mathrm{Sh}\,(H_0)$,

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & H_0 & \end{array}$$

which commutes with the actions:

$$\begin{array}{ccc} H_1 \times_{H_0} A & \xrightarrow{\alpha} & A \\ {\scriptstyle 1_{H_1} \times_{H_0} h}\downarrow & & \downarrow{\scriptstyle h} \\ H_1 \times_{H_0} B & \xrightarrow[\beta]{} & B \end{array}$$

For a topological groupoid $G_1 \rightrightarrows G_0$, the forgetful functor

$$u : \mathrm{Sh}_{G_1}(G_0) \longrightarrow \mathrm{Sh}\,(G_0)$$

(forgetting the action) is an inverse image functor. Being faithful, it is in particular a conservative coherent functor. Next, if $f : \mathbb{G} \longrightarrow \mathbb{H}$ is a morphism of topological groupoids, with components the two continuous functions,

$$\begin{array}{ccc} G_1 & \xrightarrow{f_1} & H_1 \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} & & {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ G_0 & \xrightarrow[f_0]{} & H_0 \end{array}$$

then $f : \mathbb{G} \longrightarrow \mathbb{H}$ induces a geometric morphism, which we, abusing notation, denote $f : \mathrm{Sh}\,(\mathbb{G}) \longrightarrow \mathrm{Sh}\,(\mathbb{H})$. The inverse image $f^* : \mathrm{Sh}\,(\mathbb{G}) \longrightarrow \mathrm{Sh}\,(\mathbb{H})$ sends an equivariant sheaf $\langle a : A \to H_0, \alpha\rangle$ in $\mathrm{Sh}\,(\mathbb{H})$ to the equivariant sheaf on $\mathbb{G}$ consisting of the local homeomorphism $f_0^*(a) : f_0^*(A) \to G_0$,

$$\begin{array}{ccc} f_0^*(A) & \longrightarrow & A \\ {\scriptstyle f_0^*(a)}\downarrow & \lrcorner & \downarrow{\scriptstyle a} \\ G_0 & \xrightarrow[f_0]{} & H_0 \end{array}$$

the fiber of which over $x \in G_0$ is the fiber of a over $f_0(x)$, up to an isomorphism which we fix according to the following convention:

Remark 2.3.3.1 For the sake of accuracy and notational convenience, and because we sometimes (in Section 2.4 in particular) want to e.g. define certain functors on the nose and not just up to isomorphism, we fix certain choices regarding the geometric structure in the topoi of sheaves on a space and the actions of certain inverse image functors between them. Given a canonical choice of geometric structure in **Sets** (a choice of cartesian n-products, choosing inclusions to represent subobjects, etc.), we take as the canonical choice of the same structure in sheaves on a space to be induced, fiberwise, by that in **Sets**. And given a continuous function $f : X \to Y$ between spaces, we take the inverse image of the induced geometric morphism $f : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y)$ to work by pullback along f in a way that preserves fibers. When we need to be specific, we take the category $\mathrm{Sh}(X)$ of sheaves on a space X to consist of those local homeomorphisms $a : A \to X$ over X which are left projections, i.e. such that the domain consists of ordered pairs $\langle x, y\rangle$, with $x \in X$ and with a being the projection $\langle x, y\rangle \mapsto x$. As such, $\mathrm{Sh}(X)$ is clearly equivalent to the category of (all) local homeomorphisms over X. In the context of this convention, we refer to left element as the *index*, and we refer to the fiber with index forgotten over $x \in X$ for a sheaf $\pi_1 : A \to X$ as $A_x = \pi_2(\pi_1^{-1}(x)) = \{a \mid \langle x, a\rangle \in A\}$. Accordingly, the distinguished product, for example, of two sheaves $A \to X$ and $B \to X$ is the sheaf $\{\langle x, \langle y, z\rangle\rangle \mid \langle x, y\rangle \in A \wedge \langle x, z\rangle \in B\} \to X$ the fiber (index forgotten) over $x \in X$ of which is the cartesian product $A_x \times B_x$. The distinguished representative of a subobject is the subset inclusion. Given a continuous function $f : X \to Y$, the inverse image part of the induced geometric morphism $f : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y)$ sends a local homeomorphism $b : B \to Y$ to the local homeomorphism obtained by pullback,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

where the fiber over $x \in X$ is the fiber over $f(x) \in Y$, up to reindexing. That is to say, the element of A corresponding to $\langle f(x), z\rangle \in B$ is $\langle x, z\rangle \in A$, so that $A_x = B_{f(x)}$. For an element $x \in X$, the induced point $p_x : \mathbf{Sets} \to \mathrm{Sh}(X)$ sends a sheaf $A \to X$ to the fiber over x, forgetting the index, $p_x^*(A) = A_x = \{y \mid \langle x, y\rangle \in A\}$.

Accordingly, the continuous action of $f^*(\langle a : A \to H_0, \alpha \rangle)$ is defined by

$$\langle g, \langle s(g), y \rangle \rangle \mapsto \langle t(g), \pi_2(\alpha(f_1(g), \langle f_0(s(g)), y \rangle))\rangle$$

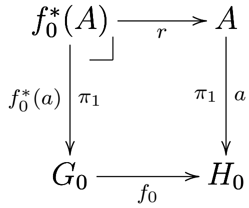
or, forgetting the indexing,

$$\langle g, y \rangle \mapsto \alpha(f_1(g), y)$$

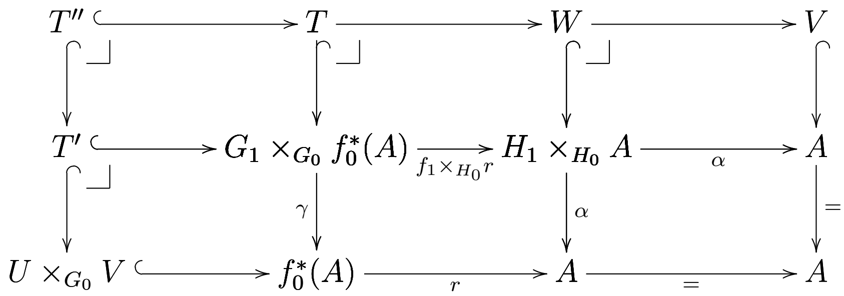
For illustration, we verify that the action so defined, call it

$$\gamma : f_0^*(A) \times_{G_0} G_1 \to f_0^*(A)$$

is indeed continuous. According to our convention, the space $f_0^*(A)$ in the pullback square



is the set $f_0^*(A) = \{\langle x, c \rangle \mid x \in G_0, \langle f_0(x), c \rangle \in A\}$—with $r : f_0^*(A) \to A$ the 'reindexing' function $\langle x, c \rangle \mapsto \langle f_0(x), c \rangle$—equipped with topology generated by basic opens of the form $U \times_{H_0} V = \{\langle x, c \rangle \in f_0^*(A) \mid x \in U, \langle f_0(x), c \rangle \in V\}$ for open sets $U \subseteq G_0$ and $V \subseteq A$. We leave the indexing implicit. Given an element $\langle g, c \rangle \in G_1 \times_{G_0} f_0^*(A)$ and a basic open neighborhood $U \times_{H_0} V \subseteq f_0^*(A)$ of $\gamma(\langle g, c \rangle)$, we must find an open neighborhood of $\langle g, c \rangle$ the image under γ of which is contained in $U \times_{H_0} V$. Then chase through the diagram:



Since $\gamma(g, c) \in U \times_{G_0} V$, we have $\langle g, c \rangle \in T'$ and $r(\gamma(g, c)) = \alpha(f_1(g), c) \in V$. Then $\langle f_1(g), c \rangle \in W$, so $\langle g, c \rangle \in T$ as well. Hence $\langle g, c \rangle \in T''$, and since all squares commute, $\gamma(T'') \subseteq U \times_{G_0} V$.

2.3.4 Equivariant Sheaves on the Semantical Groupoid

It is clear that if we are presented with a basic open set $U_{[\vec{y}:\vec{B}\,|\,\phi],\vec{b}} \subseteq X_{\mathbb{T}}$ or $V_{[\vec{x}:\vec{A},\vec{y}:\vec{B}\,|\,\psi],\vec{b}} \subseteq E_{[\vec{x}:\vec{A}\,|\,\phi]}$, we can assume without loss of generality that, for $i \neq j$, $B_i = B_j$ implies $b_i \neq b_j$. We say that $U_{[\vec{y}:\vec{B}\,|\,\phi],\vec{b}}$ is presented in *reduced form* if this condition is satisfied.

Lemma 2.3.4.1 *Let a list of sorts $\vec{A}$ of $\mathbb{T}$ and two tuples $\vec{a}$ and $\vec{b}$ of $\mathbf{Sets}_\kappa$ be given, of the same length as $\vec{A}$, and satisfying the requirement that whenever $i \neq j$, $A_i = A_j$ implies $a_i \neq a_j$ and $b_i \neq b_j$. Then for any $\mathbf{M} \in X_{\mathbb{T}}$, if $\vec{a} \in [\![\vec{x} : \vec{A} \mid \top]\!]^{\mathbf{M}}$, there exists an $\mathbf{N} \in X_{\mathbb{T}}$ and an isomorphism $f : \mathbf{M} \Rightarrow \mathbf{N}$ in $G_{\mathbb{T}}$ such that $f_{\vec{A}}(\vec{a}) = \vec{b}$.*

Proof Let $\mathfrak{S}$ be the set of sorts of $\mathbb{T}$. For each $S \in \mathfrak{S}$, we can choose a set, c_S, in $\mathbf{Sets}_\kappa$ not in the transitive closure of $\{b_1, \ldots, b_n\} \cup [\![S]\!]^{\mathbf{M}}$. Construct sets $[\![S]\!]^{\mathbf{N}'}$ and (the obvious) bijections $[\![S]\!]^{\mathbf{N}'} \cong [\![S]\!]^{\mathbf{M}}$ by setting $[\![S]\!]^{\mathbf{N}'} = [\![S]\!]^{\mathbf{M}} \times \{c_S\}$ for all $S \in \mathfrak{S}$. Next, construct sets $[\![S]\!]^{\mathbf{N}}$ and bijections $[\![S]\!]^{\mathbf{N}} \cong [\![S]\!]^{\mathbf{N}'}$ by setting $[\![S]\!]^{\mathbf{N}} = [\![S]\!]^{\mathbf{N}'}$ if $S \notin \vec{A}$, and by setting $[\![A_i]\!]^{\mathbf{N}}$ to be the set $[\![A_i]\!]^{\mathbf{N}'}$ with any element $\langle a_i, c_{A_i}\rangle \in [\![A_i]\!]^{\mathbf{N}'}$ replaced by b_i. Then for all $S \in \mathfrak{S}$, we have a bijection $f_S : [\![S]\!]^{\mathbf{M}} \to [\![S]\!]^{\mathbf{N}}$. These bijections induce a $\mathbb{T}$-model $\mathbf{N}$ with an isomorphism $f : \mathbf{M} \Rightarrow \mathbf{N}$ such that $f_{\vec{A}}(\vec{a}) = \vec{b}$. ⊣

Lemma 2.3.4.2 *The groupoid $\mathbb{G}_{\mathbb{T}}$ is open.*

Proof We verify that the source map is open, from which it follows that the target map is open as well. Let a basic open subset

$$V = \begin{pmatrix} \left[\vec{x} : \vec{A} \,\middle|\, \phi\right], \vec{a} \\ \vec{B} : \vec{b} \mapsto \vec{c} \\ \left[\vec{y} : \vec{D} \,\middle|\, \psi\right], \vec{d} \end{pmatrix}$$

of $G_{\mathbb{T}}$ be given, and suppose $f : \mathbf{M} \Rightarrow \mathbf{N}$ is in V. We must find an open neighborhood around $\mathbf{M}$ which is contained in $s(V)$. We claim that

$$U = U_{[\vec{x}:\vec{A},\vec{y}:\vec{D},\vec{z}:\vec{B}\,|\,\phi\wedge\psi],\vec{a}*f_{\vec{D}}^{-1}(\vec{d})*\vec{b}}$$

does the trick. Clearly, $\mathbf{M} \in U$. Suppose $\mathbf{K} \in U$. Consider the tuples $f_{\vec{D}}^{-1}(\vec{d}) * \vec{b}$ and $\vec{d} * \vec{c}$ together with the list of sorts $\vec{D} * \vec{B}$. Since $f_{\vec{D}*\vec{B}}$ sends the first tuple to the second, we can assume that the conditions of Lemma 2.3.4.1 are satisfied (or a simple rewriting will see that they are), and so there exists a $\mathbb{T}$-model $\mathbf{L}$ and an isomorphism $g : \mathbf{K} \Rightarrow \mathbf{L}$ such that $g \in V$. So $U \subseteq s(V)$. ⊣

For each $\left[\vec{x} : \vec{A} \,\middle|\, \phi\right] \in \mathcal{C}_{\mathbb{T}}$ there is an action

$$\theta_{\left[\vec{x}:\vec{A} \,\middle|\, \phi\right]} : G_{\mathbb{T}} \times_{X_{\mathbb{T}}} E_{\left[\vec{x}:\vec{A} \,\middle|\, \phi\right]} \longrightarrow E_{\left[\vec{x}:\vec{A} \,\middle|\, \phi\right]} \tag{2.4}$$

of $G_{\mathbb{T}}$ on $E_{\left[\vec{x}:\vec{A} \,\middle|\, \phi\right]}$ defined by $\langle f, \langle \mathbf{M}, \vec{a} \rangle \rangle \mapsto \langle \mathbf{N}, f_{\vec{A}}(\vec{a}) \rangle$. The subscript on θ will usually be left implicit.

Lemma 2.3.4.3 *The action*

$$\theta : G_{\mathbb{T}} \times_{X_{\mathbb{T}}} E_{\left[\vec{x}:\vec{A} \,\middle|\, \phi\right]} \longrightarrow E_{\left[\vec{x}:\vec{A} \,\middle|\, \phi\right]}$$

is continuous.

Proof Let a basic open

$$U = V_{\left[\vec{x}:\vec{A}, \vec{y}:\vec{B} \,\middle|\, \psi\right], \vec{b}} \subseteq E_{\left[\vec{x}:\vec{A} \,\middle|\, \phi\right]}$$

be given, and suppose $\theta(f, \langle \mathbf{M}, \vec{a} \rangle) = \langle \mathbf{N}, f_{\vec{A}}(\vec{a}) \rangle \in U$ for $\mathbf{M}, \mathbf{N} \in X_{\mathbb{T}}$ and $f : \mathbf{M} \Rightarrow \mathbf{N}$ in $G_{\mathbb{T}}$. Then we can specify an open neighborhood around $\langle f, \langle \mathbf{M}, \vec{a} \rangle \rangle$ which θ maps into U as:

$$\langle f, \langle \mathbf{M}, \vec{a} \rangle \rangle \in \left(\begin{array}{c} - \\ \vec{B} : f_{\vec{B}}^{-1}(\vec{b}) \mapsto \vec{b} \\ - \end{array} \right) \times_{X_{\mathbb{T}}} V_{\left[\vec{x}:\vec{A}, \vec{y}:\vec{B} \,\middle|\, \psi\right], f_{\vec{B}}^{-1}(\vec{b})}$$

⊣

For a subobject (represented by an inclusion) $\left[\vec{x} : \vec{A} \,\middle|\, \xi\right] \hookrightarrow \left[\vec{x} : \vec{A} \,\middle|\, \phi\right]$ in $\mathcal{C}_{\mathbb{T}}$, the open subset $E_{\left[\vec{x}:\vec{A} \,\middle|\, \xi\right]} \subseteq E_{\left[\vec{x}:\vec{A} \,\middle|\, \phi\right]}$ is closed under the action θ. We call a subset that is closed under the action of $G_{\mathbb{T}}$ *stable* so as to reserve "closed" to topologically closed. We claim that the only stable opens of $E_{\left[\vec{x}:\vec{A} \,\middle|\, \phi\right]}$ come from (joins of) subobjects of $\left[\vec{x} : \vec{A} \,\middle|\, \phi\right]$:

Lemma 2.3.4.4 *Let* $\left[\vec{x}:\vec{A}\,\middle|\,\phi\right]$ *in* $\mathcal{C}_{\mathbb{T}}$ *and* U *a basic open subset of* $E_{[\vec{x}:\vec{A}\,|\,\phi]}$ *of the form*

$$U = V_{[\vec{x}:\vec{A},\vec{y}:\vec{B}\,|\,\psi],\vec{b}}$$

be given. Then the stabilization (closure) of U *under the action* θ *of* $G_{\mathbb{T}}$ *on* $E_{[\vec{x}:\vec{A}\,|\,\phi]}$ *is a subset of the form* $E_{[\vec{x}:\vec{A}\,|\,\xi]} \subseteq E_{[\vec{x}:\vec{A}\,|\,\phi]}$.

Proof We can assume without loss that U is on reduced form. Let φ be the formula expressing the conjunction of inequalities $y_i \neq y_j$ for all pairs of indices $i \neq j$ such that $B_i = B_j$ in $\vec{B}$. We claim that the stabilization of U is $E_{[\vec{x}:\vec{A}\,|\,\xi]}$ where ξ is the formula $\exists \vec{y}{:}\vec{B}.\,\phi \wedge \psi \wedge \varphi$. First, $E_{[\vec{x}:\vec{A}\,|\,\xi]}$ is a stable set containing U. Next, suppose $\langle \mathbf{M}, \vec{a}\rangle \in E_{[\vec{x}:\vec{A}\,|\,\xi]}$. Then there exists $\vec{c}$ such that $\vec{a} * \vec{c} \in [\![\vec{x}:\vec{A}, \vec{y}:\vec{B} \mid \phi \wedge \psi \wedge \varphi]\!]^{\mathbf{M}}$. Then $\vec{b}$ and $\vec{c}$ (with respect to $\vec{B}$) satisfy the conditions of Lemma 2.3.4.1, so there exists a $\mathbb{T}$-model $\mathbf{N}$ with isomorphism $f : \mathbf{M} \Rightarrow \mathbf{N}$ such that $f_{\vec{B}}(\vec{c}) = \vec{b}$. Then $\theta(f, \langle \mathbf{M}, \vec{a}\rangle) \in U$, and hence $\langle \mathbf{M}, \vec{a}\rangle$ is in the stabilization of U. ⊣

We call a subset of the form $E_{[\vec{x}:\vec{A}\,|\,\xi]} \subseteq E_{[\vec{x}:\vec{A}\,|\,\phi]}$, for a subobject

$$\left[\vec{x}:\vec{A}\,\middle|\,\xi\right] \hookrightarrow \left[\vec{x}:\vec{A}\,\middle|\,\phi\right]$$

in $\mathcal{C}_{\mathbb{T}}$, a *definable* subset of $E_{[\vec{x}:\vec{A}\,|\,\phi]}$.

Corollary 2.3.4.5 *Any open stable subset of* $E_{[\vec{x}:\vec{A}\,|\,\phi]}$ *is a union of definable subsets.*

We also note the following:

Lemma 2.3.4.6 *Let* $U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}}$ *be a basic open of* $X_{\mathbb{T}}$ *in reduced form. Then there exists a sheaf* $E_{[\vec{x}:\vec{A}\,|\,\xi]}$ *and a (continuous) section*

$$s : U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}} \longrightarrow E_{[\vec{x}:\vec{A}\,|\,\xi]}$$

such that $E_{[\vec{x}:\vec{A}\,|\,\xi]}$ *is the stabilization of the open set* $s(U_{[\vec{x}:\vec{A}\,|\,\phi]}) \subseteq E_{[\vec{x}:\vec{A}\,|\,\xi]}$.

Proof Let φ be the formula expressing the inequalities $x_i \neq x_j$ for all pairs of indices $i \neq j$ such that $A_i = A_j$ in $\vec{A}$. Let $\xi := \phi \wedge \varphi$ and consider the function $s : U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}} \longrightarrow E_{[\vec{x}:\vec{A}\,|\,\xi]}$ defined by $\mathbf{M} \mapsto \langle \mathbf{M}, \vec{a}\rangle$. The image of s is the open set $V_{[\vec{x}:\vec{A},\vec{y}:\vec{A}\,|\,\vec{x}=\vec{y}],\vec{a}}$, so s is a (continuous) section. And by the proof of Lemma 2.3.4.4, the stabilization of $V_{[\vec{x}:\vec{A},\vec{y}:\vec{A}\,|\,\vec{x}=\vec{y}],\vec{a}}$ is exactly $E_{[\vec{x}:\vec{A}\,|\,\xi]}$. ⊣

Consider now the topos of equivariant sheaves $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$. For an object $E_{[\vec{x}:\vec{A}\,|\,\phi]}$, with action $\theta : G_{\mathbb{T}} \times_{X_{\mathbb{T}}} E_{[\vec{x}:\vec{A}\,|\,\phi]} \to E_{[\vec{x}:\vec{A}\,|\,\phi]}$ as in (2.4), the assignment

$$\left[\vec{x} : \vec{A} \,\middle|\, \phi\right] \mapsto \left\langle\, E_{[\vec{x}:\vec{A}\,|\,\phi]} \longrightarrow X_{\mathbb{T}} \,, \theta \right\rangle$$

defines, by Lemma 2.3.4.3 and since the actions clearly commute with definable arrows between definable sheaves, a functor $\mathcal{M}^{\dagger} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ which factors $\mathcal{M} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ through $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$:

$$\begin{array}{ccc} & & \mathrm{Sh}\,(X_{\mathbb{T}}) \\ & \overset{\mathcal{M}}{\nearrow} & \uparrow u^{*} \\ \mathcal{C}_{\mathbb{T}} & \xrightarrow[\mathcal{M}^{\dagger}]{} & \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) \end{array} \tag{2.5}$$

where u^{*} is the forgetful functor. We call the image of $\mathcal{M}^{\dagger}$ the *definable* objects and arrows of $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$. Since $\mathcal{M}$ is coherent and the forgetful functor u^{*} reflects coherent structure, $\mathcal{M}^{\dagger}$ is coherent. Therefore, (2.5) induces a commuting diagram of geometric morphisms:

$$\begin{array}{ccc} \mathrm{Sh}\,(X_{\mathbb{T}}) & & \\ \Big\downarrow m & \overset{u}{\searrow} & \\ & & \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) \\ & \underset{m^{\dagger}}{\swarrow} & \\ \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}}) & & \end{array}$$

where $m^{\dagger}$ is a surjection because m is. We state these facts for reference:

Lemma 2.3.4.7 $\mathcal{M}^{\dagger} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ *is coherent, conservative (i.e. faithful and reflects isomorphisms) , and reflects covers.*

The purpose of this section is to establish that $m^{\dagger}$ is an equivalence. We begin by establishing that the definable objects generate $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$. First, it is a known consequence that any equivariant sheaf on an open topological groupoid has an open action:

Lemma 2.3.4.8 *For any object in* $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$,

$$\left\langle\, R \xrightarrow{r} X_{\mathbb{T}} \,, \rho \right\rangle$$

the projection $\pi_2 : G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R \longrightarrow R$ *is open.*

PROOF By Lemma 2.3.4.2, since pullback preserves open maps of spaces. ⊣

Corollary 2.3.4.9 *For any object $\langle r : R \to X_{\mathbb{T}}, \rho\rangle$ in $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$, the action*

$$\rho : G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R \longrightarrow R$$

is open. Consequently, the stabilization of any open subset of R is again open.

PROOF Let a basic open $V \times_{X_{\mathbb{T}}} U \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R$ be given (so that $U \subseteq R$ and $V \subseteq G_{\mathbb{T}}$ are open). Observe that, since the inverse map $i : G_{\mathbb{T}} \longrightarrow G_{\mathbb{T}}$ is a homeomorphism, $i(V)$ is open, and

$$\begin{aligned}\rho(V \times_{X_{\mathbb{T}}} U) &= \{y \in R \mid \exists \langle f, x\rangle \in V \times_{X_{\mathbb{T}}} U.\, \rho(f,x) = y\}\\ &= \{y \in R \mid \exists f^{-1} \in i(V).\, s(f^{-1}) = r(y) \wedge \rho(f^{-1}, y) \in U\}\\ &= \pi_2(\rho^{-1}(U) \cap (i(V) \times_{X_{\mathbb{T}}} R))\end{aligned}$$

is open by Lemma 2.3.4.8. Finally, for any open $U \subseteq R$, the stabilization of U is $\rho(G_{\mathbb{T}} \times_{X_{\mathbb{T}}} U)$. ⊣

Lemma 2.3.4.10 *For any object $\left\langle R \xrightarrow{r} X_{\mathbb{T}}, \rho \right\rangle$ in $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$, and any element $x \in R$, there exists a basic open $U_{[\vec{x}:\vec{A} \mid \phi],\vec{a}} \subseteq X_{\mathbb{T}}$ and a section $v : U_{[\vec{x}:\vec{A} \mid \phi],\vec{a}} \to R$ containing x such that for any $f : \mathbf{M} \Rightarrow \mathbf{N}$ in $G_{\mathbb{T}}$ such that $\mathbf{M} \in U_{[\vec{x}:\vec{A} \mid \phi],\vec{a}}$ and $f_{\vec{A}}(\vec{a}) = \vec{a}$ (hence $\mathbf{N}$ is also in $U_{[\vec{x}:\vec{A} \mid \phi],\vec{a}}$), we have $\rho(f, v(\mathbf{M})) = v(\mathbf{N})$.*

PROOF Given $x \in R$, choose a section $s : U_{[\vec{y}:\vec{B} \mid \psi],\vec{b}} \to R$ such that $x \in s(U_{[\vec{y}:\vec{B} \mid \psi],\vec{b}})$. Pull the open set $s(U_{[\vec{y}:\vec{B} \mid \psi],\vec{b}})$ back along the continuous action ρ,

$$\begin{array}{ccc} V & \longrightarrow & s(U_{[\vec{y}:\vec{B} \mid \psi],\vec{b}}) \\ \subseteq \downarrow & \lrcorner & \downarrow \subseteq \\ G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R & \xrightarrow{\rho} & R \end{array}$$

to obtain an open set V containing $\langle 1_{r(x)}, x\rangle$. Since V is open, we can find a box of basic opens around $\langle 1_{r(x)}, x\rangle$ contained in V:

$$\langle 1_{r(x)}, x\rangle \in W := \begin{pmatrix} \left[\vec{z} : \vec{C} \,\middle|\, \xi\right], \vec{c} \\ \vec{K} : \vec{k} \mapsto \vec{k} \\ \left[\vec{z'} : \vec{C'} \,\middle|\, \eta\right], \vec{c'} \end{pmatrix} \times_{X_{\mathbb{T}}} v'(U_{[\vec{y'}:\vec{D} \mid \theta],\vec{d}}) \subseteq V$$

where v' is a section $v' : U_{[\vec{y}':\vec{D}\,|\,\theta],\vec{d}} \to R$ with x in its image. Notice that the preservation condition of W (i.e. $\vec{K} : \vec{k} \mapsto \vec{k}$) must have the same sets on both the source and the target side, since it is satisfied by $1_{r(x)}$. Now, restrict v' to the subset

$$U := U_{[\vec{z}:\vec{C},\vec{z}'':\vec{K},\vec{z}':\vec{C}',\vec{y}':\vec{D}\,|\,\xi\wedge\eta\wedge\theta],\vec{c}*\vec{k}*\vec{c}'*\vec{d}}$$

to obtain a section $v = v' \restriction_U : U \to R$. Notice that $x \in v(U)$. Furthermore, $v(U) \subseteq s(U_{[\vec{y}:\vec{B}\,|\,\psi],\vec{b}})$, for if $v(\mathbf{M}) \in v(U)$, then $\langle 1_{\mathbf{M}}, v(\mathbf{M})\rangle \in W$, and so $\rho(\langle 1_{\mathbf{M}}, v(\mathbf{M})\rangle) = v(\mathbf{M}) \in s(U_{[\vec{y}:\vec{B}\,|\,\psi],\vec{b}})$. Finally, if $\mathbf{M} \in U$ and $f : \mathbf{M} \Rightarrow \mathbf{N}$ is an isomorphism in $G_{\mathbb{T}}$ such that

$$f_{\vec{C}*\vec{K}*\vec{C}'*\vec{D}}(\vec{c} * \vec{k} * \vec{c}' * \vec{d}) = \vec{c} * \vec{k} * \vec{c}' * \vec{d}$$

then $\langle f, v(\mathbf{M})\rangle \in W$, and so $\rho(f, v(\mathbf{M})) \in s(U_{[\vec{y}:\vec{B}\,|\,\psi],\vec{b}})$. But we also have $v(\mathbf{N}) \in v(U) \subseteq s(U_{[\vec{y}:\vec{B}\,|\,\psi],\vec{b}})$, and $r(\rho(f, v(\mathbf{M}))) = r(v(\mathbf{N})$, so $\rho(f, v(\mathbf{M})) = v(\mathbf{N})$. $\dashv$

Lemma 2.3.4.11 *For any object in* $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$,

$$\langle\ R \xrightarrow{r} X_{\mathbb{T}}\ , \rho\rangle$$

and any element $x \in R$, there exists a function over $X_{\mathbb{T}}$ with definable domain and with x in its image.

Proof Choose a section $v : U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}} \to R$ with the property described in Lemma 2.3.4.10 such that $x \in v(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}})$. We can assume that $U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}}$ is on reduced form. Then, by Lemma 2.3.4.6 there exists an object $\left[\vec{x} : \vec{A}\,\middle|\,\xi\right]$ in $\mathcal{C}_{\mathbb{T}}$ and a section $s : U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}} \to E_{[\vec{x}:\vec{A}\,|\,\xi]}$ such that $E_{[\vec{x}:\vec{A}\,|\,\xi]}$ is the stabilization of $s(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}})$. Define a mapping $\hat{v} : E_{[\vec{x}:\vec{A}\,|\,\xi]} \to R$ as follows: for an element $\langle \mathbf{N}, \vec{c}\rangle \in E_{[\vec{x}:\vec{A}\,|\,\xi]}$, there exists $\langle \mathbf{M}, \vec{a}\rangle \in s(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}}) \subseteq E_{[\vec{x}:\vec{A}\,|\,\xi]}$ and $f : \mathbf{M} \Rightarrow \mathbf{N}$ in $G_{\mathbb{T}}$ such that $f_{\vec{A}}(\vec{a}) = \vec{c}$. Set $\hat{v}(\langle \mathbf{N}, \vec{c}\rangle) = \rho(f, v(\mathbf{M}))$. We verify that $\hat{v}$ is well defined: suppose $\langle \mathbf{M}', \vec{a}\rangle \in s(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}}) \subseteq E_{[\vec{x}:\vec{A}\,|\,\xi]}$ and $g : \mathbf{M}' \Rightarrow \mathbf{N}$ in $G_{\mathbb{T}}$ is such that $g_{\vec{A}}(\vec{a}) = \vec{c}$. Then $g^{-1} \circ f : \mathbf{M} \Rightarrow \mathbf{M}'$ sends $\vec{a} \in [\![\vec{x} : \vec{A} \mid \phi]\!]^{\mathbf{M}}$ to $\vec{a} \in [\![\vec{x} : \vec{A} \mid \phi]\!]^{\mathbf{M}'}$, and so by the choice of section $v : U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}} \to R$, we have that $\rho(g^{-1} \circ f, v(\mathbf{M})) = v(\mathbf{M}')$. But then

$$\rho(g, v(\mathbf{M}')) = \rho(g, \rho(g^{-1} \circ f, v(\mathbf{M}))) = \rho(f, v(\mathbf{M}))$$

so the value of $\hat{v}$ at $\langle \mathbf{N}, \vec{c} \rangle$ is indeed independent of the choice of $\langle \mathbf{M}, \vec{a} \rangle$ and f. Finally, the following triangle commutes,

$$\begin{array}{ccc} E_{[\vec{x}:\vec{A} \,|\, \xi]} & \xrightarrow{\hat{v}} & R \\ {\scriptstyle s}\nwarrow & & \nearrow{\scriptstyle v} \\ & U_{[\vec{x}:\vec{A} \,|\, \phi],\vec{a}} & \end{array}$$

and so x is in the image of $\hat{v}$. ⊣

Lemma 2.3.4.12 *The function* $\hat{v} : E_{[\vec{x}:\vec{A} \,|\, \xi]} \to R$ *of Lemma 2.3.4.11 is the underlying function of a morphism,*

$$\begin{array}{ccc} E_{[\vec{x}:\vec{A} \,|\, \xi]} & \xrightarrow{\hat{v}} & R \\ {\scriptstyle p}\searrow & & \swarrow{\scriptstyle r} \\ & X_{\mathbb{T}} & \end{array}$$

of $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$*, where the action on* $E_{[\vec{x}:\vec{A} \,|\, \xi]}$ *is denoted* θ *(recall 2.4 on page 51).*

Proof The definition of $\hat{v}$ makes it straightforward to see that $\hat{v}$ commutes with the actions θ and ρ of $E_{[\vec{x}:\vec{A} \,|\, \xi]}$ and R, respectively. Remains to show that $\hat{v}$ is continuous. Recall the commuting triangle from the proof of Lemma 2.3.4.11:

$$\begin{array}{ccc} E_{[\vec{x}:\vec{A} \,|\, \xi]} & \xrightarrow{\hat{v}} & R \\ {\scriptstyle s}\nwarrow & & \nearrow{\scriptstyle v} \\ & U_{[\vec{x}:\vec{A} \,|\, \phi],\vec{a}} & \end{array}$$

Let $x \in \hat{v}(E_{[\vec{x}:\vec{A} \,|\, \xi]})$ be given, and suppose U is a open neighborhood of x. By Corollary 2.3.4.9, we can assume that $U \subseteq \hat{v}(E_{[\vec{x}:\vec{A} \,|\, \xi]})$. Suppose $x = \hat{v}(\langle \mathbf{N}, \vec{c} \rangle) = \rho(f, v(\mathbf{M}))$ for a $f : \mathbf{M} \Rightarrow \mathbf{N}$ such that $\theta(f, s(\mathbf{M})) = \langle \mathbf{N}, \vec{c} \rangle$. We must find an open neighborhood W around $\langle \mathbf{N}, \vec{c} \rangle$ such that $\hat{v}(W) \subseteq U$. First, define the open neighborhood $T \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R$ around $\langle f, v(\mathbf{M}) \rangle$ by

$$T := \rho^{-1}(U) \cap \Big(G_{\mathbb{T}} \times_{X_{\mathbb{T}}} v(U_{[\vec{x}:\vec{A} \,|\, \phi],\vec{a}}) \Big)$$

Since there is a homeomorphism $v(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}}) \cong s(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}})$, we have $G_{\mathbb{T}} \times_{X_{\mathbb{T}}} v(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}}) \cong G_{\mathbb{T}} \times_{X_{\mathbb{T}}} s(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}})$. Set $T' \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} s(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}})$ to be the open subset corresponding to T under this homeomorphism,

$$\begin{array}{c} \langle f, v(\mathbf{M})\rangle \in T \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} v(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}}) \\ \cong \\ \langle f, s(\mathbf{M})\rangle \in T' \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} s(U_{[\vec{x}:\vec{A}\,|\,\phi],\vec{a}}) \end{array}$$

Then $\langle \mathbf{N}, \vec{c}\rangle = \theta(f, s(\mathbf{M})) \in \theta(T')$, and by Corollary 2.3.4.9, $\theta(T')$ is open. We claim that $\hat{v}(\theta(T')) \subseteq U$: for suppose $\theta(g, s(P)) \in \theta(T')$. Then $\langle g, v(P)\rangle \in T \subseteq \rho^{-1}(U)$, and so $\hat{v}(\theta(g, s(P))) = \rho(\langle g, v(P)\rangle) \subseteq U$. Thus $\theta(T')$ is the required W. $\dashv$

Through Lemmas 2.3.4.11–2.3.4.12 we have established the following:

Lemma 2.3.4.13 *The definable objects generate the topos* $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$.

We are thus in a position to conclude:

Theorem 2.3.4.14 *For a decidable coherent theory* $\mathbb{T}$ *with a saturated set of* $< \kappa$ *models, we have an equivalence*

$$\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}}) \simeq \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$$

of topoi.

Proof Since, by Lemma 2.3.4.13, the definable objects form a generating set, the full subcategory of definable objects is a site for $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ when equipped with the canonical coverage inherited from $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ (see e.g. [10, C2.2.16]). We argue first that $\mathcal{M}^{\dagger} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ is full: because $\mathcal{M}^{\dagger}$ is coherent (Lemma 2.3.4.7), definable objects are decidable. Therefore, any graph of a morphism between definable objects is complemented. Because $\mathcal{M}^{\dagger}$ reflects covers and any subobject of a definable object is a join of definable subobjects (Lemma 2.3.4.5), definable objects are compact in $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ (in the sense that any covering family of subobjects has a finite covering subfamily). But then every complemented subobject of a definable object is a finite join of definable subobjects, and therefore definable. Hence $\mathcal{M}^{\dagger}$ is full. By Lemma 2.3.4.7, $\mathcal{M}^{\dagger}$ is also faithful. Finally, the canonical coverage inherited from $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ coincides with the coherent coverage since $\mathcal{M}^{\dagger}$

reflects covers precisely with respect to the canonical coverage on $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ and the coherent coverage on $\mathcal{C}_{\mathbb{T}}$. Therefore, $\mathcal{C}_{\mathbb{T}}$ equipped with the coherent coverage is a site for $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$, so $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}}) \simeq \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$. $\dashv$

Note, furthermore, that if $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$ is an arrow of $\mathbf{T}$, then composition with F induces a morphism of groupoids $f : \mathbb{G}_{\mathbb{S}} \longrightarrow \mathbb{G}_{\mathbb{T}}$ which is straightforwardly seen to be continuous. For future use, we verify:

Lemma 2.3.4.15 *The square*

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \xrightarrow{\ F\ } & \mathcal{C}_{\mathbb{S}} \\ {\scriptstyle \mathcal{M}^{\dagger}_{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}^{\dagger}_{\mathbb{S}}} \\ \mathrm{Sh}(\mathbb{G}_{\mathbb{T}}) & \xrightarrow[f^*]{} & \mathrm{Sh}(\mathbb{G}_{\mathbb{S}}) \end{array}$$

commutes.

Proof Consider, for an object $[\vec{x} \mid \phi]$ in $\mathcal{C}_{\mathbb{T}}$, the square

$$\begin{array}{ccc} E_{[\vec{x} \mid \phi]} & \longleftarrow & E_{[\vec{x} \mid F(\phi)]} \\ \downarrow & & \downarrow \\ X_{\mathbb{T}} & \xleftarrow[f_0]{} & X_{\mathbb{S}} \end{array}$$

Since f_0 is composition with F, the fiber $F(\phi)(\mathbf{M}) = \mathbf{M}(F([\vec{x} \mid \phi]))$ over $\mathbf{M} \in X_{\mathbb{S}}$ is the fiber $\phi(f_0(\mathbf{M})) = \mathbf{M} \circ F([\vec{x} \mid \phi])$ over $f_0(\mathbf{M}) \in X_{\mathbb{T}}$, so the square is a pullback of sets. A basic open $V_{[\vec{x},\vec{y} \mid \psi],\vec{b}}$ is pulled back to a basic open $V_{[\vec{x} \mid F(\psi)],\vec{b}}$, so the pullback topology is contained in the logical topology. For an element $\langle \mathbf{M}, \vec{a} \rangle$ in basic open $V_{[\vec{x},\vec{y} \mid \psi],\vec{b}} \subseteq E_{[\vec{x} \mid F(\phi)]}$, the set $V = V_{[\vec{x} \mid \vec{x}=\vec{y}],\vec{a}} \subseteq E_{[\vec{x} \mid \phi]}$ is open and $\langle \mathbf{M}, \vec{a} \rangle \in V \times_{X_{\mathbb{T}}} U_{[\vec{x},\vec{y} \mid \psi],\vec{a}*\vec{b}} \subseteq V_{[\vec{x},\vec{y} \mid \psi],\vec{b}}$, so the logical topology is contained in the pullback topology. With $f_1 : G_{\mathbb{S}} \to G_{\mathbb{T}}$ being just a restriction function, we conclude that $f^* \circ \mathcal{M}^{\dagger}_{\mathbb{T}} = \mathcal{M}^{\dagger}_{\mathbb{S}} \circ F$. $\dashv$

2.4 Syntax-Semantics Adjunction

2.4.1 The Representation Theorem

Recall Section 2.2 and Lemma 2.2.2.3. Having presented a representation theorem (Theorem 2.3.4.14) for strict decidable coherent categories, in the form of syntactic categories, we return to the category $\mathbf{DC}_\kappa$ of (non-strict) decidable coherent categories (with a saturated set of $< \kappa$-models) and apply this result. Let $\mathcal{D}$ be a decidable coherent category in $\mathbf{DC}_\kappa$, and consider the equivalence $\eta_\mathcal{D} : \mathcal{D} \longrightarrow T\mathcal{D}$ as described in Section 2.2.2. Recall Definition 2.2.1.3. Composition with $\eta_\mathcal{D}$

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\mathbf{M}\circ\eta\mathcal{D}} & \mathbf{Sets}_\kappa \\ {\scriptstyle \eta\mathcal{D}}\downarrow & \nearrow{\scriptstyle \mathbf{M}} & \\ T\mathcal{D} = \mathcal{C}_{\mathbb{T}_\mathcal{D}} & & \end{array}$$

induces restriction functions

$$\begin{array}{ccc} G_{\mathbb{T}_\mathcal{D}} & \xrightarrow{\phi_1} & G_\mathcal{D} \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} & & {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ X_{\mathbb{T}_\mathcal{D}} & \xrightarrow[\phi_0]{} & X_\mathcal{D} \end{array}$$

commuting with source and target (as well as composition and insertion of identities) maps.

Lemma 2.4.1.1 *The maps ϕ_0 and ϕ_1 are homeomorphisms of spaces.*

Proof Any coherent functor $M : \mathcal{D} \longrightarrow \mathbf{Sets}_\kappa$ lifts to a unique $\mathbb{T}_\mathcal{D}$-model $\mathbf{M} : \mathcal{C}_{\mathbb{T}_\mathcal{D}} \longrightarrow \mathbf{Sets}_\kappa$, to yield an inverse $\psi_0 : X_\mathcal{D} \to X_{\mathbb{T}_\mathcal{D}}$ to ϕ_0. Similarly, an invertible natural transformation of functors $f : M \Rightarrow N$ lifts to a unique $\mathbb{T}_\mathcal{D}$-isomorphism $f : \mathbf{M} \Rightarrow \mathbf{N}$ to yield an inverse $\psi_1 : G_\mathcal{D} \to G_{\mathbb{T}_\mathcal{D}}$ to ϕ_1. We verify that these four maps are all continuous. For a subbasic open

$$U = (\langle f_1 : A \to B_1, \ldots, f_n : A \to B_n \rangle, \langle a_1, \ldots, a_n \rangle) \subseteq X_\mathcal{D}$$

we have

$$\phi_0^{-1}(U) = U_{\left[y_1:B_1,\ldots,y_n:B_n \,\middle|\, \exists x:A. \bigwedge_{1\leq i\leq n} f_i(x)=y_i\right],\vec{a}}$$

so ϕ_0 is continuous. To verify that ψ_0 is continuous, there are two cases to consider, namely non-empty and empty context. For basic open

$$U_{[x:A_1,\ldots,x_n:A_n \,|\, \phi],\langle a_1,\ldots,a_n\rangle} \subseteq X_{\mathbb{T}_\mathcal{D}}$$

the canonical interpretation of $\mathbb{T}_{\mathcal{D}}$ in $\mathcal{D}$ yields a subobject of a product in $\mathcal{D}$,

$$[\![x : A_1, \ldots, x_n : A_n \mid \phi]\!] \rightarrowtail A_1 \times \ldots \times A_n \xrightarrow{\pi_i} A_i.$$

Choose a monomorphism $r : R \rightarrowtail A_1 \times \ldots \times A_n$ representing that subobject. Then

$$\begin{aligned} &\psi_0^{-1}(U_{[x:A_1,\ldots,x_n:A_n \mid \phi],\langle a_1,\ldots,a_n\rangle}) \\ &= (\langle \pi_1 \circ r : R \to A_1, \ldots, \pi_n \circ r : R \to A_n\rangle, \langle a_1, \ldots, a_n\rangle) \end{aligned}$$

and it is clear that this is independent of the choice of product diagram and of representing monomorphism. For the empty context case, consider a basic open $U = U_{[\,\mid \varphi],\star}$, where φ is a sentence of $\mathbb{T}_{\mathcal{D}}$ and $\star$ is the element of the distinguished terminal object of **Sets** (traditionally $\star = \emptyset$, notice that any $U_{[\,\mid \varphi],a}$ with $a \neq \star$ is automatically empty). The canonical interpretation of φ in $\mathcal{D}$ yields, depending on the necessary choices for the interpretation, a subobject of a terminal object, $[\![\varphi]\!] \rightarrowtail 1$. Choose a representative monomorphism $r : R \rightarrowtail 1$. Then, independently of the choices made,

$$\psi_0^{-1}(U) = \bigcup_{a \in \mathbf{Sets}_\kappa} (r : R \to 1, a).$$

So ψ_0 is continuous. With ϕ_0 continuous, it is sufficient to check ϕ_1 on subbasic opens of the form $U = (A, a \mapsto b) \subseteq G_{\mathcal{D}}$. But

$$\phi_1^{-1}(U) = \left(\begin{array}{c} \overline{} \\ [x : A \mid \top] : a \mapsto b \\ \underline{} \end{array} \right)$$

so ϕ_1 is continuous. Similarly, it is sufficient to check ψ_1 on subbasic opens of the form

$$U = \left(\begin{array}{c} \overline{} \\ [x : A \mid \top] : a \mapsto b \\ \underline{} \end{array} \right)$$

but $\psi_1^{-1}(U) = (A, a \mapsto b)$, so ψ_1 is continuous. $\dashv$

Corollary 2.4.1.2 *Definition 2.2.1.3 yields, for a decidable coherent category $\mathcal{D}$, a topological groupoid $\mathbb{G}_{\mathcal{D}}$ such that*

$$\mathbb{G}_{\mathcal{D}} \cong \mathbb{G}_{\mathbb{T}_{\mathcal{D}}}$$

in the category **Gpd**.

We can now state the main representation result of this chapter:

Theorem 2.4.1.3 *For a decidable coherent category with a saturated set of $< \kappa$ models, the topos of coherent sheaves on $\mathcal{D}$ is equivalent to the topos of equivariant sheaves on the topological groupoid $\mathbb{G}_{\mathcal{D}}$ of models and isomorphisms equipped with the coherent topology,*

$$\mathrm{Sh}(\mathcal{D}) \simeq \mathrm{Sh}\left(\mathbb{G}_{\mathcal{D}}\right).$$

Proof The equivalence $\eta_{\mathcal{D}} : \mathcal{D} \longrightarrow T\mathcal{D} = \mathcal{C}_{\mathbb{T}_{\mathcal{D}}}$ yields an equivalence $\mathrm{Sh}(\mathcal{D}) \simeq \mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}_{\mathcal{D}}}\right)$, whence

$$\mathrm{Sh}(\mathcal{D}) \simeq \mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}_{\mathcal{D}}}\right) \simeq \mathrm{Sh}\left(\mathbb{G}_{\mathbb{T}_{\mathcal{D}}}\right) \cong \mathrm{Sh}\left(\mathbb{G}_{\mathcal{D}}\right)$$

by Theorem 2.3.4.14. $\dashv$

2.4.2 The Semantical Functor

We proceed to construct a 'syntax-semantics' adjunction between the category $\mathbf{DC}_\kappa$ (syntax) and a subcategory of topological groupoids (semantics). Given a coherent functor

$$F : \mathcal{A} \longrightarrow \mathcal{D}$$

between two objects of $\mathbf{DC}_\kappa$, composition with F,

$$\mathcal{A} \xrightarrow{\ F\ } \mathcal{D} \underset{N}{\overset{M}{\rightrightarrows}}\!\!\Downarrow\ \mathbf{Sets}_\kappa$$

yields a morphisms of (discrete) groupoids

$$\begin{array}{ccc} G_{\mathcal{D}} & \xrightarrow{\ f_1\ } & G_{\mathcal{A}} \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} & & {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ X_{\mathcal{D}} & \xrightarrow[\ f_0\ ]{} & X_{\mathcal{A}} \end{array} \tag{2.6}$$

We verify that f_0 and f_1 are both continuous. For basic open

$$U = (\langle g_1 : A \to B_1, \ldots, g_n : A \to B_n \rangle, \langle a_1, \ldots, a_n \rangle) \subseteq X_{\mathcal{A}},$$

we see that

$$f_0^{-1}(U) = (\langle F(g_1) : FA \to FB_1, \ldots, F(g_n) : FA \to FB_n \rangle, \langle a_1, \ldots, a_n \rangle) \subseteq X_{\mathcal{D}}.$$

And for basic open $U = (C, a \mapsto b) \subseteq G_{\mathcal{A}}$, we see that

$$f_1^{-1}(U) = (F(C), a \mapsto b) \subseteq G_{\mathcal{D}}$$

Thus composition with F yields a morphism of topological groupoids, $f : \mathbb{G}_{\mathcal{D}} \longrightarrow \mathbb{G}_{\mathcal{A}}$, and thereby we get a contravariant functor, which we shall refer to as the *semantical* functor,

$$\mathcal{G} : \mathbf{DC}_\kappa^{\mathrm{op}} \longrightarrow \mathbf{Gpd}.$$

2.4.3 The Syntactical Functor

We proceed to construct an adjoint to the semantical functor $\mathcal{G}$ from a subcategory of **Gpd** containing the image of $\mathcal{G}$. As in the propositional (distributive lattices) case, there are several such subcategories to choose from. We start by casting our net as wide as we can, whereafter we describe ways of restricting the adjunction, in the end also to a duality for first-order logic, i.e. to Boolean coherent categories.

The Decidable Object Classifier

The category $\mathbf{DC}_\kappa$ contains a category $\mathcal{D}(1)$ which classifies objects in decidable coherent categories, in the sense that objects of a decidable coherent category $\mathcal{D}$ correspond, up to natural isomorphism, to coherent functors $\mathcal{D}(1) \longrightarrow \mathcal{D}$. More precisely, $\mathcal{D}(1)$ contains a generic decidable object U, and any object $D \in \mathcal{D}$ determines, up to natural isomorphism, a coherent functor $F_D : \mathcal{D}(1) \longrightarrow \mathcal{D}$ such that $F_D(U) = D$. In this sense, $\mathcal{D}(1)$ is the *free decidable coherent category on one generator*, whence the notation "$\mathcal{D}(1)$".

In particular, sets of hereditarily less than κ size correspond to equivalence classes of coherent functors $\mathcal{D}(1) \longrightarrow \mathbf{Sets}_\kappa$. Taking sheaves for the coherent coverage on $\mathcal{D}(1)$, we obtain the decidable object classifier in the category of topoi and geometric morphisms. This topos can equivalently be described as the topos of presheaves on the category consisting of finite sets and injections, $\mathbf{Sets}^{\mathrm{Fin}_i}$ (see e.g. [14, VIII, Exc.7–9]), in which the generic decidable object, $\mathcal{U}$, can be taken to be the inclusion $\mathrm{Fin}_i \hookrightarrow \mathbf{Sets}$. The category $\mathcal{D}(1)$

can then be recovered as the full subcategory of finite powers of $\mathcal{U}$ together with their definable subobjects, that is, the subobjects in the closure of the diagonal and its complement under finite joins, meets, images, and pullbacks along projections. A more direct description of $\mathcal{D}(1)$ can be given as the syntactic category of the single sorted coherent theory, $\mathbb{T}_{\neq}$, consisting only of the predicate $\neq$ (in addition to $=$), and having no other axioms than those expressing that $\neq$ is the complement of $=$, with the 'generic decidable object' $\mathcal{U}$ being $[x \mid \top]$ in $\mathcal{D}(1)$. As such, $\mathcal{D}(1) = \mathcal{C}_{\mathbb{T}_{\neq}}$ is also an object of the category $\mathbf{T}$ of decidable coherent categories with distinguished structure (Definition 2.2.2.2), and objects of a category $\mathcal{D}$ in $\mathbf{DC}_\kappa$ correspond, on the nose, to strict functors $\mathcal{C}_{\mathbb{T}_{\neq}} \longrightarrow T(\mathcal{D})$. Taking semantical groupoids for the corresponding theories, we obtain, for any object of $\mathcal{D}$, a morphism of semantical groupoids, $\mathbb{G}_{\mathcal{D}} \cong \mathbb{G}_{\mathbb{T}_{\mathcal{D}}} \longrightarrow \mathbb{G}_{\mathbb{T}_{\neq}}$.

In recovering a decidable coherent category from a (suitable) topological groupoid $\mathbb{G}$, we may therefore usefully consider (suitable) morphisms $\mathbb{G} \longrightarrow \mathbb{G}_{\mathbb{T}_{\neq}}$. We proceed to give a direct characterization of the latter groupoid, and of the equivariant sheaf over it that we distinguish as the generic decidable object.

Definition 2.4.3.1 The topological groupoid $\mathbb{S}$ consists of hereditarily less than κ sets with bijections between them, equipped with topology as follows. The topology on the set of objects, S_0, is generated by the empty set and basic opens of the form

$$(a_1, \ldots, a_n) := \{A \in \mathbf{Sets}_\kappa \mid a_1, \ldots, a_n \in A\}$$

while the topology on the set, S_1 of bijections between hereditarily less than κ sets is the coarsest topology such that the source and target maps $s, t : S_1 \rightrightarrows S_0$ are both continuous, and containing all sets of the form

$$(a \mapsto b) := \{f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B \text{ in } \mathbf{Sets}_\kappa \mid a \in A \wedge f(a) = b\}$$

Lemma 2.4.3.2 *There is an isomorphism* $\mathbb{S} \cong \mathbb{G}_{\mathbb{T}_{\neq}}$ *in* $\mathbf{Gpd}$.

Proof Any set A in $\mathbf{Sets}_\kappa$ is the underlying set of a canonical $\mathbb{T}_{\neq}$-model, and any bijection $f : A \to B$ is the underlying function of a $\mathbb{T}_{\neq}$-model isomorphism, and thereby we obtain bijections $S_0 \cong X_{\mathbb{T}_{\neq}}$ and $S_1 \cong G_{\mathbb{T}_{\neq}}$ which commute with source, target, composition, and embedding of identities maps. Remains to show that the topologies correspond. Clearly, any basic open $(\vec{a}) \subseteq S_0$ corresponds to the open set $U_{[\vec{x}\,|\,\top],\vec{a}} \subseteq X_{\mathbb{T}_{\neq}}$. We show that, conversely, any basic open $U_{[\vec{x}\,|\,\phi],\vec{a}} \subseteq X_{\mathbb{T}_{\neq}}$ corresponds to an open set of S_0 by induction on $[\vec{x}\,|\,\phi]$. First, $U_{[\,|\,\top],a}$ is $X_{\mathbb{T}_{\neq}}$ if $a = \star$ and empty otherwise, where $\{\star\}$ is the distinguished terminal object of **Sets**, and $U_{[\vec{x}\,|\,\top],\vec{a}} \cong (\vec{a})$. Next, $U_{[x,y\,|\,x=y],a,b}$ corresponds to $(a) \subseteq S_0$ if $a = b$, and the empty set otherwise. Similarly, $U_{[x,y\,|\,x\neq y],a,b}$ corresponds to $(a,b) \subseteq S_0$ if $a \neq b$ and the empty set otherwise. Now, suppose $U_{[\vec{x}\,|\,\phi],\vec{a}}$ corresponds to an open set $U \subseteq S_0$. Then $U_{[\vec{x},y\,|\,\phi],\vec{a},b} \cong U \cap (b)$. Next, if $U_{[x,\vec{y}\,|\,\phi],a,\vec{b}}$ corresponds to an open set $U_a \subseteq S_0$ for each $a \in \mathbf{Sets}_\kappa$, then $U_{[\vec{y}\,|\,\exists x.\phi],\vec{b}} \cong \bigcup_{a\in\mathbf{Sets}_\kappa} U_a$. Finally, if $U_{[\vec{x}\,|\,\phi],\vec{a}}$ and $U_{[\vec{x}\,|\,\psi],\vec{a}}$ correspond to open sets $U, V \subseteq S_0$, then $U_{[\vec{x}\,|\,\phi\wedge\psi],\vec{a}} \cong U \cap V$, and $U_{[\vec{x}\,|\,\phi\vee\psi],\vec{a}} \cong U \cup V$. Therefore, $S_0 \cong X_{\mathbb{T}_{\neq}}$ as spaces. For the spaces of arrows, it remains only to observe that open subsets of the form $(a \mapsto b) \subseteq S_1$ correspond to open subsets of the form

$$\begin{pmatrix} - \\ a \mapsto b \\ - \end{pmatrix} \subseteq G_{\mathbb{T}_{\neq}}$$

and we can conclude that $\mathbb{S}$ is a topological groupoid isomorphic to $\mathbb{G}_{\mathbb{T}_{\neq}}$ in **Gpd**. ⊣

The category $\mathrm{Sh}(\mathbb{S})$ of equivariant sheaves on $\mathbb{S}$, therefore, classifies decidable objects, as $\mathrm{Sh}(\mathbb{S}) \cong \mathrm{Sh}\left(\mathbb{G}_{\mathbb{T}_{\neq}}\right) \simeq \mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}_{\neq}}\right)$.

Corollary 2.4.3.3 *There is an equivalence of topoi,*

$$\mathbf{Sets}^{\mathrm{Fin}_i} \simeq \mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}_{\neq}}\right) \simeq \mathrm{Sh}(\mathbb{S})$$

PROOF $\mathbf{Sets}^{\mathrm{Fin}_i} \simeq \mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}_{\neq}}\right)$ by [14, VIII,Exc.7–9]. ⊣

We fix the generic decidable object, $\mathcal{U}$, in $\mathrm{Sh}(\mathbb{S})$ to be the definable sheaf $\langle E_{[x\,|\,\top]} \to X_{\mathbb{T}_{\neq}} \cong S_0, \theta_{[x\,|\,\top]}\rangle$. We point out, however, that the object $\mathcal{U}$ can be constructed without reference to definable sheaves as follows. We use the results and notation of [21]. We have also taken the liberty to presented the relevant parts in Section 3.4.1.

Proposition 2.4.3.4 *There exists open set $U \subseteq S_0$ and an open set $N \subseteq S_1$ with N closed under inverses and composition and $s(N), t(N) \subseteq U$ such that the generic decidable object $\mathcal{U}$ in $\mathrm{Sh}(\mathbb{S})$ is isomorphic to the equivariant sheaf $\langle \mathbb{S}, U, N\rangle$ in $\mathrm{Sh}(\mathbb{S})$,*

$$\mathcal{U} \cong \langle \mathbb{S}, U, N\rangle$$

Proof Choose a set $a \in \mathbf{Sets}_\kappa$. Then the open subsets $U = (a) \subseteq S_0$ and $N = (a \mapsto a) \subseteq S_1$ are such that N is closed under inverses and composition and $s(N), t(N) \subseteq U$. Following [21, 6] we have an equivariant sheaf $\langle \mathbb{S}, U, N\rangle$ in $\mathrm{Sh}(\mathbb{S})$ consisting of the sheaf $t : S_1 \cap s_{-1}(U)/N \longrightarrow S_0$, where $S_1 \cap s^{-1}(U)/N$ is the set of arrows of S_1 with source in U factored out by the equivalence relation,

$$g \sim_N h \Leftrightarrow t(g) = t(h) \wedge g^{-1} \circ h \in N$$

and S_1 acts on $S_1 \cap s_{-1}(U)/N$ by composition. We replace the fibers of this sheaf by the mapping

$$[g] \mapsto g(a)$$

Following [21, 6], again, it is now straightforward to verify that this mapping is an isomorphism $\langle \mathbb{S}, U, N\rangle \cong \mathcal{U}$ in $\mathrm{Sh}(\mathbb{S})$, by observing that (by Lemma 2.3.4.6) the stabilization of the image of the section $u : U = U_{[x \mid \top],a} \to E_{[x \mid \top]}$ defined by $m \mapsto a$ is all of $E_{[x \mid \top]}$; that the induced morphism of equivariant sheaves $\tilde{u} : S_1 \cap s^{-1}(U)/N_u \longrightarrow E_{[x \mid \top]}$ is therefore an isomorphism; that $N = N_u$; and finally, that $\tilde{u}([g]) = g(a)$. ⊣

Finally, we remark on the connection between $\mathbb{S}$ and the dual $\mathcal{G}(\mathcal{D}(1))$ of the object classifier $\mathcal{D}(1) = \mathcal{C}_{\mathbb{T}_{\neq}}$ in $\mathbf{DC}$. As sets in $\mathbf{Sets}_\kappa$ correspond up to natural isomorphism to coherent functors $\mathcal{D}(1) \longrightarrow \mathbf{Sets}_\kappa$, one would expect $\mathbb{S}$ to somehow represent isomorphism classes of $\mathcal{G}(\mathcal{D}(1))$. Consider the the unique morphism U in $\mathbf{T}$ classifying the object $[x : \mathcal{U} \mid \top]$, i.e. classifying the object which is the value of $\eta : \mathcal{D}(1) \longrightarrow T\mathcal{D}(1)$ at the generic decidable object $\mathcal{U} = [x \mid \top]$ in $\mathcal{D}(1)$,

$$\mathcal{C}_{\mathbb{T}_{\neq}} \xrightarrow{U} T\mathcal{D}(1)$$

Composition with U yields a morphism

$$u : \mathcal{G}(\mathcal{D}(1)) \cong \mathbb{G}_{\mathbb{T}_{\mathcal{D}(1)}} \longrightarrow \mathbb{G}_{\mathbb{T}_{\neq}} \cong \mathbb{S}$$

of topological groupoids, which is a Morita equivalence since U is an equivalence of categories. On the other hand, every $\mathbb{T}_{\neq}$-model is, in particular, a coherent functor $\mathcal{D}(1) \longrightarrow \mathbf{Sets}_\kappa$, so there is a forgetful inclusion of discrete groupoids $v : \mathbb{S} \longrightarrow \mathcal{G}(\mathcal{D}(1))$. Together, u and v form an equivalence of (underlying) categories, with $u \circ v = 1_{\mathbb{S}}$. Moreover, $v : \mathbb{S} \longrightarrow \mathcal{G}(\mathcal{D}(1))$ is continuous, for given a subbasic open $U \subseteq X_{\mathcal{D}(1)}$ given by a finite span $[\vec{x}, \vec{y_i} \mid \sigma_i] : [\vec{x} \mid \phi] \longrightarrow [\vec{y_i} \mid \psi_i]$ in $\mathcal{D}(1)$ and sets a_i, for $1 \leq i \leq n$, we have that $v_0^{-1}(U) = U_{[\vec{y_1}, \ldots, \vec{y_n} \mid \exists \vec{x}. \sigma_1 \wedge \ldots \wedge \sigma_n], \vec{a}}$. A quick inspection also verifies that $v_1 : S_1 \hookrightarrow G_{\mathcal{D}(1)}$ is continuous. Therefore we have:

Proposition 2.4.3.5 *The topological groupoids* $\mathbb{S}$ *and* $\mathcal{G}(\mathcal{D}(1))$ *are equivalent in the sense that there are morphisms of topological groupoids*

$$\mathbb{S} \underset{u}{\overset{v}{\rightleftarrows}} \mathcal{G}(\mathcal{D}(1))$$

which form an equivalence of underlying categories.

Compact and Coherent Objects of a Topos

An object A of a topos $\mathcal{E}$ is called *compact* ([10, D3.3]) if every covering family of morphism into A has a finite covering subfamily, or equivalently, if every covering family of subobjects of A has a finite covering subfamily. A is called *coherent* (ibidem) if, in addition, it has the property that for any morphism $f : B \longrightarrow A$ with B compact, the domain K of the kernel pair of f,

$$K \underset{k_2}{\overset{k_1}{\rightrightarrows}} B \xrightarrow{f} A$$

is again compact.

Lemma 2.4.3.6 *A decidable compact object* A *of a topos* $\mathcal{E}$ *is coherent.*

Proof Let $f : B \longrightarrow A$ be given with B compact. The domain K of the kernel pair of f is the pullback of the diagonal along $f \times f$,

$$\begin{array}{ccc} K & \longrightarrow & \Delta \\ {\scriptstyle \langle k_1, k_2 \rangle} \downarrow & \lrcorner & \downarrow \\ B \times B & \xrightarrow[f \times f]{} & A \times A \end{array}$$

so K is complemented. But any complemented subobject of a compact is again compact. ⊣

If $\mathcal{C}$ is a (small) coherent category, then the full subcategory of coherent objects in $\mathrm{Sh}(\mathcal{C})$ form a pretopos containing $\mathcal{C}$ (i.e. containing the representables). Since the decidable objects (in any coherent category) are always closed under finite products, finite coproducts, and subobjects, the compact decidable objects in $\mathrm{Sh}(\mathcal{C})$ form a full coherent subcategory. However, in order to extract a coherent subcategory from a topos $\mathcal{E}$, a weaker assumption on $\mathcal{E}$ will do.

Definition 2.4.3.7 We say that an object A of a topos $\mathcal{E}$ is stably compact decidable if for any compact decidable object B, the product $A \times B$ is again compact.

Lemma 2.4.3.8 *Let $\mathcal{E}$ be a topos, and assume that the terminal object is compact. The (full) subcategory of stably compact decidable objects is a (positive) decidable coherent category.*

Proof The terminal object is compact by assumption, automatically decidable, and also automatically stably compact decidable. If A and B are stably compact decidable, then $A \times B$ is compact decidable, and stably so by associativity of the product. An equalizer $e : E \rightarrowtail A$ of a parallel pair of arrows $f, g : A \rightrightarrows B$ is decidable because it is a subobject of a decidable object, compact because it is a complemented subobject of a compact object,

$$\begin{array}{ccc} E & \longrightarrow & \Delta \\ {\scriptstyle e}\downarrow & \lrcorner & \downarrow \\ A & \xrightarrow[\langle f,g\rangle]{} & B \times B \end{array}$$

and stably so because for any compact decidable C, the product $C \times E$ is the equalizer of $1_C \times f, 1_C \times g : C \times A \rightrightarrows C \times B$. An image $\mathrm{Im}\,(f) \rightarrowtail B$ of an arrow $f : A \longrightarrow B$ is decidable because B is, compact because A is, and stably compact decidable because the product $C \times \mathrm{Im}\,(f)$ is the image of $1_C \times f : C \times A \longrightarrow C \times B$. Finally, the coproduct $A + B$ is decidable and compact, and stably so because $C \times (A + B) \cong C \times A + C \times B$ for any C.$\dashv$

Notice that stably compact decidable objects are also closed under complemented subobjects, since if A is stably compact decidable and $K \rightarrowtail A$ has a complement $\neg K \rightarrowtail A$, then K is decidable since it is a subobject of a decidable object, compact because it is complemented, and stably compact decidable since $K \times C \rightarrowtail A \times C$ has a complement in $\neg K \times C \rightarrowtail A \times C$. Moreover, stably compact decidable objects are closed under quotients of complemented equivalence relations, since if

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} B \xrightarrow{k} K$$

is a kernel pair-coequalizer diagram with $\langle r_1, r_2 \rangle : R \rightarrowtail B \times B$ complemented, then K is decidable since R is complemented, compact since it is a quotient of a compact, and stably compact decidable since

$$(B \times_K B) \times C \underset{r_2 \times 1_C}{\overset{r_1 \times 1_C}{\rightrightarrows}} B \times C \xrightarrow{k \times 1_C} K \times C$$

is a kernel pair-coequalizer diagram. We record these facts:

Lemma 2.4.3.9 *Stably compact decidable objects are closed under complemented subobjects and quotients of complemented equivalence relations.*

Lemma 2.4.3.10 *Let* $\mathrm{Sh}(\mathcal{D})$ *be a coherent topos on a decidable coherent category* $\mathcal{D}$*. Let* $f : \mathcal{E} \longrightarrow \mathrm{Sh}(\mathcal{D})$ *be a geometric morphism, and suppose* $f^* \circ y : \mathcal{D} \longrightarrow \mathcal{E}$ *factors through the subcategory of stably compact decidable objects in* $\mathcal{E}$*. Then* f^* *preserves stably compact decidable objects.*

PROOF Let K be a compact decidable object in $\mathrm{Sh}(\mathcal{D})$. Since K is compact, there is a cover

$$A = yD_1 + \ldots + yD_n \longrightarrow K$$

from a finite coproduct of representables, A, which is compact decidable by Lemma 2.4.3.8. Since K is decidable, the domain of the kernel pair $k_1, k_2 : R \rightrightarrows A$ is a complemented subobject of $A \times A$. Applying the inverse image functor f^*, we obtain a kernel pair-coequalizer diagram

$$f^*(R) \rightrightarrows f^*(yD_1) + \ldots + f^*(yD_n) \longrightarrow f^*(K)$$

and since f^* sends representables to stably compact decidable objects (and preserves complemented subobjects), an application of Lemma 2.4.3.8 and of the remark immediately following it shows that $f^*(K)$ is stably compact decidable. ⊣

The Syntactical Functor

We generalize the method by which a decidable coherent category category can be recovered from its topos of coherent sheaves—and hence from the topos of equivariant sheaves on its semantical groupoid—in order to extract decidable coherent categories from a more inclusive class of categories. Recall that for a topological groupoid $G_1 \rightrightarrows G_0$, we say that a subset of G_0 is open in the stable topology iff it is open and stable under the action of G_1. Recall also that the frame of stable open subsets of G_0 is the frame of subobjects of the terminal object in $\mathrm{Sh}_{G_1}(G_0)$.

Definition 2.4.3.11 Let **ctGpd** ('compact groupoids') be the subcategory of **Gpd** consisting of:

- groupoids $\mathbb{G}$ such that the stable topology on G_0 is compact, or equivalently, such that the terminal object in $\mathrm{Sh}(\mathbb{G})$ is a compact object;
- morphisms of groupoids $f : \mathbb{G} \longrightarrow \mathbb{H}$ such that pullback along f preserves stably compact decidable objects, i.e. such that the inverse image $f^* : \mathrm{Sh}(\mathbb{H}) \longrightarrow \mathrm{Sh}(\mathbb{G})$ of the induced geometric morphism preserves stably compact decidable objects.

For $\mathbb{G}$ in **ctGpd**, by Lemma 2.4.3.8, the full subcategory $\mathcal{C}$ of $\mathrm{Sh}(\mathbb{G})$ consisting of stably compact decidable objects form a decidable coherent category. However, $\mathcal{C}$ need not be an object of $\mathbf{DC}_\kappa$, as it need not have a saturated set of $< \kappa$ models.

Definition 2.4.3.12 Let $\mathbb{G}$ be an object of **ctGpd**. The category $\Theta(\mathbb{G})$ is the full subcategory of *firm* objects in $\mathrm{Sh}(\mathbb{G})$, consisting of those equivariant sheaves $\mathcal{A} = \langle A \to G_0, \alpha \rangle$ such that

- $\mathcal{A}$ is stably compact decidable;
- each fibre A_x for $x \in G_0$ is an element of $\mathbf{Sets}_\kappa$ (recall that A_x denotes the fiber over x with, subject to the conventions of Remark 2.3.3.1, index forgotten);
- for each set $a \in \mathbf{Sets}_\kappa$, the set $U_{\mathcal{A},a} = \{x \in G_0 \mid a \in A_x\} \subseteq G_0$ is open, and the function $s_{\mathcal{A},a} : \{x \in G_0 \mid a \in A_x\} \to A$ defined by $s(x) = a$ is a continuous section; and
- for any $a, b \in \mathbf{Sets}_\kappa$, the set

$$U_{\mathcal{A},a\mapsto b} = \{g : x \to y \mid a \in A_x \wedge \alpha(g,a) = b\} \subseteq G_1$$

  is open.

The definable objects in the category of equivariant sheaves on the groupoid of models and isomorphisms of a theory are readily seen to be a prime (and guiding) example of objects satisfying Definition 2.4.3.12:

Lemma 2.4.3.13 *For any $\mathcal{C}_\mathbb{T}$ in $\mathbf{DC}_\kappa$, the functor $\mathcal{M}^\dagger$ factors through $\Theta(\mathbb{G})$,*

$$\mathcal{M}^\dagger : \mathcal{C}_\mathbb{T} \longrightarrow \Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G}_\mathbb{T})$$

Lemma 2.4.3.14 *Let $\mathbb{G}$ be an object of* **ctGpd**. *Then $\Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$ is a (positive) decidable coherent category.*

Proof We verify that $\Theta(\mathbb{G})$ is closed under the relevant operations. By Lemma 2.4.3.8, it suffices to show that the three last properties of Definition 2.4.3.12 are closed under finite limits, images, and finite coproducts.

Terminal object. The canonical terminal object, write $\langle X' \to X, \alpha\rangle$, is such that the fiber over any $x \in G_0$ is $\{\star\} \in \mathbf{Sets}_\kappa$, whence the set $\{x \in G_0 \mid a \in X'_x\}$ is X if $a = \star$ and empty otherwise. Similarly, the set $\{g : x \to y \mid a \in X'_x \wedge \alpha(g, a) = b\} \subseteq G_1$ is G_1 if $a = \star = b$ and empty otherwise.

Finite products. We do the binary product $\mathcal{A} \times \mathcal{B}$. The fiber over $x \in G_0$ is the product $A_x \times B_x$, and so it is in $\mathbf{Sets}_\kappa$. Let a set $U_{\mathcal{A}\times\mathcal{B},c}$ be given. We may assume that c is a pair, $c = \langle a, b\rangle$, or $U_{\mathcal{A}\times\mathcal{B},c}$ is empty. Then,

$$U_{\mathcal{A}\times\mathcal{B},\langle a,b\rangle} = U_{\mathcal{A},a} \cap U_{\mathcal{B},b}$$

and the function $s_{\mathcal{A}\times\mathcal{B},\langle a,b\rangle} : U_{\mathcal{A}\times\mathcal{B},\langle a,b\rangle} \to A \times_{G_0} B$ is continuous by the following commutative diagram:

$$\begin{array}{ccccc}
U_{\mathcal{A},a} & \supseteq & U_{\mathcal{A}\times\mathcal{B},\langle a,b\rangle} & \subseteq & U_{\mathcal{B},b} \\
\downarrow{\scriptstyle s_{\mathcal{A},a}} & & \downarrow{\scriptstyle s_{\mathcal{A}\times\mathcal{B},\langle a,b\rangle}} & & \downarrow{\scriptstyle s_{\mathcal{B},b}} \\
A & \xleftarrow{\pi_1} & A \times_{G_0} B & \xrightarrow{\pi_2} & B
\end{array}$$

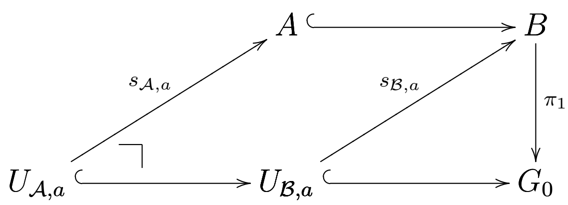
Similarly, the set $U_{\mathcal{A}\times B,c\mapsto d}$ is either empty or of the form $U_{\mathcal{A}\times B,\langle a,b\rangle\mapsto\langle a',b'\rangle}$, in which case

$$U_{\mathcal{A}\times B,\langle a,b\rangle\mapsto\langle a',b'\rangle} = U_{\mathcal{A},a\mapsto a'} \cap U_{\mathcal{B},b\mapsto b'}.$$

Equalizers and Images. Let $\mathcal{A}$ be a subobject of $\mathcal{B} = \langle \pi_1 : B \to G_0, \beta\rangle$, with $A \subseteq B$, and $\mathcal{B}$ satisfying the properties of Definition 2.4.3.12. Then given a set $U_{\mathcal{A},a}$,

$$U_{\mathcal{A},a} = \pi_1(A \cap s_{\mathcal{B},a}(U_{\mathcal{B},a}))$$

and we obtain $s_{\mathcal{A},a}$ as the restriction

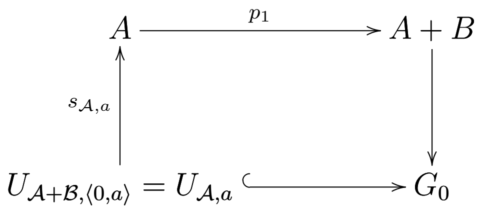


Similarly, given a set $U_{\mathcal{A},a\mapsto b} \subseteq G_1$,

$$U_{\mathcal{A},a\mapsto b} = U_{\mathcal{B},a\mapsto b} \cap s^{-1}(U_{\mathcal{A},a})$$

where s is the source map $s : G_1 \to G_0$. We conclude that $\Theta(\mathbb{G})$ is closed under both equalizers and images.

Finite coproducts. We do the binary coproduct $\mathcal{A} + \mathcal{B}$. We have not specified what the canonical coproducts in $\mathbf{Sets}_\kappa$ consist of exactly, but suppose, say, that $X+Y = \{\langle 0, x\rangle, \langle 1, y\rangle \mid x \in X \wedge y \in Y\}$. Then if $U_{\mathcal{A}+\mathcal{B},c}$ is non-empty, c is a pair $c = \langle 0, a\rangle$ or $c = \langle 1, b\rangle$. If the former, then $U_{\mathcal{A}+\mathcal{B},\langle 0,a\rangle} = U_{\mathcal{A},a}$, and the section is given by composition:



The latter case is similar, and so is verifying that the set $U_{\mathcal{A}+\mathcal{B},c\mapsto d}$. ⊣

Lemma 2.4.3.15 *Let* $\mathbb{G}$ *be an object of* **ctGpd**. *Then* $\Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$ *has a saturated set of* κ*-models.*

Proof This follows from the fact that the coherent inclusion

$$\Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$$

reflects covers, since every object of $\Theta(\mathbb{G})$ is compact, and any point, given by an element $x \in G_0$,

$$\mathbf{Sets} \longrightarrow \mathbf{Sets}/G_0 \twoheadrightarrow \mathrm{Sh}(G_0) \twoheadrightarrow \mathrm{Sh}(\mathbb{G}) \twoheadrightarrow \mathrm{Sh}(\Theta(\mathbb{G}))$$

yields a coherent functor $\Theta(\mathbb{G}) \longrightarrow \mathbf{Sets}_\kappa \hookrightarrow \mathbf{Sets}$, since the value of the point at an equivariant sheaf is the fiber over x, and objects in $\Theta(\mathbb{G})$ have fibers in $\mathbf{Sets}_\kappa$. ⊣

Lemma 2.4.3.16 *Let $\mathbb{G}$ be a groupoid in* **ctGpd**. *Then objects in $\Theta(\mathbb{G})$ are in 1-1 correspondence with morphisms $\mathbb{G} \longrightarrow \mathbb{S}$ in* **ctGpd**.

Proof Recall the four itemized conditions of Definition 2.4.3.12. Let a morphism $f : \mathbb{G} \longrightarrow \mathbb{S}$ be a morphism in **ctGpd**, inducing a geometric morphism $f : \mathrm{Sh}(\mathbb{G}) \longrightarrow \mathrm{Sh}(\mathbb{S})$. Then $f^*(\mathcal{U})$ is a stably compact decidable object with fibers in $\mathbf{Sets}_\kappa$; the set $U_{f^*(\mathcal{U}),a} = f_0^{-1}((a)) \subseteq G_0$ is open; the continuous section $(a) \to E_{[x\,|\,\top]}$ defined by $M \mapsto a$ pulls back along f_0 to yield the required section; and the set $U_{f^*(\mathcal{U}),a\mapsto b} = f_1^{-1}((a \mapsto b)) \subseteq G_1$ is open. So $f^*(\mathcal{U})$ is an object of $\Theta(\mathbb{G})$. Conversely, suppose that $\mathcal{A} = \langle A \to G_0, \alpha \rangle$ is an object of $\Theta(\mathbb{G})$. Define the function $f_0 : G_0 \to S_0$ by $x \mapsto A_x$, which is possible since $A_x \in \mathbf{Sets}_\kappa$. Then for a subbasic open set $(a) \subseteq S_0$, we have

$$f_0^{-1}((a)) = \{x \in G_0 \mid a \in A_x\} = U_{\mathcal{A},a}$$

so f_0 is continuous. Next, define $f_1 : G_1 \to S_1$ by

$$g : x \to y \quad \mapsto \quad \alpha(g, -) : A_x \to A_y.$$

Then for a subbasic open $(a \mapsto b) \subseteq S_1$, we have

$$f_1^{-1}((a \mapsto b)) = \big\{g \in G_1 \;\big|\; a \in A_{s(g)} \wedge \alpha(g,a) = b\big\} = U_{\mathcal{A},a\mapsto b}$$

so f_1 is continuous. It remains to show that $f^*(\mathcal{U}) = \mathcal{A}$. First, we must verify that what is a pullback of sets:

$$\begin{array}{ccc} A & \longrightarrow & E_{[x\,|\,\top]} \\ \big\downarrow & \lrcorner & \big\downarrow \\ G_0 & \xrightarrow[f_0]{} & S_0 \end{array}$$

is also a pullback of spaces. Let $a \in A$ with $V \subseteq A$ an open neighborhood. We must find an open box around a contained in V. Intersect V with the image of the section $s_{\mathcal{A},a}(U_{\mathcal{A},a})$ to obtain an open set U containing a and homeomorphic to a subset $W \subseteq G_0$. Then we can write U as the box $W \times_{S_0} V_{[x,y\,|\,x=y],a}$ for the open set $V_{[x,y\,|\,x=y],a} \subseteq E_{[x\,|\,\top]}$. Conversely, let a basic open $V_{[x,\vec{y}\,|\,\phi],\vec{b}} \subseteq E_{[x\,|\,\top]}$ be given, for ϕ a formula of $\mathbb{T}_{\neq}$. We must show that it pulls back to an open subset of A. Let $a \in A_z$ be given and assume that

a (in the fiber over $f_0(z)$) is in $V_{[x,\vec{y}\,|\,\phi],\vec{b}}$. Now, since $\mathcal{A}$ is decidable, there is a canonical interpretation of $[x, \vec{y}\,|\,\phi]$ in $\mathrm{Sh}\,(\mathbb{G})$ obtained by interpreting $\mathcal{A}$ as the single sort, and using the canonical coherent structure of $\mathrm{Sh}\,(\mathbb{G})$. Thereby, we obtain an object

$$\mathcal{B} := [\![x, \vec{y}\,|\,\phi]\!]^{\mathcal{A}} \hookrightarrow \mathcal{A} \times \ldots \times \mathcal{A} \xrightarrow{\pi_1} \mathcal{A}$$

in $\Theta\mathbb{G} \hookrightarrow \mathrm{Sh}\,(\mathbb{G})$ with an underlying open subset $B \subseteq A \times_{G_0} \ldots \times_{G_0} A \xrightarrow{\pi_1} A$. Let $W \subseteq B$ be the image of the continuous section $s_{\mathcal{B},a,\vec{b}}(U_{\mathcal{B}}, a, \vec{b})$. The image of W which along the projection $\pi_1 : \mathcal{A} \times \ldots \times \mathcal{A} \longrightarrow \mathcal{A}$ is an open subset of A. We claim that it has a as an element over z and that it is contained in the pullback of $V_{[x,\vec{y}\,|\,\phi],\vec{b}}$ along f_0. For an element $v \in G_0$ corresponds to a point

$$p_v : \mathbf{Sets} \longrightarrow \mathbf{Sets}/G_0 \twoheadrightarrow \mathrm{Sh}\,(G_0) \twoheadrightarrow \mathrm{Sh}\,(\mathbb{G})$$

such that value of p_v^* at an object is the fiber over v. Now, $p_v^*(\mathcal{A}) = A_v$ is a set which we can consider as the underlying set of a $\mathbb{T}_{\neq}$-model $\mathbf{M}_v = f_0(v)$ in the canonical way. And since $\mathcal{B} = [\![x, \vec{y}\,|\,\phi]\!]^{\mathcal{A}}$ is computed using the fibre-wise set-induced canonical structure in $\mathrm{Sh}\,(\mathbb{G})$, we have that $p_v(\mathcal{B}) = [\![x, \vec{y}\,|\,\phi]\!]^{\mathbf{M}_v}$. Therefore (index remembered for notational convenience),

$$\begin{aligned}
\pi_1(W) &= \left\{ \langle v, c\rangle \in A \;\middle|\; \exists \vec{d} \in \mathbf{Sets}_\kappa.\, \langle v, c, \vec{d}\rangle \in W \subseteq A \times_{G_0} \ldots \times_{G_0} A \right\} \\
&= \left\{ \langle v, a\rangle \in A \;\middle|\; \langle v, a, \vec{b}\rangle \in B \right\} \\
&= \left\{ \langle v, a\rangle \in A \;\middle|\; a, \vec{b} \in [\![x, \vec{y}\,|\,\phi]\!]^{\mathbf{M}_v} \right\} \\
&= \left\{ \langle v, a\rangle \in A \;\middle|\; a, \vec{b} \in [\![x, \vec{y}\,|\,\phi]\!]^{f_0(v)} \right\}
\end{aligned}$$

so that $\langle z, a\rangle \in \pi(W) \subseteq G_0 \times_{S_0} V_{[x,\vec{y}\,|\,\phi],\vec{b}}$. It follows that $f^*(\mathcal{U}) = \mathcal{A}$, and therefore, by Lemma 2.4.3.10, that $f^* : \mathrm{Sh}\,(\mathbb{S}) \longrightarrow \mathrm{Sh}\,(\mathbb{G})$ preserves stably compact decidable objects. We conclude that objects in $\Theta(\mathbb{G})$ are in 1-1 correspondence with morphisms $\mathbb{G} \longrightarrow \mathbb{S}$ in **ctGpd**. $\dashv$

Lemma 2.4.3.17 *If* $f : \mathbb{G} \longrightarrow \mathbb{H}$ *is a morphism of* **ctGpd**, *then the induced coherent inverse image functor* $f^* : \mathrm{Sh}\,(\mathbb{H}) \longrightarrow \mathrm{Sh}\,(\mathbb{G})$ *restricts to a coherent* *functor* $\Theta(f) = F : \Theta(\mathbb{H}) \longrightarrow \Theta(\mathbb{G})$,

$$\begin{array}{ccc} \Theta(\mathbb{H}) & \xrightarrow{F} & \Theta(\mathbb{G}) \\ \downarrow & & \downarrow \\ \mathrm{Sh}(\mathbb{H}) & \xrightarrow[f^*]{} & \mathrm{Sh}(\mathbb{G}) \end{array}$$

PROOF By Lemma 2.4.3.16. If $\mathcal{A}$ is an object of $\Theta(\mathbb{H})$ classified by $h : \mathbb{H} \longrightarrow \mathbb{S}$, then $f^*(\mathcal{A}) = F(\mathcal{A})$ is classified by $h \circ f : \mathbb{G} \longrightarrow \mathbb{S}$ in **ctGpd**. $\dashv$

This concludes the construction of the 'syntactical' functor:

Definition 2.4.3.18 The functor

$$\Theta : \mathbf{ctGpd} \longrightarrow \mathbf{DC}_\kappa^{\mathrm{op}}$$

is defined by sending a groupoid $\mathbb{G}$ to the coherent decidable category

$$\Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$$

of firm objects, and a morphism $f : \mathbb{G} \longrightarrow \mathbb{H}$ to the restricted inverse image functor $F : \Theta(\mathbb{H}) \longrightarrow \Theta(\mathbb{G})$.

2.4.4 The Syntax-Semantics Adjunction

We now show that the syntactical functor is left adjoint to the semantical functor:

$$\mathbf{DC}_\kappa{}^{\mathrm{op}} \underset{\Theta}{\overset{\mathcal{G}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{ctGpd}$$

First, we isolate a counit candidate. Given $\mathcal{D}$ in $\mathbf{DC}_\kappa$, we recognize a functor

$$\mathcal{N}_\mathcal{D} : \mathcal{D} \longrightarrow \mathrm{Sh}(\mathcal{G}_\mathcal{D})$$

which sends an object D to an equivariant sheaf with underlying set

$$D \mapsto \{\langle M, a\rangle \mid M \in X_\mathcal{D} \wedge a \in M(D)\} \xrightarrow{\pi_1} X_\mathcal{D}$$

over $X_\mathcal{D}$, and with action corresponding to the action of the natural isomorphisms of $G_\mathcal{D}$ on functors of $X_\mathcal{D}$, as the composite of coherent functors

$$\mathcal{N}_\mathcal{D} : \mathcal{D} \xrightarrow{\iota_\mathcal{D}} T\mathcal{D} \xhookrightarrow{\mathcal{M}^\dagger} \mathrm{Sh}(\mathbb{G}_{\mathbb{T}_\mathcal{D}}) \cong \mathrm{Sh}(\mathbb{G}_\mathcal{D}).$$

As such, $\mathcal{N}_{\mathcal{D}}$ factors through $\Theta(\mathbb{G}_{\mathcal{D}})$, by Lemma 2.4.3.13, to yield a coherent functor

$$\epsilon_{\mathcal{D}} : \mathcal{D} \longrightarrow \Theta(\mathbb{G}_{\mathcal{D}}) = \Theta \circ \mathcal{G}(\mathcal{D})$$

. And if $F : \mathcal{A} \longrightarrow \mathcal{D}$ is an arrow of $\mathbf{DC}_{\kappa}$, the square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\epsilon_{\mathcal{A}}} & \Theta \circ \mathcal{G}(\mathcal{A}) \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \Theta\circ\mathcal{G}(F)} \\ \mathcal{D} & \xrightarrow[\epsilon_{\mathcal{D}}]{} & \Theta \circ \mathcal{G}(\mathcal{D}) \end{array}$$

commutes by Lemma 2.3.4.15 and the observation of (2.1) on page 37.

Next, we consider the unit. Let $\mathbb{H}$ be a groupoid in **ctGpd**. We construct a morphism

$$\eta_{\mathbb{H}} : \mathbb{H} \longrightarrow \mathbb{G}_{\Theta(\mathbb{H})} = \mathcal{G}(\Theta(\mathbb{G})).$$

First, as previously noticed, each $h \in H_0$ induces a coherent functor $M_h : \Theta(\mathbb{H}) \longrightarrow \mathbf{Sets}_{\kappa}$. This defines a function $\eta_0 : H_0 \to X_{\Theta(\mathbb{H})}$. Similarly, any $a : x \to y$ in H_1 induces an invertible natural transformation $f_a : M_x \Rightarrow M_y$. This defines a function $\eta_1 : H_1 \to G_{\Theta(\mathbb{H})}$, such that $\langle \eta_1, \eta_0 \rangle$ is a morphism of discrete groupoids. We argue that η_0 and η_1 are continuous. Let a subbasic open $U = (\langle g_1 : \mathcal{A} \to \mathcal{B}_1, \ldots, g_n : \mathcal{A} \to \mathcal{B}_n \rangle, \langle a_1, \ldots, a_n \rangle) \subseteq X_{\Theta(\mathbb{H})}$ be given, with $g_i : \mathcal{A} = \langle A \to H_0, \alpha \rangle \longrightarrow \mathcal{B}_i = \langle B_i \to H_0, \beta_i \rangle$ an arrow of $\theta(\mathbb{H})$ and $a_i \in \mathbf{Sets}_{\kappa}$, for $1 \leq i \leq n$. Form the canonical product $\mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ in $\mathrm{Sh}\,(\mathbb{H})$, so as to get an arrow $g = \langle g_1, \ldots, g_n \rangle : \mathcal{A} \longrightarrow \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ in $\Theta(\mathbb{H})$. Denote by $\mathcal{C}$ the canonical image of g in $\mathrm{Sh}\,(\mathbb{H})$ (and thus in $\Theta(\mathbb{H})$), such that the underlying set C (over H_0) of $\mathcal{C}$ is a subset of $B_1 \times_{H_0} \ldots \times_{H_0} B_n$. Then

$$\begin{aligned} \eta_0^{-1}(U) &= \{x \in H_0 \mid \exists y \in M_x(\mathcal{A}).\, M_x(g_i)(y) = a_i \text{ for } 1 \leq i \leq n\} \\ &= \{x \in H_0 \mid \exists y \in A_x.\, g_i(y) = a_i \text{ for } 1 \leq i \leq n\} \\ &= \{x \in H_0 \mid \langle a_1, \ldots, a_n \rangle \in M_x(\mathcal{C})\} \\ &= \{x \in H_0 \mid \langle a_1, \ldots, a_n \rangle \in C_x\} \end{aligned}$$

which is an open subset of H_0, since $\mathcal{C}$ is in $\Theta(\mathbb{H})$. Thus η_0 is continuous. Next, consider a subbasic open of $G_{\Theta(\mathbb{H})}$ of the form $U = (\mathcal{A}, a \mapsto b) \subseteq G_{\Theta(\mathbb{H})}$, for $\mathcal{A} = \langle A \to H_0, \alpha \rangle$ in $\Theta(\mathbb{H})$. Then

$$\begin{aligned} \eta_1^{-1}(U) &= \{g : x \to y \mid a \in M_x(\mathcal{A}) \wedge (f_g)_{\mathcal{A}}(a) = b\} \subseteq H_1 \\ &= \{g : x \to y \mid a \in A_x \wedge \alpha(g, a) = b\} \subseteq H_1 \end{aligned}$$

which is an open subset of H_1, since $\mathcal{A}$ is in $\Theta(\mathbb{H})$. Thus η_1 is also continuous, so that $\langle \eta_1, \eta_0 \rangle$ is a morphism of continuous groupoids.

Lemma 2.4.4.1 *The triangle*

$$\begin{array}{ccc} & & \mathrm{Sh}\,(\mathbb{H}) \\ & \nearrow & \uparrow {\scriptstyle \eta^*_{\Theta(\mathbb{H})}} \\ \Theta(\mathbb{H}) & \xrightarrow[\mathcal{N}_{\Theta(\mathbb{H})}]{} & \mathrm{Sh}\left(\mathbb{G}_{\Theta(\mathbb{H})}\right) \end{array} \tag{2.7}$$

commutes.

Proof Let $\mathcal{A} = \langle A \to H_0, \alpha \rangle$ in $\Theta(\mathbb{H})$ be given, and write $E_{\mathcal{A}} \to X_{\Theta(\mathbb{H})}$ for the underlying sheaf of $\mathcal{N}_{\Theta(\mathbb{H})}(\mathcal{A})$. Write $a : \mathbb{H} \longrightarrow \mathbb{S}$ and $a' : \mathbb{G}_{\Theta(\mathbb{H})} \longrightarrow \mathbb{S}$, respectively, for the **ctGpd** morphisms classifying these objects. Then the triangle

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\eta_{\Theta(\mathbb{H})}} & \mathbb{G}_{\Theta(\mathbb{H})} \\ & {\scriptstyle a} \searrow \quad \swarrow {\scriptstyle a'} & \\ & \mathbb{S} & \end{array}$$

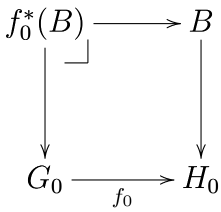
in **Gpd** commutes by the construction of the classifying morphisms (Lemma 2.4.3.16) and $\eta_{\Theta(\mathbb{H})}$. Briefly, for $x \in H_0$, we have $a(x) = A_x = M_x(\mathcal{A}) = (E_{\mathcal{A}})_{M_x} = (E_{\mathcal{A}})_{\eta_0(x)} = a'(\eta_0(x))$ and similarly for elements of H_1. ⊣

It follows from (2.7) and Lemma 2.4.3.10 that the inverse image functor $\eta^*_{\Theta(\mathbb{H})}$ preserves stably compact decidable objects, and so $\eta_{\Theta(\mathbb{H})} : \mathbb{H} \longrightarrow G_{\theta(\mathbb{H})}$ is indeed a morphism of **ctGpd**. It remains to verify that it is the component of a natural transformation. Given a morphism $f : \mathbb{G} \longrightarrow \mathbb{H}$ of **ctGpd**, we must verify that the square

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\eta_{\Theta(\mathbb{G})}} & \mathcal{G} \circ \Theta(\mathbb{G}) \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle \mathcal{G} \circ \Theta(f)} \\ \mathbb{H} & \xrightarrow[\eta_{\Theta(\mathbb{H})}]{} & \mathcal{G} \circ \Theta(\mathbb{H}) \end{array}$$

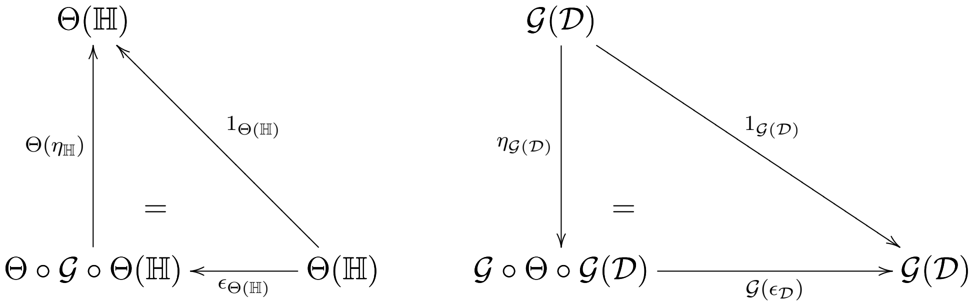
commutes. Let $x \in G_0$ be given. We chase it around the square. Applying $\eta_{\Theta(\mathbb{G})}$, we obtain the functor $M_x : \Theta(\mathbb{G}) \longrightarrow$ **Sets** which sends an object

$\mathcal{A} = \langle A \to G_0, \alpha\rangle$ to A_x. Composing with $\Theta(f) : \Theta(\mathbb{H}) \longrightarrow \Theta(\mathbb{G})$, we obtain the functor $\Theta(\mathbb{H}) \longrightarrow$ **Sets** which sends an object $\mathcal{B} = \langle B \to H_0, \beta\rangle$ to the fiber over x of the pullback



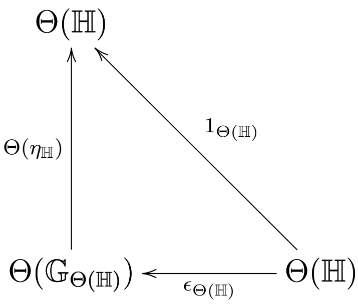
which is the same as the fiber $B_{f_0(x)}$. And this is the same functor that results from sending x to $f_0(x)$ and applying $\eta_{\Theta(\mathbb{H})}$. For $a : x \to y$ in G_1, a similar check establishes that $\eta_1 \circ f_1(a) : M_{f_0(x)} \Rightarrow M_{f_0(y)}$ equals $\eta_1(a) \circ \Theta(f) : M_x \circ \Theta(f) \Rightarrow M_y \circ \Theta(f)$. It remains to verify the triangular identities:

Lemma 2.4.4.2 *The triangular identities,*



hold.

Proof We begin with the left triangle, which we write:



This triangle commutes by the definition of $\epsilon_{\Theta(\mathbb{H})}$ and Lemma 2.4.4.1, as can be seen by the following diagram:

$$\begin{array}{ccc}
\Theta(\mathbb{H}) & \xrightarrow{\epsilon_{\Theta(\mathbb{H})}} & \Theta(\mathbb{G}_{\Theta(\mathbb{H})}) \\
\big\downarrow & \searrow^{\mathcal{N}_{\Theta(\mathbb{H})}} \quad = & \big\downarrow \\
 & = & \\
\mathrm{Sh}\,(\mathbb{H}) & \xleftarrow[\eta^*_{\mathbb{H}}]{} & \mathrm{Sh}\left(\mathbb{G}_{\Theta(\mathbb{H})}\right)
\end{array}$$

We pass to the right triangle, which can be written as:

$$\begin{array}{ccc}
\mathbb{G}_{\mathcal{D}} & & \\
\big\downarrow{\scriptstyle \eta_{\mathbb{G}_{\mathcal{D}}}} & \searrow^{1_{\mathbb{G}_{\mathcal{D}}}} & \\
\mathbb{G}_{\Theta(\mathbb{G}_{\mathcal{D}})} & \xrightarrow[\mathcal{G}(\epsilon_{\mathcal{D}})]{} & \mathbb{G}_{\mathcal{D}}
\end{array}$$

Let $N : \mathcal{D} \longrightarrow \mathbf{Sets}$ in $X_{\mathcal{D}}$ be given. As an element in $X_{\mathcal{D}}$, it determines a coherent functor $M_N : \Theta(\mathbb{G}_{\mathcal{D}}) \longrightarrow \mathbf{Sets}$, the value of which at $\mathcal{A} = \langle A \to X_{\mathcal{D}}, \alpha \rangle$ is the fiber A_N. Applying $\mathcal{G}(\epsilon_{\mathcal{D}})$ is composing with the functor $\epsilon_{\mathcal{D}} : \mathcal{D} \longrightarrow \Theta(\mathbb{G}_{\mathcal{D}})$, to yield the functor $M_N \circ \epsilon_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathbf{Sets}$, the value of which at an object B in $\mathcal{D}$ is the fiber over N of E_B, which of course is just $N(B)$. For an invertible natural transformation $f : M \Rightarrow N$ in $G_{\mathcal{D}}$, the chase is entirely similar, and we conclude the the triangle commutes. $\dashv$

Theorem 2.4.4.3 *The contravariant functors $\mathcal{G}$ and Θ are adjoint,*

$$\mathbf{DC}_{\kappa}{}^{\mathrm{op}} \underset{\Theta}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{ctGpd} \qquad (\top)$$

*where $\mathcal{G}$ is the functor that sends a decidable coherent category $\mathcal{D}$ to the groupoid $\underline{\mathrm{Hom}}^*(\mathcal{D}, \mathbf{Sets}_{\kappa})$ equipped with the coherent topology, and Θ sends a groupoid $\mathbb{G}$ in **ctGpd** to the full subcategory $\Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}\,(\mathbb{G})$ of firm objects, corresponding to the set of morphisms $\mathrm{Hom}_{\mathbf{ctGpd}}(\mathbb{G}, \mathbb{S})$.*

Notice that if $\mathcal{D}$ is an object of $\mathbf{DC}_\kappa$, then the counit component $\epsilon_\mathcal{D}$: $\mathcal{D} \longrightarrow \Theta \circ \mathcal{G}(\mathcal{D})$ is a Morita equivalence of categories, in the sense that it induces an equivalence $\mathrm{Sh}(\mathcal{D}) \simeq \mathrm{Sh}(\Theta \circ \mathcal{G}(\mathcal{D}))$. In the case where $\mathcal{D}$ is a pretopos, the counit is, moreover, also an equivalence of categories, since any decidable compact object in $\mathrm{Sh}(\mathcal{D})$ is coherent and therefore isomorphic to a representable in that case. Furthermore, for any $\mathcal{D}$ in $\mathbf{DC}_\kappa$, we have that the unit component $\eta_{\mathbb{G}_\mathcal{D}} : \mathbb{G}_\mathcal{D} \longrightarrow \mathbb{G}_{\Theta(\mathbb{G}_\mathcal{D})}$ is a Morita equivalence of categories, in the sense that it induces an equivalence $\mathrm{Sh}(\mathbb{G}_\mathcal{D}) \simeq \mathrm{Sh}\left(\mathbb{G}_{\Theta(\mathbb{G}_\mathcal{D})}\right)$. We refer to the image of $\mathcal{G}$ in $\mathbf{Gpd}$ as $\mathbf{SemGpd}$.

Corollary 2.4.4.4 *The adjunction of Proposition 2.4.4.3 restricts to an adjunction*

$$\mathbf{DC}_\kappa{}^{\mathrm{op}} \underset{\Theta}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{SemGpd} \quad (\top)$$

with the property that the unit and counit components are Morita equivalences of categories and topological groupoids respectively. Moreover, counit components at pretopoi are equivalences, so that the adjunction further restricts to the subcategory of pretoposes

$$\mathbf{DCPTop}_\kappa \hookrightarrow \mathbf{DC}_\kappa$$

and its image along $\mathcal{G}$ to form an 'equivalence' (in a suitable higher order sense).

2.4.5 Restricting the Adjunction

Closing in on Semantical Groupoids

Recall the adjunction between distributive lattices and **ctTop**, i.e. compact spaces and continuous functions between them such that the inverse image of a stably compact open is again stably compact. That adjunction could be restricted to those compact spaces where all compact opens are stably compact—i.e. those spaces such that the compact opens form a distributive lattice—and functions such that the inverse image of a compact open is compact. If, in addition, we require the compact opens to generate the topology, we obtain the category of coherent spaces. Finally, the adjunction restricts to those distributive lattices that are Boolean and those coherent spaces where all compact opens are complemented, i.e. clopen. We restrict the adjunction between $\mathbf{DC}_\kappa$ and **ctGpd** in an entirely analogous fashion.

Definition 2.4.5.1 Let **clGpd** be the subcategory of **Gpd** the objects of which are groupoids, $\mathbb{G}$, such that the terminal object in $\mathrm{Sh}(\mathbb{G})$ is compact and any compact decidable equivariant sheaf on $\mathbb{G}$ is stably compact decidable, and the morphisms of which are those morphisms of groupoids $f : \mathbb{G} \longrightarrow \mathbb{H}$ such that pullback along which preserves compact decidable objects. That is, since decidability is always preserved by an inverse image functor, it takes a compact decidable object to a compact object.

Lemma 2.4.5.2 *A topological groupoid* $\mathbb{G}$ *is an object of* **clGpd** *if and only if the compact decidable objects form a coherent subcategory of* $\mathrm{Sh}(\mathbb{G})$

Proof By the proof of Lemma 2.4.3.8. ⊣

Since a compact decidable object in $\mathrm{Sh}(\mathbb{G})$ is stably compact decidable for $\mathbb{G}$ in **clGpd**, the inclusion

$$\mathbf{clGpd} \hookrightarrow \mathbf{ctGpd}$$

is full. Notice that since the decidable compact objects forms a coherent category in any coherent topos, there is a full inclusion into **clGpd** of the category of those groupoids $\mathbb{G}$ such that $\mathrm{Sh}(\mathbb{G})$ is a coherent topos, and those morphisms of groupoids $f : \mathbb{G} \longrightarrow \mathbb{H}$ such that the inverse image of the induced geometric morphism $f : \mathrm{Sh}(\mathbb{G}) \longrightarrow \mathrm{Sh}(\mathbb{H})$ sends a compact decidable object to a compact object. Next, if we require the compact decidable objects in $\mathrm{Sh}(\mathbb{G})$ to generate the topos, in addition to forming a coherent subcategory, then the compact decidable objects form a decidable coherent site for $\mathrm{Sh}(\mathbb{G})$.

Lemma 2.4.5.3 *For* $\mathcal{C}$ *and* $\mathcal{D}$ *decidable coherent categories and*

$$f : \mathrm{Sh}(\mathcal{C}) \longrightarrow \mathrm{Sh}(\mathcal{D})$$

a geometric morphism, an inverse image functor

$$f^* : \mathrm{Sh}(\mathcal{D}) \longrightarrow \mathrm{Sh}(\mathcal{C})$$

preserves compact decidable objects if and only if it preserves compact objects.

Proof Any compact object in $\mathrm{Sh}(\mathcal{C})$ is covered by a compact decidable object, and f^* preserves covers, so if f^* preserves compact decidable objects it preserves all compacts. ⊣

Accordingly, denoting by **dcGpd** the category having as objects topological groupoids $\mathbb{G}$ such that $\mathrm{Sh}(\mathbb{G})$ has a decidable coherent site, and having as arrows those morphisms of topological groupoids that pull compact objects back to compact objects, yields a full inclusion

$$\mathbf{dcGpd} \hookrightarrow \mathbf{clGpd}.$$

Since the semantical functor $\mathcal{G} : \mathbf{DC}_\kappa{}^{\mathrm{op}} \longrightarrow \mathbf{ctGpd}$ factors through **dcGpd**, the adjunction $\mathcal{G} \vdash \Theta$ restricts to the full subcategories

$$\mathbf{dcGpd} \hookrightarrow \mathbf{clGpd} \hookrightarrow \mathbf{ctGpd}.$$

Note that one can also introduce the requirement that the groupoids be open, since semantical groupoids are open by Lemma 2.3.4.2. We end by summarizing the remarks above in a proposition:

Proposition 2.4.5.4 *The 'syntax-semantics' adjunction of Theorem 2.4.4.3 restricts to the subcategories*

$$\mathbf{dcGpd} \hookrightarrow \mathbf{clGpd} \hookrightarrow \mathbf{ctGpd}.$$

Posets

Let $\mathcal{L}$ be an object of $\mathbf{DC}_\kappa$ which is a poset, that is to say, $\mathcal{L}$ is an arbitrary distributive lattice. Then the topos of equivariant sheaves on the groupoid $\mathcal{G}(\mathcal{L}) = \mathbb{G}_\mathcal{L}$ is spatial, i.e. $\mathrm{Sh}(\mathbb{G}_\mathcal{L}) \simeq \mathrm{Sh}(\mathcal{L})$ is equivalent to a topos of sheaves on a topological space (the name *spatial* for such topological groupoids is somewhat unfortunate, but we will restrict its use to this paragraph). Thus $\mathcal{G}$ restricted to the full subcategory $\mathbf{DLat} \hookrightarrow \mathbf{DC}_\kappa$ factors through the full subcategory of **dcGpd** consisting of spatial groupoids. On the other hand, for any groupoid $\mathbb{G}$ in **dcGpd**, even if spatial, it is not the case that $\Theta(\mathbb{G})$ is a poset, since, e.g., it contains all finite coproducts of the terminal object. Moreover, the subterminal objects of $\Theta(\mathbb{G})$ form a preorder, not a poset. However, the subobject lattice of the terminal object, $\mathrm{Sub}_{\Theta(\mathbb{G})}(1)$, is a distributive lattice, and a distributive lattice $\mathcal{L}$ occurs up to isomorphism as the subobject lattice of the terminal object in $\Theta(\mathbb{G}_\mathcal{L})$,

$$\mathcal{L} \cong \mathrm{Sub}_{\Theta(\mathbb{G}_\mathcal{L})}(1)\,.$$

Now, the adjunction $\mathcal{G} \vdash \Theta$ specializes to distributive lattices in the following way and sense. Let $\mathcal{G}_2 : \mathbf{DLat} \longrightarrow \mathbf{ctGpd}$ be the contravariant functor that sends a distributive lattice $\mathcal{L}$ to the groupoid consisting of the set of 2-valued models—that is coherent functors $\mathcal{L} \longrightarrow 2 \hookrightarrow \mathbf{Sets}_\kappa$—with invertible natural transformations between them, equipped with the coherent topology. Notice that an invertible natural transformation must be an identity, and that the coherent topology in this case is the topology generated by basic open sets of the form $U_A = \{F : \mathcal{L} \longrightarrow 2 \mid F(A) = 1\}$ for $A \in \mathcal{L}$. A coherent functor between distributive lattices is sent to the continuous morphism of composition between the corresponding groupoids. In the other direction, define the contravariant functor $\Theta_2 : \mathbf{ctGpd} \longrightarrow \mathbf{DLat}$ by sending a groupoid $\mathbb{G}$ to those subterminal objects of $\mathrm{Sh}(\mathbb{G})$ that are stably compact decidable and such that all fibers are elements of 2 (that is, the fiber over any $x \in G_0$ is either $0 = \emptyset$ or $1 = \{\star\}$). Then $\Theta_2(\mathbb{G})$ is a distributive lattice isomorphic to the subobject lattice of the decidable coherent category $\Theta(\mathbb{G})$. A morphism $f : \mathbb{G} \longrightarrow \mathbb{H}$ induces an inverse image functor f^* which preserves stably compact decidable subterminal objects, and so restricts to a morphism of distributive lattices $F = \Theta_2(f) : \Theta_2(\mathbb{H}) \longrightarrow \Theta_2(\mathbb{G})$.

Proposition 2.4.5.5 *The functor Θ_2 is adjoint to $\mathcal{G}_2$:*

$$\mathbf{DLat}^{\mathrm{op}} \underset{\Theta_2}{\overset{\mathcal{G}_2}{\rightleftarrows}} \mathbf{ctGpd} \qquad (\top)$$

with counit components lattice isomorphisms.

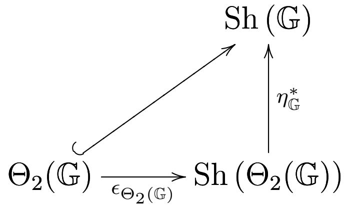
Proof We repeat the argument of Theorem 2.4.4.3 with the necessary changes. The counit component candidate at a distributive lattice

$$\varepsilon_{\mathcal{L}} : \mathcal{L} \longrightarrow \Theta_2 \circ \mathcal{G}_2(\mathcal{L})$$

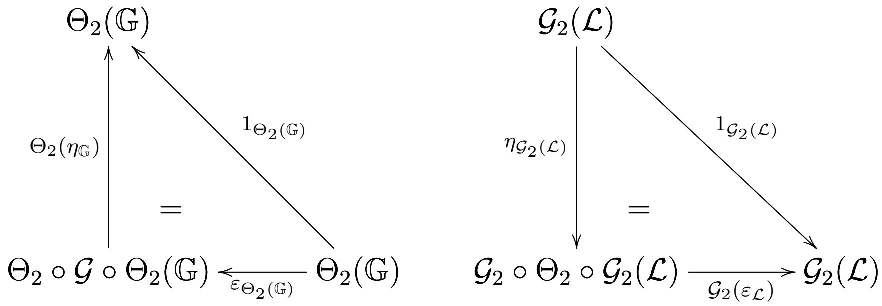
is the lattice isomorphism obtained by sending $A \in \mathcal{L}$ to the (stable) compact open set $U_A = \{\langle M, \star\rangle \mid M \in X_{\mathcal{L}} \wedge M(A) = \{\star\} = 1\}$ over $X_{\mathcal{L}}$. For a groupoid $\mathbb{G}$ in $\mathbf{ctGpd}$, the unit component candidate is the following. For each $x \in G_0$, the element x determines an $\Theta_2(\mathbb{G})$-model, as before, by sending an object A in $\Theta_2(\mathbb{G})$ to A_x, which is $1 = \{\star\}$ if A_x is inhabited and $0 = \emptyset$ otherwise (so that $M_x \in \mathrm{Hom}_{\mathbf{DLat}}(\Theta_2(\mathbb{G}), 2)$). This defines a function $\eta_0 : G_0 \to X_{\Theta_2(\mathbb{G})}$. We verify that it is continuous. Let a basic open U_A be given, for $A \in \Theta_2(\mathbb{G})$. Then

$$\eta_0^{-1}(U_A) = \{x \in G_0 \mid M_x(A) = 1\}$$

which is just the set A considered as a subset of G_0, so it is open. The function $\eta_1 : G_1 \to G_{\Theta_2(\mathbb{G})}$ is defined by sending an element $g : x \to y$ to the identity 1_{M_x} on $M_x = M_y$, and it is readily seen to be continuous, so we obtain a morphism $\eta_{\mathbb{G}} : \mathbb{G} \longrightarrow \mathcal{G}(\Theta_2(\mathbb{G}))$. In consequence, the triangle



commutes, so that $\eta_{\mathbb{G}}$ is a morphism of **ctGpd** by Lemma 2.4.3.10. It is also straightforward to verify that $\eta_{\mathbb{G}}$ is the component at $\mathbb{G}$ of a natural transformation. Remains to verify the triangular identities:



The left triangle commutes since it follows from the above that $\Theta_2(\eta_{\mathbb{G}})$ and $\varepsilon_{\Theta_2(\mathbb{G})}$ are both isomorphisms, and the only automorphism on a distributive lattice is the identity. For the right triangle, let $F : \mathcal{L} \longrightarrow 2 \hookrightarrow \mathbf{Sets}_\kappa$ be an element of $X_{\mathcal{L}}$. Applying η_0 we obtain the $\Theta_2(\mathcal{G}(\mathcal{L}))$-model M_F the value of which at A is the fiber over F, and composing M_F with the isomorphism $\varepsilon_{\mathcal{L}}$ returns F. $\dashv$

The adjunction then restricts to the subcategories **clGpd** and **dcGpd** as before, and to an adjunction between $\mathbf{BA} \hookrightarrow \mathbf{DLat}$ and $\mathbf{bcGpd} \hookrightarrow \mathbf{dcGpd}$. Of course, one can also further restrict to spatial groupoids. In particular, since the counit of the adjunction is an isomorphism, $\mathcal{G}_2$ is full and faithful (see e.g. [14, IV.3]) and so the adjunction restricts to an equivalence on the image of $\mathcal{G}_2$:

$$\mathbf{DLat}^{\mathrm{op}} \simeq \mathcal{G}_2(\mathbf{DLat}^{\mathrm{op}}) \hookrightarrow \mathbf{ctGpd}$$

Now, for a distributive lattice $\mathcal{L}$, the groupoid $\mathcal{G}_2(\mathcal{L})$ consists of the sober space of objects $X_{\mathcal{L}} = \mathrm{Hom}_{\mathbf{DLat}}(\mathcal{L}, 2)$ together with the homeomorphic space of identity arrows. We note the following:

Lemma 2.4.5.6 *For a topological groupoid $\mathbb{G}$ such that $\mathrm{Sh}(\mathbb{G}) \simeq \mathrm{Sh}(X)$ for a topological space X, the groupoid $\mathbb{G}$ is an object of* **dcGpd** *if and only if X is an element of* **CohTop**. *For a morphisms of topological groupoids $f : \mathbb{G} \longrightarrow \mathbb{H}$ such that $\mathrm{Sh}(\mathbb{G}) \simeq \mathrm{Sh}(X)$ and $\mathrm{Sh}(\mathbb{H}) \simeq \mathrm{Sh}(Y)$ with X and Y sober coherent spaces, f is an arrow of* **dcGpd** *if and only if the unique continuous map $f' : X \to Y$ induced by*

$$\mathrm{Sh}(X) \simeq \mathrm{Sh}(\mathbb{G}) \xrightarrow{f} \mathrm{Sh}(\mathbb{H}) \simeq \mathrm{Sh}(Y)$$

is a coherent map.

PROOF Recall that there is a isomorphism of frames $\mathcal{O}(X) \cong \mathrm{Sub}_{\mathrm{Sh}(1)}(X)$ for any space X. Now, if $\mathrm{Sh}(\mathbb{G}) \simeq \mathrm{Sh}(X)$ has a coherent site $\mathcal{D}$, then the compact subterminal objects in $\mathrm{Sh}(X)$ correspond to the subterminal objects in $\mathcal{D}$, so X is compact and the compact opens of X form a distributive lattice which generate X. Conversely, whenever X is compact and the compact opens form a distributive lattice that generate X, the lattice of compact opens, which are decidable since they are subterminal objects, form a site for $\mathrm{Sh}(\mathbb{G}) \simeq \mathrm{Sh}(X)$. Whence $\mathbb{G}$ is an object of **dcGpd** if and only if X is an element of **CohTop**. Next, an inverse image functor $g^* : \mathcal{E} \longrightarrow \mathcal{F}$ that preserves compact objects preserves compact subterminal objects in particular. And conversely, if $\mathcal{E}$ and $\mathcal{F}$ are generated by the compact subterminal objects and g^* preserves them, then g^* must preserve any compact object. ⊣

Accordingly, we get as a corollary of Proposition 2.4.5.5 the classical duality theorems:

Corollary 2.4.5.7 *The category of distributive lattices is dual to the category of sober coherent spaces and coherent maps between them,*

$$\mathbf{DLat}^{\mathrm{op}} \simeq \mathbf{sCohTop}$$

by an equivalence which further restricts to the category of Boolean algebras and the category of Stone spaces:

$$\mathbf{BA}^{\mathrm{op}} \simeq \mathbf{Stone}$$

Proof Given a sober coherent space X, we can consider it a groupoid $\mathbb{G}(X)$ by letting the space of arrows consist of identity arrows and be homeomorphic to X. A continuous map of spaces $X \to Y$ extends to a morphism of topological groupoid $\mathbb{G}(X) \longrightarrow \mathbb{G}(Y)$ in the obvious way. Then $\mathrm{Sh}(X) \cong \mathrm{Sh}(\mathbb{G}(X))$, so that by Lemma 2.4.5.6, we have a full and faithful embedding

$$\mathbf{sCohTop} \hookrightarrow \mathbf{dcGpd}$$

Since $\mathcal{G}_2$ is full and faithful and factors through the subcategory **sCohTop**,

$$\begin{array}{ccc} \mathbf{DLat}^{\mathrm{op}} & \xrightarrow{\mathcal{G}_2} & \mathbf{dcGpd} \\ & {\scriptstyle \mathcal{G}_2'} \searrow & \uparrow \\ & & \mathbf{sCohTop} \end{array}$$

a straightforward verification that $\mathcal{G}_2'$ is essentially surjective on objects establishes the equivalence. $\dashv$

Boolean Coherent Categories

Consider the full subcategory $\mathbf{BC}_\kappa \hookrightarrow \mathbf{DC}_\kappa$ of Boolean coherent categories. The category, **bcGpd**, of topological groupoids $\mathbb{G}$ such that $\mathrm{Sh}(\mathbb{G})$ has a Boolean coherent site, with morphisms $f : \mathbb{G} \longrightarrow \mathbb{H}$ those morphisms of topological groupoids such that the inverse image of the induced geometric morphism $f : \mathrm{Sh}(\mathbb{G}) \longrightarrow \mathrm{Sh}(\mathbb{H})$ preserves compact objects, is a full subcategory of **dcGpd**. And, by Theorem 2.4.1.3, $\mathcal{G}$ restricted to $\mathbf{BC}_\kappa$ factors through **bcGpd**. Moreover, the syntactical functor Θ restricted to **bcGpd** also factors through $\mathbf{BC}_\kappa$. For any compact subobject of a decidable compact object in a coherent topos with a Boolean site is complemented. Therefore, we have:

Proposition 2.4.5.8 *The adjunction $\mathcal{G} \vdash \Theta$ restricts to the full subcategories $\mathbf{BC}_\kappa \hookrightarrow \mathbf{DC}_\kappa$ and $\mathbf{bcGpd} \hookrightarrow \mathbf{ctGpd}$ of Boolean coherent categories (with a saturated set of $< \kappa$ models) and Boolean coherent categories, respectively,*

$$\mathbf{BC}_\kappa{}^{\mathrm{op}} \underset{\mathcal{G}\restriction_{\mathbf{BC}_\kappa}}{\overset{\Theta\restriction_{\mathbf{bcGpd}}}{\rightleftarrows}} \mathbf{bcGpd} \quad (\bot)$$

On the other hand, we can also adjust the definitions and constructions to suit the Boolean coherent case in order to get an adjunction between $\mathbf{BC}_\kappa$ and a more inclusive category of groupoids. We outline how, as the details are little different from the decidable coherent case.

Definition 2.4.5.9 An object A in a topos $\mathcal{E}$ is called *Stone* if it is compact, decidable, and its subobject lattice is generated by a set of complemented subobjects. It is called *stably* Stone if it is Stone, and for any Stone object B, the product $A \times B$ is again Stone.

Now, let **StGpd** be the category the objects of which are topological groupoids $\mathbb{G}$ such that the terminal object of $\mathrm{Sh}(\mathbb{G})$ is Stone, or equivalently, such that the stable topology on G_0 is compact with clopen basis. The arrows of **StGpd** are those morphisms of topological groupoids $f : \mathbb{G} \longrightarrow \mathbb{H}$ such that pullback along f preserves stably Stone objects. If $\mathbb{G}$ is a groupoid of **StGpd**, we can then extract a Boolean coherent subcategory $\Theta'(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$ by letting $\Theta'(\mathbb{G})$ be the full subcategory consisting of stably Stone objects that satisfy the itemized conditions of Definition 2.4.3.12, with stably Stone replacing stably compact decidable. Proceeding with few changes from the decidable compact case, it can be shown that the objects of $\Theta'(\mathbb{G})$ correspond to **StGpd** morphisms from $\mathbb{G}$ to the semantical groupoid of the object classifier of Boolean coherent categories—the groupoid $\mathbb{O}$ of Definition 2.5.0.11 below—and that Θ' is functorial and adjoint to $\mathcal{G} : \mathbf{BC}_\kappa \longrightarrow \mathbf{StGpd}$. Restricted to the category **bcGpd**, this is the same adjunction as in Proposition 2.4.5.8, since compact decidable objects are Stone in a topos with a Boolean coherent site.

Theorem 2.4.5.10 *The functor* $\mathcal{G}' = \mathcal{G} \restriction_{\mathbf{BC}_\kappa} : {\mathbf{BC}_\kappa}^{\mathrm{op}} \longrightarrow \mathbf{StGpd}$ *has a left adjoint,*

$$\mathbf{BC}_\kappa{}^{\mathrm{op}} \underset{\Theta'}{\overset{\mathcal{G}'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{StGpd}$$

which sends a topological groupoid $\mathbb{G}$ *to* $\mathrm{Hom}_{\mathbf{StGpd}}(\mathbb{G}, \mathbb{O})$ *considered as a full subcategory of* $\mathrm{Sh}(\mathbb{G})$.

As the proof is essentially a repetition of the proof for Theorem 2.4.4.3, we do not write it out, but proceed to give a detailed description of the groupoid $\mathbb{O}$.

2.5 The Boolean Object Classifier

We highlight the descriptions and role of the 'Boolean object classifier' in Boolean coherent categories, topoi, and topological groupoids. As mentioned in Section 2.4.3, the category $\mathbf{DC}_\kappa$ contains an object $\mathcal{D}(1)$, the 'decidable object classifier', that classifies objects, in the sense that objects of a decidable coherent category, $\mathcal{D}$, corresponds bijectively to natural isomorphism classes of coherent functors from $\mathcal{D}(1)$ into $\mathcal{D}$. The decidable object classifier $\mathcal{D}(1)$ can be specified as the syntactic category of the decidable coherent theory of equality, $\mathbb{T}_{\neq}$, i.e. the coherent theory of the predicate $\neq$. The topos $\mathrm{Sh}\,(\mathcal{D}(1))$ then classifies decidable objects in the category of topoi and geometric morphisms.

In groupoids, we defined the groupoid $\mathbb{S}$ by equipping the groupoid of (hereditarily of size less than κ) sets and bijection with a topology, and identified it, up to isomorphism, with the groupoid of classical models of $\mathbb{T}_{\neq}$ equipped with the logical topology, and, accordingly, up to equivalence with the dual $\mathcal{G}(\mathcal{D}(1))$ consisting of all $\mathbf{Sets}_\kappa$-valued coherent functors on $\mathcal{D}(1)$ equipped with the coherent topology. Thereby, we have that $\mathrm{Sh}\,(\mathbb{S}) \simeq \mathrm{Sh}\,(\mathcal{D}(1))$, yielding an equivalent characterization of the decidable object classifier in topoi (Corollary 2.4.3.3). The (decidable coherent) groupoid $\mathbb{S}$, therefore, plays the role corresponding to that of the (coherent) Sierpiński space in the adjunction between distributive lattices and coherent spaces: in being the dual of the 'object classifier'—that is, the free distributive lattice of one genrator—and the object we 'hom' into to recover a decidable coherent category; and in being defined by equipping the 'schizophrenic' object, $\mathbf{Sets}_\kappa$, with a topology.

Our goal is now to characterize the corresponding groupoid for Boolean coherent categories, which is, then, the object that plays the role corresponding to that of the *discrete* space 2—the dual of the free Boolean algebra in one generator—in the adjunction between Boolean algebras and Stone spaces (see also the table following page 92).

Definition 2.5.0.11 The topological groupoid $\mathbb{O}$ consists of hereditarily less than κ sets with bijections between them, equipped with a topology as follows. The topology on the set of objects, O_0, is the coarsest topology in which sets of the form

$$(a) := \{A \in \mathbf{Sets}_\kappa \mid a \in A\}$$

are open, for $a \in \mathbf{Sets}_\kappa$, and in which sets of the form

$$(n) := \{A \in \mathbf{Sets}_\kappa \mid \text{the cardinality of } A \text{ is } n\}$$

are clopen, for $n \in \mathbb{N}$. The topology on the set, O_1 of bijections between hereditarily less than κ sets is the coarsest topology such that the source and target maps $s, t : O_1 \rightrightarrows O_0$ are both continuous, and such that all sets of the form

$$(a \mapsto b) := \left\{ f : A \xrightarrow{\cong} B \text{ in } \mathbf{Sets}_\kappa \;\middle|\; a \in A \wedge f(a) = b \right\}$$

are open.

As the category $\mathbf{DC}_\kappa$, the category $\mathbf{BC}_\kappa$ of Boolean coherent categories also contains an object classifier. That is, there is a Boolean coherent category, $\mathcal{B}(1)$, such that objects of an arbitrary Boolean coherent category $\mathcal{B}$ correspond, up to natural isomorphism, to coherent functors from $\mathcal{B}(1)$ into $\mathcal{B}$. It can be specified as the syntactic category of the first-order theory of equality, which we denote $\mathbb{T}_=$, and as such, objects of $\mathcal{B}$ correspond on the nose to functors in $\mathbf{T}$ from $\mathcal{C}_{\mathbb{T}_=}$ to $T\mathcal{B} = \mathcal{C}_{\mathbb{T}_\mathcal{B}}$. Accordingly, $\mathrm{Sh}\,(\mathcal{B}(1)) = \mathrm{Sh}\,(\mathbb{T}_=)$ classifies those objects E in topoi that have the property that any subobject of any power E^n which is definable in the first-order language of equality is complemented. We call such objects *classical objects*. By Theorem 2.3.4.14, we can then characterize the classical object classifier as equivariant sheaves on the semantical groupoid $\mathbb{G}_{\mathbb{T}_=}$.

Lemma 2.5.0.12 *There is an isomorphism $\mathbb{O} \cong \mathbb{G}_{\mathbb{T}_=}$ in* $\mathbf{Gpd}$.

Proof Any set A in $\mathbf{Sets}_\kappa$ is the underlying set of a canonical $\mathbb{T}_=$-model, and any bijection $f : A \to B$ is the underlying function of a $\mathbb{T}_=$-model isomorphism, and thereby we obtain bijections $O_0 \cong X_{\mathbb{T}_=}$ and $O_1 \cong G_{\mathbb{T}_=}$ which commute with source, target, composition, and embedding of identities maps. We need to show that these bijections are homeomorphisms of spaces. Any open set of the form $(a) \subseteq O_0$ corresponds to the open set $U_{[x \mid \top],a} \subseteq X_{\mathbb{T}_=}$, while $(n) \subseteq O_0$ corresponds to the clopen set $U_{[\,\mid \phi],\star} \subseteq X_{\mathbb{T}_=}$ where ϕ is the first-order sentence in $=$ expressing that there are exactly n elements. We show that, conversely, any basic open $U_{[\vec{x} \mid \phi],\vec{a}} \subseteq X_{\mathbb{T}_=}$ corresponds to an open set of O_0. Suppose $\mathbf{M} \in U_{[\vec{x} \mid \phi],\vec{a}}$, i.e. $\langle a_1, \ldots, a_n\rangle \in [\![\vec{x} \mid \phi]\!]^{\mathbf{M}}$. We may assume without loss that the a_i are all distinct. Then $a_i \in |\mathbf{M}|$ for $1 \leq i \leq n$, and

$$\mathbf{M} \vDash \exists x_1, \ldots, x_n. \left(\bigwedge_{i \neq j} x_i \neq x_j \right) \wedge \phi. \tag{2.8}$$

Conversely, if $\vec{a} \in |\mathbf{M}|$ and (2.8) holds, then we can choose a witness

$$\langle b_1, \ldots, b_n \rangle$$

in $\mathbf{M}$, and since the assignment $a_i \mapsto b_i$, for $1 \leq i \leq n$, extends to a permutation and therefore an automorphism of $\mathbf{M}$, we have that $\langle a_1, \ldots, a_n \rangle \in [\![\vec{x} \mid \phi]\!]^{\mathbf{M}}$. Now, all infinite $\mathbb{T}_=$-models are elementary equivalent, as are all finite models of the same size. Therefore, by compactness, for any $\mathbb{T}_=$-sentence, ψ, there exists a $k \in \mathbb{N}$ such that either ψ is true in all models of size $> k$ or ψ is false in all models of size $> k$. Therefore, there exists a finite set $K \subset \mathbb{N}$ such that, in the latter case, $\mathbf{N} \vDash \psi$ if and only if $|\mathbf{N}| \in K$, and in the former case, such that $\mathbf{N} \vDash \psi$ if and only if $|\mathbf{N}| \notin K$. Since

$$U := \{A \in \mathbf{Sets}_\kappa \mid |A| \in K\} = \bigcup_{n \in K} (n)$$

is a clopen subset of O_0, the set of $\mathbb{T}_=$-models in which ψ is true corresponds to a clopen set in O_0. Hence $U_{[\vec{x} \mid \phi], \vec{a}}$ corresponds to an open subset of O_0, since it is the intersection of those models that contain a_i, for $1 \leq i \leq n$, and those models in which the sentence

$$\exists x_1, \ldots, x_n. (\bigwedge_{i \neq j} x_i \neq x_j) \wedge \phi$$

is true. For the spaces of arrows, it remains only to observe that open subsets of the form $(a \mapsto b) \subseteq O_1$ correspond to open subsets of the form

$$\left(\begin{array}{c} - \\ a \mapsto b \\ - \end{array} \right) \subseteq G_{\mathbb{T}_{\neq}}$$

and we can conclude that $\mathbb{O}$ is a topological groupoid isomorphic to $\mathbb{G}_{\mathbb{T}_=}$ in **Gpd**. ⊣

The category $\mathrm{Sh}(\mathbb{O})$ of equivariant sheaves on $\mathbb{O}$, therefore, classifies classical objects. The generic classical object, $\mathcal{U}$, in $\mathrm{Sh}(\mathbb{O})$ can be taken to be the definable sheaf $\langle E_{[x \mid \top]} \to X_{\mathbb{T}_=} \cong O_0, \theta_{[x \mid \top]} \rangle$. Similarly to the generic decidable sheaf over $\mathbb{S}$ (see Proposition 2.4.3.4), the object $\mathcal{U}$ can be constructed without reference to definable sheaves using the terminology and results of [21, §6] (we have taken the liberty to present the relevant parts in Section 3.4.1):

Proposition 2.5.0.13 *There exists open set $U \subseteq O_0$ and an open set $M \subseteq O_1$ with M closed under inverses and composition and $s(M), t(M) \subseteq U$ such that the generic classical object $\mathcal{U}$ in $\mathrm{Sh}(\mathbb{O})$ is isomorphic to the equivariant sheaf $\langle \mathbb{S}, U, M\rangle$ in $\mathrm{Sh}(\mathbb{O})$,*

$$\mathcal{U} \cong \langle \mathbb{O}, U, M\rangle$$

Proof Choose a set $a \in \mathbf{Sets}_\kappa$. We have open subsets $U = (a) \subseteq O_0$ and $M = (a \mapsto a) \subseteq O_1$, with M closed under identities of objects in U, inverses, and composition, and $s(M), t(M) \subseteq U$. Following [21, §6], we have an equivariant sheaf $\langle \mathbb{O}, U, M\rangle$ in $\mathrm{Sh}(\mathbb{O})$ consisting of the sheaf $t : O_1 \cap s_{-1}(U)/M \longrightarrow O_0$, where $O_1 \cap s^{-1}(U)/M$ is the set of arrows of O_1 with source in U factored out by the equivalence relation,

$$g \sim_M h \Leftrightarrow t(g) = t(h) \wedge g^{-1} \circ h \in M$$

and O_1 acts on $O_1 \cap s_{-1}(U)/M$ by composition. We replace the fibers of this sheaf by the mapping

$$[g] \mapsto g(a)$$

Following [21, 6], again, it is now straightforward to verify that this mapping is an isomorphism $\langle \mathbb{O}, U, M\rangle \cong \mathcal{U}$ in $\mathrm{Sh}(\mathbb{O})$, by observing that (by Lemma 2.3.4.6) the stabilization of the image of the section $u : U = U_{[x\,|\,\top],a} \to E_{[x\,|\,\top]}$ defined by $\mathbf{M} \mapsto a$ is all of $E_{[x\,|\,\top]}$; that the induced morphism of equivariant sheaves $\tilde{u} : O_1 \cap s^{-1}(U)/N_u \longrightarrow E_{[x\,|\,\top]}$ is therefore an isomorphism; that $M = N_u$; and finally, that $\tilde{u}([g]) = g(a)$. ⊣

The groupoid $\mathbb{O}$ relates to the dual $\mathcal{G}(\mathcal{C}_{\mathbb{T}_=}) = \mathcal{G}(\mathcal{B}(1))$ of the object classifier $\mathcal{B}(1)$ in the same way as the groupoid $\mathbb{S}$ relates to the decidable object classifier $\mathcal{D}(1)$ (recall Proposition 2.4.3.5), that is, there is a morphism of topological groupoids $u : \mathcal{G}(O) \longrightarrow \mathbb{O}$ which sends a coherent functor $M : \mathcal{B}(1) = \mathcal{C}_{\mathbb{T}_=} \longrightarrow \mathbf{Sets}_\kappa$ to the value of M at the object $[x \mid \top]$. The morphism u is a Morita equivalence, and in fact, one half of an equivalence of categories, with the other half being the morphism of topological groupoids $v : \mathbb{O} \longrightarrow \mathcal{G}(\mathcal{B}(1))$ which sends a set to the canonical $\mathbb{T}_=$-model on it. We record this fact.

Proposition 2.5.0.14 *The topological groupoids $\mathbb{O}$ and $\mathcal{G}(\mathcal{B}(1))$ are equivalent in the sense that there are morphisms of topological groupoids*

$$\mathbb{O} \underset{u}{\overset{v}{\rightleftarrows}} \mathcal{G}(\mathcal{B}(1))$$

which form an equivalence of underlying categories.

PROOF Same as for Proposition 2.4.3.5. ⊣

Since $\mathbb{S}$ and $\mathbb{O}$ share the same underlying groupoid, with O_1 and O_0 having finer topologies that S_1 and S_0, respectively, the identity maps form a morphism of topological groupoids,

$$\mathbb{O} \xrightarrow{u} \mathbb{S}$$

(similar to the continuous identity map $2 \to 2$ from the Stone space 2 to the Sierpiński space 2.)

Proposition 2.5.0.15 *For* $\mathbb{G}$ *in* **ctGpd** *and an object* $\mathcal{A}$ *in* $\Theta(\mathbb{G})$*, if (and only if)* $\mathcal{A}$ *has classical diagonal, then the classifying morphism* $a : \mathbb{G} \longrightarrow \mathbb{S}$ *of* $\mathcal{A}$ *factors through* $\mathbb{O}$*, that is, through the morphism* $u : \mathbb{O} \longrightarrow \mathbb{S}$ *the components of which are identities.*

PROOF With reference to the proof of Lemma 2.4.3.16, the only thing to check is that subsets of the form $(n) \subseteq S_0$ pull back to clopen sets of G_0, and that follows from $\mathcal{A}$ having a classical diagonal, the corresponding clopen set of G_0 being the image of the complemented subobject which interprets the classical sentence in $=$ expressing that there is exactly n elements interpreted over $\mathcal{A}$. ⊣

The groupoid $\mathbb{O}$ thus plays a role analogous to the role played by the discrete space 2 in the propositional case: by being defined of the 'schizophrenic object' of $\mathbf{Sets}_\kappa$; by being the dual (up to equivalence) of the object classifier $\mathcal{B}(1)$; and by being the object we 'hom into' to recover a Boolean coherent category from its dual. We summarize the relationship between $\mathcal{B}(1)$, $\mathbb{O}$, and their categories of sheaves:

- The category $\mathcal{B}(1)$, with generic object U, is the object classifier in the category of Boolean coherent categories, i.e. it is the free Boolean coherent category on one object. It can be given as $\langle \mathcal{C}_{\mathbb{T}_=}, [x \mid \top] \rangle$. The (Morita equivalent) pretopos completion $p : \mathcal{B}(1) \hookrightarrow \mathbf{Pretop}(\mathcal{B}(1))$ is, then, the object classifier in the category of Boolean pretopoi.

- The topological groupoid $\mathbb{O}$ is specified by equipping the sets of hereditarily less than κ size and bijections between them with a topology. It can be obtained as the groupoid of $\mathbb{T}_=$-models and isomorphisms or, up to equivalence, as the groupoid $\mathcal{G}(\mathcal{B}(1)) = \underline{\mathrm{Hom}}^*_{\mathbf{DC}}(\mathcal{B}(1), \mathbf{Sets}_\kappa)$ equipped with the coherent topology.

- The topos $\mathfrak{O}$,with generic object $\mathfrak{U}$, classifying classical objects can be characterized as $\langle \mathrm{Sh}\left(\mathcal{B}(1)\right), yU\rangle$, or equivalently as $\langle \mathrm{Sh}\left(\mathbb{O}\right), \mathcal{U}\rangle$. (Other equivalent characterizations are then
  - $\langle \mathrm{Sh}\left(\mathbf{Pretop}(\mathcal{B}(1))\right), y(p(U))\rangle$;
  - $\langle \mathrm{Sh}\left(\mathcal{G}(\mathcal{B}(1))\right), u^*(\mathcal{U})\rangle$; or
  - $\langle \mathrm{Sh}\left(\mathcal{G}(\mathbf{Pretop}(\mathcal{B}(1)))\right), \mathcal{G}(p)^*(u^*(\mathcal{U}))\rangle$.)
- The category $\mathbf{Pretop}(\mathcal{B}(1))$ can be recovered, up to equivalence, from $\mathfrak{O}$ as the compact decidable objects, corresponding to the compact object preserving geometric endomorphisms of $\mathfrak{O}$. The compact decidable objects can also be identified, via the equivalence

$$\mathfrak{O} \simeq \mathrm{Sh}\left(\mathcal{G}(\mathbf{Pretop}(\mathcal{B}(1)))\right)$$

  as $\mathrm{Hom}_{\mathbf{bcGpd}}\left(\mathcal{G}(\mathbf{Pretop}(\mathcal{B}(1))), \mathbb{O}\right)$. The category $\mathcal{B}(1)$ can be recovered, up to equivalence, from the topos $\mathfrak{O}$ as the subcategory consisting of the finite powers of $\mathfrak{U}$ and their definable subobjects—i.e. the subobjects obtained from the diagonal by complements, finite meets and joins, and pullbacks and images along projections. The groupoid $\mathbb{O}$ can be recovered from the topos $\mathfrak{O}$ as the points $\mathrm{Hom}_{\mathcal{TOP}}\left(\mathbf{Sets}_\kappa, \mathfrak{O}\right)$, and invertible geometric transformations between them, modulo agreement on $\mathfrak{U}$, and equipped with the appropriate topology.

In the topological groupoid $\mathbb{O}$, we obtain from the category of sets a mathematical object, given in terms of an independent and quite natural characterization, which relates thus to first-order logic in both a syntactical and a semantical aspect via the properties of its topos of equivariant sheaves. Closely related groupoids, both in their properties and their construction, are the already described groupoid $\mathbb{S}$, and the restricted topological group(oid) of permutations on a single infinite set, the topos of sheaves on which (called the *Schanuel topos*) classifies infinite decidable objects, which corresponds to the theory of equality and inequality on an infinite set (see [10, A2.1.11, C5.2.14, D3.4.10], [14, VIII, Exc.7–9]). In Chapter 3, where we focus on first-order single sorted logic, we construct a groupoid $\mathbb{N}/_\sim$ similar to $\mathbb{O}$, but consisting of the 'enumerated' models of the classical theory of equality, in order to obtain a duality between first-order single sorted logic and a subcategory of groupoids over $\mathbb{N}/_\sim$.

Category	Free on one object	Dual - by homming into 2 or $\mathbf{Sets}_\kappa$
DLat	3-element lattice	2 (Sierpiński)
BA	4-element B. algebra	2 (Stone/discrete)
DC	$\mathcal{D}(1)$	$\mathbb{S}$
BC	$\mathcal{B}(1)$	$\mathbb{O}$

Chapter 3

First-Order Logical Duality

The 'syntax-semantics' adjunction of Chapter 2 restricts to an adjunction between Boolean coherent categories and semantical groupoids such that unit and counit components are Morita equivalences of groupoids and categories respectively. Thus one recovers a first-order theory from its groupoid of models and isomorphisms up to Morita equivalence, or equivalence if the theory is presented as a pretopos. For a single-sorted theory, however, we can do better. We show how the techniques and results of the 'syntax-semantics' adjunction can be applied to the single sorted case to yield an adjunction between single-sorted theories and a full subcategory of a slice of topological groupoids, such that the counit components on the syntactical side are isomorphisms. We also show how a suitable slice category of topological groupoids can be specified intrinsically. We have made the current chapter self-contained, allowing us to show a variation of the setup which in some aspects more resembles [5]. For the reader familiar with Chapter 2, Section 3.2 in particular will seem familiar, and the most interesting parts might be the alternative construction of the space of models for a theory with no empty models of Section 3.1.1, the use of slices over the object classifier to obtain an adjunction with isomorphism counits in Section 3.3, and the intrinsic characterization of the category of Stone fibrations over the object classifier in Section 3.4. The goal of Sections 3.1 and 3.2 is the representation of a theory $\mathbb{T}$ in terms of its topological groupoid $G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}}$ of 'enumerated' models and model isomorphisms in Theorem 3.2.2.10, which states that the topos of coherent sheaves on the syntactic category of $\mathbb{T}$ is equivalent to the topos of equivariant sheaves on the semantical groupoid of $\mathbb{T}$:

$$\mathrm{Sh}(\mathcal{C}_{\mathbb{T}}) \simeq \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$$

The structure of the argument is similar to the structure of the argument for Theorem 2.3.4.14 in Chapter 2, and resembles that sketched in Section 1.2.1 for the representation of a Boolean algebra in terms of its space of models: Corresponding to the Stone Representation Theorem, we embed $\mathcal{C}_{\mathbb{T}}$ in the topos of sets over the set of 'enumerated' models, $\mathbf{Sets}/X_{\mathbb{T}}$, in Lemma 3.1.1.7. The introduction of a logical topology on the set $X_{\mathbb{T}}$ allows us to factor that embedding through the topos $\mathrm{Sh}(X_{\mathbb{T}})$ of sheaves on the space of 'enumerated' models in Lemma 3.1.1.10. The goal of Section 3.1.2 is then to show Proposition 3.1.2.8 to the effect that the induced geometric morphism

$$\mathrm{Sh}(X_{\mathbb{T}}) \longrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$$

is an open surjection. Section 3.2.1 introduces $\mathbb{T}$-model isomorphisms and describes the topological groupoid $G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}}$ of 'enumerated' models and model isomorphisms. We then use the results of Section 3.1.2 to show that the embedding $\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathrm{Sh}(X_{\mathbb{T}})$ factors through the topos $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ of equivariant sheaves on $G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}}$ and, in Lemma 3.2.2.9, that the image of the embedding generates $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$. We are then in a position to conclude that the embedding $\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ lifts to an equivalence $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}}) \simeq \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ in Theorem 3.2.2.10.

3.1 Sheaves on the Space of Models

For a theory $\mathbb{T}$ we construct an open cover $\mathrm{Sh}(X_{\mathbb{T}}) \twoheadrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ of the topos of coherent sheaves on $\mathbb{T}$ (i.e. on $\mathcal{C}_{\mathbb{T}}$), where $X_{\mathbb{T}}$ is a space of $\mathbb{T}$-models equipped with a certain 'logical' topology. This is an adaptation of the construction of [5] (see also [3], [7], and [6]) of a locally connected cover from a space of points and enumerations.

3.1.1 The Space of Models

Let a (classical, single sorted) theory $\mathbb{T}$ in language $\mathcal{L}_{\mathbb{T}}$ be given. As usual in classical first-order logic, we assume that $\mathbb{T} \vdash \exists x.\, x = x$, that is, that the empty set is not a model of $\mathbb{T}$. We may think of the language $\mathcal{L}_{\mathbb{T}}$ as being countable, although we shall not be needing this assumption until Lemma 3.1.1.7, and immediately thereafter (Remark 3.1.1.9) we will introduce a more general countability condition on the theory $\mathbb{T}$. The reason is to be able to cut down the class of $\mathbb{T}$-models to a set which is nevertheless 'large enough', in a sense that is made clear in Lemma 3.1.1.7.

Definition 3.1.1.1 For a theory $\mathbb{T}$, let the set of all *enumerated* $\mathbb{T}$*-models*, $X_{\mathbb{T}}$, consist of those models of $\mathbb{T}$ which have underlying set a quotient, $\mathbb{N}/_{\sim}$, of the set of natural numbers. In the context of a fixed theory $\mathbb{T}$ (as in the current section), we forget the subscript and just write X.

Extend $\mathcal{L}_{\mathbb{T}}$ to $\mathcal{L}_{\mathbb{T}_{\mathbb{N}}}$ by adding the natural numbers as constants. We assume without loss that such constants do not already exist in $\mathcal{L}_{\mathbb{T}}$. Let $\mathbb{T}_{\mathbb{N}}$ be the closure of $\mathbb{T}$ in $\mathcal{L}_{\mathbb{T}_{\mathbb{N}}}$. The models in X extend to $\mathbb{T}_{\mathbb{N}}$-models by interpreting a constant n by its equivalence class. (In fact, $\mathbb{T}_{\mathbb{N}}$ is the theory of sentences of $\mathcal{L}_{\mathbb{T}_{\mathbb{N}}}$ which are true in all models in X under this interpretation, as justified by Corollary 3.1.2.2 below.)

Definition 3.1.1.2 The *logical* topology (with respect to $\mathbb{T}_{\mathbb{N}}$) on X is generated by basic open sets

$$U_{\phi} := \{\mathbf{M} \in X \mid \mathbf{M} \vDash \phi\}$$

for ϕ a sentence in $\mathcal{L}_{\mathbb{T}_{\mathbb{N}}}$.

We shall usually write sentences in $\mathcal{L}_{\mathbb{T}_{\mathbb{N}}}$ with their distinct natural number constants displayed. E.g. $U_{\phi(\vec{n}/\vec{x})}$, where $[\vec{x} \mid \phi]$ is a formula-in-canonical-context of $\mathcal{L}_{\mathbb{T}}$—where "canonical" indicates that $\vec{x}$ occurs free in ϕ—and the constants occurring in $\vec{n}$ are distinct. Now and then, however, we shall not bother with displaying the number constants if there is no need to. For instance, it is clear that sets of the form U_{ϕ} form a basis on X, since

$$U_{\top} = X$$
$$U_{\phi} \cap U_{\psi} = U_{\phi \wedge \psi}$$

Remark 3.1.1.3 Concerning the difference between the current chapter and Chapter 2, we emphasize again that the current chapter only considers theories with non-empty models, which Chapter 2 did not. Further, in addition to restricting the size of the models, we 'enumerate' their elements in insisting that the underlying sets of the models we consider are quotients of $\mathbb{N}$, which one can regard as giving the underlying set A of a model together with a surjection $\mathbb{N} \twoheadrightarrow A$. The topology is then, in a sense, given in terms of the enumeration, rather than in terms of elements as was the case in Chapter 2. The set-up of the current chapter is therefore distinctly different—note e.g. that the topology on the set of models X is given in terms of a *clopen* basis—but, as we shall see, the formal structure of the arguments are very similar.

For a formula-in-context $[\vec{x} \,|\, \phi]$ in $\mathcal{L}_{\mathbb{T}}$, define

$$E_{[\vec{x}\,|\,\phi]} := \left\{ \langle \mathbf{M}, [\vec{n}] \rangle \;\middle|\; \mathbf{M} \in X \wedge [\vec{n}] \in |\mathbf{M}| \wedge \mathbf{M} \vDash \phi(\vec{n}/\vec{x}) \right\} \cong \coprod_{\mathbf{M} \in X} [\![\vec{x} \mid \phi]\!]^{\mathbf{M}}$$

These sets are of interest to us together with their projections as sets over X, and we shall often let context make clear whether we are thinking of them as sets or as sets over X. Thus, whenever we talk about *a set* $E_{[\vec{x}\,|\,\phi]}$ *over* X, we mean the set together with the function

$$\pi_1 : E_{[\vec{x}\,|\,\phi]} \longrightarrow X$$

defined by $\pi_1 \left(\mathbf{M}, [\vec{n}] \right) = \mathbf{M}$.

Definition 3.1.1.4 The *logical topology* on a set of the form $E_{[\vec{x}\,|\,\phi]}$ has as basis sets of the form

$$V_{[\vec{x}\,|\,\phi \wedge \psi]} := \left\{ \langle \mathbf{M}, [\vec{n}] \rangle \;\middle|\; \mathbf{M} \in X \wedge [\vec{n}] \in |\mathbf{M}| \wedge \mathbf{M} \vDash (\phi \wedge \psi)(\vec{n}/\vec{x}) \right\}$$

where $[\vec{x} \,|\, \psi]$ is a formula-in-context of $\mathcal{L}_{\mathbb{T}_{\mathbb{N}}}$.

Again, we shall usually display the distinct number constants, and write $V_{[\vec{x}\,|\,\phi \wedge \psi(m_1,\ldots,m_k/y_1,\ldots,y_k)]}$ where $[\vec{x}, \vec{y} \,|\, \psi]$ is a formula-in-context of $\mathcal{L}_{\mathbb{T}}$ such that no y_i occurs in $\vec{x}$, $\vec{y}$ is free in ψ, and $m_i \neq m_j \in \mathbb{N}$ for $1 \leq i \lneqq j \leq k$.

Lemma 3.1.1.5 *For any formulas-in-context* $[\vec{x} \,|\, \phi]$ *and* $[\vec{y} \,|\, \psi]$ *of* $\mathcal{L}_{\mathbb{T}}$*, the following is a pullback square in* **Top***:*

$$\begin{array}{ccc} E_{[\vec{x},\vec{y}\,|\,\phi \wedge \psi]} & \longrightarrow & E_{[\vec{y}\,|\,\psi]} \\ \downarrow & \lrcorner & \downarrow \pi_1 \\ E_{[\vec{x}\,|\,\phi]} & \xrightarrow[\pi_1]{} & X \end{array}$$

(with the unlabeled morphisms being the evident projections).

Proof It is clear that as far as the underlying sets are concerned, $E_{[\vec{x},\vec{y}\,|\,\phi\wedge\psi]} \cong E_{[\vec{x}\,|\,\phi]} \times_X E_{[\vec{y}\,|\,\psi]}$ by the bijection $f(\langle \mathbf{M}, [\vec{n}], [\vec{m}]\rangle) = \langle\langle \mathbf{M}, [\vec{n}]\rangle, \langle \mathbf{M}, [\vec{m}]\rangle\rangle$. It remains to verify that the logical topology on the former corresponds to the pullback topology on the latter: let a basic open $V_{\left[\vec{x},\vec{y}\,\middle|\,\phi\wedge\psi\wedge\sigma(\vec{i}/\vec{z})\right]} \subseteq E_{[\vec{x},\vec{y}\,|\,\phi\wedge\psi]}$ be given, and suppose that $\langle \mathbf{M}, [\vec{n}], [\vec{m}]\rangle \in V_{\left[\vec{x},\vec{y}\,\middle|\,\phi\wedge\psi\wedge\sigma(\vec{i}/\vec{z})\right]}$. Then

$$\begin{aligned}\langle\langle \mathbf{M}, [\vec{n}]\rangle, \langle \mathbf{M}, [\vec{m}]\rangle\rangle &\in V_{\left[\vec{x}\,\middle|\,\phi\wedge\sigma(\vec{m}/\vec{y},\vec{i}/\vec{z})\wedge\vec{x}=\vec{n}\right]} \times_X V_{\left[\vec{y}\,\middle|\,\psi\wedge\sigma(\vec{n}/\vec{x},\vec{i}/\vec{z})\wedge\vec{y}=\vec{m}\right]}\\ &\subseteq f(V_{\left[\vec{x},\vec{y}\,\middle|\,\phi\wedge\psi\wedge\sigma(\vec{i}/\vec{z})\right]})\\ &\subseteq E_{[\vec{x}\,|\,\phi]} \times_X E_{[\vec{y}\,|\,\psi]}\end{aligned}$$

In the other direction, given a box of basics

$$V_{[\vec{x}\,|\,\phi\wedge\sigma]} \times_X V_{[\vec{y}\,|\,\psi\wedge\delta]} \subseteq E_{[\vec{x}\,|\,\phi]} \times_X E_{[\vec{y}\,|\,\psi]}$$

we have that

$$f^{-1}(V_{[\vec{x}\,|\,\phi\wedge\sigma]} \times_X V_{[\vec{y}\,|\,\psi\wedge\delta]}) = V_{[\vec{x},\vec{y}\,|\,\phi\wedge\psi\wedge\sigma\wedge\delta]} \subseteq E_{[\vec{x},\vec{y}\,|\,\phi\wedge\psi]} \qquad \dashv$$

Lemma 3.1.1.6 *For any formula-in-context $[\vec{x}\,|\,\phi]$ in $\mathcal{L}_{\mathbb{T}}$ the projection $\pi_1 : E_{[\vec{x}\,|\,\phi]} \to X$ is a local homeomorphism.*

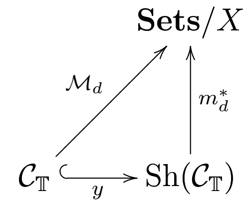
Proof The function π_1 is continuous since the inverse image of a basic open $U_{\psi(\vec{n}/\vec{y})}$ is $V_{[\vec{x}\,|\,\phi\wedge\psi(\vec{n}/\vec{y})]}$. π_1 is open since the image of a basic open $V_{[\vec{x}\,|\,\phi\wedge\psi(\vec{n}/\vec{y})]}$ is $U_{\exists\vec{x}.\psi(\vec{n}/\vec{y})\wedge\phi}$. And the diagonal is open by Lemma 3.1.1.5, since $V_{[\vec{x},\vec{y}\,|\,\vec{x}=\vec{y}]} = \Delta \subseteq E_{[\vec{x}\,|\,\phi]} \times_X E_{[\vec{y}\,|\,\phi]}$. $\dashv$

We carry over our habit of suppressing mention of the projection, and so, whenever we talk of *an (etale) space $E_{[\vec{x}\,|\,\phi]}$ over X*, we mean the sets $E_{[\vec{x}\,|\,\phi]}$ and X equipped with their respective logical topologies and with the locally homeomorphic left projection between them:

$$E_{[\vec{x}\,|\,\phi]} \xrightarrow{\pi_1} X$$

Lemma 3.1.1.7 *The assignment $[\vec{x}\,|\,\phi] \mapsto E_{[\vec{x}\,|\,\phi]}$ determines a model $\mathcal{M}_d$ of $\mathbb{T}$ in* **Sets**$/X$*. If $\mathcal{L}_{\mathbb{T}}$ is countable, then the classifying geometric morphism,*

m_d, of $\mathcal{M}_d$ to the topos of coherent sheaves, $\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$*, on* $\mathbb{T}$ *is a surjection, that is, the inverse image functor m_d^* is faithful:*



Proof We define the functor $\mathcal{M}_d : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}/X$ by sending an object $[\vec{x}\,|\,\phi]$ of $\mathcal{C}_{\mathbb{T}}$ to the set $E_{[\vec{x}\,|\,\phi]}$ over X, and an arrow

$$[\vec{x}, \vec{y}\,|\,\sigma] : [\vec{x}\,|\,\phi] \longrightarrow [\vec{y}\,|\,\psi]$$

of $\mathcal{C}_{\mathbb{T}}$ to the function $\mathrm{f}_\sigma : E_{[\vec{x}\,|\,\phi]} \longrightarrow E_{[\vec{y}\,|\,\psi]}$ over X whose value at

$$\langle \mathbf{M}, \vec{[n]} \rangle \in E_{[\vec{x}\,|\,\phi]}$$

is the pair

$$\langle \mathbf{M}, \vec{[m]} \rangle \in E_{[\vec{y}\,|\,\psi]}$$

where $\vec{[m]}$ are the ($\mathbb{T}$-provably) unique elements of $\mathbf{M}$ such that

$$\mathbf{M} \vDash \sigma(\vec{n}/\vec{x}, \vec{m}/\vec{y}).$$

With $\mathbf{Sets}/X \cong \prod_{\mathbf{M}\in X} \mathbf{Sets}$, we recognise that this functor is, up to isomorphism, the tuple $\langle \mathbf{M} \rangle_{\mathbf{M}\in X}$ of the coherent functors $\mathbf{M} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$ corresponding to the models in X, and so $\mathcal{M}_d : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}/X$ is itself coherent. Now, suppose $\mathcal{L}_{\mathbb{T}}$ is countable. To verify that the corresponding geometric morphism $m_d : \mathbf{Sets}/X \longrightarrow \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ is surjective, it is sufficient to establish that $\mathcal{M}_d$ reflects covers. Consider a set $E_{[\vec{x}\,|\,\phi]}$ over X. Let a covering, i.e. jointly surjective, family in the image of $\mathcal{M}_d$ be given. Because we can factor out the image of each morphism in the family, we can assume without loss that it is of the form $\{E_{[\vec{x}\,|\,\psi_i]} \subseteq E_{[\vec{x}\,|\,\phi]} \mid i \in I\}$. We need to show that there exists a finite selection of formulas $\psi_{i_1} \ldots \psi_{i_k}$ such that $\mathbb{T} \vdash \forall \vec{x}.\, \phi \rightarrow \psi_{i_1} \vee \ldots \vee \psi_{i_k}$. Suppose not. Thus for any finite set of indices, $i_1, \ldots, i_k$, $\mathbb{T} \bigcup \{\exists \vec{x}.\, \phi \wedge \neg\psi_{i_1} \wedge \ldots \wedge \neg\psi_{i_k}\}$ is consistent. Then $p(\vec{x}) := \{\phi, \neg\psi_i \mid i \in I\}$ is a (possibly incomplete) $\mathbb{T}$-type, so it is realized by a countable model, and thus by a model $\mathbf{M}$ in X. But then there exists elements $[\vec{n}] \in \mathbf{M}$ such that $\mathbf{M} \vDash \phi(\vec{n})$ and $\mathbf{M} \vDash \neg\psi_i(\vec{n})$ for all $i \in I$, contradicting that we were given a *covering* family $\{E_{[\vec{x}\,|\,\psi_i]} \subseteq E_{[\vec{x}\,|\,\phi]} \mid i \in I\}$. So $\mathcal{M}_d$ does indeed reflect covers. $\dashv$

Definition 3.1.1.8 We say that a theory $\mathbb{T}$ has *a saturated set of countable models* if the geometric morphism $m_d : \mathbf{Sets}/X \longrightarrow \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ of Lemma 3.1.1.7 is a surjection.

Thus the second part of Lemma 3.1.1.7 states that countable theories have a saturated set of countable models.

Remark 3.1.1.9 We assume that the theory $\mathbb{T}$ which we have fixed for the current section has a saturated set of countable models.

The identity function is continuous and onto between X as a discrete space and X equipped with the logical topology, and thus induces a surjective geometric morphism $\iota : \mathbf{Sets}/X \twoheadrightarrow \mathrm{Sh}\,(X)$.

Lemma 3.1.1.10 *There is a surjective geometric morphism*

$$m : \mathrm{Sh}\,(X) \twoheadrightarrow \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$$

such that m_d factors as $m \circ \iota$:

$$\begin{array}{ccc} \mathbf{Sets}/X & \overset{\iota}{\twoheadrightarrow} & \mathrm{Sh}\,(X) \\ & {\scriptstyle m_d}\searrow & \downarrow {\scriptstyle m} \\ & & \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}}) \end{array}$$

Proof The inverse image $\iota^* : \mathrm{Sh}\,(X) \longrightarrow \mathbf{Sets}/X$ is the forgetful functor sending an etale space over X to its underlying set over X. Similar to Lemma 3.1.1.10, we define a functor $\mathcal{M} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}\,(X)$ by sending an object $[\vec{x}\,|\,\phi]$ of $\mathcal{C}_{\mathbb{T}}$ to the space $E_{[\vec{x}\,|\,\phi]}$ over X, and an arrow

$$[\vec{x}, \vec{y}\,|\,\sigma] : [\vec{x}\,|\,\phi] \longrightarrow [\vec{y}\,|\,\psi]$$

of $\mathcal{C}_{\mathbb{T}}$ to the function $\mathrm{f}_\sigma : E_{[\vec{x}\,|\,\phi]} \longrightarrow E_{[\vec{y}\,|\,\psi]}$ over X, which is continuous since if $V = V_{[\vec{y}\,|\,\psi\wedge\eta]} \subseteq E_{[\vec{y}\,|\,\psi]}$, then

$$f_\sigma^{-1}(V) = V_{[\vec{x}\,|\,\phi\wedge\exists\vec{y}.\eta\wedge\sigma]}.$$

It follows that $\mathcal{M}_d = \iota^* \circ \mathcal{M}$,

$$\begin{array}{ccc} \mathbf{Sets}/X & \xleftarrow{\iota^*} & \mathrm{Sh}\,(X) \\ \uparrow{\scriptstyle \mathcal{M}_d} & \nearrow{\scriptstyle \mathcal{M}} & \\ \mathcal{C}_{\mathbb{T}} & & \end{array}$$

and since ι^* is geometric and conservative and therefore reflects the coherent structure, we can conclude that $\mathcal{M}$ is coherent. Thus it corresponds to a geometric morphism $m : \mathrm{Sh}\,(X) \longrightarrow \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ such that $m_d \cong m \circ \iota$. Finally, m is surjective since $m \circ \iota$ is. $\dashv$

3.1.2 An Open Surjection

We show in this section that the surjection $m : \mathrm{Sh}\,(X) \twoheadrightarrow \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ of Lemma 3.1.1.10 is open. Recall the following necessary and sufficient condition for m to be open (e.g. from [14, IX.6], see also [4] and [10, C3.1]): the image under m^* of the universal monomorphism $\top : 1 \to \Omega_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}$ in $\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ has a classifying map in $\mathrm{Sh}\,(X)$ that we may call τ:

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle m^*(\top)}\downarrow & \lrcorner & \downarrow{\scriptstyle \top} \\ m^*(\Omega_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}) & \xrightarrow[\tau]{} & \Omega_{\mathrm{Sh}(X)} \end{array}$$

Denote its transpose by $\tilde{\tau} : \Omega_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})} \longrightarrow m_*(\Omega_{\mathrm{Sh}(X)})$. Then m is open if and only if $\tilde{\tau}$ has an internal left adjoint γ:

$$\Omega_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})} \underset{\tilde{\tau}}{\overset{\gamma}{\rightleftarrows}} m_*(\Omega_{\mathrm{Sh}(X)}) \qquad (\gamma \dashv \tilde{\tau})$$

We shall construct γ by constructing an operation on subobjects of the etale spaces $E_{[\vec{x}\,|\,\phi]} = \mathcal{M}([\vec{x}\,|\,\phi])$ in $\mathrm{Sh}\,(X)$. Such subobjects are just open subsets of the spaces, some of which, however, are in the image of $\mathcal{M}$ and some of which are in the image of m^*

$$\begin{array}{ccc} & & \mathrm{Sh}\,(X) \\ & \overset{\mathcal{M}}{\nearrow} & \uparrow{\scriptstyle m^*} \\ \mathcal{C}_{\mathbb{T}} & \underset{y}{\hookrightarrow} & \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}}) \end{array}$$

The former subobjects we shall call *definable* and latter we shall call *ideals*. We shall also call the objects and arrows in the image of $\mathcal{M}$ *definable*, whereas we continue to call the objects and arrows in the image of Yoneda *representable*. The reason for using "ideals" is, briefly, that a subobject of a representable object, $S \rightarrowtail y([\vec{x} \mid \phi])$, in $\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ corresponds to a closed sieve on $[\vec{x} \mid \phi]$ in $\mathcal{C}_{\mathbb{T}}$, which, being closed, is generated by an ideal, I_S in $\mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}([\vec{x} \mid \phi])$. Since m^* preserves colimits, $m^*(S)$ can be computed from I_S by taking the union of the definable subobjects of $E_{[\vec{x} \mid \phi]}$ coming from elements of I_S, thus

$$m^*(S) \cong \bigcup_{[\vec{x} \mid \xi] \in I_S} E_{[\vec{x} \mid \xi]} \subseteq E_{[\vec{x} \mid \phi]}$$

In slightly more detail and introducing some notation: for an object $C \in \mathcal{C}_{\mathbb{T}}$, we have the lattice $\mathrm{ClSieve}(C) \cong \Omega_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}\,(C)$ of closed sieves on C ordered by inclusion, the lattice $\mathrm{Idl}(C)$ of ideals in $\mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}\,(C)$—i.e. downward closed sets of subobjects closed under finite joins—also ordered by inclusion, and a lattice isomorphism between them which sends an ideal to the sieve generated by it and a sieve to the ideal obtained by factoring out images. Sieves on C in $\mathcal{C}_{\mathbb{T}}$ correspond to subobjects of $y(C)$ in $\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$, and so we have lattice isomorphisms

$$\begin{array}{ccc} \mathrm{ClSieve}(C) & & \\ & \overset{Y'_C}{\searrow} & \\ \updownarrow{\scriptstyle \cong} & & \mathrm{Sub}_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}\,(y(C)) \\ & \underset{Y_C}{\nearrow} & \\ \mathrm{Idl}(C) & & \end{array}$$

natural in C. Furthermore, $m^* : \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}}) \longrightarrow \mathrm{Sh}\,(X)$ being conservative and the inverse image part of a geometric morphism, it restricts to order reflecting lattice morphisms $\widehat{m^*_C} : \mathrm{Sub}_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}\,(yC) \longrightarrow \mathrm{Sub}_{\mathrm{Sh}(X)}\,(m^*(yC))$. And thus, by composing we have order reflecting lattice morphisms $\widehat{\mathcal{M}_C} := \widehat{m_C^*} \circ Y_C : \mathrm{Idl}(C) \longrightarrow \mathrm{Sub}_{\mathrm{Sh}(X)}(\mathcal{M}(C))$, natural in C. It is thus the subobjects in the image of these morphisms that we call ideal subobjects. We show through the following lemmas that there is a closure operation—'least including ideal'—on $\mathrm{Sub}_{\mathrm{Sh}(X)}(\mathcal{M}([\vec{x} \mid \phi]))$ which is natural in $[\vec{x} \mid \phi]$.

Lemma 3.1.2.1 *For any* $\mathbf{M}$ *in* X*, any finite list* $m_1, \ldots, m_k \in \mathbb{N}$*, and any finite list of distinct numbers* $n_1, \ldots, n_k \in \mathbb{N}$*, there exists a model* $\mathbf{N}$ *in* X *and an isomorphism* $\mathrm{f} : \mathbf{M} \to \mathbf{N}$ *such that*

$$\mathrm{f}([m_i]) = [n_i]$$

for all $1 \leq i \leq k$.

Proof Write $|\mathbf{M}| = \mathbb{N}/_\sim$. Since $n_1, \ldots, n_k$ are all distinct, $n_i \mapsto [m_i]$ defines a partial function from $\mathbb{N}$ to $\mathbb{N}/_\sim$. Choose any surjective extension $\mathrm{p} : \mathbb{N} \twoheadrightarrow \mathbb{N}/_\sim$, and write $\equiv$ for the induced equivalence relation on $\mathbb{N}$. Then $[m] \mapsto \mathrm{p}^{-1}([m])$ defines a bijection $\mathrm{f} : \mathbb{N}/_\sim \to \mathbb{N}/_\equiv$ such that $\mathrm{f}([m_i]) = [n_i]$ for all $1 \leq i \leq k$. Thus we can let $\mathbf{N}$ be the $\mathbb{T}$-model induced by f and $\mathbf{M}$ on $\mathbb{N}/_\equiv$.⊣

Corollary 3.1.2.2 *For any sentence* $\phi(\vec{n}/\vec{x}) \in \mathcal{L}_{\mathbb{T}_\mathbb{N}}$*—where, following our covention,* $[\vec{x} \mid \phi] \in \mathcal{L}_\mathbb{T}$ *and* $\vec{n}$ *is a sequence of distinct numbers—if* $\mathbf{M} \vDash \phi(\vec{n}/\vec{x})$ *for all* $\mathbf{M} \in X$ *then* $\mathbb{T} \vdash \forall \vec{x}.\, \phi(\vec{x})$*. Consequently,*

$$\mathbb{T}_\mathbb{N} = \{\phi \in \mathcal{L}_{\mathbb{T}_\mathbb{N}} \mid \mathbf{M} \vDash \phi \quad \textit{for all } \mathbf{M} \textit{ in } X\}$$

Proof Suppose $\mathbb{T} \nvdash \forall \vec{x}.\, \phi$. Then there exists a model $\mathbf{M}$ in X such that $\mathbf{M} \vDash \exists \vec{x}.\, \neg\phi$, and therefore $\mathbf{M} \vDash \neg\phi(\vec{m}/\vec{x})$ for some $\vec{m}$. By Lemma 3.1.2.1 there exists a model $\mathbf{N}$ in X and an isomorphism $\mathrm{f} : \mathbf{M} \to \mathbf{N}$ such that $\mathrm{f}\left(\vec{[m]}\right) = \vec{[n]}$, so $\mathbf{N} \vDash \neg\phi(\vec{n}/\vec{x})$. ⊣

Lemma 3.1.2.3 *Let a formula-in-context* $[\vec{x} \mid \phi]$ *of* $\mathcal{L}_\mathbb{T}$ *be given, and suppose we have a basic open set* $V_{[\vec{x} \mid \phi \wedge \psi(m_1 \ldots m_k / y_1 \ldots y_k)]} \subseteq E_{[\vec{x} \mid \phi]}$*, where we assume as usual that* $m_i \neq m_j$ *for* $1 \leq i \neq j \leq k$*. Then there exists a formula-in-context* $[\vec{x} \mid \eta]$ *in* $\mathcal{L}_\mathbb{T}$ *such that* η *is the least formula in* $\mathcal{L}_\mathbb{T}$ *covering* $V_{[\vec{x} \mid \phi \wedge \psi(m_1 \ldots m_k / y_1 \ldots y_k)]}$*, i.e. such that* $V_{[\vec{x} \mid \phi \wedge \psi(m_1 \ldots m_k / y_1 \ldots y_k)]} \subseteq E_{[\vec{x} \mid \eta]}$.

Proof We claim that $\eta :\equiv \exists y_1 \ldots y_k.\, \phi \wedge \psi$ will do the trick. First, we immediately see that

$$V_{[\vec{x} \mid \phi \wedge \psi(m_1 \ldots m_k / y_1 \ldots y_k)]} \subseteq E_{[\vec{x} \mid \exists y_1 \ldots y_k . \phi \wedge \psi]}$$

Now assume $[\vec{x} \mid \delta]$ is such that for all $\langle \mathbf{M}, [\vec{n}] \rangle$ in $V_{[\vec{x} \mid \phi \wedge \psi(\vec{m}/\vec{y})]}$ we have $\mathbf{M} \models \delta(\vec{n}/\vec{x})$. Then we claim that

$$\mathbb{T} \vdash \forall \vec{x}. (\exists y_1 \dots y_k. \phi \wedge \psi) \to \delta$$

For suppose not. Since $\mathcal{M} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}(X)$ preserves and reflects the order on subobjects of $[\vec{x} \mid \top]$, there must exist $\langle \mathbf{M}, [\vec{n}] \rangle$ such that $\mathbf{M} \models \exists y_1 \dots y_k. (\phi \wedge \psi)(\vec{n}/\vec{x})$ and $\mathbf{M} \models \neg\delta(\vec{n}/\vec{x})$. Choose $m'_1 \dots m'_k$ in $\mathbb{N}$ such that $\mathbf{M} \models (\phi \wedge \psi)(\vec{m'}/\vec{y}, \vec{n}/\vec{x})$. Then by Lemma 3.1.2.1 there is a model $\mathbf{N}$ in X and an isomorphism $\mathrm{f} : \mathbf{M} \to \mathbf{N}$ such that $f([\vec{m'}]) = [\vec{m}]$. But then $\mathbf{N} \models (\phi \wedge \psi)(\vec{m}/\vec{y}, \mathrm{f}(\vec{n})/\vec{x})$ and $\mathbf{N} \not\models \delta(\mathrm{f}(\vec{n})/\vec{x})$, contrary to assumption. $\dashv$

Lemma 3.1.2.4 *Let a formula-in-context $[\vec{x} \mid \phi]$ of $\mathcal{L}_{\mathbb{T}}$ be given, and suppose we have a basic open $V_{[\vec{x} \mid \phi \wedge \psi(m_1 \dots m_k/y_1 \dots y_k)]} \subseteq E_{[\vec{x} \mid \phi]}$, where we assume as usual that $m_i \neq m_j$ for $1 \leq i \neq j \leq k$. Then there exists a principal ideal I on $[\vec{x} \mid \phi]$ which is the least ideal covering $V_{[\vec{x} \mid \phi \wedge \psi(m_1 \dots m_k/y_1 \dots y_k)]}$, i.e. such that*

$$V_{[\vec{x} \mid \phi \wedge \psi(\vec{m}/\vec{y})]} \subseteq \widehat{\mathcal{M}_{[\vec{x} \mid \phi]}}(I) = \bigcup_{[\vec{x} \mid \xi] \in I} E_{[\vec{x} \mid \xi]} \subseteq E_{[\vec{x} \mid \phi]}$$

Proof We show that the principal ideal I in $\mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}([\vec{x} \mid \phi])$ generated by $(\exists y_1 \dots y_k. \phi \wedge \psi)$ is the least ideal covering $V_{[\vec{x} \mid \phi \wedge \psi(\vec{m}/\vec{y})]}$. First, clearly

$$V_{[\vec{x} \mid \phi \wedge \psi(m_1 \dots m_k/y_1 \dots y_k)]} \subseteq E_{[\vec{x} \mid \exists y_1 \dots y_k. \phi \wedge \psi]} = \widehat{\mathcal{M}_{[\vec{x} \mid \phi]}}(I)$$

Let J be an ideal in $\mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}([\vec{x} \mid \phi])}$ such that

$$V_{[\vec{x} \mid \phi \wedge \psi(\vec{m}/\vec{y})]} \subseteq \widehat{\mathcal{M}_{[\vec{x} \mid \phi]}}(J) = \bigcup_{[\vec{x} \mid \xi] \in J} E_{[\vec{x} \mid \xi]} \subseteq E_{[\vec{x} \mid \phi]}$$

Then

$$\widehat{\mathcal{M}_{[\vec{x} \mid \phi]}}(I) = E_{[\vec{x} \mid \exists y_1 \dots y_k. \phi \wedge \psi]} \subseteq \bigcup_{[\vec{x} \mid \xi] \in J} E_{[\vec{x} \mid \xi]} = \widehat{\mathcal{M}_{[\vec{x} \mid \phi]}}(J)$$

by the same reasoning as above. Namely, if we assume that $\widehat{\mathcal{M}_{[\vec{x} \mid \phi]}}(I) \nsubseteq \widehat{\mathcal{M}_{[\vec{x} \mid \phi]}}(J)$, then we can choose a model $\mathbf{M} \in X$ and elements $[\vec{n}] \in \mathbf{M}$ such that

$$\mathbf{M} \models \exists y_1 \dots y_k. (\phi \wedge \psi)(\vec{n}/\vec{x})$$

and

$$\mathbf{M} \vDash \neg\xi(\vec{n}/\vec{x})$$

for all $[\vec{x} \,|\, \xi] \in J$. Next, we can choose $m'_1, \ldots, m'_k \in \mathbf{M}$ such that

$$\mathbf{M} \vDash (\phi \wedge \psi)(\vec{n}/\vec{x}, m'_1, \ldots, m'_k/y_1, \ldots, y_k)$$

But then by Lemma 3.1.2.1 there is a model $\mathbf{N} \in X$ and an isomorphism $\mathrm{f} : \mathbf{M} \to \mathbf{N}$ such that $\mathrm{f}\left([\vec{m'}]\right) = [\vec{m}]$. Whence

$$\mathbf{N} \vDash (\phi \wedge \psi)(\mathrm{f}\,(\vec{n})/\vec{x}, \vec{m}/\vec{y})$$

and

$$\mathbf{N} \vDash \neg\xi(\mathrm{f}\,(\vec{n})/\vec{x})$$

for all $[\vec{x} \,|\, \xi] \in J$, contradicting that

$$V_{[\vec{x} \,|\, \phi \wedge \psi(\vec{m}/\vec{y})]} \subseteq \widehat{\mathcal{M}_{[\vec{x} \,|\, \phi]}}(J) = \bigcup_{[\vec{x} \,|\, \xi] \in J} E_{[\vec{x} \,|\, \xi]}$$

and thus establishing our claim. ⊣

Corollary 3.1.2.5 *Let a formula-in-context $[\vec{x} \,|\, \phi]$ of $\mathcal{L}_{\mathbb{T}}$ be given, and suppose we have a subobject—that is, an open subset—$U \subseteq E_{[\vec{x} \,|\, \phi]}$. Then there exists a least ideal I on $[\vec{x} \,|\, \phi]$ covering U.*

Proof Write U as a union of basic opens

$$U = \bigcup_{\alpha \in \mathfrak{A}} V_{[\vec{x} \,|\, \phi \wedge \psi(\vec{m}/\vec{y})]_\alpha}$$

Supose J is an ideal in $\mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}([\vec{x} \,|\, \phi])$ such that $U \subseteq \widehat{\mathcal{M}_{[\vec{x} \,|\, \phi]}}(J)$. Then, by Lemma 3.1.2.4, J must contain the pricipal ideal generated by

$$[\vec{x} \,|\, \exists y_1 \ldots y_k.\, \phi \wedge \psi]_\alpha$$

for all $\alpha \in \mathfrak{A}$. Thus the ideal generated by

$$\left\{[\vec{x} \,|\, \exists y_1 \ldots y_k.\, \phi \wedge \psi]_\alpha \;\middle|\; \alpha \in \mathfrak{A}\right\}$$

in $\mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}([\vec{x} \,|\, \phi])$ is the least ideal covering U. ⊣

Definition 3.1.2.6 For a formula-in-context $[\vec{x}\,|\,\phi]$ of $\mathcal{L}_{\mathbb{T}}$ a subobject $U \subseteq E_{[\vec{x}\,|\,\phi]}$ in $\mathrm{Sh}\,(X)$, we denote the least ideal on $[\vec{x}\,|\,\phi]$ covering U by I_U. Its corresponding subobject of $E_{[\vec{x}\,|\,\phi]}$ in $\mathrm{Sh}\,(X)$ we refer to as the *closure* of U and denote by $\overline{U}$, so that $\overline{U} = \widehat{\mathcal{M}}(I_U)$ in $\mathrm{Sub}_{\mathrm{Sh}(X)}\left(E_{[\vec{x}\,|\,\phi]}\right)$.

Note from the proof of Corollary 3.1.2.5 that for a basic open subset

$$V_{[\vec{x}\,|\,\phi\wedge\psi(m_1,\ldots,m_k/y_1,\ldots,y_k)]} \subseteq E_{[\vec{x}\,|\,\phi]}$$

we have that

$$\overline{V_{[\vec{x}\,|\,\phi\wedge\psi(m_1,\ldots,m_k/y_1,\ldots,y_k)]}} = E_{[\vec{x}\,|\,\exists y_1\ldots y_k.\phi\wedge\psi]}$$

and that for a union of basic opens $U = \bigcup_{\alpha\in\mathfrak{A}} V_\alpha \subseteq E_{[\vec{x}\,|\,\phi]}$ we have that

$$\overline{\bigcup_{\alpha\in\mathfrak{A}} V_\alpha} = \bigcup_{\alpha\in\mathfrak{A}} \overline{V_\alpha}$$

Lemma 3.1.2.7 *For an arrow $[\vec{x},\vec{y}\,|\,\sigma] : [\vec{x}\,|\,\phi] \longrightarrow [\vec{y}\,|\,\psi]$ in $\mathcal{C}_{\mathbb{T}}$ and a subobject $U \subseteq E_{[\vec{y}\,|\,\psi]} = \mathcal{M}([\vec{y}\,|\,\psi])$ in $\mathrm{Sh}\,(X)$, we have that $f_\sigma^*(\overline{U}) = \overline{f_\sigma^*(U)}$ in $\mathrm{Sub}_{\mathrm{Sh}(X)}\left(E_{[\vec{x}\,|\,\phi]}\right)$, that is*

$$\begin{array}{ccc}
\overline{f_\sigma^*(U)} \cong f_\sigma^*(\overline{U}) & \longrightarrow & \overline{U} \\
\big\downarrow & & \big\downarrow \\
E_{[\vec{x}\,|\,\phi]} = \mathcal{M}([\vec{x}\,|\,\phi]) & \xrightarrow[f_\sigma = \mathcal{M}(\sigma)]{} & \mathcal{M}([\vec{y}\,|\,\psi]) = E_{[\vec{y}\,|\,\psi]}
\end{array}$$

PROOF Suppose U is a basic open, thus of the form

$$U = V_{[\vec{y}\,|\,\psi\wedge\xi(n_1,\ldots,n_k/z_1,\ldots,z_k)]}$$

Then $\overline{U} = E_{[\vec{y}\,|\,\exists z_1,\ldots,z_k.\psi\wedge\xi]}$. In $\mathcal{C}_{\mathbb{T}}$ the pullback along σ of the subobject

$$[\vec{y}\,|\,\exists z_1,\ldots,z_k.\,\psi\wedge\xi] \rightarrowtail [\vec{y}\,|\,\psi]$$

is the subobject

$$[\vec{x}\,|\,\exists\vec{y}.\,\sigma\wedge\exists z_1,\ldots,z_k.\,\xi] \rightarrowtail [\vec{x}\,|\,\phi]$$

and since $\mathcal{M}$ preserves pullbacks,

$$\mathrm{f}^*_\sigma\left(\overline{U}\right) = E_{[\vec{x}\,|\,\exists\vec{y}.\sigma\wedge\exists z_1,\dots,z_k.\xi]}$$

in $\mathrm{Sub}_{\mathrm{Sh}(X)}\left(E_{[\vec{x}\,|\,\phi]}\right)$. On the other hand, the pullback along f^*_σ of U is the subset

$$\begin{aligned}
&\mathrm{f}^*_\sigma(U) && (3.1)\\
&\quad=\left\{\langle\mathbf{M},\vec{m}\rangle\in E_{[\vec{x}\,|\,\phi]}\;\middle|\;\exists\vec{p}\in\mathbf{M}.\,\mathbf{M}\vDash\xi(\vec{n}/\vec{z},\vec{p}/\vec{y})\wedge\sigma(\vec{m}/\vec{x},\vec{p}/\vec{y})\right\} && (3.2)\\
&\quad=\left\{\langle\mathbf{M},\vec{m}\rangle\in E_{[\vec{x}\,|\,\phi]}\;\middle|\;\mathbf{M}\vDash\exists\vec{y}.\,\xi(\vec{n}/\vec{z})\wedge\sigma(\vec{m}/\vec{x})\right\} && (3.3)\\
&\quad=V_{[\vec{x}\,|\,\phi\wedge(\exists\vec{y}.(\sigma\wedge\xi)(n_1,\dots,n_k/z_1,\dots,z_k))]} && (3.4)
\end{aligned}$$

which is a basic open, the closure of which is

$$\begin{aligned}
\overline{f^*_\sigma(U)} &= \overline{V_{[\vec{x}\,|\,\phi\wedge(\exists\vec{y}.(\sigma\wedge\xi)(n_1,\dots,n_k/z_1,\dots,z_k))]}}\\
&= E_{[\vec{x}\,|\,\exists z_1,\dots,z_k.\exists\vec{y}.\sigma\wedge\xi]}\\
&= E_{[\vec{x}\,|\,\exists\vec{y}.\sigma\wedge\exists z_1,\dots,z_k.\xi]}\\
&= f^*_\sigma(\overline{U})
\end{aligned}$$

Now, for an arbitrary open $U\subseteq E_{[\vec{x}\,|\,\phi]}$, we can write U as a union of basic opens $U=\bigcup_{\alpha\in\mathfrak{A}}V_\alpha$. Then, using, as observed in rows 3.1 to 3.4 above, that the pullback of a basic open along a definable morphism is again a basic open; that the closure of a union of basic opens is the union of the closure of the basic opens (row 3.6 and 3.9 below); that f^*_σ preserves unions (3.7 and 3.10); and that closure is stable under pullback for basic opens (3.8), we have that

$$
\begin{aligned}
f_\sigma^*(\overline{U}) &= f_\sigma^*(\overline{\bigcup_{\alpha\in\mathfrak{A}} V_\alpha}) && (3.5)\\
&= f_\sigma^*(\bigcup_{\alpha\in\mathfrak{A}} \overline{V_\alpha}) && (3.6)\\
&= \bigcup_{\alpha\in\mathfrak{A}} f_\sigma^*(\overline{V_\alpha}) && (3.7)\\
&= \bigcup_{\alpha\in\mathfrak{A}} \overline{f_\sigma^*(V_\alpha)} && (3.8)\\
&= \overline{\bigcup_{\alpha\in\mathfrak{A}} f_\sigma^*(V_\alpha)} && (3.9)\\
&= \overline{f_\sigma^*(\bigcup_{\alpha\in\mathfrak{A}} V_\alpha)} && (3.10)\\
&= \overline{f_\sigma^*(U)} && (3.11)
\end{aligned}
$$

in $\mathrm{Sub}_{\mathrm{Sh}(X)}\left(E_{[\vec{x}\,|\,\phi]}\right)$ also for general U. ⊣

Proposition 3.1.2.8 *The geometric morphism* $m : \mathrm{Sh}(X) \twoheadrightarrow \mathrm{Sh}(\mathcal{C}_\mathbb{T})$ *is open.*

Proof We wrap up the construction of a internal left adjoint to $\tilde{\tau}$,

$$\Omega_{\mathrm{Sh}(\mathcal{C}_\mathbb{T})} \underset{\tilde{\tau}}{\overset{\gamma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} m_*(\Omega_{\mathrm{Sh}(X)})$$

The following commutative square displays the action of $\tilde{\tau}$ as a natural transformation between contravariant functors on $\mathcal{C}_{\mathbb{T}}$:

$$
\begin{array}{ccc}
\Omega_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}(-) & \xrightarrow{\quad\tilde{\tau}\quad} & m_*\Omega_{\mathrm{Sh}(X)}(-) \\
\updownarrow\cong & & \updownarrow\cong \\
\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}\big(y(-),\Omega_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}\big) & & \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}\big(y(-),m_*\Omega_{\mathrm{Sh}(X)}\big) \\
 & & \updownarrow\cong \\
\updownarrow\cong & & \mathrm{Hom}_{\mathrm{Sh}(X)}\big(m^*\circ y(-),\Omega_{\mathrm{Sh}(X)}\big) \\
 & & \updownarrow \\
 & & \mathrm{Hom}_{\mathrm{Sh}(X)}\big(\mathcal{M}(-),\Omega_{\mathrm{Sh}(X)}\big) \\
 & & \updownarrow\cong \\
\mathrm{Sub}_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}\left(y(-)\right) & \xrightarrow[\quad\hat{\tau}\quad]{} & \mathrm{Sub}_{\mathrm{Sh}(X)}\big(E_{(-)}\big)
\end{array}
$$

$$
\begin{array}{ccc}
S & & m^*(S) = \widehat{\mathcal{M}_{[\vec{x}\,|\,\phi]}}(I_S) \\
\downarrow & \overset{\hat{\tau}_{[\vec{x}\,|\,\phi]}}{\longmapsto} & \downarrow \\
y([\vec{x}\,|\,\phi]) & & m^*(y([\vec{x}\,|\,\phi])) = E_{[\vec{x}\,|\,\phi]}
\end{array}
$$

Now, by Corollary 3.1.2.5 and Lemma 3.1.2.7 together with the fact that m^* reflects pullbacks, we have a natural transformation

$$
\begin{array}{ccc}
\mathrm{Sub}_{\mathrm{Sh}(X)}\big(E_{(-)}\big) & \xrightarrow{\quad\hat{\gamma}\quad} & \mathrm{Sub}_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}\left(y(-)\right) \\
 & & \\
U \subseteq E_{[\vec{x}\,|\,\phi]} & \longmapsto & Y_{[\vec{x}\,|\,\phi]}(I_U) \rightarrowtail y([\vec{x}\,|\,\phi])
\end{array}
$$

where I_U is the least ideal on $[\vec{x}\,|\,\phi]$ covering U as per Corollary 3.1.2.5 and Definition 3.1.2.6. Finally, since

$$Y_{[\vec{x}\,|\,\phi]}(I_U) \leq S \Leftrightarrow U \leq \widehat{m^*}(S)$$

$\hat{\gamma}$ is an internal left adjoint to $\hat{\tau}$, and so the corresponding natural transformation $\gamma : m_*(\Omega_{\mathrm{Sh}(X)}) \longrightarrow \Omega_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})}$ is an internal left adjoint to $\tilde{\tau} : \Omega_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})} \longrightarrow \Omega_{\mathrm{Sh}(X)})$. $\dashv$

3.2 Equivariant Sheaves on the Groupoid of Models

To the space $X_{\mathbb{T}}$ of models, we add the space $G_{\mathbb{T}}$ of $\mathbb{T}$-isomorphisms to obtain a topological groupoid $\mathbb{G}_{\mathbb{T}}$. We show that the topos of coherent sheaves $\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ on $\mathbb{T}$ is equivalent to the topos of equivariant sheaves on $\mathbb{G}_{\mathbb{T}}$. Again inspired by the similar result of [5] (and by the fundamental representation theorem of Joyal and Tierney [11], also presented in [10, C5.2]), we proceed by a different argument. The functor from (localic) groupoids to topoi which assigns a groupoid to its topos of equivariant sheaves was studied in [21] and [23].

3.2.1 The Groupoid of Models

Fix a theory $\mathbb{T}$ with a saturated set of countable models. Recall the space X of $\mathbb{T}$-models from Section 3.1.1. Let G be the set of $\mathbb{T}$-isomorphisms between elements of X, with source/domain (s), target/codomain (t), identity (e), and inverse functions (i):

$$\overset{\mathrm{i}}{\circlearrowleft} G \;\underset{\mathrm{t}}{\overset{\mathrm{s}}{\rightrightarrows}}\; X, \qquad G \xleftarrow{\;\mathrm{e}\;} X$$

Define a topology $\mathfrak{T}$ on G generated by a basis $\mathfrak{B}$ obtained by taking as a subbasis all the subsets of G which are of the form $\mathrm{s}^{-1}(V)$ or $\mathrm{t}^{-1}(V)$, for V a basic open of X, or of the form $\{f \in G \mid f([n]) = [m]\}$ for $n, m \in \mathbb{N}$. Thus a basic open of G is given by three data, and we shall write basic opens of G displaying these data, in the form

$$V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y})}(\vec{i},\vec{j}) = \left\{ f \in G \;\middle|\; \mathrm{s}\,(f) \vDash \phi(\vec{m}/\vec{x}), \mathrm{t}\,(f) \vDash \psi(\vec{n}/\vec{y}), f([\vec{i}]) = [\vec{j}] \right\}$$

where the source and target data are governed by the same conventions as for basic opens of X, with the added convention that $\vec{x}$ and $\vec{y}$ has no variable in common. The vectors of natural numbers $\vec{i}$ and $\vec{j}$ must of course have the same length. We can always assume, without loss of generality, that no number constant occurs more than once in the tuple $\vec{i}$ nor more than once in the tuple $\vec{j}$, since e.g.

$$V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y})}(\langle i, i\rangle, \langle j, k\rangle) = V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y}) \wedge j=k}(i, j)$$

and

$$V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y})}(\langle i,j\rangle, \langle k,k\rangle) = V^{\phi(\vec{m}/\vec{x})\wedge i=j}_{\psi(\vec{n}/\vec{y})}(i,k)$$

The space of models and isomorphisms now forms a topological groupoid:

Lemma 3.2.1.1 *With the logical topology on X and the topology $\mathfrak{T}$ on G the source, target, inverse, and identity functions are continous, as is the composition* (c) *function:*

$$G \times_X G \xrightarrow{\quad \mathrm{c} \quad} G$$

Proof The source and target functions $\mathrm{s}, \mathrm{t} : G \rightrightarrows X$ are continuous by definition of $\mathfrak{T}$. Let a basic open $V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y})}(\vec{i},\vec{j}) \in \mathfrak{B}$ be given.

$$\mathrm{i}^{-1}\left(V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y})}(\vec{i},\vec{j})\right) = V^{\psi(\vec{n}/\vec{y})}_{\phi(\vec{m}/\vec{x})}(\vec{j},\vec{i})$$

and

$$\mathrm{e}^{-1}\left(V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y})}(\vec{i},\vec{j})\right) = U_{(\phi\wedge\psi)(\vec{m}/\vec{x},\vec{n}/\vec{y})\wedge\vec{i}=\vec{j}}$$

so the inverse and identity functions are also continuous. and so only the composition function remains. Let basic open $V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y})}(\vec{i},\vec{j})$ be given and suppose we have two isomorphisms

$$\mathbf{L} \xrightarrow{f} \mathbf{M} \xrightarrow{g} \mathbf{N}$$

such that $g \circ f \in V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y})}(\vec{i},\vec{j})$. Choose $\vec{k} \in f([\vec{i}])$. Then

$$\langle g,f\rangle \in V^{\top}_{\psi(\vec{n}/\vec{y})}(\vec{k},\vec{j}) \times_X V^{\phi(\vec{m}/\vec{x})}_{\top}(\vec{i},\vec{k})$$

and

$$\mathrm{c}\left(V^{\top}_{\psi(\vec{n}/\vec{y})}(\vec{k},\vec{j}) \times_X V^{\phi(\vec{m}/\vec{x})}_{\top}(\vec{i},\vec{k})\right) \subseteq V^{\phi(\vec{m}/\vec{x})}_{\psi(\vec{n}/\vec{y})}(\vec{i},\vec{j})$$

so the composition function is continuous as well. ⊣

Lemma 3.2.1.2 $\mathbb{G}_{\mathbb{T}}$ *is an open groupoid, that is, the source and target maps* $s,t : G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}}$ *are both open.*

Proof Suffice to show that, say, the source map is open. Let

$$V = V^{\phi(\vec{m})}_{\psi(\vec{n})}(\vec{i}, \vec{j}) \subseteq G_{\mathbb{T}}$$

be given, and suppose $\mathbf{M} \in s(V)$. Assume, without loss of generality, that no number constant occurs more than once in the tuple $\vec{i}$ nor more than once in the tuple $\vec{j}$. Let an isomorphism $f : \mathbf{M} \to \mathbf{N}$ in V be given. Choose, for each number constant n in the tuple $\vec{n}$, a number k such that $f([k]) = [n]$ in such a way that if for some j in the tuple $\vec{j}$ we have $n = j$ then $k = i$. Now $\mathbf{M} \in U_{\phi(\vec{m}) \wedge \psi(\vec{k})}$. And if $\mathbf{K} \in U_{\phi(\vec{m}) \wedge \psi(\vec{k})}$, then we can find a $\mathbb{T}$-model $\mathbf{L}$ and isomorphism $g : \mathbf{K} \to \mathbf{L}$ such that $g([\vec{k}]) = [\vec{n}]$ and $g([\vec{i}]) = [\vec{j}]$, by Lemma 3.1.2.1. Hence $\mathbf{K} \in s(V)$. ⊣

Remark 3.2.1.3 In general, we use blackboard bold letters $\mathbb{G}$, $\mathbb{H}$, etc. to refer to (topological) groupoids, with G_1, H_1 denoting the set (space) of arrows, and G_0, H_0 the set (space) of objects. However, a groupoid of models for a theory $\mathbb{T}$ will be denoted $\mathbb{G}_{\mathbb{T}}$ with $G_{\mathbb{T}}$ denoting the space of arrows, and $X_{\mathbb{T}}$ the space of objects. The subscript may be dropped when the theory is fixed, as it is in the current section. We denote the components of a morphism $f : \mathbb{G} \longrightarrow \mathbb{H}$ of (topological) groupoids as $f_1 : G_1 \to H_1$ and $f_0 : G_0 \to H_0$.

We thus refer to the topological groupoid of models of $\mathbb{T}$ as $\mathbb{G}_{\mathbb{T}}$, or simply $\mathbb{G}$ in this section as long as the theory $\mathbb{T}$ is fixed:

$$G \times_X G \xrightarrow{\ \mathrm{c}\ } G \circlearrowleft^{\mathrm{i}} \quad G \underset{\mathrm{t}}{\overset{\mathrm{s}}{\rightrightarrows}} X, \quad X \xrightarrow{\ \mathrm{e}\ } G$$

The purpose of this section is to describe the topos of coherent sheaves, $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$, on $\mathbb{T}$ in terms of $\mathbb{G}$. To this end, one can proceed from the fact that the geometric morphism $\mathrm{Sh}(X) \twoheadrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ is an open surjection, and therefore descent ([11], see also [22], [10, C5.1]). Such a line of argument is presented in the appendix. In this section, we offer a different proof that $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ is the topos of equivariant sheaves on $\mathbb{G}$, proceeding more directly from the reasoning that established Proposition 3.1.2.8. This argument is less conceptual, but shorter, and it has the advantage of not assuming that $\mathbb{T}$ is countable, or even (modulo the set used to construct the groupoid of models) that it has countable points. That is to say, the argument given in the current and preceding section can be repeated for a theory of arbitrary size by replacing $\mathbb{N}$ as the 'index set' for the set of models $X_{\mathbb{T}}$ by another sufficiently large set (see Remark 3.2.2.11).

3.2.2 Equivariant Sheaves on the Groupoid of Models

Consider the groupoid of $\mathbb{T}$-models $G \rightrightarrows X$. For an object $[\vec{x} \,|\, \phi]$ in $\mathcal{C}_{\mathbb{T}}$, we have the functor $\mathcal{M} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}\,(X)$ assigning $[\vec{x} \,|\, \phi]$ to the etale space $E_{[\vec{x} \,|\, \phi]}$ over X. There is an obvious action 'of application',

$$\theta_{[\vec{x} \,|\, \phi]} : G \times_X E_{[\vec{x} \,|\, \phi]} \longrightarrow E_{[\vec{x} \,|\, \phi]} \tag{3.12}$$

defined by sending a $\mathbb{T}$-isomorphism $f : \mathbf{M} \longrightarrow \mathbf{N}$ and an element $[\vec{n}] \in [\![\vec{x} \,|\, \phi]\!]^{\mathbf{M}}$ to the element $f([\vec{n}]) \in [\![\vec{x} \,|\, \phi]\!]^{\mathbf{N}}$. We shall mostly leave the subscript implicit.

Lemma 3.2.2.1 *The etale space $E_{[\vec{x} \,|\, \phi]}$ together with the function $\theta : G \times_X E_{[\vec{x} \,|\, \phi]} \longrightarrow E_{[\vec{x} \,|\, \phi]}$ is an object of $\mathrm{Sh}_G(X)$.*

Proof We must verify that θ is continuous and satisfies the axioms for being an action. The latter is straightforward, so we do the former. Let $V = V_{[\vec{x} \,|\, \phi \wedge \psi(\vec{m}/\vec{y})]} \subseteq E_{[\vec{x} \,|\, \phi]}$ be given, and suppose $\langle f : \mathbf{M} \to \mathbf{N}, [\vec{n}] \rangle \in \theta^{-1}(V) \subseteq G \times_X E_{[\vec{x} \,|\, \phi]}$. Choose $\vec{k}$ such that $f([\vec{k}]) = [\vec{m}]$. Then

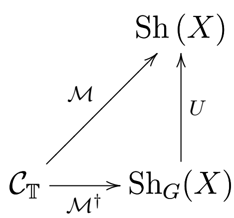
$$\langle f : \mathbf{M} \to \mathbf{N}, [\vec{n}] \rangle \in V_{\top}^{\top}(\vec{k}, \vec{m}) \times_X V_{\phi \wedge \psi(\vec{k}/\vec{y})}$$

and $\theta \left(V_{\top}^{\top}(\vec{k}, \vec{m}) \times_X V_{\phi \wedge \psi(\vec{k}/\vec{y})} \right) \subseteq V$. So θ is continuous. $\dashv$

It is clear that any definable morphism of etale spaces $f_\sigma : E_{[\vec{x} \,|\, \phi]} \longrightarrow E_{[\vec{y} \,|\, \psi]}$, for $\sigma : [\vec{x} \,|\, \phi] \longrightarrow [\vec{y} \,|\, \psi]$ in $\mathcal{C}_{\mathbb{T}}$ commutes with their respective actions of application, and so we have a functor

$$\mathcal{M}^{\dagger} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}_G(X) = \mathrm{Sh}\,(\mathbb{G})$$

Composed with the faithful forgetful functor $U : \mathrm{Sh}_G(X) \longrightarrow \mathrm{Sh}\,(X)$, which is the inverse image part of a geometric morphism $u : \mathrm{Sh}\,(X) \longrightarrow\!\!\!\!\!\!\rightarrow \mathrm{Sh}_G(X)$, we get a commuting triangle:



from which we conclude that $\mathcal{M}^\dagger$ is coherent, and that we have a factorization,

$$\begin{array}{ccc} \mathrm{Sh}\,(X) & & \\ & \overset{u}{\searrow} & \\ m \Big\downarrow & & \mathrm{Sh}_G(X) \\ & \swarrow{\scriptstyle m^\dagger} & \\ \mathrm{Sh}\,(\mathcal{C}_\mathbb{T}) & & \end{array}$$

We show that the geometric morphism $m^\dagger : \mathrm{Sh}_G(X) \twoheadrightarrow \mathrm{Sh}\,(\mathcal{C}_\mathbb{T})$ is an equivalence by showing that $\mathcal{C}_\mathbb{T}$ is a site for $\mathrm{Sh}_G(X)$. Notice first that subobjects of an equivariant sheaf $\langle a : A \to X, \alpha\rangle$ can be thought of, or correspond to, open subsets of A that are closed under the action α. The word "closed" being badly overworked already, we call a subset $S \subseteq A$ that is closed under the action *stable*, and the least stable subset containing S we call the *stabilization* of A. We call the objects and arrows in the image of $\mathcal{M}^\dagger : \mathcal{C}_\mathbb{T} \longrightarrow \mathrm{Sh}_G(X)$ *definable*. From Section 3.1.2, we have the following:

Lemma 3.2.2.2 *The stabilization of a basic open subset* $V_{[\vec{x}\,|\,\phi\wedge\psi(\vec{n}/\vec{y})]} \subseteq E_{[\vec{x}\,|\,\phi]}$ *is the definable subset* $E_{[\vec{x}\,|\,\phi\wedge\exists\vec{y}.\psi]}$.

PROOF This is really a scholium of Lemma 3.1.2.4. Recall that unless otherwise stated, we always assume that the number constants (the n's) in a presentation $V_{[\vec{x}\,|\,\phi\wedge\psi(\vec{n}/\vec{y})]}$ of a basic open are distinct. ⊣

Lemma 3.2.2.3 *Any subobject of a definable object in* $\mathrm{Sh}_G(X)$ *is a join of definable objects. As a consequence, definable objects are compact, and subobjects of definable objects are definable if and only if complemented.*

PROOF Consider a definable object $\langle E_{[\vec{x}\,|\,\phi]} \to X, \theta\rangle$. It follows from Lemma 3.2.2.2 that a stable open subset $S \subseteq E_{[\vec{x}\,|\,\phi]}$ is a union of definable subsets. Since $\mathcal{M}^\dagger : \mathcal{C}_\mathbb{T} \longrightarrow \mathrm{Sh}_G(X)$ is coherent, definable subobjects are complemented. Finally, $m^\dagger : \mathrm{Sh}_G(X) \twoheadrightarrow \mathrm{Sh}\,(\mathcal{C}_\mathbb{T})$ being a surjection means that $\mathcal{M}^\dagger$ reflects covers. It follows that a definable object must be compact, in the sense that any covering family of subobjects contains a finite covering subfamily. A complemented subobject of a compact object is again compact, and so a complemented subobject of a definable object must be a finite join of definable subobjects, which means it is itself definable. ⊣

Lemma 3.2.2.4 $\mathcal{M}^\dagger : \mathcal{C}_\mathbb{T} \longrightarrow \mathrm{Sh}_G(X)$ *is full and faithful.*

PROOF Faithful follows from $m^\dagger : \mathrm{Sh}_G(X) \twoheadrightarrow \mathrm{Sh}(\mathcal{C}_\mathbb{T})$ being a surjection. Full follows from Lemma 3.2.2.3 as follows: since definable objects are decidable, graphs of arrows between definable objects are complemented. Therefore, such graphs are definable. $\dashv$

The following sequence of lemmas serve to establish that the definable objects form a generating set for $\mathrm{Sh}_G(X)$.

Lemma 3.2.2.5 *Let a basic open $U_{\phi(\vec{n}/\vec{x})} \subseteq X$ be given. As always, we assume that the n's are distinct. Then there exists a (continuous) section $s : U_{\phi(\vec{n}/\vec{x})} \to E_{[\vec{x}\,|\,\phi]}$ such that $E_{[\vec{x}\,|\,\phi]}$ is the stabilization of $s(U_{\phi(\vec{n}/\vec{x})})$ (with respect to the action of application θ).*

PROOF The section $s : U_{\phi(\vec{n}/\vec{x})} \to E_{[\vec{x}\,|\,\phi]}$ is defined by

$$\mathbf{M} \mapsto \langle \mathbf{M}, [\vec{n}] \rangle \in E_{[\vec{x}\,|\,\phi]}.$$

s is continuous since $s(U_{\phi(\vec{n}/\vec{x})})$ is the basic open $V_{[\vec{x}\,|\,\phi\wedge\vec{x}=\vec{n}]} \subseteq E_{[\vec{x}\,|\,\phi]}$. By Lemma 3.2.2.2, $E_{[\vec{x}\,|\,\phi]}$ is the stabilization of $s(U_{\phi(\vec{n}/\vec{x})}) = V_{[\vec{x}\,|\,\phi\wedge\vec{x}=\vec{n}]}$. $\dashv$

Recall that the groupoid $\mathbb{G}$ is open (Lemma 3.2.1.2). We record the following consequences thereof [21]:

Lemma 3.2.2.6 *Let $\langle a : A \to X, \alpha \rangle$ be an object of $\mathrm{Sh}_G(X)$. Then the projection $\pi_2 : G \times_X A \longrightarrow A$ is an open map.*

PROOF By Lemma 3.2.1.2, since the projection is the pullback of the open source map,

$$\begin{array}{ccc} G \times_X A & \xrightarrow{\ \pi_2\ } & A \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle a} \\ G & \xrightarrow[\ s\ ]{} & X \end{array}$$

$\dashv$

Corollary 3.2.2.7 *For any object $\langle a : A \to X, \alpha \rangle$ of $\mathrm{Sh}_G(X)$, the action $\alpha : G \times_X A \longrightarrow A$ is an open map.*

Proof Let a basic open $V \times_X U \subseteq G \times_X A$ be given. Since $i : G \to G$ is a homeomorphism, $V^{-1} := i(V)$ is open, and so

$$\begin{aligned}
\alpha(V \times_X U) &= \{y \in A \mid \exists g \in V, x \in U.\, s(g) = a(x) \wedge \alpha(g, x) = y\} \\
&= \left\{y \in A \;\middle|\; \exists g^{-1} \in V^{-1}, x \in U.\, t(g^{-1}) = a(x) \wedge \alpha(g^{-1}, y) = x\right\} \\
&= \pi_2(\alpha^{-1}(U) \cap V^{-1} \times_X A)
\end{aligned}$$

is open by Lemma 3.2.2.6. (Alternatively, the result follows directly from Lemma 3.2.1.2 by a simple diagram chase, using that the action is a surjection.) $\dashv$

Lemma 3.2.2.8 *Let $\langle a : A \to X, \alpha\rangle$ be an equivariant sheaf in $\mathrm{Sh}_G(X)$, and let $x \in A$. Then there exists a (continuous) section $s : U_{\phi(\vec{n}/\vec{y})} \longrightarrow A$ such that x is in the image of s and for any $f : \mathbf{M} \longrightarrow \mathbf{N}$ in G, if $\mathbf{M} \in U_{\phi(\vec{n}/\vec{y})}$ and $f([\vec{n}]) = [\vec{n}]$, then $\alpha(f, s(\mathbf{M})) = s(\mathbf{N})$.*

Proof Choose a section $s : U_{\psi(\vec{m})} \to A$ such that $x \in s(U_{\psi(\vec{m})})$. Pull the open set $s(U_{\psi(\vec{m})})$ back along the continuous action,

$$\begin{array}{ccc}
W & \longrightarrow & s(U_{\psi(\vec{m})}) \\
{\scriptstyle\subseteq}\downarrow & \lrcorner & \downarrow{\scriptstyle\subseteq} \\
G \times_X A & \xrightarrow[\alpha]{} & A
\end{array}$$

to obtain an open neighborhood $W \subseteq G \times_X A$ around $\langle 1_a(x), x\rangle$. Since W is open, there exists an open box

$$V = V^{\vartheta(\vec{k})}_{\varrho(l)}(\vec{i}, \vec{j}) \times_X t(U_{\varphi(\vec{p})}) \subseteq W$$

containing $\langle 1_a(x), x\rangle$. Notice that because $\langle 1_a(x), x\rangle \in V$, it must be the case that $a(x) \vDash \vec{i} = \vec{j}$. For the same reason, the open set

$$U = U_{\vartheta(\vec{k}) \wedge \vec{i} = \vec{j} \wedge \varrho(l) \wedge \varphi(\vec{p}) \wedge \psi(\vec{m})} \subseteq U_{\psi(\vec{m})} \subseteq X$$

contains $a(x)$. We claim that

$$s : U = U_{\vartheta(\vec{k}) \wedge \vec{i} = \vec{j} \wedge \varrho(l) \wedge \varphi(\vec{p}) \wedge \psi(\vec{m})} \to A$$

has the required property. For assume that $\mathbf{M} \in U$ and $f : \mathbf{M} \to \mathbf{N}$ is an isomorphism such that $f([\vec{k}] * [\vec{i}] * [\vec{j}] * [\vec{l}] * [\vec{p}] * [\vec{m}]) = [\vec{k}] * [\vec{i}] * [\vec{j}] * [\vec{l}] *$

$[\vec{p}] * [\vec{m}]$. First, this implies that $\mathbf{N} \in U$. Second, $\langle 1_{\mathbf{M}}, t(\mathbf{M})\rangle \in V \subseteq W$, so $\alpha(1_{\mathbf{M}}, t(\mathbf{M})) = t(\mathbf{M}) \in s(U_{\psi(\vec{m})})$, whence $t(\mathbf{M}) = s(\mathbf{M})$. Finally, $\langle f, t(\mathbf{M})\rangle \in V \subseteq W$, so $\alpha(f, t(\mathbf{M})) = \alpha(f, s(\mathbf{M})) \in s(U_{\psi(\vec{m})})$. But $a(\alpha(f, s(\mathbf{M}))) = \mathbf{N} = a(s(\mathbf{N}))$, and a restricted to $s(U_{\psi(\vec{m})})$ is 1-1, so $\alpha(f, s(\mathbf{M})) = s(N)$. $\dashv$

Lemma 3.2.2.9 *The set of definable objects in* $\mathrm{Sh}_G(X)$ *is a generating set.*

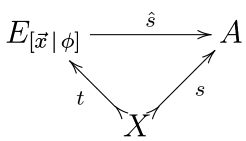
Proof Let $\mathcal{A} = \langle a : A \to X, \alpha\rangle$ be an equivariant sheaf in $\mathrm{Sh}_G(X)$. We must show that the arrows with definable domain into $\mathcal{A}$ are jointly epimorphic. Let $x \in A$. We construct a morphism from a definable equivariant sheaf which has x in its image. By Lemma 3.2.2.8, we can choose a section $s : U_{\phi(\vec{n}/\vec{x})} \to A$ such that x is in the image of s, and for any $g : \mathbf{K} \to \mathbf{L}$ in G, if $g([\vec{n}]) = [\vec{n}]$ and $\mathbf{K} \in U_{\phi(\vec{n})}$, then $\alpha(g, s(\mathbf{K})) = s(\mathbf{L})$. We can assume without loss that all the n's are distinct. By Lemma 3.2.2.2, we have a section $t : U_{\phi(\vec{n}/\vec{x})} \to E_{[\vec{x}\,|\,\phi]}$, defined by $\mathbf{K} \mapsto \langle \mathbf{K}, [\vec{n}]\rangle$, such that $E_{[\vec{x}\,|\,\phi]}$ is the stabilization of the image $t(U_{\phi(\vec{n}/\vec{x})}) = V_{[\vec{x}\,|\,\phi\wedge\vec{x}=\vec{n}]}$. We define a function $\hat{s} : E_{[\vec{x}\,|\,\phi]} \to A$ over X as follows: for $\langle \mathbf{L}, [\vec{m}]\rangle \in E_{[\vec{x}\,|\,\phi]}$, we choose a $g : \mathbf{K} \to \mathbf{L}$ in G such that $\mathbf{K} \in U_{\phi(\vec{n}/\vec{x})}$ and $\theta(g, t(\mathbf{K})) = \theta(g, \langle \mathbf{K}, [\vec{n}]\rangle) = \langle \mathbf{L}, [\vec{m}]\rangle$. Set

$$\hat{s}(\langle \mathbf{L}, [\vec{m}]\rangle) = \alpha(g, s(\mathbf{K})).$$

We must verify that $\hat{s}$ is well-defined. Suppose $h : \mathbf{M} \to \mathbf{L}$ is an isomorphism in G such that $\mathbf{M} \in U_{\phi(\vec{n}/\vec{x})}$ and $\theta(h, t(\mathbf{M})) = \theta(h, \langle \mathbf{M}, [\vec{n}]\rangle) = \langle \mathbf{L}, [\vec{m}]\rangle$. Then $h^{-1} \circ g : \mathbf{K} \to \mathbf{L} \to \mathbf{M}$ is such that $h^{-1} \circ g([\vec{n}]) = [\vec{n}]$, and so $\alpha(h^{-1} \circ g, s(\mathbf{K})) = s(\mathbf{M})$. Thus

$$\begin{aligned} \alpha(h, s(\mathbf{M})) &= \alpha(h, \alpha(h^{-1} \circ g, s(\mathbf{K}))) \\ &= \alpha(g, s(\mathbf{K})) \end{aligned}$$

so $\hat{s}$ is indeed well-defined, and we have a commuting triangle (of functions),



By construction, $\hat{s}$ commutes with the actions,

$$\begin{array}{ccc} G \times_X E_{[\vec{x}\,|\,\phi]} & \xrightarrow{\theta} & E_{[\vec{x}\,|\,\phi]} \\ {\scriptstyle 1_G \times_X \hat{s}}\downarrow & & \downarrow{\scriptstyle \hat{s}} \\ G \times_X A & \xrightarrow[\alpha]{} & A \end{array}$$

and so it remains to verify that $\hat{s}$ is continuous. Given $\hat{s}(\langle \mathbf{L}, [\vec{m}] \rangle) \in A$ and an open neighborhood $U \subseteq A$ around $\hat{s}(\langle \mathbf{L}, [\vec{m}] \rangle)$, we must find an open neighborhood around $\langle \mathbf{L}, [\vec{m}] \rangle \in E_{[\vec{x}\,|\,\phi]}$ which $\hat{s}$ sends into U. First, pull back along the continuous action:

$$\begin{array}{ccc} \alpha^{-1}(U) & \longrightarrow & U \\ {\scriptstyle \subseteq}\downarrow & \lrcorner & \downarrow{\scriptstyle \subseteq} \\ G \times_X A & \xrightarrow[\alpha]{} & A \end{array}$$

Set

$$W := \alpha^{-1}(U) \cap (G \times_X s(U_{\phi(\vec{n}/\vec{x})})) \subseteq G \times_X A.$$

Choose $g : \mathbf{K} \to \mathbf{L}$ such that $\mathbf{K} \in U_{\phi(\vec{n}/\vec{x})}$ and $\theta(g, t(\mathbf{K})) = \langle \mathbf{L}, [\vec{m}] \rangle$. Notice that $\langle g, s(\mathbf{K}) \rangle \in W$. Now, we have a homeomorphism $t \circ a \restriction_{s(U_{\phi(\vec{n}/\vec{x})})} : s(U_{\phi(\vec{n}/\vec{x})}) \to t(U_{\phi(\vec{n}/\vec{x})})$, and we can take the image

$$\begin{array}{ccc} W & \twoheadrightarrow & W' \\ {\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle \subseteq} \\ G \times_X s(U_{\phi(\vec{n}/\vec{x})}) & \xrightarrow[1_G \times_X t \circ a \restriction_{s(U_{\phi(\vec{n}/\vec{x})})}]{\cong} & G \times_X t(U_{\phi(\vec{n}/\vec{x})}) \end{array}$$

to obtain $W' := 1_G \times_X (t \circ a \restriction_{s(U_{\phi(\vec{n}/\vec{x})})})(W)$, which is, then, an open neighborhood of $\langle g, t(\mathbf{K}) \rangle$. By Corollary 3.2.2.7, we have that $\theta(W')$ is an open neighborhood of $\langle \mathbf{L}, [m] \rangle = \theta(g, t(\mathbf{K}))$. And, finally, we see that $\hat{s}(\theta(W')) = \alpha(W) \subseteq U$, and we can conclude that $\hat{s}$ is continuous. Thus we have constructed a morphism of equivariant sheaves

$$\begin{array}{ccc} E_{[\vec{x}\,|\,\phi]} & \xrightarrow{\hat{s}} & A \\ & \searrow \quad \swarrow{\scriptstyle a} & \\ & X & \end{array}$$

such that $x \in A$ is in the image of $\hat{s}$. $\dashv$

We have reached the main result of this chapter. In stating it, we reintroduce the subscript indicating the relevant theory.

Theorem 3.2.2.10 *For any theory* $\mathbb{T}$ *with a saturated set of countable models, there is an equivalence*

$$\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}}) \simeq \mathrm{Sh}\,(\mathbb{G}_{\mathbb{T}}) = \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$$

where $\mathbb{G}_{\mathbb{T}} = (G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}})$ *is the topological groupoid of enumerated* $\mathbb{T}$*-models.*

Proof By Lemma 3.2.2.9, the definable objects form a generating set for $\mathrm{Sh}\,(\mathbb{G}_{\mathbb{T}})$. Therefore, the full subcategory of definable objects equipped with the coverage inherited from the canonical coverage of $\mathrm{Sh}\,(\mathbb{G}_{\mathbb{T}})$ is a site for $\mathrm{Sh}\,(\mathbb{G}_{\mathbb{T}})$ [10, C2.2.16]. By Lemma 3.2.2.4, the full subcategory of definable objects is equivalent (in fact isomorphic, see Section 3.3.1) to $\mathcal{C}_{\mathbb{T}}$, and by Lemma 3.2.2.3, the canonical coverage coincides with the coherent coverage on definables. Therefore, $\mathcal{C}_{\mathbb{T}}$ with the coherent coverage is a site for $\mathrm{Sh}\,(\mathbb{G}_{\mathbb{T}})$, whence $\mathrm{Sh}\,(\mathbb{G}_{\mathbb{T}}) \simeq \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$. $\dashv$

Remark 3.2.2.11 Although Theorem 3.2.2.10 only holds for theories with a saturated set of countable models, nothing in our argument for it depends on countability. Therefore, for an arbitrary theory $\mathbb{T}$, one can choose an 'index set' $\mathbb{I}$ to replace $\mathbb{N}$ which is large enough to have Lemma 3.1.1.7 hold with respect to the topological groupoid of models with underlying set a quotient of $\mathbb{I}$, and repeat the argument to the effect that $\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ is equivalent to the category of equivariant sheaves on that groupoid.

3.3 Syntax-Semantics Adjunction

3.3.1 The Category of FOL

Recall that we construct the *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of a single-sorted theory $\mathbb{T}$ as follows. The objects of $\mathcal{C}_{\mathbb{T}}$ are equivalence classes of (α-equivalence classes of) formulas-in-context, $|\,[\vec{x}\,|\,\phi]\,|$, where $[\vec{x}\,|\,\phi] \sim [\vec{x}\,|\,\psi]$ iff $\mathbb{T} \vdash \forall \vec{x}.\, \phi \leftrightarrow \psi$. Arrows between such objects are as usual given by $\mathbb{T}$-provable equivalence classes of formulas-in-context,

$$|\,[\vec{x}, \vec{y}\,|\,\sigma]\,| : |\,[\vec{x}\,|\,\phi]\,| \longrightarrow |\,[\vec{y}\,|\,\psi]\,|$$

such that $\mathbb{T} \vdash (\forall \vec{x}, \vec{y}.\, \sigma \to \phi \wedge \psi) \wedge (\forall \vec{x}.\, \phi \to \exists! \vec{y}.\, \sigma)$. This definition of $\mathcal{C}_\mathbb{T}$ is clearly equivalent, in the sense of producing equivalent categories, to the usual one where objects are just α-equivalence classes (and not $\mathbb{T}$-provable equivalence classes) of formulas-in-context, but is more convenient as long as we are mostly interested in $\mathbb{T}$-models in **Sets**. In what follows, we usually drop the vertical bars indicating equivalence class in our notation (i.e. we write $[\vec{x} \,|\, \phi]$ but mean $|\,[\vec{x} \,|\, \phi]\,|$). With this definition of syntactical category, every syntactic category has the properties:

- There is a distinguished object, U, $([x \,|\, \top])$ with distinguished finite powers $([\,|\, \top], [x \,|\, \top], [x_1, x_2 \,|\, \top], \ldots)$.
- $\mathcal{C}_\mathbb{T}$ has a *system of inclusions*, that is, a set $\mathfrak{I} \subseteq (\mathcal{C}_\mathbb{T})_1$ of distinguished monomorphisms which is closed under composition and identities, and such that every object has a unique inclusion into a finite power of U. Moreover (and this is not the case with the alternative definition of $\mathcal{C}_\mathbb{T}$) every subobject, considered as a set of monomorphisms, of an object contains a unique inclusion. (We can take $\mathfrak{I}$ in $\mathcal{C}_\mathbb{T}$ to be the set of all arrows $[x_1, \ldots, x_n \,|\, \phi] \longrightarrow [y_1, \ldots, y_n \,|\, \psi]$ which contain the formula-in-context $[\vec{x}, \vec{y} \,|\, \phi \wedge \psi \wedge \vec{x} = \vec{y}]$.)

We claim that this characterizes syntactical categories (for single-sorted classical theories) up to isomorphism. Suppose $\mathcal{B}$ is a Boolean coherent category with a distinguished object and a system of inclusions. Let the language $\mathcal{L}_\mathbb{B}$ of $\mathcal{B}$ consist of, for each inclusion $R \hookrightarrow U^n$ a n-ary relation symbol. Set the theory of $\mathcal{B}$, $\mathbb{T}_\mathcal{B}$, to be the set of true sentences in $\mathcal{L}_\mathbb{B}$ under the canonical interpretation of $\mathcal{L}_\mathbb{B}$ in $\mathcal{B}$.

Lemma 3.3.1.1 *There is an isomorphism*

$$\mathcal{B} \cong \mathcal{C}_{\mathbb{T}_\mathcal{B}}$$

Proof Define a functor $F : \mathcal{B} \longrightarrow \mathcal{C}_\mathbb{T}$ by sending an object A in $\mathcal{B}$ to $[\vec{x} \,|\, \alpha]$, where α is the predicate in $\mathcal{L}_\mathbb{B}$ corresponding to its unique inclusion, $A \hookrightarrow U^n$, into a power of U. For an arrow $f : A \longrightarrow B$ in $\mathcal{B}$, $F(B) = [\vec{y} \,|\, \beta]$ there is an inclusion $\mathrm{Grph}(f) \hookrightarrow U^{n+m}$ corresponding to a relation symbol σ such that $[\vec{x}, \vec{y} \,|\, \sigma]$ is $\mathbb{T}_\mathcal{B}$-provably functional from $[\vec{x} \,|\, \alpha]$ to $[\vec{y} \,|\, \beta]$, so set $F(f) = [\vec{x}, \vec{y} \,|\, \sigma]$. In the other direction, define a functor $G : \mathcal{C}_\mathbb{T} \longrightarrow \mathcal{B}$ by sending an object $[\vec{x} \,|\, \phi]$ to the domain of the inclusion representing the subobject $[\![\vec{x}|\phi]\!]^{\mathcal{B}}$ under the canonical interpretation of $\mathcal{L}_{\mathbb{B}}$ in $\mathcal{B}$. Then $G \circ F = 1_{\mathcal{B}}$. And if $[\vec{x} \mid \phi] \in \mathcal{C}_{\mathbb{T}}$, with $F([\vec{x} \mid \phi]) = R \hookrightarrow U^n$, and $[\vec{x} \mid \rho]$ is the predicate of $\mathcal{L}_{\mathbb{B}}$ representing $R \hookrightarrow U^n$, then $\mathbb{T}_{\mathcal{B}} \vdash \forall \vec{x}.\, \phi \leftrightarrow \rho$, so $F \circ G = 1_{\mathcal{C}_{\mathbb{T}}}$. ⊣

We also note the following:

Lemma 3.3.1.2 *For a theory* $\mathbb{T}$ *with a saturated set of countable models, the functor* $\mathcal{M}^{\dagger} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}\,(\mathbb{G}_{\mathbb{T}})$ *is an isomorphism on its image.*

PROOF $\mathcal{M}^{\dagger}$ is full and faithful, and therefore an equivalence on its image. That it is an isomorphism follows as long as $\mathcal{M}^{\dagger}$ is 1-1 on objects. But that much is clear from the construction of $\mathcal{C}_{\mathbb{T}}$ above, and from the construction of the definables in $\mathrm{Sh}\,(\mathbb{G}_{\mathbb{T}})$. ⊣

Because we have assumed that our theories all satisfy the axiom $\exists x.\, x = x$, we shall assume, when we say that a category has a distinguished object, that it is well-supported. Last, if $F : \mathcal{B} \longrightarrow \mathcal{D}$ is a coherent functor that preserves the distinguished object, or synonymously the *single sort*, then it is naturally isomorphic to one that moreover preserves the distinguished finite powers of U on the nose, and that preserves inclusions.

Definition 3.3.1.3 The category **FOL** consists of Boolean coherent categories with a saturated set of countable models, and with a distinguished, well-supported object and a system of inclusions. Arrows in **FOL** are coherent functors that preserve the distinguished object (and its distinguished finite powers) and inclusions on the nose.

We write $\mathcal{C}_{\mathbb{T}}$ for an object of **FOL**, since it is (isomorphic to) a syntactic category for a classical theory $\mathbb{T}$ by Lemma 3.3.1.1. By a $\mathbb{T}$-model, we mean a coherent functor $M : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$ that sends $M(U^n)$ to the n-fold cartesian product of $M(U)$, and inclusions to subset inclusions.

3.3.2 The Object Classifier

The (classical, single-sorted) theory with no constant, function, or relation symbols and no axioms (except $\exists x.\, x = x$)—i.e. the classical theory of equality—we denote by $\mathbb{T}_{=}$ (abusing notation, as it is the same name as we used for the same theory without the axiom $\exists x.\, x = x$ in Chapter 2, but hopefully without causing confusion). The theory $\mathbb{T}_{=}$ is a well-supported object classifier in the category of Boolean coherent categories, in the sense that well-supported objects, $B \twoheadrightarrow 1$, in a Boolean coherent category $\mathcal{B}$ correspond to coherent functors $\mathcal{C}_{\mathbb{T}_=} \longrightarrow \mathcal{B}$, up to natural isomorphism. More precisely, there is an equivalence of categories, natural in $\mathcal{B}$,

$$\mathrm{Hom}^*_{\mathbf{BC}}(\mathcal{C}_{\mathbb{T}_=}, \mathcal{B}) \simeq \mathcal{B}^*$$

where $\mathcal{B}^*$ is the groupoid consisting of well-supported objects and isomorphisms in $\mathcal{B}$. Accordingly, the topos of coherent sheaves $\mathrm{Sh}(\mathbb{T}_=)$ classifies well-supported classical objects with in the category of topoi and geometric morphisms (compare with Section 2.5). In the category **FOL**, there are only distinguished object preserving functors, whence there exists exactly one arrow from $\mathcal{C}_{\mathbb{T}_=}$ to any $\mathcal{C}_{\mathbb{T}}$ in **FOL**, that is, $\mathcal{C}_{\mathbb{T}_=}$ classifies the distinguished object. In accordance with this, we consider topological groupoids over the dual of $\mathcal{C}_{\mathbb{T}_=}$—the semantical groupoid $\mathbb{G}_{\mathbb{T}_=}$—in order to construct an adjoint to the semantical groupoid functor.

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \longrightarrow & \mathcal{C}_{\mathbb{T}'} \\ \nwarrow & & \nearrow \\ & \mathcal{C}_{\mathbb{T}_=} & \end{array} \qquad \mapsto \qquad \begin{array}{ccc} \mathbb{G}_{\mathbb{T}} & \longleftarrow & \mathbb{G}_{\mathbb{T}'} \\ \searrow & & \swarrow \\ & \mathbb{G}_{\mathbb{T}_=} & \end{array}$$

First, we give a characterization of $\mathbb{G}_{\mathbb{T}_=}$.

Definition 3.3.2.1 The topological groupoid $\mathbb{N}_\sim$ consists of the set N_0 of quotients of $\mathbb{N}$ with the set N_1 of bijections between them, equipped with topology as follows. The topology on the set of objects,

$$N_0 = \{\mathbb{N}/_\sim \mid \sim \text{ an equivalence relation on } \mathbb{N}\}$$

is the coarsest topology in which both sets of the form

$$(n \sim m) := \{A \in N_0 \mid [n] = [m] \text{ in } A\}$$

and

$$(n) := \{A \in N_0 \mid \text{the cardinality of } A \text{ is } n\}$$

are clopen, for $n, m \in \mathbb{N}$. The topology on the set, N_1 of bijections is the coarsest topology such that the source and target maps $s, t : N_1 \rightrightarrows N_0$ are both continuous, and such that all sets of the form

$$(m \mapsto n) := \left\{ f : A \xrightarrow{\cong} B \text{ in } N_1 \;\middle|\; f([m]) = [n] \right\}$$

are open.

Lemma 3.3.2.2 *There is an isomorphism* $\mathbb{N}_\sim \cong \mathbb{G}_{\mathbb{T}_=}$ *in* **Gpd**.

PROOF Any set A in N_0 is the underlying set of a canonical $\mathbb{T}_=$-model, and any bijection $f : A \to B$ is the underlying function of a $\mathbb{T}_=$-model isomorphism, and thereby we obtain bijections $N_0 \cong X_{\mathbb{T}_=}$ and $N_1 \cong G_{\mathbb{T}_=}$ which commute with source, target, composition, and embedding of identities maps. Remains to show that the topologies correspond. Clearly, any open set of the form $(n \sim m) \subseteq N_0$ corresponds to the open set $U_{[x,y\,|\,x=y],n,m} \subseteq X_{\mathbb{T}_=}$, while $(n) \subseteq N_0$ corresponds to the clopen set $U_{[\,|\,\phi],\star} \subseteq X_{\mathbb{T}_=}$ where ϕ is the first-order sentence in $=$ expressing that there are exactly n elements. Conversely, let a basic open $U_{[x_1,\ldots,x_n\,|\,\phi],m_1,\ldots,m_n} \subseteq X_{\mathbb{T}_=}$ be given. Consider the list of variables $x_1, \ldots, x_n$. Let $\sigma_1, \ldots, \sigma_k$ be a list of all the possible equality relations between these variables, that is to say, such that each σ_i is a conjunction of the form

$$\bigwedge_{1 \le i,j \le n} P_{i,j}$$

with $P_{i,j}$ either the formula $x_i = x_j$ or the formula $x_i \neq x_j$. First, we claim that

$$\mathbf{M} \in U_{[x_1,\ldots,x_n\,|\,\phi],m_1,\ldots,m_n} \Leftrightarrow \mathbf{M} \vDash \bigvee_{1 \le i \le k} (\sigma_i(\vec{m}/\vec{x}) \wedge \exists \vec{x}.\, \sigma_i \wedge \phi)$$

where we can refer to the disjunction occurring on the right as Φ, for short. The left-to-right implications is clear. Suppose that $\mathbf{M} \vDash \Phi$, say that the jth disjunct is true in $\mathbf{M}$. Choose a witness $\vec{a} \in |\mathbf{M}|^n$ for the existential quantifier. Since $\mathbf{M} \vDash \sigma_j(\vec{a}/\vec{x})$ and $\mathbf{M} \vDash \sigma_j(\vec{[m]}/\vec{x})$, the assignment $[m_i] \mapsto a_i$, for $1 \le i \le n$, is well-defined and extends to a permutation of $|\mathbf{M}|$, and thus to an automorphism of $\mathbf{M}$. Whence $\mathbf{M} \vDash \phi(\vec{m}/\vec{x})$, so the right-to-left implication holds. Next, we claim that the set of models satisfying the statement Φ corresponds to an open (and closed) subset of N_0. For the set of models satisfying $m_i = m_j$ corresponds to the subset $(m \sim n)$ and the models satisfying $m_i \neq m_j$ corresponds to the complement of $(m \sim n)$, so the set of models satisfying $\sigma_i(\vec{m}/\vec{x})$ is a finite intersection of clopen sets. Now, all infinite $\mathbb{T}_=$-models are elementary equivalent, as are all finite models of the same size. Therefore, by compactness, for any $\mathbb{T}_=$-sentence, ψ, there exists a $k \in \mathbb{N}$ such that either ψ is true in all models of size $> k$ or ψ is false in all models of size $> k$. Therefore, there exists a finite set $K \subset \mathbb{N}$ such that, in the latter case, $\mathbf{N} \vDash \psi$ if and only if $|\mathbf{N}| \in K$, and in the former case, such that $\mathbf{N} \vDash \psi$ if and only if $|\mathbf{N}| \notin K$. Since

$$U := \{A \in N_0 \mid |A| \in K\} = \bigcup_{n \in K} (n)$$

is a clopen subset of N_0, the set of $\mathbb{T}_=$-models in which ψ is true corresponds to an clopen set in N_0. In particular, the set of models satisfying $\exists \vec{x}.\, \sigma_i \wedge \phi$ corresponds to a clopen set of N_0. Therefore, the set of models satisfying Φ corresponds to a finite union of clopen subsets of N_0, and so is clopen. For the spaces of arrows, it remains only to observe that open subsets of the form $(m \mapsto n) \subseteq N_1$ correspond to open subsets of the form

$$V_=^-(m, n)$$

and we can conclude that $\mathbb{N}_\sim$ is a topological groupoid isomorphic to $\mathbb{G}_{\mathbb{T}_=}$ in **Gpd**. $\dashv$

The category $\mathrm{Sh}(\mathbb{N}_\sim)$ of equivariant sheaves on $\mathbb{N}_\sim$, therefore, classifies well-supported classical objects. The generic decidable object, $\mathcal{U}$, in $\mathrm{Sh}(\mathbb{N}_\sim)$ can be taken to be the definable sheaf $\langle E_{[x \mid \top]} \to X_{\mathbb{T}_=} \cong N_0, \theta_{[x \mid \top]}\rangle$. Similar to Proposition 2.4.3.4 and Proposition 2.5.0.13, the object $\mathcal{U}$ can be constructed without reference to definable sheaves as follows, following [21, §6] (the relevant parts of which we have taken the liberty to present in Section 3.4.1).

Proposition 3.3.2.3 *There exists an open set $M \subseteq N_1$ with M closed under inverses and composition and $s(M), t(M) \subseteq N_0$ such that the equivariant sheaf $\mathcal{U}$ is isomorphic to the equivariant sheaf $\langle \mathbb{N}_\sim, N_0, M\rangle$ in* $\mathrm{Sh}(\mathbb{N}_\sim)$*,*

$$\mathcal{U} \cong \langle \mathbb{N}_\sim, N_0, M\rangle$$

Proof Choose a $n \in \mathbb{N}$. We have an open subset $M = (n \mapsto n) \subseteq N_1$, with M closed under identities, inverses, and composition, and $s(M), t(M) \subseteq N_0$. Accordingly, following [21, §6], we have an equivariant sheaf $\langle \mathbb{N}_\sim, N_0, M\rangle$ in $\mathrm{Sh}(\mathbb{N}_\sim)$ consisting of the sheaf $t : N_1/M \longrightarrow N_0$, where N_1/M is the set of arrows of N_1 factored out by the equivalence relation,

$$g \sim_M h \Leftrightarrow t(g) = t(h) \wedge g^{-1} \circ h \in M$$

and N_1 acts on N_1/M by composition. Following [21, 6], again, it is then straightforward to verify that the mapping

$$[g] \mapsto g([n])$$

defines an isomorphism

$$\mathcal{U} \cong \langle \mathbb{N}_{\sim}, N_0, M \rangle$$

by observing that the stabilization of the image of the global section $u : N_0 \to E_{[x\,|\,\top]}$ defined by $A \mapsto [n]$ is all of $E_{[x\,|\,\top]}$; that the induced morphism of equivariant sheaves $\tilde{u} : O_1/N_u \longrightarrow E_{[x\,|\,\top]}$ is therefore an isomorphism; that $M = N_u$; and finally, that $\tilde{u}([g]) = g([n])$. ⊣

Accordingly, a morphism $f : \mathbb{G} \longrightarrow \mathbb{N}_{\sim}$ of topological groupoids identifies a well-supported classical object $f^*(\mathcal{U})$ in $\mathrm{Sh}(\mathbb{G})$. This allows us to construct an adjunction between **FOL** and a full subcategory of $\mathbf{Gpd}/\mathbb{N}_{\sim}$. We formulate this adjunction first in terms of the most inclusive subcategory of $\mathbf{Gpd}/\mathbb{N}_{\sim}$ for which it works. Section 3.4 then identifies a condition on morphisms over $\mathbb{N}_{\sim}$ more directly in terms of the morphism involved and its domain, yielding a smaller category to which the adjunction restricts.

3.3.3 The Semantical Groupoid Functor

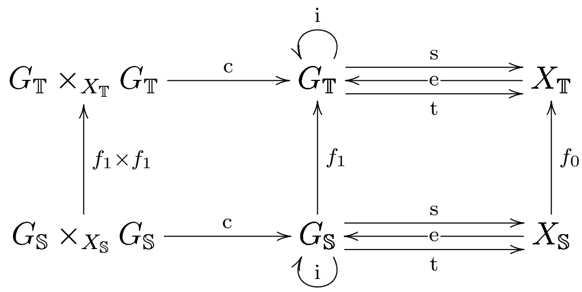
For $\mathbb{T}$ a single-sorted classical theory with a saturated set of countable models, we can construct the topological groupoid, $\mathbb{G}_{\mathbb{T}}$, of $\mathbb{T}$-models with underlying set a quotient of $\mathbb{N}$ and $\mathbb{T}$-isomorphisms between them,

$$G_{\mathbb{T}} \times_{X_{\mathbb{T}}} G_{\mathbb{T}} \xrightarrow{\ \mathrm{c}\ } G_{\mathbb{T}} \circlearrowleft^{\mathrm{i}} \quad G_{\mathbb{T}} \underset{\mathrm{t}}{\overset{\mathrm{s}}{\rightrightarrows}} X_{\mathbb{T}}, \quad X_{\mathbb{T}} \xrightarrow{\ \mathrm{e}\ } G_{\mathbb{T}}$$

as in Section 3.2.1.

Lemma 3.3.3.1 *The assignment $\mathbb{T} \mapsto \mathbb{G}_{\mathbb{T}}$ is functorial (in $\mathbb{T}$).*

Proof Let two theories $\mathbb{T}$ and $\mathbb{S}$ be given, and let $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$ be a coherent, single-sort preserving functor. Then any $\mathbb{S}$-model, $\mathcal{C}_{\mathbb{S}} \longrightarrow \mathbf{Sets}$, 'restricts' along F to a $\mathbb{T}$-model with the same underlying set, and any $\mathbb{S}$-isomorphism restricts to a $\mathbb{T}$-isomorphism with the same underlying function, and so we get functions $f_0 : X_{\mathbb{S}} \to X_{\mathbb{T}}$ and $f_1 : G_{\mathbb{S}} \to G_{\mathbb{T}}$ such that the following commutes:



Given a basic open $U_{\phi(\vec{n}/\vec{x})} \subseteq X_{\mathbb{T}}$, we see that its inverse image is given by translating ϕ along F,

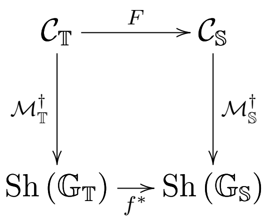
$$f_0^{-1}\left(U_{\phi(\vec{n}/\vec{x})}\right) = U_{F(\phi)(\vec{n}/\vec{x})} \subseteq X_{\mathbb{S}}$$

and so f_0 is continuous. Similarly, f_1 is continuous because

$$f_1^{-1}\left(V_{\psi}^{\phi}(\vec{m},\vec{n})\right) = V_{F(\psi)}^{F(\phi)}(\vec{m},\vec{n}) \subseteq G_{\mathbb{S}}$$

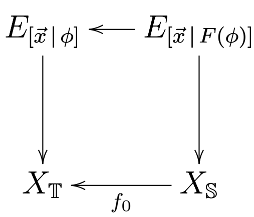
(suppressing mention of number constants in ϕ and ψ). Thus we obtain a morphism of continuous groupoids $f : \mathbb{G}_{\mathbb{S}} \longrightarrow \mathbb{G}_{\mathbb{T}}$ ⊣

Lemma 3.3.3.2 *The square*



commutes.

Proof Consider, for an object $[\vec{x} \mid \phi]$ in $\mathcal{C}_{\mathbb{T}}$, the square



Since f_0 is composition with F, the fiber $F(\phi)(\mathbf{M}) = \mathbf{M}(F([\vec{x} \,|\, \phi]))$ over $\mathbf{M} \in X_{\mathbb{S}}$ is the fiber $\phi(f_0(\mathbf{M})) = \mathbf{M} \circ F([\vec{x} \,|\, \phi])$ over $f_0(\mathbf{M}) \in X_{\mathbb{T}}$, so the square is a pullback of sets. A basic open $V_{[\vec{x} \,|\, \psi(\vec{m})]}$ is pulled back to a basic open $V_{[\vec{x} \,|\, F(\psi)(\vec{m})]}$, so the pullback topology is contained in the logical topology. For an element $\langle \mathbf{M}, [\vec{n}] \rangle$ in basic open $V_{[\vec{x} \,|\, \psi]}$, the set $V = V_{[\vec{x} \,|\, \vec{x} = \vec{n}]} \subseteq E_{[\vec{x} \,|\, \phi]}$ is open and $\langle \mathbf{M}, [\vec{n}] \rangle \in V \times_{X_{\mathbb{T}}} U_{\psi(\vec{n}/\vec{x})} \subseteq V_{[\vec{x} \,|\, \psi]}$, so the logical topology is contained in the pullback topology. With $f_1 : G_{\mathbb{S}} \to G_{\mathbb{T}}$ being just a restriction function, we conclude that $f^* \circ \mathcal{M}^{\dagger}_{\mathbb{T}} = \mathcal{M}^{\dagger}_{\mathbb{S}} \circ F$. ⊣

For any theory $\mathbb{T}$, there is a unique arrow $U_{\mathbb{T}} : \mathcal{C}_{\mathbb{T}_=} \longrightarrow \mathcal{C}_{\mathbb{T}}$ in **FOL**, that is, $\mathcal{C}_{\mathbb{T}_=}$ is an initial object. Applying Lemma 3.3.3.1 to $U_{\mathbb{T}} : \mathcal{C}_{\mathbb{T}_=} \longrightarrow \mathcal{C}_{\mathbb{T}}$ and shifting to the isomorphic groupoid $\mathbb{N}_{\sim}$, we obtain a forgetful morphism of topological groupoids $u_{\mathbb{T}} : \mathbb{G}_{\mathbb{T}} \longrightarrow \mathbb{G}_{\mathbb{T}_=} \cong \mathbb{N}_{\sim}$, sending a $\mathbb{T}$-model to its underlying set and a $\mathbb{T}$-isomorphism to its underlying bijection. For any arrow $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$ in **FOL** we obtain a commuting triangle of morphisms of topological groupoids:

$$\begin{array}{ccc} \mathbb{G}_{\mathbb{S}} & \xrightarrow{\;f\;} & \mathbb{G}_{\mathbb{T}} \\ & {\scriptstyle u_{\mathbb{S}}} \searrow \quad \swarrow {\scriptstyle u_{\mathbb{T}}} & \\ & \mathbb{N}_{\sim} & \end{array}$$

Definition 3.3.3.3 The contravariant *semantical groupoid functor*

$$\Gamma : \mathbf{FOL}^{\mathrm{op}} \longrightarrow \mathbf{Gpd}/\mathbb{N}_{\sim}$$

sends a theory to its topological groupoid (over $\mathbb{N}_{\sim}$) of models:

$$\begin{array}{c} \mathcal{C}_{\mathbb{T}} \\ {\scriptstyle F}\big\downarrow \\ \mathcal{C}_{\mathbb{S}} \end{array} \qquad \mapsto \qquad \begin{array}{ccc} \mathbb{G}_{\mathbb{S}} & \xrightarrow{\;f\;} & \mathbb{G}_{\mathbb{T}} \\ & {\scriptstyle u_{\mathbb{S}}} \searrow \quad \swarrow {\scriptstyle u_{\mathbb{T}}} & \\ & \mathbb{N}_{\sim} & \end{array}$$

Remark 3.3.3.4 As long as there is little danger of confusion, we will continue the practice above of denoting the semantical morphism of groupoids obtained by applying the semantical groupoid functor to a morphism of **FOL** simply by switching from upper case to lower case letters. E.g. for $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$,

$$\Gamma(F) = f : \mathbb{G}_{\mathbb{S}} \longrightarrow \mathbb{G}_{\mathbb{T}}$$

3.3.4 The Theory Functor

We construct an adjoint to the contravariant semantical groupoid functor $\Gamma : \mathbf{FOL} \longrightarrow \mathbf{Gpd}/\mathbb{N}_\sim$ which restricts to an equivalence on the image of Γ. As noted in Section 3.3.3, any morphism of topological groupoids $f : \mathbb{G} \longrightarrow \mathbb{N}_\sim$ identifies an object $U_\mathbb{G} := f^*(\mathcal{U})$ in $\mathrm{Sh}(\mathbb{G})$, with finite powers $1, U_\mathbb{G}, U_\mathbb{G}^2, \ldots$ obtained by taking the (canonical) finite fiberwise products in $\mathrm{Sh}(\mathbb{G})$.

Definition 3.3.4.1 A groupoid $f : \mathbb{G} \longrightarrow \mathbb{N}_\sim$ over $\mathbb{N}_\sim$ is called *classical* if the finite powers $1, U_\mathbb{G}, U_\mathbb{G}^2, \ldots$ of $U_\mathbb{G}$ in $\mathrm{Sh}(\mathbb{G})$ are compact and such that their complemented subobjects are closed under images along projections. That is to say, for any complemented $S \rightarrowtail U_\mathbb{G}^{n+1}$, the image $\exists_{\pi_i}(S) \rightarrowtail U_\mathbb{G}^n$ along the projection $\pi_i : U_\mathbb{G}^{n+1} \longrightarrow U_\mathbb{G}^n$ is again complemented, for $1 \leq i \leq n+1$.

The full subcategory of $\mathbf{Gdp}/\mathbb{N}_\sim$ consisting of classical groupoids over $\mathbb{N}_\sim$ is denoted

$$\mathbf{Class} \hookrightarrow \mathbf{Gpd}/\mathbb{N}_\sim$$

Definition 3.3.4.2 For $\mathbb{G} \longrightarrow \mathbb{N}_\sim$ a classical groupoid over $\mathbb{N}_\sim$, let

$$\Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$$

be the full subcategory consisting of the (canonical) finite powers $1, U_\mathbb{G}, U_\mathbb{G}^2, \ldots$ of $U_\mathbb{G}$ in $\mathrm{Sh}(\mathbb{G})$ together with any equivariant sheaf A such that there exists an arrow $A \longrightarrow U_\mathbb{G}^n$ with underlying function a subset inclusion and such that A is complemented as a subobject of $U_\mathbb{G}^{n+1}$.

Lemma 3.3.4.3 *For a classical groupoid $\mathbb{G} \longrightarrow \mathbb{N}_\sim$, the subcategory*

$$\Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$$

is a Boolean coherent category with distinguished object and a system of inclusions. Moreover, $\Theta(\mathbb{G})$ has a saturated set of countable models. That is, $\Theta(\mathbb{G}) \in \mathbf{FOL}$.

Proof $\Theta(\mathbb{G})$ is clearly Boolean. It inherits the terminal object from $\mathrm{Sh}(\mathbb{G})$. Complemented subobjects of $1, U_\mathbb{G}, U_\mathbb{G}^2, \ldots$ in $\mathrm{Sh}(\mathbb{G})$ are closed under finite joins and meets and pullbacks along projections, from which we can conclude that $\Theta(\mathbb{G})$ also inherits equalizers, finite products, and finite joins. From the fact that the image of any complemented subobject $S \rightarrowtail U_\mathbb{G}^{n+1}$ along the projection $\pi : U_\mathbb{G}^{n+1} \to U_\mathbb{G}^n$ is again complemented, we can infer that $\Theta(\mathbb{G})$

inherits images from $\mathrm{Sh}(\mathbb{G})$. Thus $\Theta(\mathbb{G})$ is coherent. Finally, $U_{\mathbb{G}}$ is a distinguished object, and subset inclusions form a system of inclusions. Remains to show that $\Theta(\mathbb{G})$ has a saturated set of countable models. Consider the inclusion of categories $w : \Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$. w is coherent, and since every object in $\Theta(\mathbb{G})$ is compact, w reflects covers when $\Theta(\mathbb{G})$ is equipped with the coherent coverage. Thus w induces a surjective geometric morphism $w : \mathrm{Sh}(\mathbb{G}) \twoheadrightarrow \mathrm{Sh}(\Theta(\mathbb{G}))$. Consider the composite,

$$\mathbf{Sets}/G_0 \xrightarrow{u_1} \mathrm{Sh}(G_0) \xrightarrow{u_2} \mathrm{Sh}(\mathbb{G}) \xrightarrow{w} \mathrm{Sh}(\Theta(\mathbb{G}))$$

where the two leftmost geometric morphisms have forgetful inverse image parts. Now, any point $m : \mathbf{Sets} \longrightarrow \mathbf{Sets}/G_0$ corresponds, by considering m^* to be pullback along $x : 1 \longrightarrow G_0$ for some $x \in G_0$, to a countable model $\mathbf{M}_x : \Theta(\mathbb{G}) \longrightarrow \mathbf{Sets}$, in the form of a coherent functor which sends inclusions to subset inclusions, and whose value at $U_{\mathbb{G}}$ is $\{[n] \mid \langle x, [n]\rangle \in e^{-1}(x)\}$ and so is a quotient of $\mathbb{N}$. And these points jointly reflects covers. $\dashv$

Lemma 3.3.4.4 *For a theory $\mathbb{T}$ with a saturated set of countable models, the semantical groupoid $u_{\mathbb{T}} : \mathbb{G}_{\mathbb{T}} \longrightarrow \mathbb{N}_{\sim}$ is classical.*

Proof $u_{\mathbb{T}}^*(\mathcal{U})$ is the sheaf $\langle E_{[x\,|\,\top]} \to X_{\mathbb{T}}, \theta_{[x\,|\,\top]}\rangle$ which is compact, as are its finite powers—of the form $\langle E_{[\vec{x}\,|\,\top]} \to X_{\mathbb{T}}, \theta_{[\vec{x}\,|\,\top]}\rangle$—and any complemented subobject thereof is definable and therefore closed under images along projections. $\dashv$

The assignment $\mathbb{G} \mapsto \Theta(\mathbb{G})$ of classical groupoids to theories is functorial. For a morphism of classical groupoids,

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{f} & \mathbb{H} \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & \mathbb{N}_{\sim} & \end{array}$$

induces a commuting triangle of geometric morphisms,

$$\begin{array}{ccc} \mathrm{Sh}(\mathbb{G}) & \xrightarrow{f} & \mathrm{Sh}(\mathbb{H}) \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & \mathrm{Sh}(\mathbb{N}_{\sim}) & \end{array}$$

and since inverse image morphisms are coherent and preserve complemented subobjects, f^* restricts to a coherent single sort preserving functor $F : \Theta(\mathbb{H}) \longrightarrow \Theta(\mathbb{G})$.

Definition 3.3.4.5 Let $\Theta : \mathbf{Class} \longrightarrow \mathbf{FOL}^{\mathrm{op}}$ be the functor which assigns a classical groupoid $\mathbb{G}$ over $\mathbb{N}_{\sim}$ to the theory $\Theta(\mathbb{G})$ and a morphism $f : \mathbb{G} \longrightarrow \mathbb{H}$ to the functor $F : \Theta(\mathbb{H}) \longrightarrow \Theta(\mathbb{G})$ obtained by restricting the induced inverse image functor.

$$\mathbf{FOL}^{\mathrm{op}} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Class}$$

Lemma 3.3.4.6 *For any $\mathcal{C}_{\mathbb{T}}$ in* $\mathbf{FOL}$*,*

$$\mathcal{C}_{\mathbb{T}} \cong \Theta \circ \Gamma(\mathcal{C}_{\mathbb{T}}) = \Theta(\mathbb{G}_{\mathbb{T}})$$

Proof The functor $M^{\dagger} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}(\mathbb{G}_{\mathbb{T}}) \cong \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ is an isomorphism on its image, by Lemma 3.3.1.2, and preserves the distinguished object and inclusions by construction. The image is $\Theta(\mathbb{G}_{\mathbb{T}})$ by Lemma 3.3.3.2. ⊣

The isomorphism obtained by factoring $\mathcal{M}^{\dagger}$ through its image

$$\begin{array}{ccc} & & \Theta(\mathbb{G}_{\mathbb{T}}) \\ & \overset{\epsilon_{\mathbb{T}}}{\nearrow} & \big\downarrow \\ \mathcal{C}_{\mathbb{T}} & \xrightarrow[\mathcal{M}^{\dagger}_{\mathbb{T}}]{} & \mathrm{Sh}(\mathbb{G}_{\mathbb{T}}) \end{array} \tag{3.13}$$

is our counit component candidate at $\mathbb{T}$.

Lemma 3.3.4.7 *There is a natural transformation,*

$$\epsilon : 1_{\mathbf{FOL}} \longrightarrow \Theta \circ \Gamma$$

whose component at an object $\mathcal{C}_{\mathbb{T}}$ in $\mathbf{FOL}$ *is the isomorphism*

$$\epsilon_{\mathbb{T}} : \mathcal{C}_{\mathbb{T}} \longrightarrow \Theta(\mathbb{G}_{\mathbb{T}})$$

of (3.13).

Proof By Lemma 3.3.3.2. ⊣

Next, we construct the unit candidate. For a given classical groupoid $h : \mathbb{H} \longrightarrow \mathbb{N}_\sim$ over $\mathbb{N}_\sim$, we define an arrow

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\eta_{\mathbb{H}}} & \Gamma(\Theta(\mathbb{H})) = \mathbb{G}_{\Theta(\mathbb{H})} \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle u_{\Theta(\mathbb{H})}} & \\ & \mathbb{N}_\sim & \end{array}$$

of **Class** (Definition 3.3.4.8–Lemma 3.3.4.10), and show that it is natural in $\mathbb{H}$ (Lemma 3.3.4.12). Just to reduce subscripts in the construction, choose a theory $\mathbb{T}$ so that $\mathcal{C}_{\mathbb{T}} \cong \Theta(\mathbb{H})$, and write the above triangle as

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\eta_{\mathbb{H}}} & \mathbb{G}_{\mathbb{T}} \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle u} & \\ & \mathbb{N}_\sim & \end{array}$$

and write the components of $\eta_{\mathbb{H}}$ as

$$\begin{array}{ccc} H_1 & \xrightarrow{\eta_1} & G_{\mathbb{T}} \\ {\scriptstyle t}\downarrow\ \uparrow{\scriptstyle e}\ \downarrow{\scriptstyle s} & & {\scriptstyle t}\downarrow\ \uparrow{\scriptstyle e}\ \downarrow{\scriptstyle s} \\ H_0 & \xrightarrow{\eta_0} & X_{\mathbb{T}} \end{array} \tag{3.14}$$

Definition 3.3.4.8 We define the function $\eta_0 : H_0 \to X_{\mathbb{T}}$ over N_0 as follows: Consider an element $x \in H_0$. It corresponds to a point

$$\mathbf{Sets} \longrightarrow \mathrm{Sh}\,(H_0) \twoheadrightarrow \mathrm{Sh}\,(\mathbb{H}) \twoheadrightarrow \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$$

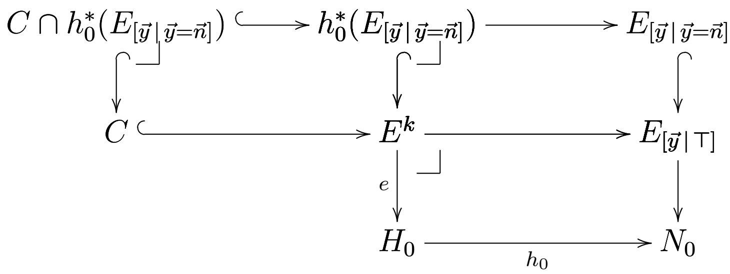
and thereby (as in the proof of Lemma 3.3.4.3) to a $\mathbb{T}$-model $\mathbf{M}_x$ with underlying set a quotient of $\mathbb{N}$. Specifically, the underlying set is $h_0(x)$. Accordingly, set $\eta_0(x) = \mathbf{M}_x$.

Lemma 3.3.4.9 *The map $\eta_0 : H_0 \to X_{\mathbb{T}}$ is continuous.*

Proof Given a basic open $V \subseteq X_{\mathbb{T}}$ we may think of it as being presented in terms of an inclusion $C \hookrightarrow U^k_{\mathbb{H}}$ in $\mathcal{C}_{\mathbb{T}} \cong \Theta(\mathbb{H}) \hookrightarrow \mathrm{Sh}\,(\mathbb{H})$ and a tuple of numbers $\vec{n}$ with length k, as

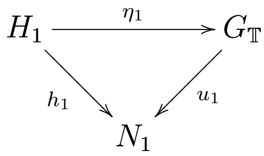
$$V = \left\{ \mathbf{M} \in X_{\mathbb{T}} \;\middle|\; [\vec{n}] \in \mathbf{M}(C) \subseteq \mathbf{M}(U^k_{\mathbb{H}}) \right\}.$$

Pulled back along η_0, this is the set of those $x \in H_0$ such that the tuple of equivalence classes $[\vec{n}]$ in the fiber over x in $U^k_{\mathbb{H}}$ is in C. But this set is open, for it is the image along the local homeomorphism $e : E^k \longrightarrow H_0$ of the pullback of $E_{[\vec{y}\,|\,\vec{y}=\vec{n}]}$ along $h_0 : H_0 \longrightarrow N_0$ intersected with C:



so that $\eta_0^{-1}(V) = \exists_e(C \cap h_0^*(E_{[\vec{y}\,|\,\vec{y}=\vec{n}]}))$. ⊣

Next, a point $a : x \to y$ in H_1 gives us a $\mathbb{T}$-isomorphism between $\eta_0(x)$ and $\eta_0(y)$—the underlying function of which is the bijection of fibers $\theta_h(a, -)$—and so we obtain a function $\eta_1 : H_1 \to G_{\mathbb{T}}$ over N_1,



(3.15)

so that (3.14) commutes.

Lemma 3.3.4.10 *The map $\eta_1 : H_1 \to G_{\mathbb{T}}$ is continuous, and η_1 together with η_0 constitute a morphism of topological groupoids $\eta_{\mathbb{G}} : \mathbb{G} \longrightarrow \mathbb{H}$ over $\mathbb{N}_\sim$.*

Proof A sub-basic open of $G_{\mathbb{T}}$ is either an an open set of $X_{\mathbb{T}}$ pulled back along $s : G_{\mathbb{T}} \to X_{\mathbb{T}}$, an open set of $X_{\mathbb{T}}$ pulled back along $t : G_{\mathbb{T}} \to X_{\mathbb{T}}$, or an open set of N_1 pulled back along $u_1 : G_{\mathbb{T}} \to N_1$. Therefore, diagrams (3.14) and (3.15) and the continuity of η_0 implies that η_1 is continuous, and thereby that η_1 together with η_0 constitute a morphism of topological groupoids $\eta_{\mathbb{G}} : \mathbb{G} \longrightarrow \mathbb{H}$ over $\mathbb{N}_\sim$. ⊣

Lemma 3.3.4.11 *Given a classical groupoid $h : \mathbb{H} \longrightarrow \mathbb{N}_\sim$, if we apply the theory functor Θ to the comparison morphism $\eta_{\mathbb{H}} : \mathbb{H} \longrightarrow \mathbb{G}_{\Theta(\mathbb{H})}$, then*

$$\Theta(\eta_{\mathbb{H}}) : \Theta(\mathbb{G}_{\Theta(\mathbb{H})}) \longrightarrow \Theta(\mathbb{H})$$

is an isomorphism, with inverse $\epsilon_{\Theta(\mathbb{H})}$,

$$\begin{array}{ccc} \Theta(\mathbb{H}) & \xleftarrow{\Theta(\eta_{\mathbb{H}})} & \Theta(\mathbb{G}_{\Theta(\mathbb{H})}) \\ \big\downarrow & \searrow^{\mathcal{M}^{\dagger}} & \big\downarrow \\ \mathrm{Sh}\,(\mathbb{H}) & \xleftarrow[\eta_{\mathbb{H}}^{*}]{} & \mathrm{Sh}\left(\mathbb{G}_{\Theta(\mathbb{H})}\right) \end{array}$$

Proof Straightforward. ⊣

Lemma 3.3.4.12 *The morphism of classical groupoids over* $\mathbb{N}_{\sim}$

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\eta_{\mathbb{G}}} & \Gamma\,(\Theta\,(\mathbb{G})) \\ & {}_{g}\searrow \quad \swarrow & \\ & \mathbb{N}_{\sim} & \end{array}$$

is natural in $\mathbb{G}$.

Proof Given a morphism of classical groupoids, with their induced geometric morphisms of topoi,

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{f} & \mathbb{H} \\ & {}_{g}\searrow \quad \swarrow_{h} & \\ & \mathbb{N}_{\sim} & \end{array} \qquad \begin{array}{ccc} \mathrm{Sh}\,(\mathbb{G}) & \xrightarrow{f} & \mathrm{Sh}\,(\mathbb{H}) \\ & {}_{g}\searrow \quad \swarrow_{h} & \\ & \mathrm{Sh}\,(\mathbb{N}_{\sim}) & \end{array}$$

we need to verify that the following squares commute:

$$\begin{array}{ccc} G_0 & \xrightarrow{\eta_0} & \Gamma\circ\Theta(\mathbb{G})_0 \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma\circ\Theta(f)_0} \\ H_0 & \xrightarrow[\eta_0]{} & \Gamma\circ\Theta(\mathbb{H})_0 \end{array} \qquad \begin{array}{ccc} G_1 & \xrightarrow{\eta_1} & \Gamma\circ\Theta(\mathbb{G})_1 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma\circ\Theta(f)_1} \\ H_1 & \xrightarrow[\eta_1]{} & \Gamma\circ\Theta(\mathbb{H})_1 \end{array}$$

(abusing notation somewhat). We do the left square first: given an element $x \in G_0$, $\eta_0(x)$ is the $\Theta(\mathbb{G})$-model which sends an $C \hookrightarrow U^n_{\mathbb{G}}$ in $\Theta(\mathbb{G}) \hookrightarrow \mathrm{Sh}\,(\mathbb{G})$ to the fibre of C over x. Applying $\Gamma \circ \Theta(f_0)$ means composing this model with $f^* \restriction_{\Theta(\mathbb{H})} \colon \Theta(\mathbb{H}) \longrightarrow \Theta(\mathbb{G})$ to obtain the $\Theta(\mathbb{H})$-model which sends an $D \hookrightarrow U^n_{\mathbb{H}}$ in $\Theta(\mathbb{H}) \hookrightarrow \mathrm{Sh}(\mathbb{H})$ to the fibre of $f^*(D)$ over x. But this is precisely the fibre of D over $f_0(x)$. So $\Gamma \circ \Theta(f_0) \circ \eta_0 = \eta_0 \circ f_0$. For the right square, suppose $a : x \to y$ is in G_1. The underlying function of the $\Theta(\mathbb{G})$-isomorphism $\eta_1(a) : \eta_0(x) \to \eta_0(y)$ is the bijections of fibers $\theta_g(a, -) : (e^1_{\mathbb{G}})^{-1}(x) \to (e^1_{\mathbb{G}})^{-1}(y)$, which (modulo reindexing) is the bijection of fibers $\theta_h(f_1(a), -) : (e^1_{\mathbb{H}})^{-1}(f_0(x)) \to (e^1_{\mathbb{H}})^{-1}(f_0(y))$ and it is now straightforward to see that $\Gamma \circ \Theta(f)_1 = \eta_1 \circ f_1$. ⊣

Proposition 3.3.4.13 *Θ is left adjoint to Γ,*

$$\mathbf{FOL}^{\mathrm{op}} \underset{\Gamma}{\overset{\Theta}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Class}$$

Proof We need to verify the triangular identities,

$$\begin{array}{ccc} \Theta(\mathbb{G}) & & \\ \uparrow{\scriptstyle \Theta(\eta_{\mathbb{G}})} & = & \nwarrow{\scriptstyle 1_{\Theta(\mathbb{G})}} \\ \Theta \circ \Gamma \circ \Theta(\mathbb{G}) & \xleftarrow{\varepsilon_{\Theta(\mathbb{G})}} & \Theta(\mathbb{G}) \end{array} \qquad \begin{array}{ccc} \Gamma(\mathcal{C}_{\mathbb{T}}) = \mathbb{G}_{\mathbb{T}} & & \\ \downarrow{\scriptstyle \eta_{\Gamma(\mathcal{C}_{\mathbb{T}})}} & = & \searrow{\scriptstyle 1_{\Gamma(\mathcal{C}_{\mathbb{T}})}} \\ \Gamma \circ \Theta \circ \Gamma(\mathcal{C}_{\mathbb{T}}) & \xrightarrow{\Gamma(\varepsilon_{\Gamma(\mathcal{C}_{\mathbb{T}})})} & \Gamma(\mathcal{C}_{\mathbb{T}}) = \mathbb{G}_{\mathbb{T}} \end{array}$$

That the left triangle commutes is Lemma 3.3.4.11. The right triangle is equally straightforward. ⊣

Definition 3.3.4.14 Let **Sem** be the image of Γ in **Class**, i.e. the subcategory of semantical groupoids over $\mathbb{N}_\sim$ and semantical morphisms between them.

From Lemma 3.3.4.6 and the remarks preceding it, it is clear that **Sem** is a full subcategory of **Class**. We record the following as a now easy consequence of the above:

Proposition 3.3.4.15 *The adjunction $\Gamma \vdash \Theta$ restricts to an equivalence*

$$\mathbf{FOL}^{\mathrm{op}} \simeq \mathbf{Sem}$$

3.4 Stone Fibrations

In this section we give a characterization of a subcategory of $\mathbf{Gpd}/\mathbb{N}_{\sim}$ which contains the image of the 'semantical' functor $\Gamma : \mathbf{FOL} \longrightarrow \mathbf{Gpd}/\mathbb{N}_{\sim}$, and to which the adjunction of Proposition 3.3.4.13 can, therefore, be restricted.

Recall that by a *Stone object* in a topos $\mathcal{E}$, we mean a decidable object such that its frame of subobjects is a Stone frame, that is to say, a compact frame (in the sense that its top element is compact) which has a generating set of complemented elements. Any definable object in a topos of equivariant sheaves on a semantical groupoid $\mathrm{Sh}(\mathbb{G}_{\mathbb{T}})$ is, accordingly, a Stone object, since it is compact decidable and its frame of subobjects is generated by the definable subobjects, which are complemented. Notice that if $f : \mathbb{G} \longrightarrow \mathbb{N}_{\sim}$ is a groupoid over $\mathbb{N}$ such that the finite powers of $f^*(\mathcal{U})$ are Stone objects, then f is in **Class**, as any compact subobject of a Stone object is complemented. We characterize in direct terms a notion of Stone fibration over $\mathbb{N}_{\sim}$ so that the adjunction of Section 3.3 restricts to a contravariant adjunction between the category **FOL** of first-order theories and the full subcategory of $\mathbf{Gpd}/\mathbb{N}_{\sim}$ consisting of Stone fibrations over $\mathbb{N}_{\sim}$. We begin by reviewing the description of a site for the topos of equivariant sheaves on a topological groupoid $\mathbb{G}$ given by Moerdijk in [21, §6]. All groupoids considered in the current section (3.4) are open (recall that all semantical groupoids are open by Lemma 3.2.1.2).

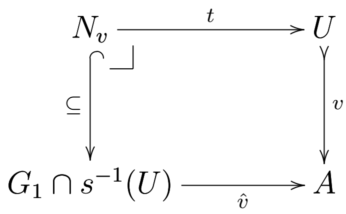
3.4.1 Sites for Groupoids

In the current section, we present the parts of [21, §6] which describe a site for the topos of equivariant sheaves on an open groupoid. The presentation and proofs have been modified to suit our situation—e.g. we are only concerned with topological groupoids and not localic groupoids in general—but the results up to Corollary 3.4.1.9 are essentially Moerdijk's.

Let $\mathbb{G}$ be an open topological groupoid. It follows that composition is an open map, and that the continuous action of any equivariant sheaf over $\mathbb{G}$ is an open map. Let $\langle A \to G_0, \alpha \rangle$ be an equivariant sheaf over $\mathbb{G}$, and $v : U \to A$ a continuous section, for an open subset $U \subseteq G_0$. With

$$\hat{v} = \alpha \circ \langle 1_{G_1}, v \circ s \rangle : G_1 \cap s^{-1}(U) \longrightarrow G_1 \times_{G_0} A \longrightarrow A$$

we take a pullback of topological spaces,

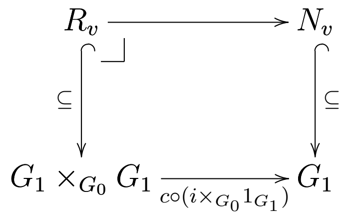


whence $N_v = \{g : x \to y \mid x, y \in U \wedge \alpha(g, v(x)) = v(y)\}$ is such that:

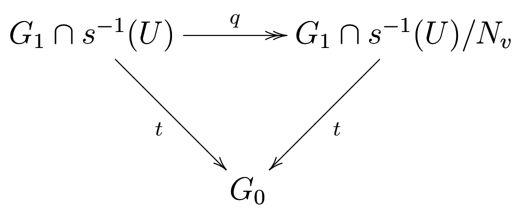
1. N_v is an open subset of $G_1 \cap s^{-1}(U)$ such that $s(N_v), t(N_v) \subseteq U$;

2. N_v is closed under identities of objects in U, as well as under inverses and composition.

We make the following construction from the properties (1) and (2). Take the pullback of spaces



such that $R_v = \{\langle h : y \to z, g : x \to z\rangle \mid h^{-1} \circ g \in N_v\}$ is an open subset of $G_1 \times_{G_0} G_1$. Then R_v is an open equivalence relation on $G_1 \cap s^{-1}(U)$ over G_0, and we take the (fibrewise) quotient,



so that $G_1 \cap s^{-1}(U)/N_v$ has the quotient topology. Then the right t is continuous, so $G_1 \cap s^{-1}(U)/N_v$ is a space over G_0. The right t is also open, since the left t is and q is a surjection. And q is open, since $\pi_1, \pi_2 : R_v \rightrightarrows G_1 \cap s^{-1}(U)$ are open.

Lemma 3.4.1.1 $t : G_1 \cap s^{-1}(U)/N_v \longrightarrow G_0$ *is a local homeomorphism.*

PROOF The diagonal subset of $G_1 \cap s^{-1}(U)/N_v \times_{G_0} G_1 \cap s^{-1}(U)/N_v$ is open because it is the image of the open subset $R_v \subseteq G_1 \cap s^{-1}(U) \times_{G_0} G_1 \cap s^{-1}(U)$ along the open map $q \times_{G_0} q$,

$$\begin{array}{ccc} R_v & \overset{\subseteq}{\hookrightarrow} & G_1 \cap s^{-1}(U) \times_{G_0} G_1 \cap s^{-1}(U) \\ \downarrow & & \downarrow {\scriptstyle q \times_{G_0} q} \\ G_1 \cap s^{-1}(U)/N_v & \underset{\Delta}{\rightarrowtail} & G_1 \cap s^{-1}(U)/N_v \times_{G_0} G_1 \cap s^{-1}(U)/N_v \end{array}$$

⊣

By restricting the continuous insertion on identities, e : $G_0 \to G_1$ and composing with q, we get a continuous section

$$\mathrm{Id} : U \longrightarrow G_1 \cap s^{-1}(U)/N_v.$$

Moreover, there is an action

$$G_1 \times_{G_0} G_1 \cap s^{-1}(U)/N_v \xrightarrow{c} G_1 \cap s^{-1}(U)/N_v$$

defined by composition, which is continuous since the square

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 \cap s^{-1}(U) & \xrightarrow{c} & G_1 \cap s^{-1}(U) \\ {\scriptstyle 1_{G_1} \times_{G_0} q} \downarrow & & \downarrow {\scriptstyle q} \\ G_1 \times_{G_0} G_1 \cap s^{-1}(U)/N_v & \xrightarrow[c]{} & G_1 \cap s^{-1}(U)/N_v \end{array}$$

commutes and q and $1_{G_1} \times_{G_0} q$ are open surjections. Thus $G_1 \cap s^{-1}(U)/N_v$ is an equivariant sheaf over $\mathbb{G}$ as a consequence of the properties (1) and (2) above. Now, the continuous map $\hat{v} : G_1 \cap s^{-1}(U) \to A$ equalizes the parallel pair $\pi_1, \pi_2 : R_v \rightrightarrows G_1 \cap s^{-1}(U)$ so that we have a continuous function $\tilde{v} : G_1 \cap s^{-1}(U)/N_v \to A$, acting by $\tilde{v}([g : x \to y]) = \alpha(g, v(x))$, whence it is clear that $\tilde{v}$ is a morphism of equivariant sheaves such that the triangle

$$\begin{array}{ccc} G_1 \cap s^{-1}(U)/N_v & \xrightarrow{\tilde{v}} & A \\ & {\scriptstyle \mathrm{Id}} \nwarrow \quad \nearrow {\scriptstyle v} & \\ & U & \end{array}$$

commutes. We denote this equivariant sheaf $\langle \mathbb{G}, U, N_v \rangle$, and extend that notation to any equivariant sheaf accordingly induced by open sets $U \subseteq G_0$ and $N \subseteq G_1$ such that $s(N), t(N) \subseteq U$ and N is closed by identity arrows for objects in U, as well as closed by inverses and composition. Thus we have, following [21, 6.1],

Lemma 3.4.1.2 *For a topological groupoid $\mathbb{G}$, equivariant sheaves of the form $\langle \mathbb{G}, U, N \rangle$ form a generating set of objects for* $\mathrm{Sh}(\mathbb{G})$.

Now, consider again the equivariant sheaf $\langle \mathbb{G}, U, N_v \rangle$ induced by the section $v : U \to A$.

Lemma 3.4.1.3 *The map $\tilde{v} : G_1 \cap s^{-1}(U)/N_v \to A$ is 1-1.*

Proof Suppose $\tilde{v}([g : x \to z]) = \tilde{v}([h : y \to z])$. Then

$$\begin{aligned}\alpha(h^{-1} \circ g, v(x)) &= \alpha(h^{-1}, \alpha(g, v(x))) \\ &= \alpha(h^{-1}, \tilde{v}([g : x \to z])) \\ &= \alpha(h^{-1}, \tilde{v}([h : y \to z])) \\ &= \alpha(h^{-1}, \alpha(h, v(y))) \\ &= v(y)\end{aligned}$$

so $g \sim_{N_v} h$, i.e. $[g : x \to z] = [h : y \to z]$. ⊣

Corollary 3.4.1.4 *If the stabilization of $v(U)$ in A is all of A, then $\tilde{v} : G_1 \cap s^{-1}(U)/N_v \to A$ is a homeomorphism.*

Proof An element $\alpha(g, v(x)) \in A$ is the value of $\tilde{v}$ at $[g]$, so in that case, $\tilde{v}$ is a bijective, open, continuous map, whence a homeomorphism. ⊣

Now, if $f : \mathbb{G} \longrightarrow \mathbb{H}$ is a morphism of topological groupoids, the following condition on f ensures that f^* preserves equivariant sheaves of the form $\langle \mathbb{G}, U, N \rangle$.

Lemma 3.4.1.5 *For a morphism of topological groupoids, $f : \mathbb{G} \longrightarrow \mathbb{H}$, if the continuous function*

$$\begin{array}{ccccc} G_1 & \xrightarrow{\langle t, f_1 \rangle} & G_0 \times_{H_0} H_1 & \longrightarrow & H_1 \\ & & \downarrow & & \downarrow t \\ & & G_0 & \xrightarrow[f_0]{} & H_0 \end{array}$$

is a surjection, then $f^*(\langle \mathbb{H}, V, M\rangle) = \langle \mathbb{G}, f_0^{-1}(V), f_1^{-1}(M)\rangle$.

Proof First, consider the continuous section $\mathrm{Id} : V \to H_1 \cap s^{-1}(V)/M$. Pulling back, we get a continuous section

$$U := f_0^{-1}(V) \xrightarrow{v} G_0 \times_{H_0} H_1 \cap s^{-1}(V) =: A$$

where $v(x) = [1_{f_0(x)}]$. Give the action of A the name α, so as to abbreviate $f^*(\langle \mathbb{H}, V, M\rangle)$ to $\langle A \to G_0, \alpha\rangle$. We claim that the stabilization of $v(U)$ is all of A. Given an element of A_y, it is of the form $[g : x' \to f_0(y)]$. Then we have an element $\langle y, g\rangle \in G_0 \times_{H_0} H_1$, and by assumption, there is a $h \in G_1$ such that $f_1(h) = g$, so that, in particular, $h : x \to y$ with $f_0(x) = x'$. Then

$$\begin{aligned} \alpha(h, v(x)) &= f_1(h) \circ [1_{f_0(x)}] \\ &= g \circ [1_{x'}] \\ &= [g] \end{aligned}$$

and so the stabilization of $v(U)$ is all of A. By Corollary 3.4.1.4, the equivariant sheaf $\langle A \to G_0, \alpha\rangle$ is isomorphic to $\langle \mathbb{G}, U, N_v\rangle$, and we see that

$$\begin{aligned} N_v &= \{g : x \to y \mid x, y \in U \wedge \alpha(g, v(x)) = v(y)\} \\ &= \left\{g : x \to y \mid x, y \in f_0^{-1}(V) \wedge f_1(g) \circ [1_{f_0(x)}] = [1_{f_0(y)}]\right\} \\ &= \left\{g : x \to y \mid x, y \in f_0^{-1}(V) \wedge [f_1(g)] = [1_{f_0(y)}]\right\} \\ &= \left\{g : x \to y \mid x, y \in f_0^{-1}(V) \wedge f_1(g) \in M\right\} \\ &= f_1^{-1}(M) \end{aligned}$$

whence $f^*(\langle \mathbb{H}, V, M\rangle) \cong \langle \mathbb{G}, f_0^{-1}(V), f_1^{-1}(V)\rangle$. ⊣

We turn to consider the subobject lattices of equivariant sheaves of the form $\langle \mathbb{G}, U, N\rangle$. As usual, we identity the subobject lattice of an object $\langle A \to G_0, \alpha\rangle$ with the lattice of stable open subsets of A.

Lemma 3.4.1.6 *For an equivariant sheaf of the form* $\langle \mathbb{G}, U, N\rangle$ *over a groupoid* $\mathbb{G}$, *given a stable open subset* $K \subseteq G_1 \cap s^{-1}(U)/N$, *the pullback of spaces*

$$\begin{array}{ccc} K & \xrightarrow{\subseteq} & G_1 \cap s^{-1}(U)/N \\ \uparrow & & \uparrow \mathrm{Id} \\ U_K & \xrightarrow[\subseteq]{} & U \end{array}$$

yields an N*-stable open subset of* U.

PROOF Let $x \in U_K = \{x \in U \mid [1_x] \in K\}$ be given, and suppose $g : x \to y$ is an arrow in N with source x. Then since K is stable, it contains the element $g \circ [1_x] = [g] = [1_y]$, so U_K is N-stable. ⊣

Now, if $V \subseteq U$ is an N-stable subset of U, the restriction of the section $\mathrm{Id} : U \longrightarrow G_1 \cap s^{-1}(U)/N$ to V yields a section we can call $v : V \subseteq U \longrightarrow G_1 \cap s^{-1}(U)/N$, which induces a monomorphism $\tilde{v} : G_1 \cap s^{-1}(V)/N_v \longrightarrow G_1 \cap s^{-1}(U)/N$. Notice that

$$\begin{aligned} N_v &= \{g : x \to y \mid x, y \in V \wedge g \circ [1_x] = [1_y]\} \\ &= \{g : x \to y \mid x, y \in V \wedge g \in N\} \\ &= N \restriction_V \end{aligned}$$

Lemma 3.4.1.7 *For an equivariant sheaf of the form $\langle \mathbb{G}, U, N \rangle$ over a groupoid $\mathbb{G}$, given a stable open subset $K \subseteq G_1 \cap s^{-1}(U)/N$, the equivariant sheaf K is isomorphic to $\langle \mathbb{G}, U_K, N \restriction_{U_K} \rangle$.*

PROOF Denote by $v : U_K \subseteq U \longrightarrow G_1 \cap s^{-1}(U)/N$ the restriction of the section Id to U_K, so that we have a monomorphism

$$\tilde{v} : G_1 \cap s^{-1}(U_K)/N_v = G_1 \cap s^{-1}(U_K)/N \restriction_{U_K} \longrightarrow G_1 \cap s^{-1}(U)/N.$$

We claim that the image of $\tilde{v}$ is $K \subseteq G_1 \cap s^{-1}(U)/N$. The image is contained in K since K is stable and $U_K = \{x \in U \mid [1_x] \in K\}$. Conversely, given $[g : x \to y] \in K$. Then $g^{-1} \circ [g] = [1_x] \in K$, so $x \in U_K$ and $\tilde{v}([g]) = g \circ \tilde{v}([1_x]) = g \circ [1_x] = [g]$. ⊣

Corollary 3.4.1.8 *For an equivariant sheaf of the form $\langle \mathbb{G}, U, N \rangle$ over a groupoid $\mathbb{G}$, there is an isomorphism of frames between the subobject lattice of $\langle \mathbb{G}, U, N \rangle$ and the frame of N-stable open subsets of U.*

PROOF The assignment $K \mapsto U_K$, for stable open $K \subseteq G_1 \cap s^{-1}(U)/N$, is by pullback and so is a frame morphism which is now straightforwardly verified to be bijective. ⊣

Corollary 3.4.1.9 *A decidable equivariant sheaf of the form $\langle \mathbb{G}, U, N \rangle$ over a groupoid $\mathbb{G}$ is Stone if and only if the N-stable open subsets of U form a Stone frame, i.e. a frame which is compact and generated by complemented objects.*

3.4.2 Stone Fibrations over $\mathbb{N}_\sim$

Consider the generic classical object, $\mathcal{U}$, in $\mathbb{N}_\sim$, or isomorphically, the equivariant sheaf $\langle E_{[x\,|\,\top]} \to X_{\mathbb{T}_=}, \theta_{[x\,|\,\top]}\rangle$. For $n \in \mathbb{N}$, say $n = 0$, the global section $0 : X_{\mathbb{T}_=} \to E_{[x\,|\,\top]}$ defined by $\mathbf{M} \mapsto [0] \in |\mathbf{M}|$ is such that the stabilization of $0(X_{\mathbb{T}_=}) \subseteq E_{[x\,|\,\top]}$ is all of $E_{[x\,|\,\top]}$, by Lemma 3.1.2.1. Accordingly, $\mathcal{U}$ is isomorphic to the equivariant sheaf $\langle \mathbb{G}_{\mathbb{T}_=}, X_{\mathbb{T}_=}, N_0\rangle$, where

$$N_0 = \{f : \mathbf{M} \to \mathbf{N} \mid f([0]) = [0]\} = V_-^-(0,0) \subseteq G_{\mathbb{T}_=}.$$

Similarly, for any tuple of distinct numbers, say $\langle 0, 1, \ldots, n-1\rangle$, the section $\langle 0, 1, \ldots, n\rangle : X_{\mathbb{T}_=} \longrightarrow E_{[x_0,\ldots,x_n\,|\,\top]}$ generates all of $E_{[x_0,\ldots,x_n\,|\,\top]}$, so $\mathcal{U}^{n+1} \cong \langle \mathbb{G}_{\mathbb{T}_=}, X_{\mathbb{T}_=}, N_{\langle 0,\ldots,n\rangle}\rangle$.

Definition 3.4.2.1 Let $f : \mathbb{H} \longrightarrow \mathbb{N}_\sim$ be a groupoid over $\mathbb{N}_\sim$ with $\mathbb{H}$ an open groupoid. We say that f is an *fibration* of groupoids if the continuous function

$$\begin{array}{ccccc} G_1 & \xrightarrow{\langle t, f_1\rangle} & G_0 \times_{H_0} H_1 & \longrightarrow & H_1 \\ & & \downarrow & & \downarrow t \\ & & G_0 & \xrightarrow[f_0]{} & H_0 \end{array} \tag{3.16}$$

is a surjection. We say that $f : \mathbb{H} \longrightarrow \mathbb{N}_\sim$ is a *Stone fibration* over $\mathbb{N}_\sim$ if, in addition, for every $n \in \mathbb{N}$, the frame of $f_1^{-1}(N_{\langle 0,\ldots,n\rangle})$-stable open subsets of H_0 is a Stone frame, and the frame of H_1-stable subsets of H_0 is a Stone frame.

Note that the definition of fibration in [21] requires the map $\langle t, f_1\rangle$ of 3.16 to be open as well as surjective, while for our purposes this is not needed.

Lemma 3.4.2.2 *If $f : \mathbb{H} \longrightarrow \mathbb{N}_\sim$ is a Stone fibration over $\mathbb{N}_\sim$, then $f^*(\mathcal{U})^n$ is a Stone object for all $n \in \mathbb{N}$.*

Proof $f^*(\mathcal{U})^n$ is automatically decidable, since $\mathcal{U}^n$ is. By Lemma 3.4.1.5, the object $f^*(\mathcal{U})^{n+1}$ is isomorphic to $\langle \mathbb{H}, H_0, f_1^{-1}(N_{\langle 0,\ldots,n\rangle})\rangle$, which is a Stone object if and only if it is decidable and the frame of $f_1^{-1}(N_{\langle 0,\ldots,n\rangle})$-stable open subsets of H_0 is a Stone frame, by Lemma 3.4.1.6. Similarly, the terminal object 1 is Stone if and only if the frame of H_1-stable subsets of H_0 is a Stone frame. $\dashv$

Corollary 3.4.2.3 *Any Stone fibration over* $\mathbb{N}_\sim$ *is an object of* **Class**.

Proof For any arrow $f : A \longrightarrow B$ between Stone objects in a topos, if $S \rightarrowtail A$ is complemented then it is compact since A is compact, whence the image $f(S) \rightarrowtail B$ is compact, and therefore complemented since the subobjects of B are generated by complemented elements. $\dashv$

Lemma 3.4.2.4 *For any theory* $\mathbb{T}$ *in* **FOL**, *the forgetful morphism* $u_{\mathbb{T}} : \mathbb{G}_{\mathbb{T}} \longrightarrow \mathbb{G}_{\mathbb{T}_=} \cong \mathbb{N}_\sim$ *is a Stone fibration.*

Proof Given a $\mathbb{T}$-model $\mathbf{N}$ and a bijection $g : M \to |\mathbf{N}|$, the bijection g induces a $\mathbb{T}$-structure on M making it a $\mathbb{T}$-model, $\mathbf{M}$, with g a $\mathbb{T}$-model isomorphism $G : \mathbf{M} \to \mathbf{N}$. So $u_{\mathbb{T}}$ is an fibration. It is a Stone fibration in virtue of $u^*(\mathcal{U}^n)$ being a Stone object. $\dashv$

Definition 3.4.2.5 Let **Stone** be the full subcategory of $\mathbf{Gpd}/\mathbb{N}_\sim$ consisting of Stone fibrations.

Then, since **Stone** is a full subcategory of **Class** and Γ factors through **Stone**, the adjunction

$$\mathbf{FOL}^{\mathrm{op}} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Class} \quad (\Theta \dashv \Gamma)$$

of Proposition 3.3.4.13 restricts to **Stone**. Therefore, we have:

Theorem 3.4.2.6 *There is a contravariant adjunction*

$$\mathbf{FOL}^{\mathrm{op}} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Stone} \quad (\Theta \dashv \Gamma)$$

between the category of first-order theories (with a saturated set of hereditarily countable models) and the category of Stone fibrations over $\mathbb{N}_\sim$. *The adjunction restricts to the (full) image,* **Sem** *of* Γ *to an equivalence,*

$$\mathbf{FOL}^{\mathrm{op}} \simeq \mathbf{Sem}$$

3.5 Groupoids and Theories

> *The most interesting phenomena in model theory are conclusions concerning the syntactical structure of a first order theory drawn from the examination of the models of the theory. With these phenomena in mind, it is natural to ask if it is possible to endow the collection of models of the theory with a natural abstract structure so that from the resulting entity one can fully recover the theory as a syntactical structure.*[17, p.97]

3.5.1 Applications and Future Work

The current section indicates some areas of possible application of, and future work expanding on, the representation and duality results presented in this thesis. Section 2.5 displayed one example of possible use of these results, namely to obtain groupoid representations of certain syntactically given topoi, and thus uncovering mathematically interesting objects such as the groupoid $\mathbb{O}$ together with its topos of equivariant sheaves $\mathrm{Sh}(\mathbb{O})$. The intended primary area of application for the duality constructed in this thesis, however, is the relationship between first-order syntax and semantics, and the study of first-order theories, and of relations between first-order theories, in terms of their semantics. Using the duality between semantical groupoids and syntactical categories, questions concerning the syntactical structure of, or relations between, theories can be translated into questions about their respective semantical topological groupoids, or about continuous morphisms between such groupoids, and studied using topological and sheaf theoretical techniques. In the same vein, a motivation for the work presented in this thesis is the expectation that the algebraic and geometrical techniques employed and the interconnectedness of algebra, geometry, syntax, and semantics that we have, at least to some extent, demonstrated may be of both conceptual and technical interest for model theoretic issues concerning e.g. definability and invariance.

Consider for instance the Beth Definability Theorem, which we for present purposes can take as stating that if (1) $\mathbb{S}$ is a conservative extension of $\mathbb{T}$,

$$\mathcal{C}_{\mathbb{T}} \overset{F}{\hookrightarrow} \mathcal{C}_{\mathbb{S}}$$

and; (2) every $\mathbb{T}$-isomorphism between $\mathbb{S}$-models, $g : \mathbf{M} \to \mathbf{N}$, is a $\mathbb{S}$-

isomorphism, then $\mathbb{T}$ and $\mathbb{S}$ are the same theory, up to definability, i.e.

$$F : \mathcal{C}_{\mathbb{T}} \xrightarrow{\cong} \mathcal{C}_{\mathbb{S}}$$

The second condition in the antecedent states a property (fullness) of the semantic morphism of groupoids $f : \mathbb{G}_{\mathbb{S}} \longrightarrow \mathbb{G}_{\mathbb{T}}$ (and as we shall see in Lemma 3.5.2.6 the first condition can also be seen as a statement of such a property) while the conclusion is one of the syntactic relationship between the theories $\mathbb{S}$ and $\mathbb{T}$. Section 3.5.2 briefly indicates some first steps towards studying theory extensions in terms of morphisms of semantic groupoids; and using only the most immediate consequences of the duality, a weaker version (Proposition 3.5.2.7) of the Beth Definability Theorem is quickly arrived at. Applying the full power of the theory of Grothendieck topoi and topological groupoids, perhaps with some adjustment and development to suit this particular area of application, one can hope to obtain more substantial results.

On a related note, the connection between sub-theories of countable categorical theories and closed subgroups of the automorphism groups of their respective countable models have already been subjected to some study (see e.g. [25, 26]). Our setup offers the possibility to extent this type of investigation to arbitrary theories by considering, instead of the group of automorphisms of *a* model, the groupoid of isomorphisms of *all* (countable or suitably restricted) models. Again we indicate some first steps in Section 3.5.2, establishing a Galois connection between sub-theories of a first-order theory and certain 'intermediate' groupoids.

We hasten to point to the results obtained in these areas by Makkai and Zawadowski. Using his duality theory for Boolean pretopoi and ultragroupoids, Makkai's main result is a Descent Theorem for Boolean Pretopoi ([19]), to the effect that the coregular factorization in the groupoid enriched category of Boolean pretopoi is the quotient/conservative factorization; a result from which the Beth Definability Theorem follows. The same result for the 2-category of pretopoi was earlier shown by Zawadowski ([27]) using Makkai's duality for pretopoi and ultra-categories ([17]). Here, a morphism $F : \mathcal{C} \longrightarrow \mathcal{D}$ of (Boolean) pretopoi is a quotient if it is full on subobjects and for any object $D \in \mathcal{D}$ there is an object $C \in \mathcal{C}$ and a cover $F(C) \longrightarrow\!\!\!\!\triangleright D$ ([19]). It also falls within the category of future work to investigate to what extent and in what respect the results of Makkai and Zawadowski can be reproduced, and perhaps further developed, within the more geometrical duality theory of this thesis.

3.5.2 Groupoids and Theory Extensions

We use the 'syntax-semantics' adjunction

$$\mathbf{FOL}^{\mathrm{op}} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Class} \quad (\Gamma \dashv \Theta)$$

to investigate some properties of morphisms of groupoids corresponding to certain familiar properties of morphisms of theories. Let $\mathbb{T}$ and $\mathbb{S}$ be (countable) single-sorted theories. In this section, whenever we have a (single-sort and inclusion preserving) coherent functor

$$F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$$

we shall assume that $\mathbb{S}$ is a theory extension of $\mathbb{T}$, so that $\mathcal{L}_{\mathbb{T}} \subseteq \mathcal{L}_{\mathbb{S}}$ and $F([\vec{x} \mid \phi]) = [\vec{x} \mid \phi]$ (and we refer to F as an extension). We loose no generality in so doing, since we can construct a theory $\mathbb{S}'$ extending $\mathbb{T}$ such that $\mathcal{C}_{\mathbb{S}} \cong \mathcal{C}_{\mathbb{S}'}$ by setting $\mathcal{L}_{\mathbb{S}'} = \mathcal{L}_{\mathbb{S}} \cup \mathcal{L}_{\mathbb{T}}$ and taking the axioms of $\mathbb{S}'$ to be the axioms of $\mathbb{S}$ together with $\forall \vec{x}.\, \phi \leftrightarrow \psi$ for all formulas $\phi \in \mathcal{L}_{\mathbb{T}}$ and $\phi \in \mathcal{L}_{\mathbb{S}}$ such that $F([\vec{x} \mid \phi]) = [\vec{x} \mid \psi]$.

Definition 3.5.2.1 The *quotient-conservative factorization* of F is the factorization

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \xrightarrow{F} & \mathcal{C}_{\mathbb{S}} \\ & {}_{G}\searrow \quad \nearrow_{H} & \\ & \mathcal{C}_{\mathbb{T}^{\mathbb{S}}} & \end{array}$$

where $\mathbb{T}^{\mathbb{S}}$ is the theory $\{\phi \in \mathcal{L}_{\mathbb{T}} \mid \mathbb{S} \vdash \phi\}$ and G and H are the obvious functors, with G surjective on objects and H a conservative subcategory inclusion (up to isomorphism).

We call F a *quotient* extension if H is an isomorphism. If G is an isomorphism, we call F a conservative extension. If we pass from coherent categories into topoi by taking sheaves (for the coherent coverage), the quotient-conservative factorization of $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$ becomes the surjective-embedding factorization of the corresponding geometric morphism:

$$\begin{array}{ccc} \mathrm{Sh}(\mathcal{C}_{\mathbb{T}}) & \xleftarrow{f} & \mathrm{Sh}(\mathcal{C}_{\mathbb{S}}) \\ & {}^{g}\nwarrow \quad \swarrow^{h} & \\ & \mathrm{Sh}(\mathcal{C}_{\mathbb{T}^{\mathbb{S}}}) & \end{array}$$

Both quotients and conservative extensions can be characterized in terms of their corresponding semantical morphisms in the category of topological groupoids.

Lemma 3.5.2.2 *Let $\mathbb{T}$ be a theory, $G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}}$ its semantical groupoid. Any non-empty subset $A \subseteq X_{\mathbb{T}}$ determines a quotient theory extension*

$$F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{T}_{\mathbb{A}}}$$

such that the semantical groupoid of $\mathbb{T}_A$ is the full subgroupoid determined by the closure of A in the stable topology. That is to say, such that $\Gamma(F) : \mathbb{G}_{\mathbb{T}_A} \longrightarrow \mathbb{G}_{\mathbb{T}}$ is full as a functor of categories, the components f_1 and f_0 of $\Gamma(F)$,

$$\begin{array}{ccc} G_{\mathbb{T}_A} & \xrightarrow{f_1} & G_{\mathbb{T}} \\ \downarrow\downarrow & & \downarrow\downarrow \\ X_{\mathbb{T}_A} & \xrightarrow[f_0]{} & X_{\mathbb{T}} \end{array}$$

are both closed subset inclusions, and $f_0(X_{\mathbb{T}_A})$ is the least stable and closed set containing A.

Proof Let $A \subseteq X_{\mathbb{T}}$ be given. Set $\mathbb{T}_A$ to be the theory

$$\mathbb{T}_A := \{\phi \in \mathcal{L}_{\mathbb{T}} \mid \forall M \in A.\, M \vDash \phi\}$$

in the same language as $\mathbb{T}$, and let $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{T}_{\mathbb{A}}}$ be the obvious quotient. Then $X_{\mathbb{T}_A}$ is the stable subset $\{M \in X_{\mathbb{T}} \mid M \vDash \mathbb{T}_A\} \subseteq X_{\mathbb{T}}$. $X_{\mathbb{T}_A}$ is a closed subset since for any $M \notin X_{\mathbb{T}_A}$, there is a sentence ϕ such that $\mathbb{T}_A \vdash \phi$ but $M \nvDash \phi$, and so M is an element of $U_{\neg\phi}$ which is open and does not intersect $X_{\mathbb{T}_A}$. Now, any stable closed subset, C, of $X_{\mathbb{T}}$ that contains a model M must contain the elementary equivalence class of M. For if $N \equiv M$ but $N \notin C$, then there exists an open set $N \in U = U_{\phi(\vec{n}/\vec{x})}$ such that $U \cap C = \emptyset$. Since $N \equiv M$, $M \vDash \exists\vec{x}.\, \phi$, so $M \vDash \phi(\vec{m}/\vec{x})$ for some $\vec{m}$, and by an application of Lemma 3.1.2.1, there exists isomorphic $M' \cong M$ such that $M' \in U$, contradicting that C is stable and $U \cap C = \emptyset$. Now, $X_{\mathbb{T}_A}$ is the union of the elementary equivalence classes of elements of A, and must therefore be the least stable closed set containing A. It is clear that the logical topology on $X_{\mathbb{T}_A}$ is just the restriction of the logical topology on $X_{\mathbb{T}}$, so that the restriction function $f_0 : X_{\mathbb{T}_A} \to X_{\mathbb{T}}$ is a closed inclusion. And finally, since any $\mathbb{T}$-isomorphism between $\mathbb{T}_A$-models is a $\mathbb{T}_A$-isomorphism, we have that $f_1 : G_{\mathbb{T}_A} \to G_{\mathbb{T}}$ is the closed inclusion of the subset $\{g : M \to N \mid M, N \in X_{\mathbb{T}_A}\} \subseteq G_{\mathbb{T}}$. $\dashv$

Corollary 3.5.2.3 *If $\mathbb{H}$ is any topological groupoid with a morphism $f : \mathbb{H} \longrightarrow \mathbb{G}_{\mathbb{T}}$ which is full as a functor of categories, such that the components*

$$\begin{array}{ccc} H_1 & \xrightarrow{f_1} & G_{\mathbb{T}} \\ \downarrow\downarrow & & \downarrow\downarrow \\ H_0 & \xrightarrow[f_0]{} & X_{\mathbb{T}} \end{array}$$

are both injective and closed, and with $f_0(H_0)$ stable in $X_{\mathbb{T}}$, then $\mathbb{H}$ is, up to isomorphism, the semantical groupoid of a quotient theory extension of $\mathbb{T}$.

For reference, we also note the following from the proof of Lemma 3.5.2.2:

Scholium 3.5.2.4 *In the stable topology on $X_{\mathbb{T}}$, the irreducible closed sets are precisely the elementary equivalence classes.*

Thus we have proved the following:

Proposition 3.5.2.5 *The quotients of a theory $\mathbb{T}$ correspond bijectively to closed and stable subsets (or closed subsets in the stable topology) of $X_{\mathbb{T}}$.*

We pass to conservative theory extensions. The following observation is immediate:

Lemma 3.5.2.6 *An extension $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$ is conservative if and only if the corresponding semantical morphism $f : \mathbb{G}_{\mathbb{S}} \longrightarrow \mathbb{G}_{\mathbb{T}}$ is such that the object component $f_0 : X_{\mathbb{S}} \to X_{\mathbb{T}}$ is surjective on irreducible closed sets in the stable topology, in the sense that for any $C \subseteq X_{\mathbb{T}}$ which is irreducible closed in the stable topology, $f_0(X_{\mathbb{S}}) \cap C \neq \emptyset$.*

Proof By Scholium 3.5.2.4, f_0 is surjective on irreducible closed sets (in the stable topology) if and only if there exists a $\mathbb{T}$-model in each elementary equivalence class of $X_{\mathbb{T}}$ which is the reduct of a $\mathbb{S}$-model. Which is saying that $\mathbb{T}^{\mathbb{S}} = \mathbb{T}$. ⊣

The following statement is now an easy consequence of Proposition 3.5.2.5 and Lemma 3.5.2.6:

Proposition 3.5.2.7 *For any extension $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$, if the object component $f_0 : X_{\mathbb{S}} \to X_{\mathbb{T}}$ is a closed injection and surjective on irreducible closed sets in the stable topology then F is an isomorphism.*

PROOF Since F is surjective on irreducible closed sets in the stable topology, F is a conservative extension by Lemma 3.5.2.6. Now, the image $f_0(X_{\mathbb{S}}) \subseteq X_{\mathbb{T}}$ is closed by assumption. It is stable since if $\mathbf{M} \in f_0(X_{\mathbb{S}})$ and $\mathbf{N} \vDash \mathbb{T}$, then any bijection $g : \mathbf{M} \to \mathbf{N}$ induces a $\mathbb{S}$-structure on $\mathbf{N}$ such that g is an $\mathbb{S}$-isomorphism. The same fact combined with f_0 being injective tells us that $f : \mathbb{G}_{\mathbb{S}} \longrightarrow \mathbb{G}_{\mathbb{T}}$ is full (as well as automatically faithful) as a functor of categories. Now, since $f_0(X_{\mathbb{S}})$ is closed and stable, it corresponds to a quotient theory extension $\mathbb{T}'$ of $\mathbb{T}$ by Proposition 3.5.2.5. Since f is closed, the spaces $X_{\mathbb{S}} \cong X_{\mathbb{T}'}$ are homeomorphic, and f full and faithful, it follows that $\mathbb{G}_{\mathbb{S}} \cong \mathbb{G}_{\mathbb{T}'}$ so that $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$ is both a conservative and a quotient extension, i.e. an isomorphism. ⊣

By the Beth Definability Theorem, however, we know that it is sufficient that the object component $f_0 : X_{\mathbb{S}} \to X_{\mathbb{T}}$ is 1-1; that is, we can state the Beth Definability Theorem as follows:

Theorem 3.5.2.8 (Beth) *For a conservative theory extension $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$ if $f_0 : X_{\mathbb{S}} \to X_{\mathbb{T}}$ is injective, then F is an isomorphism. That is to say, $\mathbb{S}$ is the same theory as $\mathbb{T}$, up to definability.*

PROOF Another formulation of Beth's definability theorem is to say that F is an isomorphism if for any bijection between $\mathbb{S}$-models, $h : M \to N$, if h is a $\mathbb{T}$-isomorphism, then h is a $\mathbb{S}$-isomorphism. This implies that f_0 is injective, since if M and N are $\mathbb{S}$-models with the same underlying $\mathbb{T}$-model, then the identity is a $\mathbb{T}$-isomorphism. Conversely, suppose $h : M \to N$ is a $\mathbb{T}$-isomorphism between $\mathbb{S}$-models. h and M induces a $\mathbb{S}$-structure on the underlying set of N, which must be the $\mathbb{S}$-structure of N since f_0 is injective. So h is a $\mathbb{S}$-isomorphism. ⊣

Corollary 3.5.2.9 *An extension $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{S}}$ is an isomorphism if and only if the corresponding semantical morphism $f : \mathbb{G}_{\mathbb{S}} \longrightarrow \mathbb{G}_{\mathbb{T}}$ is such that the component $f_0 : X_{\mathbb{S}} \to X_{\mathbb{T}}$ is injective and surjective on irreducible closed sets in the stable topology.*

We conclude this section by briefly pointing out how the subtheories of a theory $\mathbb{S}$, i.e. the theories $\mathbb{T}$ of which $\mathbb{S}$ is a conservative extension, $\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{C}_{\mathbb{S}}$, relate to certain intermediate groupoids of bijections between $\mathbb{S}$-models. Let $\mathbb{S}$ be a theory. There is the unique extension $U : \mathcal{C}_{\mathbb{T}_=} \longrightarrow \mathcal{C}_{\mathbb{S}}$, which factors through a conservative extension $U : \mathcal{C}_{\mathbb{S}_=} \hookrightarrow \mathcal{C}_{\mathbb{S}}$. The theory

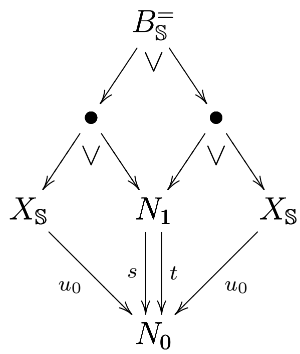
$\mathbb{S}_=$—the theory of equality in $\mathbb{S}$—is, in a sense, the least theory of which $\mathbb{S}$ is a conservative extension: for any theory $\mathbb{T}$ of which $\mathbb{S}$ is a conservative extension, $U : \mathcal{C}_{\mathbb{S}_=} \hookrightarrow \mathcal{C}_{\mathbb{S}}$ factors through $\mathcal{C}_{\mathbb{T}}$,

$$\mathcal{C}_{\mathbb{S}_=} \hookrightarrow \mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{C}_{\mathbb{S}}$$

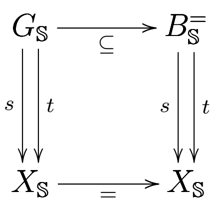
Now, consider the topological groupoid, $\mathbb{B}_{\mathbb{S}}^{=}$, of $\mathbb{S}$-models and bijections between them,

$$\{g_{M,N} : |M| \to |N| \mid g \text{ is bijective, } M, N \in X_{\mathbb{S}}\} =: B_{\mathbb{S}}^{=} \underset{t}{\overset{s}{\rightrightarrows}} X_{\mathbb{S}}$$

with the topology on $B_{\mathbb{S}}^{=}$ generated by basic opens, $V_{\psi}^{\phi}(\vec{m}, \vec{n})$, specified in the same way as for $G_{\mathbb{S}}$. $B_{\mathbb{S}}^{=}$ together with the source and target maps can also be obtained by taking the following pullbacks in **Top**:



It is straightforward, and indeed entirely similar to the proof of Lemma 3.2.1.1, to verify that $\mathbb{B}_{\mathbb{S}}^{=}$ is a topological groupoid. We also immediately see from the construction that $G_{\mathbb{S}}$ is a subspace of $B_{\mathbb{S}}^{=}$. We thereby have a morphism of topological groupoids (over $\mathbb{N}_{\sim}$):



Moreover, if $\mathbb{S}$ is a conservative extension of a theory $\mathbb{T}$,

$$\mathcal{C}_{\mathbb{S}_=} \hookrightarrow \mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{C}_{\mathbb{S}}$$

then we can construct the groupoid, $\mathbb{B}_{\mathbb{S}}^{\mathbb{T}}$ of $\mathbb{S}$-models and $\mathbb{T}$-isomorphisms in the same way,

$$\{g_{M,N} : |M| \to |N| \mid g \text{ is a } \mathbb{T}\text{-isomorphism}, M, N \in X_{\mathbb{S}}\} =: B_{\mathbb{S}}^{\mathbb{T}} \underset{t}{\overset{s}{\rightrightarrows}} X_{\mathbb{S}}$$

to obtain a subspace $G_{\mathbb{S}} \subseteq B_{\mathbb{S}}^{\mathbb{T}} \subseteq B_{\mathbb{S}}^{=}$, and morphisms of topological groupoids (over $\mathbb{N}_{\sim}$):

$$\begin{array}{ccccc}
G_{\mathbb{S}} & \xrightarrow[\subseteq]{} & B_{\mathbb{S}}^{\mathbb{T}} & \xrightarrow[\subseteq]{} & B_{\mathbb{S}}^{=} \\
s \downarrow\downarrow t & & s \downarrow\downarrow t & & s \downarrow\downarrow t \\
X_{\mathbb{S}} & \xrightarrow[=]{} & X_{\mathbb{S}} & \xrightarrow[=]{} & X_{\mathbb{S}}
\end{array}$$

Thus, $\mathbb{G}_{\mathbb{S}} = \mathbb{T}_{\mathbb{S}}^{\mathbb{S}}$ (and $\mathbb{B}_{\mathbb{S}}^{=} = \mathbb{B}_{\mathbb{S}}^{\mathbb{S}_{=}}$).

Definition 3.5.2.10 Say that $\mathbb{T}$ is a *sub-theory* of $\mathbb{S}$ if $\mathbb{S}$ is a conservative extension of $\mathbb{T}$.

We see, then, that sub-theories $\mathbb{T}$ of $\mathbb{S}$, $\mathcal{C}_{\mathbb{S}_{=}} \hookrightarrow \mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{C}_{\mathbb{S}}$, correspond to certain subspaces $B_{\mathbb{S}}^{\mathbb{T}}$ of $B_{\mathbb{S}}^{=}$ that contain $G_{\mathbb{S}}$ and are closed under inverses and composition. Conversely, given any subset $K \subseteq B_{\mathbb{S}}^{=}$ that contains $G_{\mathbb{S}}$ and is closed under inverses and composition, we can consider K as a subspace and obtain a topological groupoid, $\mathbb{K}$, and morphisms (over $\mathbb{N}_{\sim}$) as displayed in the following diagram:

$$\begin{array}{ccccccc}
G_{\mathbb{S}} & \xrightarrow[\subseteq]{} & K & \xrightarrow[\subseteq]{} & B_{\mathbb{S}}^{=} & \xrightarrow{} & N_1 \\
s \downarrow\downarrow t & & s \downarrow\downarrow t & & s \downarrow\downarrow t & & s \downarrow\downarrow t \\
X_{\mathbb{S}} & \xrightarrow[=]{} & X_{\mathbb{S}} & \xrightarrow[=]{} & X_{\mathbb{S}} & \xrightarrow[o_0]{} & N_0
\end{array} \tag{3.17}$$

Lemma 3.5.2.11 *$\mathbb{K}$ is a classical groupoid over $\mathbb{N}_{\sim}$.*

PROOF Observe that $E_{\mathbb{G}_{\mathbb{S}}}^{n} = E_{[\vec{x} \mid \top]} = E_{\mathbb{K}}^{n}$ (identity of spaces) and that an open subset of $E_{\mathbb{K}}^{n}$ which is stable with respect to the action of K must also be stable with the respect to the action of $G_{\mathbb{S}}$. Thus we can immediately conclude that $U_{\mathbb{K}}^{n}$ is compact and decidable. And a complemented subobject $S \rightarrowtriangle U_{\mathbb{K}}^{n+1}$, represented by a clopen K-stable subset $S \subseteq E_{\mathbb{K}}^{n+1}$, is also $G_{\mathbb{S}}$-stable, whence its image along a projection $\pi : U_{\mathbb{K}}^{n+1} \longrightarrow U_{\mathbb{K}}^{n}$ is a clopen set, and since the image is automatically K-stable we conclude that $\exists_{\pi}(S) \rightarrowtriangle E_{\mathbb{K}}^{n}$ is a complemented subobject. ⊣

The morphisms of topological groupoids in (3.17) induces a commuting diagram of geometric morphisms,

$$\begin{array}{ccccc} \mathrm{Sh}_{G_{\mathbb{S}}}(X_{\mathbb{S}}) & \overset{f_K}{\twoheadrightarrow} & \mathrm{Sh}_K(X_{\mathbb{S}}) & \twoheadrightarrow & \mathrm{Sh}_{B_{\mathbb{S}}^{=}}(X_{\mathbb{S}}) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Sh}\,(\mathbb{N}_{\sim}) & & \end{array}$$

with f_K a surjection. Thus the set K gives rise to a theory and a conservative subcategory inclusion $F_K : \Theta(\mathbb{K}) = \mathcal{C}_{\mathbb{K}} \hookrightarrow \mathcal{C}_{\mathbb{S}}$. We have, then, operations

$$\begin{array}{c} \mathfrak{T} := \{\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{C}_{\mathbb{S}} \mid \mathbb{T} \text{ a sub-theory of } \mathbb{S}\} \\ J \uparrow \quad \downarrow I \\ \mathfrak{G} := \{K \subseteq B_{\mathbb{S}} \mid G_{\mathbb{S}} \subseteq K \text{and } K \text{ is closed under inverses and composition}\} \end{array}$$

which are contravariant on the natural partial ordering of these sets. We refer to $\mathfrak{T}$ as the set of *sub-theories* and $\mathfrak{G}$ as the set of *intermediate groupoids.* The following are direct consequences of the setup:

Theorem 3.5.2.12 *$J \circ I$ is a closure operator on $\mathfrak{T}$ with respect to which objects in the image of J are closed. Symmetrically, $I \circ J$ is a closure operator on $\mathfrak{G}$ with respect to which objects in the image of I are closed. Consequently, I and J are adjoint,*

$$\mathfrak{T}^{\mathrm{op}} \underset{I}{\overset{J}{\rightleftarrows}} \mathfrak{G} \qquad (J \dashv I)$$

and I and J form a Galois correspondence.

Proof Let $\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{C}_{\mathbb{S}}$ be given. Consider the theory $J \circ I(\mathcal{C}_{\mathbb{T}}) = \Theta(\mathbb{B}_{\mathbb{S}}^{\mathbb{T}}) = \mathcal{C}_{\mathbb{B}_{\mathbb{S}}^{\mathbb{T}}} \hookrightarrow \mathcal{C}_{\mathbb{S}}$. Now, any $[\vec{x} \mid \phi] \in \mathcal{C}_{\mathbb{T}}$ defines a clopen subset of $E_{[\vec{x} \mid \top]} \to X_{\mathbb{S}}$ which is stable under $B_{\mathbb{S}}^{\mathbb{T}}$—i.e. stable under those bijections between $\mathbb{S}$-models which are $\mathbb{T}$-isomorphisms—and therefore $\mathbb{T}$ is a sub-theory of $\mathcal{C}_{\mathbb{B}_{\mathbb{S}}^{\mathbb{T}}}$:

$$\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{C}_{\mathbb{B}_{\mathbb{S}}^{\mathbb{T}}} \hookrightarrow \mathcal{C}_{\mathbb{S}}$$

Since any $\mathbb{T}$-isomorphism between $\mathbb{S}$-models is also a $\mathcal{C}_{\mathbb{B}_{\mathbb{S}}^{\mathbb{T}}}$-isomorphism, we see that $I(\mathcal{C}_{\mathbb{T}}) = \mathbb{B}_{\mathbb{S}}^{\mathbb{T}} = I(\mathcal{C}_{\mathbb{B}_{\mathbb{S}}^{\mathbb{T}}}) = I \circ J \circ I(\mathcal{C}_{\mathbb{T}})$, whence $J \circ I(\mathcal{C}_{\mathbb{T}}) = J \circ I \circ J \circ I(\mathcal{C}_{\mathbb{T}})$.

On the other hand, for any $G_{\mathbb{S}} \subseteq K \subseteq B_{\mathbb{S}}^{=}$ that is closed under inverses and composition, we have $K \subseteq I \circ J(K)$ as a direct consequence of the construction of I and J, as well as $J(K) = J(I \circ J(K))$, whence $I \circ J \circ I \circ J(K) = I \circ J(K)$. Thus we conclude that $J \circ I$ is a closure operation with respect to which objects in the image of J are closed and $I \circ J$ is a closure operation with respect to which objects in the image of I are closed. $\dashv$

As is always the case for a Galois correspondence, we have:

Corollary 3.5.2.13 *I and J are inverse when restricted to closed sub-theories and closed intermediate groupoids (with respect to the closure operators $J \circ I$ and $I \circ J$ respectively).*

Appendix A

An Argument by Descent

A.1 An Argument by Descent

Let a countable, classical, single-sorted theory with no empty models be given. We show how Proposition 3.2.2.10 can be derived for $\mathbb{T}$ using a descent-theoretic argument, and characterize in the process the topos of sheaves $\mathrm{Sh}\,(X_{\mathbb{T}})$ as the classifying topos for the geometric propositional theory of a $\mathbb{T}$-model on a quotient of $\mathbb{N}$. Again, we refer to the topological groupoid of models as $\mathbb{G}_{\mathbb{T}}$ (or simply $\mathbb{G}$):

$$
G \times_X G \;\begin{array}{c} \xrightarrow{\;l\;} \\ \xrightarrow{\;c\;} \\ \xrightarrow[\;r\;]{} \end{array}\; G \circlearrowleft^{i} \;\begin{array}{c} \xrightarrow{\;s\;} \\ \xleftarrow{\;\mathrm{Id}\;} \\ \xrightarrow[\;t\;]{} \end{array}\; X
$$

Since the geometric morphism $\mathrm{Sh}\,(X) \twoheadrightarrow \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ is an open surjection it is descent (in the sense of [10], see in particular sections B1.5, B3.4, C5.1, and C5.2), and thus $\mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}})$ is equivalent to the category $\mathbf{Desc}_m(\mathrm{Sh}\,(X)_\bullet)$ of descent data for the 2-truncated simplicial topos $\mathrm{Sh}\,(X)_\bullet$ obtained by taking pullbacks and triple pullbacks of m along itself in the category $\mathcal{TOP}$ of Grothendieck topoi and geometric morphisms,

$$
\begin{array}{c}
\mathrm{Sh}\,(X) \times_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})} \mathrm{Sh}\,(X) \times_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})} \mathrm{Sh}\,(X) \\
\pi_{12} \downarrow \;\; \pi_{13} \downarrow \;\; \downarrow \pi_{23} \\
\mathrm{Sh}\,(X) \times_{\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})} \mathrm{Sh}\,(X) \;\begin{array}{c} \xrightarrow{\;\pi_1\;} \\ \xleftarrow{\;\Delta\;} \\ \xrightarrow[\;\pi_2\;]{} \end{array}\; \mathrm{Sh}\,(X)
\end{array}
\tag{A.1}
$$

We show that this 2-truncated simplicial topos is in fact the truncated simplicial topos $\mathbb{G}_\bullet$ that we obtain by taking sheaves on the groupoid $\mathbb{G}$,

$$\mathrm{Sh}\,(G \times_X G) \underset{\mathrm{r}}{\overset{\mathrm{l}}{\underset{\mathrm{c}}{\rightrightarrows}}} \mathrm{Sh}\,(G) \underset{\mathrm{t}}{\overset{\mathrm{s}}{\underset{\mathrm{Id}}{\leftrightarrows}}} \mathrm{Sh}\,(X) \tag{A.2}$$

It follows that $\mathrm{Sh}\,(\mathcal{C}_\mathbb{T})$ is equivalent to the category of descent data for (A.2), $\mathbf{Desc}(\mathbb{G}_\bullet)$. But that is just the topos of equivariant $\mathcal{G}$-sheaves:

$$\mathrm{Sh}\,(\mathcal{C}_\mathbb{T}) \cong \mathbf{Desc}_m(\mathrm{Sh}\,(X)_\bullet) \cong \mathbf{Desc}(\mathbb{G}_\bullet) \cong \mathrm{Sh}_G(X)$$

To this end, we define three geometric propositional theories that the topoi $\mathrm{Sh}\,(X)$, $\mathrm{Sh}\,(G)$, and $\mathrm{Sh}\,(G \times_X G)$, respectively, classify.

A.1.1 The Theory of Countable $\mathbb{T}$-Models

Recall the theory $\mathbb{T}_\mathbb{N}$ in the language $\mathcal{L}_{\mathbb{T}_\mathbb{N}}$ defined in section 3.1.1. Let the language $\mathcal{L}_\mathbb{P}$ of the geometric propositional theory[1] $\mathbb{P}$ be generated by the set of propositional constants $\{P_\phi \mid \phi \text{ a sentence of } \mathcal{L}_{\mathbb{T}_\mathbb{N}}\}$, and take as axioms:

P1 $P_\phi \vdash P_\psi$ whenever $\mathbb{T}_\mathbb{N} \vdash \phi \to \psi$

P2
- $P_{\phi\wedge\psi} \dashv\vdash P_\psi \wedge P_\psi$
- $P_{\phi\vee\psi} \dashv\vdash P_\psi \vee P_\psi$
- $P_\bot \vdash \bot$
- $\top \vdash P_\top$

P3 $P_{\exists x.\phi} \vdash \bigvee_{n\in\mathbb{N}} P_{\phi(n/x)}$

The syntactic category of the geometric propositional theory $\mathbb{P}$ is a frame, which we denote by $\mathcal{F}_\mathbb{P}$. A two valued model of $\mathbb{P}$ is a frame morphism $\mathcal{F}_\mathbb{P} \longrightarrow \mathbf{2}$, which we also call a *point* of $\mathcal{F}_\mathbb{P}$. Clearly, any model $\mathbf{M} \in X$ determines a point of $\mathcal{F}_\mathbb{P}$, but the converse also holds:

Lemma A.1.1.1 *Suppose $A \subset \mathcal{F}_\mathbb{P}$ is such that*

[1] A coherent propositional theory is geometric if it has infinite disjunctions, i.e. for every set $\{\phi_i \mid i \in I\}$ of formulas, there is a formula $\bigvee_{i\in I} \phi_i$. See [10].

(i) A is an ideal, i.e. a proper subset which is downward closed and closed under finite joins.

(ii) For all sentences ϕ of $\mathcal{L}_{\mathbb{T}_\mathbb{N}}$, $[P_\phi] \in A$ or $[P_{\neg\phi}] \in A$.

(iii) For any sentence of $\mathcal{L}_{\mathbb{T}_\mathbb{N}}$ of the form $\exists x.\, \phi$

$$[P_{\exists x.\phi}] \in A \Leftrightarrow \left\{[P_{\phi(n/x)}] \;\middle|\; n \in \mathbb{N}\right\} \subseteq A$$

then A determines an equivalence relation $\sim$ on $\mathbb{N}$ by $n \sim m \Leftrightarrow [P_{n \neq m}] \in A$ and a $\mathbb{T}$-model $\mathbf{M}$ in X by the interpretation

$$[\![\vec{x} \mid \phi]\!]^{\mathbf{M}} = \left\{[\vec{n}] \in \mathbb{N}/_{\sim} \;\middle|\; [P_{\phi(\vec{n}/\vec{x})}] \notin A\right\} \tag{A.3}$$

for any formula-in-context $[\vec{x} \mid \phi]$ in $\mathcal{L}_{\mathbb{T}}$.

Proof If S is the complement of A, $S = A^C \subseteq \mathcal{F}_\mathbb{P}$, then we can conclude from (i)–(iii) that

a) $[P_\phi] \in S$ or $[P_{\neg\phi}] \in S$ but not both, for all sentences $\phi \in \mathcal{L}_{\mathbb{T}_\mathbb{N}}$.

b) $[P_\phi] \in S$ or $[P_\psi] \in S \Leftrightarrow [P_{\phi\vee\psi}] \in S$

c) $[P_\phi] \in S$ and $[P_\psi] \in S \Leftrightarrow [P_{\phi\wedge\psi}] \in S$

d) $[P_{\exists x.\phi(x)}] \in S \Leftrightarrow$ there exists $n \in \mathbb{N}$ such that $[P_{\phi(n)}] \in S$

e) $[P_\phi] \in S$ if $\mathbb{T}_\mathbb{N} \vdash \phi$.

It is then clear that $\sim$ is a well-defined equivalence relation on $\mathbb{N}$, and it is straightforward to verify that the proposed interpretation (A.3) defines a model of $\mathbb{T}$:

$$\begin{aligned}
[\![\phi(\vec{x}) \wedge \psi(\vec{x})]\!] &= \left\{[\vec{n}] \in \mathbb{N}/_\sim \;\middle|\; [P_{\phi(\vec{n})\wedge\psi(\vec{n})}] \in S\right\} \\
&= \left\{[\vec{n}] \in \mathbb{N}/_\sim \;\middle|\; [P_{\phi(\vec{n})}] \in S \text{ and } [P_{\psi(\vec{n})}] \in S\right\} \\
&= \left\{[\vec{n}] \in \mathbb{N}/_\sim \;\middle|\; [P_{\phi(\vec{n})}] \in S\right\} \bigcap \left\{[\vec{n}] \in \mathbb{N}/_\sim \;\middle|\; [P_{\psi(\vec{n})}] \in S\right\} \\
&= [\![\phi(\vec{x})]\!] \bigcap [\![\psi(\vec{x})]\!]
\end{aligned}$$

$$\begin{aligned}
[\![\neg\phi(\vec{x})]\!] &= \left\{[\vec{n}] \in \mathbb{N}/_\sim \;\middle|\; [P_{\neg\phi(\vec{n})}] \in S\right\} \\
&= \left\{[\vec{n}] \in \mathbb{N}/_\sim \;\middle|\; [P_{\phi(\vec{n})}] \notin S\right\} \\
&= \left\{[\vec{n}] \in \mathbb{N}/_\sim \;\middle|\; [P_{\neg\phi(\vec{n})}] \in S\right\}^C \\
&= [\![\phi(\vec{x})]\!]^C
\end{aligned}$$

$$\begin{aligned}
[\![\exists y.\, \phi(y, \vec{x})]\!] &= \left\{[\vec{n}] \in \mathbb{N}/_\sim \;\middle|\; [P_{\exists y.\phi(y,\vec{n})}] \in S\right\} \\
&= \left\{[\vec{n}] \in \mathbb{N}/_\sim \;\middle|\; \text{There exists } m \in \mathbb{N} \text{ such that } [P_{\phi(m,\vec{n})}] \in S\right\} \\
&= \exists \left(\left\{[m], [n] \in \vec{\mathbb{N}}/_\sim \;\middle|\; [P_{\phi(m,\vec{n})}] \in S\right\}\right) \\
&= \exists \left([\![\phi(y, \vec{x})]\!]\right)
\end{aligned}$$

$$\mathbb{T} \vdash \phi \Rightarrow [P_\phi] \in S$$

. ⊣

Corollary A.1.1.2 *The set X of models of $\mathbb{T}$ on $\mathbb{N}$ corresponds bijectively to the set of points of $\mathbb{P}$,*

$$X \cong \mathrm{Hom}_{\mathrm{Frame}}\left(\mathcal{F}_{\mathbb{P}}, \mathbf{2}\right)$$

Proof For a model $\mathbf{M} \in X$ the assignment

$$P_\phi \mapsto \begin{cases} 1 & \text{if } \mathbf{M} \vDash \phi \\ 0 & \text{otherwise} \end{cases}$$

satisfies the axioms of $\mathbb{P}$ and thus defines a frame morphism $\mathcal{F}_{\mathbb{P}} \longrightarrow \mathbf{2}$. In the other direction, the kernel of any frame morphism satisfies the conditions of lemma A.1.1.1. These operations are clearly mutually inverse. ⊣

Lemma A.1.1.3 *Any subset $A \subset \mathcal{F}_{\mathbb{P}}$ which is such that $\bigvee A \neq 1$ has a extension $A \subseteq \overline{A} \subset \mathcal{F}_{\mathbb{P}}$ satisfying the conditions of lemma A.1.1.1.*

Proof We construct $\overline{A}$ inductively. Notice that for any subset $B \subseteq \mathcal{F}_{\mathbb{P}}$ and any propositional constant P_ϕ, if $\bigvee B \neq 1$ then either $\bigvee \{\{[P_\phi]\} \bigcup B\} \neq 1$ or $\bigvee \{\{[P_{\neg\phi}]\} \bigcup B\} \neq 1$ since $[P_\phi]$ and $[P_{\neg\phi}]$ are complements in $\mathcal{F}_{\mathbb{P}}$. Let A_{-1} be the closure of A downward and under finite joins. Then $\bigvee A_{-1} = \bigvee A \neq 1$. Since $\mathcal{L}_{\mathbb{T}_\mathbb{N}}$ is countable, we can list the sentences of $\mathcal{L}_{\mathbb{T}_\mathbb{N}}$, $\{\phi_\alpha \mid \alpha < \omega\}$. Construct A_α as follows: consider ϕ_α. By induction hypothesis $\bigvee A_{\alpha-1} \neq 1$ so either $\bigvee \{\{[P_{\phi_\alpha}]\} \bigcup A_{\alpha-1}\} \neq 1$ or $\bigvee \{\{[P_{\neg\phi_\alpha}]\} \bigcup A_{\alpha-1}\} \neq 1$. Choose ϕ_α or $\neg\phi_\alpha$ accordingly. We temporarily refer to the chosen sentence as ψ. Let A'_α be the set $\{\{[P_\psi]\} \cup A_{\alpha-1}\}$ closed downwards and under finite joins. Now, if ψ is of the form $\neg\exists x.\, \sigma$, then A'_α cannot contain the set $\big\{[P_{\sigma(n)}] \bigm| n \in \mathbb{N}\big\}$, since that would imply that the join of A'_α is 1. Therefore, we can choose an $n \in \mathbb{N}$ such that $\bigvee \big\{\{[P_{\neg\sigma(n)}]\} \cup A'_\alpha\big\} \neq 1$, and we set A_α to be the closure downwards and under finite joins of $\big\{\{[P_{\neg\sigma(n)}]\} \cup A'_\alpha\big\}$. If ψ is not of that form, we simply let $A_\alpha := A'_\alpha$.

Now, let $\overline{A} := \bigcup_{\alpha<\omega} A_\alpha$. Then $\overline{A}$ is a proper subset, for if $[P_\phi] \in \overline{A}$, then its complement $[P_{\neg\phi}]$ cannot be in $\overline{A}$ since $\bigvee A_\alpha \neq 1$ for all $\alpha < \omega$. $\overline{A}$ is clearly closed downwards and under finite joins, and by the construction, we have made sure that $[P_\phi] \in A$ or $[P_{\neg\phi}] \in \overline{A}$, and that $[P_{\exists x.\phi}] \in A \Leftrightarrow \big\{[P_{\phi(n/x)}] \bigm| n \in \mathbb{N}\big\} \subseteq A$. ⊣

Lemma A.1.1.4 *There is an isomorphism of frames*

$$\mathcal{O}(X) \cong \mathcal{F}_{\mathbb{P}}$$

Proof The basic opens of X are given by sentences of $\mathcal{L}_{\mathbb{T}_\mathbb{N}}$, and so are the propositional constants of $\mathbb{P}$. We show that the assignment $U_\phi \mapsto [P_\phi]$ extends to a well-defined frame isomorphism $\Phi : \mathcal{O}(X) \longrightarrow \mathcal{F}_{\mathbb{P}}$. First, suppose $U_\phi \subseteq \bigcup_{i\in I} U_{\psi_i}$. If $[P_\phi] \nleq \bigvee_{i\in I}[P_{\psi_i}]$, then $\bigvee \{[P_{\neg\phi}], [P_{\psi_i}] \mid i \in I\} \neq 1$, and so by lemma A.1.1.1 and lemma A.1.1.3, there is a model $\mathbf{M} \in X$ such that $\mathbf{M} \vDash \phi$ but $\mathbf{M} \vDash \neg\psi_i$ for all $i \in I$, contrary to assumption. Thus extending the assignment from basic open sets to arbitrary open sets by setting $\Phi(\bigcup_{i\in I} U_{\psi_i}) := \bigvee_{i\in I} \Phi([P_{\psi_i}])$ yields a well-defined, order preserving map. Next, we conclude that Φ is a morphism of frames, by

$$\begin{array}{rcl}\Phi(\bot) & = & \Phi(U_{\phi\wedge\neg\phi}) = [P_{\phi\wedge\neg\phi}] = \bot \\ \Phi(\top) & = & \Phi(U_{\phi\vee\neg\phi}) = [P_{\phi\vee\neg\phi}] = \top\end{array}$$

$$\begin{array}{rcl}\Phi(A\cap B) & = & \Phi\left(\bigcup_I U_{\phi_i} \cap \bigcup_J U_{\phi_j}\right) \\ & = & \Phi\left(\bigcup_{I\times J} U_{\phi_i} \cap U_{\phi_j}\right) \\ & = & \Phi\left(\bigcup_{I\times J} U_{\phi_i\wedge\phi_j}\right) \\ & = & \bigvee_{I\times J}[P_{\phi_i\wedge\phi_j}] \\ & = & \bigvee_{I\times J}[P_{\phi_i}] \wedge [P_{\phi_j}] \\ & = & \bigvee_I [P_{\phi_i}] \wedge \bigvee_J [P_{\phi_j}] \\ & = & \Phi(A) \wedge \Phi(B)\end{array}$$

$$\begin{array}{rcl}\Phi\left(\bigcup_I (A_i)\right) & = & \Phi\left(\bigcup_I \left(\bigcup_{J_i} (U_{\phi_{ij}})\right)\right) \\ & = & \Phi\left(\bigcup_{I\times J} U_{\phi_{ij}}\right) \\ & = & \bigvee_{I\times J}[P_{\phi_{ij}}] \\ & = & \bigvee_I \left(\bigvee_J [P_{\phi_{ij}}]\right) \\ & = & \bigvee_I \Phi(A_i)\end{array}$$

Finally, we notice that corollary A.1.1.2 implies that Φ reflects the order, and so is injective, while it is clear that Φ must be surjective since every formula of $\mathbb{P}$ can be written as a join of propositional constants. Thus Φ is a frame isomorphism. $\dashv$

Corollary A.1.1.5 $\mathrm{Sh}(X)$ *classifies the geometric propositional theory* $\mathbb{P}$ *of countable* $\mathbb{T}$*-models.*

A.1.2 The Theory of an Isomorphism between Two Countable $\mathbb{T}$-models

Take two copies, $\mathcal{L}^1_{\mathbb{T}_\mathbb{N}}$ and $\mathcal{L}^2_{\mathbb{T}_\mathbb{N}}$, of the language $\mathcal{L}_{\mathbb{T}_\mathbb{N}}$. We will use superscripted 1's and 2's to distinguish elements of the two copies (e.g. ϕ^1, ψ^2, etc.). Let α be a new unary function symbol. To the set of sentences of $\mathcal{L}^1_{\mathbb{T}_\mathbb{N}}$ and $\mathcal{L}^2_{\mathbb{T}_\mathbb{N}}$, add statements $\alpha(n^1) = m^2$ for all $n, m \in \mathbb{N}$. Call the resulting set STAT. Let the language $\mathcal{L}_\mathbb{S}$ of the geometric propositional theory $\mathbb{S}$ be generated by the set of propositional constants $\{P_\phi \mid \phi \in \mathrm{STAT}\}$. The following axiom schemes define $\mathbb{S}$:

S1 $P_{\phi^i} \vdash P_{\psi^i}$ whenever $\mathbb{T}^i_{\mathbb{N}} \vdash \phi^i \to \psi^i$, for $i = 1, 2$.

S2
- $P_{\phi^i \wedge \psi^i} \dashv\vdash P_{\psi^i} \wedge P_{\psi^i}$
- $P_{\phi^i \vee \psi^i} \dashv\vdash P_{\psi^i} \vee P_{\psi^i}$
- $P_{\bot^i} \vdash \bot$
- $\top \vdash P_{\top^i}$

S3 $P_{\exists x.\phi^i} \vdash \bigvee_{n \in \mathbb{N}} P_{\phi(n/x)^i}$

S4 Axioms for α a bijective well-defined function.

i $P_{n^1=k^1} \wedge P_{\alpha(n^1)=m^2} \vdash P_{\alpha(k^1)=m^2}$

ii $P_{m^2=k^2} \wedge P_{\alpha(n^1)=m^2} \vdash P_{\alpha(n^1)=k^2}$

iii $P_{\alpha(n^1)=m^2} \wedge P_{\alpha(n^1)=k^2} \vdash P_{m^2=k^2}$ for all $m, n, k \in \mathbb{N}$.

iv $P_{\alpha(n^1)=m^2} \wedge P_{\alpha(k^1)=m^2} \vdash P_{n^1=k^1}$ for all $m, n, k \in \mathbb{N}$.

v $\top \vdash \bigvee_{n \in \mathbb{N}} P_{\alpha(m^1)=n^2}$ for all $m \in \mathbb{N}$.

vi $\top \vdash \bigvee_{m \in \mathbb{N}} P_{\alpha(m^1)=n^2}$ for all $n \in \mathbb{N}$.

S5 Axioms for α a $\mathbb{T}$-model morphism.

i $P_{\phi(\vec{n}/\vec{x})^1} \wedge P_{\alpha(\vec{n}^1)=\vec{m}^2} \vdash P_{\phi(\vec{m}/\vec{x})^2}$ for all $\vec{n}, \vec{m} \in \mathbb{N}$, and formulas ϕ of $\mathcal{L}_{\mathbb{T}}$ in context $\vec{x}$.

ii $P_{\phi(\vec{m}/\vec{x})^2} \wedge P_{\alpha(\vec{n}^1)=\vec{m}^2} \vdash P_{\phi(\vec{n}/\vec{x})^1}$ for all $\vec{n}, \vec{m} \in \mathbb{N}$, and formulas ϕ of $\mathcal{L}_{\mathbb{T}}$ in context $\vec{x}$.

Where $P_{\alpha(\vec{n}^1)=\vec{m}^2}$ is shorthand for $P_{\alpha\left(n^1_1\right)=m^2_1} \wedge \ldots \wedge P_{\alpha\left(n^1_k\right)=m^2_k}$.

We show that there is a frame isomorphism $\mathcal{F}_{\mathbb{S}} \cong \mathcal{O}(G)$.

Lemma A.1.2.1 *$G \cong \mathrm{Pt}(\mathcal{F}_{\mathbb{S}})$ in* Top.

Proof Let $f \in G$. Then the assignment

$$P_{\phi^1} = \begin{cases} 1 & \text{if } s(f) \vDash \phi \\ 0 & \text{otherwise} \end{cases}$$

$$P_{\phi^2} = \begin{cases} 1 & \text{if } t(f) \vDash \phi \\ 0 & \text{otherwise} \end{cases}$$

$$P_{\alpha(m^1)=n^2} = \begin{cases} 1 & \text{if } f([m]) = [n] \\ 0 & \text{otherwise} \end{cases}$$

satisfies the axioms of $\mathbb{S}$ and so determines a frame morphism $\mathcal{F}_{\mathbb{S}} \longrightarrow \mathbf{2}$. Conversely, given a frame morphism $p : \mathcal{F}_{\mathbb{S}} \longrightarrow \mathbf{2}$, composing with the obvious inclusions $i_1, i_2 : \mathcal{F}_{\mathbb{P}} \rightrightarrows \mathcal{F}_{\mathbb{S}}$ gives two models $\mathbf{M}$ and $\mathbf{N}$ by corollary A.1.1.2. Notice that for any $[m] \in |\mathbf{M}|$ there must, by the axioms of $\mathbb{S}$, be exactly one $[n] \in |\mathbf{N}|$ such that $p([P_{\alpha(m^1)=n^2}]) = 1$, so we can define $f : \mathrm{M} \longrightarrow \mathrm{N}$ by setting $f([m])$ to be that $[n]$. Again by the axioms of $\mathbb{S}$, it follows that f is a $\mathbb{T}$-isomorphism. Recall, next, that the topology on $\mathrm{Pt}(\mathcal{F}_{\mathbb{S}})$ is given by basic open sets $U_a = \{p \in \mathrm{Pt}(\mathcal{F}_{\mathbb{S}}) \mid p(a) = 1\}$ for $a \in \mathcal{F}_{\mathbb{S}}$. And since any element of $\mathcal{F}_{\mathbb{S}}$ can be written as a join of finite intersections of propositional constants, sets of the form $U_{[P_{\phi^1}]\wedge[P_{\psi^2}]\wedge[P_{\alpha(\vec{m}^1)=\vec{n}^2}]}$ form a basis, whence it is clear that the correspondence $G \cong \mathrm{Hom}_{Frame}(\mathcal{F}_{\mathbb{S}}, \mathbf{2})$ is a homeomorphism of spaces $G \cong \mathrm{Pt}(\mathcal{F}_{\mathbb{S}})$. ⊣

Lemma A.1.2.2 *$\mathcal{F}_{\mathbb{S}}$ is spatial.*

PROOF Recall that a frame is spatial if and only if for any two elements $a \nleq b$, there is a point p such that $p(a) = 1$ but $p(b) = 0$. We can assume without loss that a is a finite intersection of propositional constants, so suppose we are given $a = [P_{\phi^1}] \wedge [P_{\psi^2}] \wedge [P_{\alpha(\vec{m}^1)=\vec{n}^2}] \nleq b$. We construct a descending sequence of elements $(a_i)_{i<\omega}$ such that no a_i is below b. List the sentences of $\mathcal{L}^1_{\mathbb{T}_{\mathbb{N}}}$ and, separately, those of $\mathcal{L}^2_{\mathbb{T}_{\mathbb{N}}}$. Set $a_{-1} = a$. Construct a_i in 4 steps as follows:

(I) $b \ngeq a_{i-1} = a_{i-1} \wedge 1 = a_{i-1} \wedge ([P_{\phi^1_i}] \vee [P_{\neg\phi^1_i}]) = (a_{i-1} \wedge [P_{\phi^1_i}]) \vee (a_{i-1} \wedge [P_{\neg\phi^1_i}])$ so we can choose a disjunct not below b. If the chosen disjunct is $(a_{i-1} \wedge [P_{\phi^1_i}])$ and ϕ^1 is of the form $\exists x.\, \psi(x)^1$, then $b \ngeq (a_{i-1} \wedge [P_{\phi^1_i}]) = (a_{i-1} \wedge [P_{\phi^1_i}]) \wedge \left(\bigvee_{n\in\mathbb{N}} [P_{\psi(n)^1}]\right) = \bigvee_{n\in\mathbb{N}} \left(a_{i-1} \wedge [P_{\phi^1_i}] \wedge [P_{\psi(n)^1}]\right)$ so we can choose one disjunct not below b and call it a^I_i. If not, then set $a^I_i := (a_{i-1} \wedge [P_{\neg\phi^1_i}])$.

(II) $b \ngeq a^I_{i-1} = a^I_{i-1} \wedge 1 = a^I_{i-1} \wedge ([P_{\phi^2_i}] \vee [P_{\neg\phi^2_i}]) = (a^I_{i-1} \wedge [P_{\phi^2_i}]) \vee (a^I_{i-1} \wedge [P_{\neg\phi^2_i}])$ so we can choose one disjunct not below b. Proceed as in step 1 and call the result a^{II}_i.

(III) ("alpha forth") $b \ngeq a^{II}_{i-1} = a^{II}_{i-1} \wedge 1 = a^{II}_{i-1} \wedge \left(\bigvee_{n\in\mathbb{N}} [P_{\alpha(i^1)=n^2}]\right) = \bigvee_{n\in\mathbb{N}} \left(a^{II}_{i-1} \wedge [P_{\alpha(i^1)=n^2}]\right)$ so we can choose one disjunct not below b and call it a^{III}_i.

(IV) ("alpha back") $b \not\geq a_{i-1}^{III} = a_{i-1}^{III} \wedge 1 = a_{i-1}^{III} \wedge \left(\bigvee_{n\in\mathbb{N}}[P_{\alpha(n^1)=i^2}]\right) = \bigvee_{n\in\mathbb{N}}\left(a_{i-1}^{III} \wedge [P_{\alpha(n^1)=i^2}]\right)$ so we can choose one disjunct not below b and set it to be a_i.

Now define $S := \bigcup_{i<\omega} \uparrow a_i$. Then

a) $[P_{\phi^i}] \in S$ or $[P_{\neg\phi^i}] \in S$ but not both, for all sentences $\phi^i \in \mathcal{L}^i_{\mathbb{T}_\mathbb{N}}$.

b) $[P_{\phi^i}] \in S$ or $[P_{\psi^i}] \in S \Leftrightarrow [P_{(\phi\vee\psi)}{}^i] \in S$

c) $[P_{\phi^i}] \in S$ and $[P_{\psi^i}] \in S \Leftrightarrow [P_{(\phi\wedge\psi)^i}] \in S$

d) $[P_{\exists x.\phi(x)^i}] \in S \Leftrightarrow$ there exists $n \in \mathbb{N}$ such that $[P_{\phi(n)^i}] \in S$

e) $[P_{\phi^i}] \in S$ if $\mathbb{T}_\mathbb{N} \vdash \phi^i$.

so, as in lemma A.1.1.1, S determines two models, $\mathbf{M}$ and $\mathbf{N}$, in X, with the property that $\mathbf{M} \vDash \phi \Leftrightarrow [P_{\phi^1}] \in S$ and similarly for $\mathbf{N}$. Next, observe that for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $[P_{\alpha(m^1)=n^2}] \in S$, and that this n is unique up to equivalence, on pain of some a_i being 0 and thus below b. Thus S determines a function $f : |\mathbf{M}| \to |\mathbf{N}|$, which is onto by the construction of S. Furthermore, for any formula ϕ of $\mathcal{L}_\mathbb{T}$, if $f([\vec{m}]) = [\vec{n}]$ then

$$\begin{aligned} \mathbf{M} \vDash \phi(\vec{m}) &\Leftrightarrow [P_{\phi(\vec{m}^1)}] \wedge [P_{\alpha(\vec{m}^1)=\vec{n}^2}] \in S \\ &\Leftrightarrow [P_{\phi(\vec{n}^2)}] \wedge [P_{\alpha(\vec{m}^1)=\vec{n}^2}] \in S \\ &\Leftrightarrow \mathbf{N} \vDash \phi(\vec{n}) \end{aligned}$$

so f is an isomorphism, and thus an element of G. By lemma A.1.2.1 there is a corresponding frame morphism $p : \mathcal{F}_\mathbb{S} \longrightarrow \mathbf{2}$, and this frame morphism is, by our construction, such that $p(a) = 1$ and $p(b) = 0$. Thus we can conclude that $\mathcal{F}_\mathbb{S}$ is spatial. ⊣

Corollary A.1.2.3 *There is an isomorphism of frames $\mathcal{O}(G) \cong \mathcal{F}_\mathbb{S}$.*

PROOF $G \cong \mathrm{Pt}(\mathcal{F}_\mathbb{S})$ in Top by lemma A.1.2.1, so $\mathcal{O}(G) \cong \mathcal{O}(\mathrm{Pt}(\mathcal{F}_\mathbb{S})) \cong \mathcal{F}_\mathbb{S}$ in Frame. ⊣

Corollary A.1.2.4 Sh(G) *classifies the geometric propositional theory $\mathbb{S}$ of an isomorphism between two countable $\mathbb{T}$-models.*

A.1.3 Pullback of the Open Surjection against Itself

The source and target maps of the topological groupoid $\mathrm{s}, \mathrm{t} : G \rightrightarrows X$ induce geometric morphisms, $s, t : \mathrm{Sh}(G) \rightrightarrows \mathrm{Sh}(X)$, in the 2-category $\mathcal{TOP}$ of topoi, geometric morphisms, and geometric transformations. No confusion should arise from the use of the same names for the continuous functions and the geometric morphisms, but we use different fonts all the same. The two composite geometric morphisms

$$\mathrm{Sh}(G) \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Sh}(X) \xrightarrow{m} \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$$

classify to two $\mathbb{T}$-models

$$\mathcal{C}_{\mathbb{T}} \underset{\mathcal{M}_t}{\overset{\mathcal{M}_s}{\rightrightarrows}} \mathrm{Sh}(G)$$

obtained by pulling etale spaces back along s and t respectively, e.g.

$$\begin{array}{ccc} \mathcal{M}_s([\vec{x} \mid \phi]) = \mathrm{s}^*\left(E_{[\vec{x} \mid \phi]}\right) & \longrightarrow & E_{[\vec{x} \mid \phi]} \\ \downarrow \quad \lrcorner & & \downarrow \\ G & \xrightarrow[\mathrm{s}]{} & X \end{array}$$

Now, there is a natural action of G on the sets $E_{[\vec{x} \mid \phi]}$ in the form of 'application' functions

$$\begin{array}{ccc} \mathrm{s}^*\left(E_{[\vec{x} \mid \phi]}\right) \cong G \times_X E_{[\vec{x} \mid \phi]} & \xrightarrow{a\restriction_{[\vec{x} \mid \phi]}} & E_{[\vec{x} \mid \phi]} \\ \langle f, \mathrm{s}(f), [\vec{m}] \rangle & \mapsto & \langle \mathrm{t}(f), f([\vec{m}]) \rangle \end{array}$$

which are continuous, since for any $U_{\phi \wedge \psi(\vec{n}/\vec{y})}$, if $\langle t(f), f([\vec{m}]) \rangle \in U_{\phi \wedge \psi(\vec{n}/\vec{y})}$ we can choose $\vec{l} \in f^{-1}([\vec{n}])$, and then

$$\langle f, \mathrm{s}(f), [\vec{m}] \rangle \in V_{\top}^{\top}(\vec{l}, \vec{n}) \times_X U_{\phi \wedge \psi(\vec{l}/\vec{y})}$$

and

$$a\left(V_{\top}^{\top}(\vec{l}, \vec{n}) \times_X U_{\phi \wedge \psi(\vec{l}/\vec{y})}\right) \subseteq U_{\phi \wedge \psi(\vec{n}/\vec{y})}$$

Then

$$\mathrm{s}^*\left(E_{[\vec{x}\,|\,\phi]}\right) \xrightarrow{\langle 1_G, a_{[\vec{x}\,|\,\phi]}\rangle} \mathrm{t}^*\left(E_{[\vec{x}\,|\,\phi]}\right)$$

$$\langle f, \mathrm{s}(f), [\vec{m}]\rangle \quad \mapsto \quad \langle f, \mathrm{t}(f), f([\vec{m}])\rangle$$

is a continuous function over G, so an arrow of $\mathrm{Sh}(G)$. It is now easy to see that these arrows of G are the components of an invertible natural transformation $a : \mathcal{M}_s \Rightarrow \mathcal{M}_t$. By the equivalence of categories

$$\mathrm{HOM}_{\mathcal{TOP}}(\mathrm{Sh}(G), \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})) \simeq \mathrm{HOM}_{\mathrm{Coh}}(\mathcal{C}_{\mathbb{T}}, \mathrm{Sh}(G))$$

between the category of geometric morphisms $\mathrm{Sh}(G) \longrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ and geometric transformations and the category of coherent functors $\mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}(G)$ and natural transformations, the natural transformation a corresponds to an invertible geometric transformation $\alpha : m \circ s \Rightarrow m \circ t$,

$$\begin{array}{ccccc} & & \mathrm{Sh}(G) & & \\ & {}^{s}\swarrow & & \searrow^{t} & \\ \mathrm{Sh}(X) & & \overset{\alpha}{\Rightarrow} & & \mathrm{Sh}(X) \\ & {}_{m}\searrow & & \swarrow_{m} & \\ & & \mathrm{Sh}(\mathcal{C}_{\mathbb{T}}) & & \end{array} \tag{A.4}$$

We show in this section that (A.4) is a pullback diagram in $\mathcal{TOP}$. Suppose we are given a Grothendieck topos $\mathcal{E}$. On the one hand, we have the category of geometric morphisms and geometric transformations $\mathrm{HOM}_{\mathcal{TOP}}(\mathcal{E}, \mathrm{Sh}(G))$. On the other hand, the category having as objects triples of two geometric morphisms $f, g : \mathcal{E} \rightrightarrows \mathrm{Sh}(X)$ and one invertible geometric transformation $\beta : m \circ f \Rightarrow m \circ g$,

$$\begin{array}{ccccc} & & \mathcal{E} & & \\ & {}^{f}\swarrow & & \searrow^{g} & \\ \mathrm{Sh}(X) & & \overset{\beta}{\Rightarrow} & & \mathrm{Sh}(X) \\ & {}_{m}\searrow & & \swarrow_{m} & \\ & & \mathrm{Sh}(\mathcal{C}_{\mathbb{T}}) & & \end{array}$$

with an arrow between two such triples $\langle f, g, \beta\rangle$ and $\langle f', g', \beta'\rangle$ consisting of a pair of geometric transformations $\gamma : f \Rightarrow f'$ and $\delta : g \Rightarrow g'$ such that the following square commutes:

$$\begin{array}{ccc} m \circ f & \overset{\beta}{\Longrightarrow} & m \circ g \\ {\scriptstyle m\circ\gamma}\Big\Downarrow & & \Big\Downarrow{\scriptstyle m\circ\delta} \\ m \circ f' & \underset{\beta'}{\Longrightarrow} & m \circ g' \end{array}$$

We refer to this latter category as the *category of commuting pairs* between $\mathcal{E}$ and m, $\mathrm{Cpair}(\mathcal{E}, m)$. Composing with the source and target geometric morphisms $s, t : \mathrm{Sh}\,(G) \rightrightarrows \mathrm{Sh}\,(X)$ yields a functor

$$\Phi : \mathrm{HOM}_{\mathcal{TOP}}\,(\mathcal{E}, \mathrm{Sh}\,(G)) \longrightarrow \mathrm{Cpair}(\mathcal{E}, m).$$

In order to establish that $\mathrm{Sh}\,(G)$ is the pullback of m against itself, we must show that Φ is an equivalence of categories (see [10, B1.1]).

Lemma A.1.3.1 *$\Phi : \mathrm{HOM}_{\mathcal{TOP}}\,(\mathcal{E}, \mathrm{Sh}\,(G)) \longrightarrow \mathrm{Cpair}(\mathcal{E}, m)$ is essentially surjective on objects.*

PROOF Let two geometric morphisms $f, g : \mathcal{E} \rightrightarrows \mathrm{Sh}\,(X)$ with an invertible geometric transformation $\beta : m \circ f \Rightarrow m \circ g$ be given.

$$\begin{array}{ccccc} & & \mathcal{E} & & \\ & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} & \\ \mathrm{Sh}\,(X) & & \overset{\beta}{\Rightarrow} & & \mathrm{Sh}\,(X) \\ & {\scriptstyle m}\searrow & & \swarrow{\scriptstyle m} & \\ & & \mathrm{Sh}\,(\mathcal{C}_{\mathbb{T}}) & & \end{array}$$

We construct a model of the geometric propositional theory $\mathbb{S}$ in $\mathcal{E}$. Recall that $\mathrm{Sh}\,(X)$ classifies the theory $\mathbb{P}$ (lemma A.1.1.4), so there is a generic model of $\mathbb{P}$ in $\mathrm{Sh}\,(X)$ which pulls back to $\mathcal{E}$ via f^* and g^*. Now, set

$$\begin{aligned} [\![P_{\phi^1}]\!]^{\mathcal{E}} &:= f^*\left([\![P_\phi]\!]^{\mathrm{Sh}(X)}\right) = f^*\,(U_\phi) \rightarrowtail 1 \\ [\![P_{\phi^2}]\!]^{\mathcal{E}} &:= g^*\left([\![P_\phi]\!]^{\mathrm{Sh}(X)}\right) = g^*\,(U_\phi) \rightarrowtail 1 \end{aligned}$$

while for $m, n \in \mathbb{N}$,

$$\begin{array}{ccccc} [\![P_{\alpha(m^1)=n^2}]\!]^{\mathcal{E}} & \longrightarrow & & & 1 \\ \big\downarrow & & & & \big\downarrow{\scriptstyle g^*(n)} \\ 1 & \xrightarrow{f^*(m)} & f^*(m^*(y\,[x\,|\,\top])) & \xrightarrow{\beta_{y[x\,|\,\top]}} & g^*(m^*(y\,[x\,|\,\top])) \end{array}$$

The axioms schemes of **S1**–**S3** of $\mathbb{S}$ are then immediately satisfied. The axiom schemes of **S4** are satisfied since $[\![P_{\alpha(m^1)=n^2}]\!]^{\mathcal{E}}$ correspond to the canonical interpretation of the internal statement $\beta_{\mathrm{y}[\mathrm{x}\,|\,\top]}\,(f^*(m)) = g^*(n)$ of $\mathcal{E}$, $\beta_{y[x\,|\,\top]}$ has an inverse, and

$$\mathcal{E} \vDash \bigvee_{m\in\mathbb{N}} x : f^*(m^*(y\,[x\,|\,\top])) = f^*(m) : f^*(m^*(y\,[x\,|\,\top]))$$
$$\mathcal{E} \vDash \bigvee_{n\in\mathbb{N}} x : g^*(m^*(y\,[x\,|\,\top])) = g^*(n) : g^*(m^*(y\,[x\,|\,\top]))$$

It now suffices to recall that β is a natural transformation

$$\beta : f^* \circ m^* \Longrightarrow g^* \circ m^*$$

to observe that the axiom schemes of **S5** also hold. Thus $\mathcal{E} \models \mathbb{S}$. By Lemma A.1.2.4, there is a classifying geometric morphism $h : \mathcal{E} \longrightarrow \mathrm{Sh}\,(G)$, unique up to isomorphism. Now, by construction, the $\mathbb{P}$-model classified by $f : \mathcal{E} \longrightarrow \mathrm{Sh}\,(X)$ is the same as the one classified by $s \circ h$, and the $\mathbb{P}$-model classified by g is the same as the one classified by $t \circ h$, and so $f \cong s \circ h$ and $g \cong t \circ h$. ⊣

Lemma A.1.3.2 *The functor*

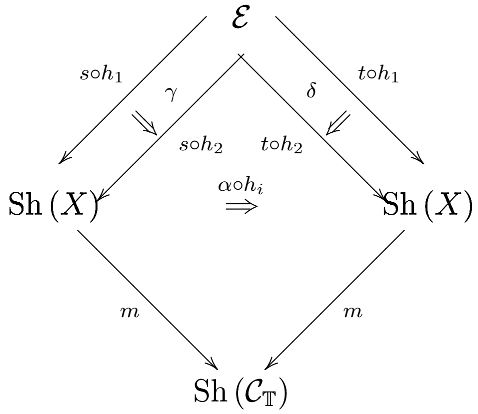
$$\Phi : \mathrm{HOM}_{\mathcal{TOP}}\,(\mathcal{E}, \mathrm{Sh}\,(G)) \longrightarrow \mathrm{Cpair}(\mathcal{E}, m)$$

is full and faithful.

Proof Since $\mathcal{F}_{\mathbb{S}}$ is generated by a set of complemented elements, any frame morphism preserves complementation, and $A \leq B$ implies $\neg B \leq \neg A$ for any frame elements A and B, the only natural transformation that can exists between two frame morphisms $\mathcal{F}_{\mathbb{S}} \rightrightarrows \mathrm{Sub}_{\mathcal{E}}\,(1)$ is the identity transformation. Thus there can at most be one geometric transformation from an object of $\mathrm{HOM}_{\mathcal{TOP}}\,(\mathcal{E}, \mathrm{Sh}\,(G))$ to another, and it must be invertible. Thus Φ is faithful. The same goes for $\mathrm{HOM}_{\mathcal{TOP}}\,(\mathcal{E}, \mathrm{Sh}\,(X))$, so in order to establish that Φ is full, it is sufficient to verify that if

$$\langle \gamma, \delta \rangle : \langle s \circ h_1, t \circ h_1, \alpha \circ h_1 \rangle \longrightarrow \langle s \circ h_2, t \circ h_2, \alpha \circ h_2 \rangle$$

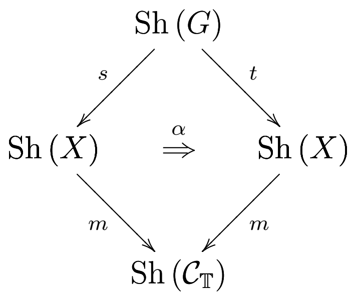
is an arrow in Cpair($\mathcal{E}, m$),



then $h_1 \cong h_2$. For this it is sufficient to check that the two models $\mathbf{H}_1$ and $\mathbf{H}_2$ of $\mathbb{S}$ in $\mathcal{E}$ corresponding to h_1 and h_2 are identical. But the existence of the invertible transformations γ and δ means that $[\![P_{\phi^1}]\!]^{\mathbf{H}_1} = [\![P_{\phi^2}]\!]^{\mathbf{H}_2}$ for any sentence ϕ of $\mathcal{L}_{\mathbb{T}_{\mathbb{N}}}$, and a straightforward diagram chase in $\mathcal{E}$ verifies that $[\![P_{\alpha(m^1)=n^2}]\!]^{\mathbf{H}_1} = [\![P_{\alpha(m^1)=n^2}]\!]^{\mathbf{H}_2}$ for any $m, n \in \mathbb{N}$. $\dashv$

We conclude:

Proposition A.1.3.3 *The following is a pullback diagram in $\mathcal{TOP}$:*



A.1.4 The Theory of Two Composable Isomorphisms and the Triple Pullback of the Open Surjection against Itself

We expand the definitions and arguments of section A.1.2 and section A.1.3 to give the promised characterization of the triple pullback of the open surjection $m : \mathrm{Sh}(X) \twoheadrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ against itself.

Take three copies, $\mathcal{L}^1_{\mathbb{T}_\mathbb{N}}$, $\mathcal{L}^2_{\mathbb{T}_\mathbb{N}}$, and $\mathcal{L}^3_{\mathbb{T}_\mathbb{N}}$, of the language $\mathcal{L}_{\mathbb{T}_\mathbb{N}}$. We will use superscripted 1's, 2's, and 3's to distinguish elements of the three copies. To the set of sentences of the three, add statements $\alpha(n^1) = m^2$ and $\beta(n^2) = m^3$ for all $n, m \in \mathbb{N}$. Call the resulting set STAT. Let the language $\mathcal{L}_\mathbb{R}$ of the geometric propositional theory $\mathbb{R}$ be generated by the set of propositional constants $\{P_\phi \mid \phi \in \text{STAT}\}$. The following axiom schemes define $\mathbb{R}$:

S1 $P_{\phi^i} \vdash P_{\psi^i}$ whenever $\mathbb{T}^i_\mathbb{N} \vdash \phi^i \to \psi^i$, for $i = 1, 2, 3$.

S2
- $P_{\phi^i \wedge \psi^i} \dashv\vdash P_{\psi^i} \wedge P_{\psi^i}$
- $P_{\phi^i \vee \psi^i} \dashv\vdash P_{\psi^i} \vee P_{\psi^i}$
- $P_{\perp^i} \vdash \perp$
- $\top \vdash P_{\top^i}$

S3 $P_{\exists x.\phi^i} \vdash \bigvee_{n \in \mathbb{N}} P_{\phi(n/x)^i}$

S4 Axioms for α a well-defined bijective function.

i $P_{n^1=k^1} \wedge P_{\alpha(n^1)=m^2} \vdash P_{\alpha(k^1)=m^2}$

ii $P_{m^2=k^2} \wedge P_{\alpha(n^1)=m^2} \vdash P_{\alpha(n^1)=k^2}$

iii $P_{\alpha(n^1)=m^2} \wedge P_{\alpha(n^1)=k^2} \vdash P_{m^2=k^2}$ for all $m, n, k \in \mathbb{N}$.

iv $P_{\alpha(n^1)=m^2} \wedge P_{\alpha(k^1)=m^2} \vdash P_{n^1=k^1}$ for all $m, n, k \in \mathbb{N}$.

v $\top \vdash \bigvee_{n \in \mathbb{N}} P_{\alpha(m^1)=n^2}$ for all $m \in \mathbb{N}$.

vi $\top \vdash \bigvee_{m \in \mathbb{N}} P_{\alpha(m^1)=n^2}$ for all $n \in \mathbb{N}$.

S5 Axioms for β a well-defined bijective function.

i $P_{n^2=k^2} \wedge P_{\alpha(n^2)=m^3} \vdash P_{\alpha(k^2)=m^3}$

ii $P_{m^3=k^3} \wedge P_{\beta(n^2)=m^3} \vdash P_{\beta(n^2)=k^3}$

iii $P_{\beta(n^2)=m^3} \wedge P_{\beta(n^2)=k^3} \vdash P_{m^3=k^3}$ for all $m, n, k \in \mathbb{N}$.

iv $P_{\beta(n^2)=m^3} \wedge P_{\beta(k^2)=m^3} \vdash P_{n^2=k^2}$ for all $m, n, k \in \mathbb{N}$.

v $\top \vdash \bigvee_{n \in \mathbb{N}} P_{\beta(m^2)=n^3}$ for all $m \in \mathbb{N}$.

vi $\top \vdash \bigvee_{m \in \mathbb{N}} P_{\beta(m^2)=n^3}$ for all $n \in \mathbb{N}$.

S6 Axioms for α a $\mathbb{T}$-model morphism.

i $P_{\phi(\vec{n}/\vec{x})^1} \wedge P_{\alpha(\vec{n}^1)=\vec{m}^2} \vdash P_{\phi(\vec{m}/\vec{x})^2}$ for all $\vec{n}, \vec{m} \in \mathbb{N}$, and formulas ϕ of $\mathcal{L}_{\mathbb{T}}$ in context $\vec{x}$.

ii $P_{\phi(\vec{m}/\vec{x})^2} \wedge P_{\alpha(\vec{n}^1)=\vec{m}^2} \vdash P_{\phi(\vec{n}/\vec{x})^1}$ for all $\vec{n}, \vec{m} \in \mathbb{N}$, and formulas ϕ of $\mathcal{L}_{\mathbb{T}}$ in context $\vec{x}$.

S7 Axioms for β a $\mathbb{T}$-model morphism.

i $P_{\phi(\vec{n}/\vec{x})^2} \wedge P_{\beta(\vec{n}^2)=\vec{m}^3} \vdash P_{\phi(\vec{m}/\vec{x})^3}$ for all $\vec{n}, \vec{m} \in \mathbb{N}$, and formulas ϕ of $\mathcal{L}_{\mathbb{T}}$ in context $\vec{x}$.

ii $P_{\phi(\vec{m}/\vec{x})^3} \wedge P_{\beta(\vec{n}^2)=\vec{m}^3} \vdash P_{\phi(\vec{n}/\vec{x})^2}$ for all $\vec{n}, \vec{m} \in \mathbb{N}$, and formulas ϕ of $\mathcal{L}_{\mathbb{T}}$ in context $\vec{x}$.

We show that there is a frame isomorphism $\mathcal{F}_{\mathbb{R}} \cong \mathcal{O}(G \times_X G)$.

Lemma A.1.4.1 $G \times_X G \cong \mathrm{Pt}(\mathcal{F}_{\mathbb{R}})$ *in* Top.

Proof The argument of lemma A.1.2.1 is easily expanded to go through for $\mathcal{F}_{\mathbb{R}}$. ⊣

Lemma A.1.4.2 $\mathcal{F}_{\mathbb{R}}$ *is spatial.*

Proof The argument of lemma A.1.2.2 is easily expanded to go through for $\mathcal{F}_{\mathbb{R}}$. ⊣

Corollary A.1.4.3 $\mathrm{Sh}(G \times_X G)$ *classifies the theory* $\mathbb{R}$ *of two composable isomorphisms.*

From the space $G \times_X G$ to X we have three continuous maps

$$\pi_1, \pi_2, \pi_3 : G \times_X G \rightrightarrows X$$

such that $\pi_1(\langle g, f\rangle) = \mathrm{s}(f)$, $\pi_2(\langle g, f\rangle) = \mathrm{t}(f) = \mathrm{s}(g)$, and $\pi_3(\langle g, f\rangle) = \mathrm{t}(g)$. In $\mathcal{TOP}$, we get the diagram

$$\pi_1, \pi_2, \pi_3 : \mathrm{Sh}(G \times_X G) \rightrightarrows \mathrm{Sh}(X) \xrightarrow{m} \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$$

and, correspondingly, three $\mathbb{T}$-models

$$\mathcal{M}_{\pi_1}, \mathcal{M}_{\pi_2}, \mathcal{M}_{\pi_3} : \mathcal{C}_{\mathbb{T}} \rightrightarrows \mathrm{Sh}(G \times_X G)$$

There are two invertible $\mathbb{T}$-model morphisms $a : \mathcal{M}_{\pi_1} \Rightarrow \mathcal{M}_{\pi_2}$ and $b : \mathcal{M}_{\pi_2} \Rightarrow \mathcal{M}_{\pi_3}$,

$$\mathcal{M}_{\pi_1}([\vec{x}\,|\,\phi]) \xrightarrow{a_{[\vec{x}\,|\,\phi]}} \mathcal{M}_{\pi_2}([\vec{x}\,|\,\phi])$$

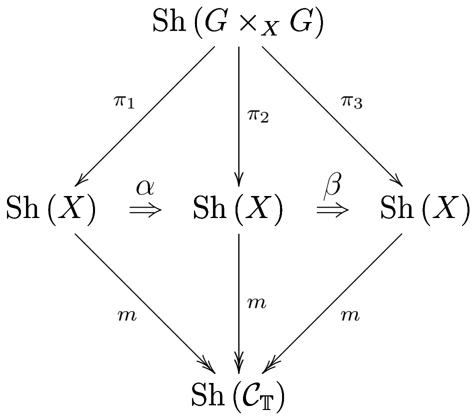
$$\langle\langle g \circ f\rangle, \mathrm{s}(f), [\vec{m}]\rangle \quad \mapsto \quad \langle\langle g \circ f\rangle, \mathrm{t}(f), f([\vec{m}])\rangle$$

$$\mathcal{M}_{\pi_2}([\vec{x}\,|\,\phi]) \xrightarrow{b_{[\vec{x}\,|\,\phi]}} \mathcal{M}_{\pi_3}([\vec{x}\,|\,\phi])$$

$$\langle\langle g \circ f\rangle, \mathrm{s}(g), [\vec{m}]\rangle \quad \mapsto \quad \langle\langle g \circ f\rangle, \mathrm{t}(g), g([\vec{m}])\rangle$$

similar to the invertible $\mathbb{T}$-model morphism described in section A.1.3 (in fact, we are using the same name, a, as we did there, an abuse of notation that is restricted to this section and should cause no harm). The $\mathbb{T}$-model morphisms a and b correspond to geometric transformations $\alpha : m \circ \pi_1 \Rightarrow \pi_2$ and $\beta : m \circ \pi_2 \Rightarrow \pi_3$, and we claim:

Lemma A.1.4.4 *The diagram*



is a triple-pullback diagram in $\mathcal{TOP}$.

Proof This follows from lemma A.1.4.3 in very much the same way as proposition A.1.3.3 followed from lemma A.1.2.4. ⊣

A.1.5 The Category of G-Equivariant Sheaves on X

As pointed out in section 3.2.1, it now follows that the topos of coherent sheaves, $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$, on $\mathbb{T}$ is equivalent to the category of sheaves on X equipped with descent data,

$$\mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}}\right) \simeq \mathbf{Desc}\left(\mathbb{G}_{\bullet}\right)$$

where $\mathbb{G}_{\bullet}$ is the 2-truncated simplicial topos that we obtain by taking sheaves on the groupoid $\mathbb{G}$,

$$\mathrm{Sh}\left(G \times_X G\right) \underset{r}{\overset{l}{\underset{c}{\rightrightarrows}}} \mathrm{Sh}\left(G\right) \underset{t}{\overset{s}{\underset{Id}{\rightleftarrows}}} \mathrm{Sh}\left(X\right)$$

Recall that the category $\mathbf{Desc}\left(\mathbb{G}_{\bullet}\right)$ has as objects pairs $\langle B, \theta\rangle$ where B is an object of $\mathrm{Sh}\left(X\right)$ and θ is descent data for B, that is an arrow $\theta : s^*(B) \to t^*(B)$ in $\mathrm{Sh}\left(G\right)$ such that (modulo coherence isomorphisms) we have $Id^*(\theta) = 1_B$ and the diagram

$$\begin{array}{ccc} \pi_1^*\left(B\right) & \xrightarrow{\pi_{12}^*(\theta)} & \pi_2^*\left(B\right) \\ & \searrow^{\pi_{13}^*(\theta)} & \downarrow{\scriptstyle \pi_{23}^*(\theta)} \\ & & \pi_3^*\left(B\right) \end{array}$$

commutes in $\mathrm{Sh}\left(G \times_X G\right)$. An arrow $h : \langle B, \theta\rangle \to \langle C, \vartheta\rangle$ in $\mathbf{Desc}\left(\mathbb{G}_{\bullet}\right)$ is an arrow $h : B \to C$ of $\mathrm{Sh}\left(X\right)$ such that

$$\begin{array}{ccc} s^*(B) & \xrightarrow{\theta} & t^*(B) \\ {\scriptstyle s^*(h)}\downarrow & & \downarrow{\scriptstyle t^*(h)} \\ s^*(C) & \xrightarrow[\vartheta]{} & t^*(C) \end{array}$$

commutes in $\mathrm{Sh}\left(G\right)$. It is easy to see (and well known, see e.g. [10, B3.4.14] that this is equivalent to the category $\mathrm{Sh}_G(X)$ of G-equivariant sheaves over X, whose objects are pairs $\langle b, \theta\rangle$ where $b : B \to X$ is a local homeomorphism over X and θ is a continuous map

$$\mathrm{s}^*\left(B\right) = G \times_X B \xrightarrow{\theta} B$$

satisfying the conditions of an action on B. An arrow $h : b \to c$ of $\mathrm{Sh}_G(X)$ is a continuous map over X which commutes with the action. The equivalence $\mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}}\right) \xrightarrow{\simeq} \mathbf{Desc}(\mathbb{G}_{\bullet}) \simeq \mathrm{Sh}_G(X)$ sends an object A in $\mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}}\right)$ to the pair $\langle m^*(A), \alpha_A\rangle$, where $\alpha : m{\circ}s \Rightarrow m{\circ}t$ is the geometric transformation of section

A.1.3. In particular, a representable object $y\left([\vec{x}\mid\phi]\right)$ of $\mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}}\right)$ is sent to the space $E_{[\vec{x}\mid\phi]}$ over X equipped with the previously mentioned (A.1.3) action

$$\begin{array}{ccc} G\times_X E_{[\vec{x}\mid\phi]} & \xrightarrow{\quad a\restriction_{[\vec{x}\mid\phi]}\quad} & E_{[\vec{x}\mid\phi]} \\ \langle f,\mathrm{s}(f),[\vec{m}]\rangle & \mapsto & \langle \mathrm{t}(f),f([\vec{m}])\rangle \end{array}$$

We end this section by collecting our result in the form:

Theorem A.1.5.1 *If $\mathbb{T}$ is a countable theory and $\mathbb{G}$ ($G\rightrightarrows X$) is the groupoid of countable $\mathbb{T}$-models with the logical topology, then the category $\mathrm{Sh}_G(X)$ of equivariant $\mathbb{G}$-sheaves is equivalent to the topos of coherent sheaves on $\mathbb{T}$:*

$$\mathrm{Sh}\left(\mathcal{C}_{\mathbb{T}}\right)\simeq\mathbf{Desc}_m(\mathrm{Sh}\left(X\right)_\bullet)\simeq\mathbf{Desc}(\mathbb{G}_\bullet)\simeq\mathrm{Sh}_G(X)$$

Appendix B

Table of Categories

B.1 A Table of Categories

For reference, we list the various categories referred to by abbreviated names in the text:

	‘Algebraic’ categories
DC	-(Small) decidable coherent categories and coherent functors.
$\mathbf{DC}_\kappa$	-Full subcategory of **DC** of decidable coherent categories with a saturated set of $< \kappa$ models (2.2.1.2).
T	-Subcategory of $\mathbf{DC}_\kappa$ of syntactic categories and strict functors between them (2.2.2.2).
DLat	-Full subcategory of $\mathbf{DC}_\kappa$ of distributive lattices.
BC	-(Small) Boolean coherent categories and coherent functors.
$\mathbf{BC}_\kappa$	-Full subcategory of **BC** of Boolean coherent categories with a saturated set of $< \kappa$ models.
FOL	-Subcategory of $\mathbf{BC}_\kappa$ of categories with a distinguished, well-supported object and a system of inclusions, with distinguished object and inclusion preserving functors between them (3.3.1.3).
BA	-Full subcategory of $\mathbf{BC}_\kappa$ of Boolean algebras.

'Geometric' categories	
Top	-Topological spaces and continuous maps.
ctTop	-Compact spaces and continuous maps that pull stably compact opens back to stably compact opens (2.1.0.2).
clTop	-Full subcategory of **ctTop** of compact spaces such that the intersection of any two compact open sets is again compact (2.1.0.3).
CohTop	-Full subcategory of **clTop** of coherent spaces and coherent continuous maps (2.1.0.3).
sCohTop	-Full subcategory of **CohTop** of sober coherent spaces.
Stone	-Full subcategory of **sCohTop** of Stone spaces.
Gpd	-Topological groupoids and continuous groupoid morphisms.
ctGpd	-Topological groupoids $\mathbb{G}$ such that the terminal object in $\mathrm{Sh}(\mathbb{G})$ is compact, and those morphisms of groupoids that pull stably compact decidable objects back to stably compact decidable objects (2.4.3.11).
clGpd	-Full subcategory of **ctGpd** of those groupoids $\mathbb{G}$ such that any compact decidable equivariant sheaf on $\mathbb{G}$ is stably compact decidable (2.4.5.1).
dcGpd	-Full subcategory of **clGpd** of those groupoids $\mathbb{G}$ such that $\mathrm{Sh}(\mathbb{G})$ has a decidable coherent site.
bcGpd	-Full subcategory of **dcGpd** of those groupoids $\mathbb{G}$ such that $\mathrm{Sh}(\mathbb{G})$ has a Boolean coherent site.
$\mathbf{Gpd}/\mathbb{N}_\sim$	-Slice category of topological groupoids over $\mathbb{N}_\sim$ (3.3.2.1).
Class	-Full subcategory of $\mathbf{Gpd}/\mathbb{N}_\sim$ of classical groupoids over $\mathbb{N}_\sim$ (3.3.4.1).
Stone	-Full subcategory of **Class** consisting of Stone fibrations (3.4.2.5).
Sem	-Full subcategory of **Class** of semantical groupoids over $\mathbb{N}_\sim$ (3.3.4.14)

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