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Ultrasheaves

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Abstract

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This thesis treats *ultrasheaves*, sheaves on the category of ultrafilters.

In the classical theory of ultrapowers, you start with an ultrafilter $(I, \mathcal{U})$ and, given a structure S, you construct the ultrapower $S^I/\mathcal{U}$. The fundamental result is Łoś's theorem for ultrapowers giving the connection between what formulas are satisfied in the ultrapower and in the original structure S. In this thesis we instead start with the category of ultrafilters (denoted $\mathbb{U}$). On this category we build the topos $\mathrm{Sh}(\mathbb{U})$ of sheaves on $\mathbb{U}$ (the ultrasheaves), which we think of as generalized ultrapowers.

The theorem for ultrasheaves corresponding to Łoś's theorem is Moerdijk's theorem, first proved by Moerdijk for the topos $\mathrm{Sh}(\mathbb{F})$ of sheaves on filters. In the thesis we prove that Łoś's theorem follows from Moerdijk's theorem. We also investigate the exact relation between the topos of ultrasheaves and Moerdijk's topos $\mathrm{Sh}(\mathbb{F})$ and prove that $\mathrm{Sh}(\mathbb{U})$ is the double negation subtopos of $\mathrm{Sh}(\mathbb{F})$.

The connection between ultrapowers and ultrasheaves is investigated in detail. We also prove some model theoretic results for ultrasheaves, for instance we prove that they are saturated models. The Rudin-Keisler ordering is a tool used in set theory to study ultrafilters. It has a strong relationship to the category $\mathbb{U}$. Blass has given a model theoretic characterization of this ordering and in the thesis we give a new proof of his result.

One common use of ultrapowers is to give non-standard models. In the thesis we prove that you can model internal set theory (IST), a nonstandard set theory, in the ultrasheaves. IST, introduced by Nelson, is an axiomatic approach to nonstandard mathematics.

Key words and phrases. ultrapower, ultrafilter, Moerdijk's topos, sheaf model, non-standard set theory, saturated model.

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PREFACE

This thesis consists of a summary and three papers. The papers are:

I. "Ultrapowers as sheaves on a category of ultrafilters", accepted for publication in *Arch. Math. Logic*.

II. "Ultrasheaves and double negation" (joint with S. Awodey), accepted for publication in *Notre Dame J. Formal Logic*.

III. "Ultrasheaves and ultrapowers", in manuscript

ULTRASHEAVES

JONAS ELIASSON

The subject of this thesis is the topos of *ultrasheaves*, sheaves on the category of ultrafilters. The study of ultrasheaves is a part of categorical logic, more specifically they are sheaf models. The outline of the thesis is this. In Section 1 we give some background to the results in the thesis. In Section 2 we give some preliminary definitions and results. In the following three section we present the main results from the Papers I, II, and III, respectively. Finally in Section 6 we give some suggestions for future research.

1. Background

First in this section we give a background in categorical logic and general topos theory. Then follows a background directly related to ultrasheaves.

1.1. Background in categorical logic.

The study of sheaf theory was pioneered by Grothendieck. He was motivated by examples of sheaves in algebraic geometry. Sheaves had for instance appeared as families of Abelian groups A_x, parametrized by points $x \in X$ of a topological space X in a nice way respecting some locality and integrality properties with respect to the topology on X.

Grothendieck gave a more general definition of sheaves by replacing the partially ordered collection of open subsets of a topological space by objects in a category $\mathbf{C}$, in which some suitable families of maps $U_i \to X$ (for $i \in I$) form "covers" of objects X in $\mathbf{C}$. For such a "Grothendieck topology" a sheaf is a functor which respects the topology on $\mathbf{C}$ in a appropriate manner. The sheaves on a category $\mathbf{C}$ with a Grothendieck topology J (a "site") form a "Grothendieck topos" $\mathrm{Sh}(\mathbf{C}, J)$.

The interest for toposes in logic has as a starting point Lawvere's thesis from 1963. In it he launched the grand project of a purely categorical foundation for all of mathematics, starting with an appropriate axiomatization of the category of sets, replacing membership by composition of functions. When Lawvere learned of the Grothendieck toposes, he soon observed that such a topos admits the basic operations of set theory, such as the formation of the exponential set Y^X (of all functions from X to Y) and the power set $P(X)$ (of all subsets of X).

At about the same time as Lawvere's work, Tierney saw that Grothendieck's work could lead to an axiomatic study of sheaves. Working together, Lawvere and Tierney discovered an axiomatization of categories of sheaves of sets (and, in particular, of the category of sets) via an appropriate formulation of the set-theoretic properties. This was the definition of an "elementary topos", which is defined without any set-theoretic assumptions. Any Grothendieck topos is an elementary topos, but not conversely.

A second input into the use of toposes in logic was from the "forcing" method used by Cohen to prove the independence of the continuum hypothesis (CH), and other set-theoretic axioms, from the rest of Zermelo-Fraenkel set theory (ZF). The method works by expanding a model of ZF to a new model by forcing some sets to exist in the new model, in the case of CH a subset $B \subset \mathbb{R}$ such that the cardinality of B is strictly between the cardinalities of $\mathbb{N}$ and $\mathbb{R}$.

This forcing technique was later rephrased by Solovay and Scott in terms of Boolean-valued models, where the truth-values are taken from an arbitrary Boolean algebra. Shortly after this, Lawvere and Tierney discovered that Cohen forcing could be performed in toposes: indeed, using Cohen's construction one obtains a topos with the desired properties.

As observed above objects in an elementary topos behave like sets. This means that, just as for sets, we can interpret logic into a topos. In set theory we can identify a formula with the set of elements for which the formula holds. Then the logical operations "and", "or" and "not" have their counterparts in the set operations union, intersection and complement. Quantification can be interpreted with the help of products of sets and projections.

The subsets of a set form a Boolean algebra. This means that if you take the complement of the complement of a set, you get the set back. In logical language this says that "not not A" is equivalent to A. This is not true in a general topos. Instead of getting back the original set after taking the complement twice you only get a subset of the set you started with. This means that the logic corresponding to a general topos is not classical. It is intuitionistic. For more background on categorical logic see Mac Lane and Moerdijk [13].

1.2. **Background to ultrasheaves.** In 1993 Moerdijk [15] presented a new model of nonstandard arithmetic. His model uses the Grothendieck topos of sheaves on the category of filters, $\mathrm{Sh}(\mathbb{F})$. Later Palmgren [17, 18, 19, 20] extended this to a model of nonstandard analysis. The models in particular make use of the sheaves ${}^*S$, whose value ${}^*S(\mathcal{F})$ at any filter $\mathcal{F}$ is the reduced power of the set S over $\mathcal{F}$.

The category of filters $\mathbb{F}$ was first studied in Koubek and Reiterman [12] and later by Blass [3]. In this thesis we consider the full subcategory $\mathbb{U}$ of ultrafilters in $\mathbb{F}$. It inherits a topology from $\mathbb{F}$ and the ultrasheaves are the sheaves on $\mathbb{U}$ for this topology. The sheaves ${}^*S$ are still sheaves when restricted to $\mathbb{U}$ and ${}^*S(\mathcal{U})$ is now the ultrapower of S over an ultrafilter $\mathcal{U}$. More on this in Paper I. In Paper II we give another description of the relationship between $\mathrm{Sh}(\mathbb{F})$ and $\mathrm{Sh}(\mathbb{U})$.

Before Moerdijk, Ellerman [9] studied ultrapowers (or more generally ultraproducts) as sheaves on ultrafilters. The ultrafilters on a set I can be thought of as a topological space, the Stone space of the Boolean algebra of the subsets of I. Ellerman then considered ultraproducts as sheaves on this topological space. There is a brief discussion of the relation between Ellerman's work and $\mathrm{Sh}(\mathbb{F})$ in Section 6.

Other work on filter categories has been done by Makkai [14], Pitts [21, 22, 23] and Butz [4]. Pitts uses the filter construction on coherent categories to prove completeness and interpolation results. Makkai's topos of types is related to the prime filters in Pitts construction. The precise relation between the two toposes is considered by Butz, who uses filters to construct generic saturated models of intuitionistic first-order theories.

The existence and qualities of ultrafilters have long been studied in set theory. The answers to many questions on ultrafilters depend on what set theoretic axioms are used, for instance what choice principle you have or what size the power set has. In this thesis we will always assume the axiom of choice which guarantees the existence of ultrafilters.

A tool in the study of ultrafilters is the Rudin-Keisler ordering, introduced independently by Rudin, Keisler and Katetov (as a general reference see Comfort and Negrepontis [6]). The Rudin-Keisler ordering is a partial order on the equivalence classes of ultrafilters. It has been studied, under various assumptions, with respect to properties like the existence of minimal elements or what orderings can be embedded into it.

The Rudin-Keisler ordering has a direct relevance for $\mathbb{U}$. We have that an ultrafilter $\mathcal{V}$ is greater than or equal to $\mathcal{U}$ in the ordering if and only if the set of morphisms from $\mathcal{U}$ to $\mathcal{V}$ is non-empty. This is proved in Paper I. In his thesis [2], Blass proved a model theoretic characterization of the Rudin-Keisler ordering. In Paper I we also give a new proof of this.

Ultrapowers and ultraproducts is a standard tool in model theory (for more on this see Chang and Keisler [5] or Hodges [10]). They give a way of finding new models from old ones. Essential for the usefulness of ultraproducts is Łoś's theorem which relates truth in the ultraproduct to truth in the original model. In Paper III we show that Łoś's theorem is a consequence of Moerdijk's theorem. Ultrapowers are also used for giving non-standard models. In Paper I we show that Sh($\mathbb{U}$) can be used to model a non-standard set theory.

One important result in model theory related to our work is the Isomorphism theorem for ultrapowers (the Keisler-Shelah theorem) which says that for two elementarily equivalent structures there is an ultrafilter such that the ultrapowers are isomorphic. It is proved by constructing ultrapowers that are saturated. That ultrasheaves are saturated is proved in paper III.

2. Preliminaries

In this section we will give some preliminary definitions and results. First we provide some general preliminaries in categorical logic. Then we give the technical background directly related to this thesis.

2.1. **Preliminaries in categorical logic.** We will assume knowledge of the basic categorical devices such as pullbacks, functors etc. For a detailed presentation, see Mac Lane and Moerdijk [13] or Johnstone [11].

Let $\mathbf{C}$ be a category. A *sieve* S on an object C in $\mathbf{C}$ is a family of morphisms in $\mathbf{C}$, all with codomain C, such that $f \in S \Rightarrow f \circ g \in S$, whenever this composition makes sense. If S is a sieve on C and $h : D \to C$ is any arrow to C, then $h^*(S) = \{g \mid \mathrm{cod}(g) = D, h \circ g \in S\}$ is a sieve on D.

Definition 2.1. A *Grothendieck topology* on a category $\mathbf{C}$ is a function J which assigns to each object C in $\mathbf{C}$ a collection $J(C)$ of sieves on C in such a way that

(i) the maximal sieve $t_C = \{f \mid \mathrm{cod}(f) = C\}$ is in $J(C)$,
(ii) (stability) if $S \in J(C)$, then $h^*(S) \in J(D)$ for any arrow $h : D \to C$,
(iii) (transitivity) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$ in S, then $R \in J(C)$.

A *site* is a pair $(\mathbf{C}, J)$, with a category $\mathbf{C}$ and a Grothendieck topology J on it. If $S \in J(C)$ one says that S *covers* C (or that S is a *covering sieve*).

An example of a site is a topological space X with the usual notion of a cover. The topology $\mathcal{O}(X)$ of X is partially ordered under inclusion. The set $(\mathcal{O}(X), \subseteq)$ can be viewed as a category, with objects the open sets U of X, and exactly one morphism $U \to V$ if and only if $U \subseteq V$. A sieve S on U is now a family of open subsets of U with the property that $V' \subseteq V \in S$ implies $V' \in S$. A Grothendieck topology on X is given by saying that S covers U if and only if U is contained in the union of the sets in S.

Usually on a topological space one describes a cover of U as a family $\{U_i \mid i \in I\}$ of open subsets of U such that $U = \bigcup U_i$. Such a family is not necessarily a sieve, but it can be used to generate a sieve - namely the collection of all open $V \subseteq U$

such that $V \subseteq U_i$, for some $i \in I$. In a similar way for an arbitrary category $\mathbf{C}$ it is often enough to consider a *basis* (for a Grothendieck topology) and then use this basis to generate the covering sieves.

Now we turn our attention to sheaves. Let $(\mathbf{C}, J)$ be a site. A *presheaf* (of sets) on $\mathbf{C}$ is a contravariant functor $P : \mathbf{C} \to \mathbf{Sets}$, i.e. a functor $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$. If P is a presheaf and the sieve S covers an object C in $\mathbf{C}$, then a *matching family* for S of elements of P is a function which assigns to each element $f : D \to C$ in S an element $x_f \in P(D)$, in such a way that $P(g)(x_f) = x_{f \circ g}$, for all $g : E \to D$ in $\mathbf{C}$. An *amalgamation* of such a matching family is an element $x \in P(C)$ such that $P(f)(x) = x_f$, for all $f \in S$.

We can now give the definition of a sheaf:

Definition 2.2. A presheaf P on a site $(\mathbf{C}, J)$ is a *sheaf* if every matching family for any cover of any object of $\mathbf{C}$ has a unique amalgamation.

An example of a sheaf on a topological space X (considered as a site as above) is the functor taking any open subset U to the set of continuous real-valued functions on U. This functor has the following properties

(i) If $f : U \to \mathbb{R}$ is continuous and $V \subseteq U$, then f restricted to V, $f|_V : V \to \mathbb{R}$, is continuous.
(ii) If U is covered by open subsets U_i (with $i \in I$), and there are continuous functions $f_i : U_i \to \mathbb{R}$ such that for every $i, j \in I$ we have $f_i(x) = f_j(x)$, for $x \in U_i \cap U_j$, then there is an unique function $f : U \to \mathbb{R}$ such that $f|U_i = f_i$.

Property (i) shows that the functor is a presheaf, while (ii) is the sheaf-condition, with $U_i \mapsto f_i$ as the matching family and f as the unique amalgamation. Property (ii) above is a variation on the matching condition, for the case when the collection of U_i is only a basis and not a sieve. All of the above is done only for sheaves of sets. Other types of sheaves are achieved by replacing the category **Sets** with some suitable category.

The category of sheaves on a site $(\mathbf{C}, J)$ with natural transformations as morphisms is called a *Grothendieck topos*, $\mathrm{Sh}(\mathbf{C}, J)$. A Grothendieck topos has some very nice properties. Many of these are expressed by saying that it is an elementary topos (we will often just say topos):

Definition 2.3. A category $\mathcal{E}$ is an *elementary topos* if:

(i) it has all finite limits,

and is equipped with

(ii) a special object Ω,
(iii) a function P, which assigns to each object B of $\mathcal{E}$ an object PB of $\mathcal{E}$,
(iv) for each object A of $\mathcal{E}$ two isomorphisms, each natural in A,

$$\mathrm{Sub}_{\mathcal{E}} A \cong \mathrm{Hom}_{\mathcal{E}}(A, \Omega), \tag{1}$$

and

$$\mathrm{Hom}_{\mathcal{E}}(B \times A, \Omega) \cong \mathrm{Hom}_{\mathcal{E}}(A, PB). \tag{2}$$

To be natural in A means that the isomorphisms are functorial with respect to A. $\mathrm{Sub}_{\mathcal{E}} A$ in (1) is the collection of subobjects of A, i.e. monomorphisms into A.

In the definition the object Ω in (ii) is the *subobject classifier* and (1) shows that the subobjects of an object A is internally represented as morphisms from A to Ω. The function P in (iii) will serve as the power set operator, and from (2) it follows that PB is the exponential Ω^B. Every topos is Cartesian closed since the existence of arbitrary exponentials A^B can be proved from the existence of the exponentials on the form Ω^B.

The perhaps simplest example of a topos is the category **Sets**. Here the subobject classifier Ω is the set $\{0, 1\}$. Equation (1) says that every subset B of a set A can be represented as a function from A to $\{0, 1\}$ ($\chi_B(x) = 1$, if $x \in B$). If you put $A = \{*\}$ (the one-point set) in equation (2), then you get that functions from B to $\{0, 1\}$ (i.e. the subsets of B) can be identified with elements in PB, i.e. PB is the power set of B.

We now use toposes for mathematical logic. Assume that we have a many-sorted language L and that we want to interpret the theories over L in a topos $\mathcal{E}$. First we associate to each sort in the language an object in $\mathcal{E}$. Formulas are interpreted as subobjects of the product of the objects of the sorts of the free variables of the formula. As an example, assume that $\Theta(x, y)$ is a formula with x of sort X and y of sort Y as its only free variables. Assume that X is interpreted in $\mathcal{E}$ by S and Y by T. Then $\Theta(x, y)$ is interpreted as a subobject, $M_{x,y}(\Theta(x, y))$ of $S \times T$, i.e. a monomorphism $m : M_{x,y}(\Theta(x, y)) \to S \times T$. This is called the Mitchell-Bénabou language for $\mathcal{E}$. It is based on the observation that a topos behaves like a "universe of sets".

The truth values for a formula are given internally by the subobject classifier, which in general is not a Boolean but a Heyting algebra. This means that we do not get the same object back after double negation, and thus the internal logic in a topos is, in general, intuitionistic. This semantic is called the "Kripke-Joyal semantics" for a topos, and it is a generalization of the semantics for intuitionistic logic presented by Kripke.

We will give the semantics for a Grothendieck topos Sh($\mathbf{C}$,J) (the *sheaf semantics*). The semantics is expressed in terms of a forcing relation $C \Vdash \Theta(\alpha)$, where C is an object in $\mathbf{C}$, Θ a formula interpreted in the topos, α is an element in $X(C)$ and X is the sort of the free variable of Θ. We read $C \Vdash \Theta(\alpha)$ as "Θ is true for α at C". For simplicity of notation, we only consider the case when Θ has one free variable (we will do this throughout the thesis). Given a sheaf X, an object C, $\alpha \in X(C)$ and a morphism $f : D \to C$ we will use $\alpha \cdot f$ as an abbreviation for $X(f)(\alpha)$.

The sheaf semantics is a simplification of the semantics in a general topos.

Theorem 2.4. *For a Grothendieck topology J on $\mathbf{C}$, let X be a sheaf on $\mathbf{C}$ and let $\Theta(x)$ and $\Psi(x)$ be formulas in the language of the topos* $\mathrm{Sh}(\mathbf{C}, J)$ *where x is a free variable of sort X. Let C be an object in $\mathbf{C}$ and let $\alpha \in X(C)$.*
Then

(i) $C \Vdash \Theta(\alpha) \wedge \Psi(\alpha)$ *if and only if* $C \Vdash \Theta(\alpha)$ *and* $C \Vdash \Psi(\alpha)$,
(ii) $C \Vdash \Theta(\alpha) \vee \Psi(\alpha)$ *if and only if there is a covering* $\{f_i : C_i \to C\}$ *such that for each index i, either* $C_i \Vdash \Theta(\alpha \cdot f_i)$ *or* $C_i \Vdash \Psi(\alpha \cdot f_i)$,
(iii) $C \Vdash \Theta(\alpha) \to \Psi(\alpha)$ *if and only if for all* $f : D \to C$, $D \Vdash \Theta(\alpha \cdot f)$ *implies* $D \Vdash \Psi(\alpha \cdot f)$,
(iv) $C \Vdash \neg\Theta(\alpha)$ *if and only if for all* $f : D \to C$, *if* $D \Vdash \Theta(\alpha \cdot f)$ *then the empty family is a cover of D.*

Moreover, if $\Theta(x, y)$ is a formula with free variables x and y of sorts X and Y, then for $\alpha \in X(C)$,

(v) $C \Vdash \exists y\Theta(\alpha, y)$ *if and only if there are a covering* $\{f_i : C_i \to C\}$ *and elements* $\{\beta_i \in Y(C_i)\}$ *such that* $C_i \Vdash \Theta(\alpha \cdot f_i, \beta_i)$ *for each index i,*

(vi) $C \Vdash \forall y \Theta(\alpha, y)$ *if and only if for all* $f : D \to C$ *and all* $\beta \in Y(D)$ *one has* $D \Vdash \Theta(\alpha \cdot f, \beta)$.

The forcing relation "$\Vdash$" has two additional properties:

(i) (Monotonicity) If $C \Vdash \Theta(\alpha)$ and $f : D \to C$ then $D \Vdash \Theta(\alpha \cdot f)$.
(ii) (Local character) If $\{f_i : C_i \to C\}$ is a cover in J such that $C_i \Vdash \Theta(\alpha \cdot f_i)$ for all i, then $C \Vdash \Theta(\alpha)$.

2.2. **Filters and reduced powers.** A *filter* $\mathcal{F}$ on a set A is a non-empty collection of subsets of A which is closed under intersections and supersets. We say that a filter is *proper* if it does not contain all subsets of A (equivalently if it does not contain the empty set). A maximal proper filter is called an *ultrafilter*. These filters $\mathcal{U}$ are distinguished by the property that for any subset B of A, either B or the complement of B is in $\mathcal{U}$. Assuming the axiom of choice we can prove that any proper filter can be extended to an ultrafilter.

Given a filter $(I, \mathcal{F})$ and a family of sets $\{S_i \mid i \in I\}$ we can form the *reduced product* of the S_i over $\mathcal{F}$,

$$\prod_{i \in I} S_i / \mathcal{F},$$

which is the product of the S_i quoted out by the equivalence relation

$$a \sim b \iff \{i \in I \mid a(i) = b(i)\} \in \mathcal{F}$$

for $a, b \in \prod_{i \in I} S_i$. If for every $i \in I$ we have $S_i = S$ then we call the reduced product for the *reduced power* of S over $\mathcal{F}$ and we will denote it ${}^*S(\mathcal{F})$. For an ultrafilter $\mathcal{U}$ we say the *ultraproduct* and *ultrapower* over $\mathcal{U}$, respectively.

Let L be a first order language and assume that the sets $\{S_i \mid i \in I\}$ are, in fact, L-structures. Then we can define an interpretation of L in the reduced product of the S_i. We will denote the interpretation of the constant, relation and function symbols from L in the reduced product with superscript S. The corresponding interpretations in S_i will be denoted in the same way.

- Constant c: $c^S = [a]$, where $a(i) = c^{S_i}$.
- Relation R: R^S defined by
$$[a] \in R^S \iff \{i \in I \mid a(i) \in R^{S_i}\} \in \mathcal{F}.$$
- Function f: $f^S([a]) = [b]$, where $b(i) = f^{S_i}(a(i))$.

Here is the fundamental Łoś's theorem.

Theorem 2.5. *Let* $(I,\mathcal{U})$ *be an ultrafilter,* $\varphi(x)$ *an* L*-formula,* $\{S_i \mid i \in I\}$ *a family of* L*-structures and* $\alpha \in \prod_{i \in I} S_i/\mathcal{U}$*. Then*

$$\prod_{i \in I} S_i/\mathcal{U} \models \varphi^S(\alpha) \text{ if and only if } \{i \in I \mid S_i \models \varphi^{S_i}(\alpha(i))\} \in \mathcal{U} \,.$$

Here is the precise definition of the Rudin-Keisler ordering.

Definition 2.6. If $(A,\mathcal{U})$ and $(B,\mathcal{V})$ are ultrafilters then

$$\mathcal{U} \leq \mathcal{V} \iff \exists f : B \to A \text{ such that } \mathcal{U} = \{X \subseteq A \mid f^{-1}(X) \in \mathcal{V}\}.$$

The Rudin-Keisler ordering is a partial ordering on the equivalence classes of ultrafilters. The ordering is very rich. For instance, assuming the generalized continuum hypothesis it can be shown that there are 2^{2^κ} non-equivalent ultrafilters greater than or equal to any given ultrafilter $\mathcal{U}$ on an infinite set A with $|A| = \kappa$ (for a proof see Comfort and Negrepontis [6, Chapter 9]).

In his thesis [2], Blass proved the following model theoretic characterization of the Rudin-Keisler ordering.

Theorem 2.7. *If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters then $\mathcal{U} \leq \mathcal{V}$ if and only if for every L-structure S there is an elementary embedding ${}^*S(\mathcal{U}) \prec {}^*S(\mathcal{V})$.*

2.3. **The category $\mathbb{F}$.** In his paper [3], Blass gives the following definition of the category of filters:

Definition 2.8. The category $\mathbb{F}$ has as *objects* pairs $(A, \mathcal{F})$, where A is a set and $\mathcal{F}$ a filter on A. The *morphisms* $\alpha : (A, \mathcal{F}) \to (B, \mathcal{G})$ are equivalence classes of partial functions $\alpha : A \to B$ such that

(i) α is defined on some $F \in \mathcal{F}$,
(ii) $\alpha^{-1}(G) \in \mathcal{F}$, for all $G \in \mathcal{G}$.

Two such partial functions $\alpha : F \to B$ and $\alpha' : F' \to B$ are equivalent if there is $E \subseteq F \cap F'$ in $\mathcal{F}$ such that $\alpha|_E = \alpha'|_E$.

Koubek and Reiterman's category has the same objects but the morphisms are equivalence classes of total functions. The proposition below is from Koubek and Reiterman [12].

Proposition 2.9. *For any morphism $\alpha : \mathcal{F} \to \mathcal{G}$ we have:*

(i) *α is mono if and only if there is an $F \in \mathcal{F}$ such that α is injective on F,*
(ii) *α is epi if and only if $\alpha(F) \in \mathcal{G}$, for all $F \in \mathcal{F}$.*

We also note that the definition of $\mathbb{F}$ gives that the reduced power of a set S over a filter $\mathcal{F}$ is equal to the set of morphisms from $\mathcal{F}$ to the trivial filter $(S, \{S\})$, i.e.

$$ {}^*S(\mathcal{F}) = \mathrm{Hom}_{\mathbb{F}}(\mathcal{F}, (S, \{S\})). $$

2.4. **The topos Sh($\mathbb{F}$).** Moerdijk [15] defined a Grothendieck topology J on $\mathbb{F}$ as follows:

Definition 2.10. A finite family $\{\alpha_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^n$ is a *J-covering* if the induced map

$$ \mathcal{G}_1 + \ldots + \mathcal{G}_n \to \mathcal{F} $$

is an epimorphism.

Moerdijk only considered filters on the natural numbers. When Palmgren later extended the model to nonstandard analysis he introduced a universe of sets into set theory, e.g. V_κ, where κ is an inaccessible cardinal, and then considered filters on the *small* sets in V_κ.

Over the resulting site Moerdijk studied, in particular, the representable sheaves of the form ${}^*S = \mathrm{Hom}_{\mathbb{F}}(-, (S, \{S\}))$. From the observation above we see that this notation is justified, since, for any filter $\mathcal{F}$, $\mathrm{Hom}_{\mathbb{F}}(\mathcal{F}, (S, \{S\}))$ is the reduced power of S over $\mathcal{F}$.

We now turn our interest to the internal logic of the topos Sh($\mathbb{F}$). Let L be a first order language and S an L-structure. Let ${}^*S$, the ${}^*(\,)$-transform of S, be the L-structure in Sh($\mathbb{F}$) defined as follows:

- Set S: ${}^*S$ is the sheaf previously defined.
- Constant c: ${}^*c$ is given at $\mathcal{F}$ by the constant function

$$ \lambda x.c^S \in {}^*S(\mathcal{F}). $$

- Relation $R \subseteq S$: ${}^*R$ is the subsheaf of ${}^*S$ given at $\mathcal{F}$ by
$$\alpha \in {}^*R(\mathcal{F}) \iff (\exists F \in \mathcal{F})(\forall x \in F)\alpha(x) \in R^S.$$
- Function $f : T \to S$: ${}^*f$ is the natural transformation from ${}^*T$ to ${}^*S$ given at $\mathcal{F}$ by
$${}^*f_{\mathcal{F}}(\alpha) = \lambda x.f^S(\alpha(x)).$$

We also define the predicate St (for *standard* element) on ${}^*S$ for an $\alpha \in {}^*S(\mathcal{F})$:

- $\mathrm{St}(\alpha)$ if and only if there is a cover $\{\beta_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^n$ such that for each i, $\alpha \circ \beta_i$ is constant on some $G \in \mathcal{G}_i$.

Thus every L-structure S (in **Sets**) gives rise to an $L \cup \{\mathrm{St}\}$-structure ${}^*S$ in $\mathrm{Sh}(\mathbb{F})$. Given an L-formula φ we will denote the interpretation of φ in ${}^*S$ by ${}^*\varphi^{{}^*S}$. When not necessary the superscript will be omitted.

The fundamental result for $\mathrm{Sh}(\mathbb{F})$ is Moerdijk's theorem.

Theorem 2.11. *Let $\mathcal{F}$ be a filter, φ an L-formula and $\alpha \in {}^*S(\mathcal{F})$. Then*
$$\mathcal{F} \Vdash {}^*\varphi(\alpha) \text{ if and only if } (\exists F \in \mathcal{F})(\forall x \in F)\varphi(\alpha(x)).$$

3. Results in Paper I

In this paper we introduce and investigate the topos of sheaves on the category of ultrafilters, the ultrasheaves. We denote the full subcategory of ultrafilters in $\mathbb{F}$ by $\mathbb{U}$.

First we note that the category $\mathbb{U}$ is not a very nice category, for instance it does not have products. In general, the product filter (in $\mathbb{F}$) of two ultrafilters is not an ultrafilter. Since $\mathbb{U}$ has a terminal object this also means that it is not closed under pullbacks either.

We get some help in studying $\mathbb{U}$ from its close relation to the Rudin-Keisler ordering. We show that:

Proposition 3.1. *For any two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ we have that:*
$$\mathcal{U} \leq \mathcal{V} \iff \text{there is a morphism } \alpha : \mathcal{V} \to \mathcal{U} \text{ in } \mathbb{U}.$$

We will consider $\mathbb{U}$ with the Grothendieck topology induced on $\mathbb{U}$ from $(\mathbb{F}, J)$. We show that any morphism in $\mathbb{U}$ is epi. So any morphism in $\mathbb{U}$ is a J-covering. Thus the topology induced on $\mathbb{U}$ is the *atomic* topology where any set of morphisms with codomain $\mathcal{U}$ is a covering of $\mathcal{U}$. We can now prove that all representable sheaves on $\mathbb{F}$ are still sheaves when restricted to $\mathbb{U}$. Thus the ${}^*S$ are ultrasheaves.

For any first order language L we introduce an interpretation of L in $\mathrm{Sh}(\mathbb{U})$ via the ${}^*(\,)$-transform just as we did for $\mathrm{Sh}(\mathbb{F})$. Thus, as before, every L-structure S (in **Sets**) gives rise to an $L \cup \{\mathrm{St}\}$-structure ${}^*S$ in $\mathrm{Sh}(\mathbb{U})$. We have the usual interpretation of the logical symbols in the Grothendieck topos. Below we give the sheaf semantics for $\mathrm{Sh}(\mathbb{U})$.

Theorem 3.2. *Let $\mathcal{U}$ be an ultrafilter, Θ and Ψ arbitrary formulas and $\alpha \in {}^*T(\mathcal{U})$. Then*

(i) $\mathcal{U} \Vdash \Theta(\alpha) \wedge \Psi(\alpha)$ *if and only if* $\mathcal{U} \Vdash \Theta(\alpha)$ *and* $\mathcal{U} \Vdash \Psi(\alpha)$,
(ii) $\mathcal{U} \Vdash \Theta(\alpha) \vee \Psi(\alpha)$ *if and only if* $\mathcal{U} \Vdash \Theta(\alpha)$ *or* $\mathcal{U} \Vdash \Psi(\alpha)$,
(iii) $\mathcal{U} \Vdash \Theta(\alpha) \to \Psi(\alpha)$ *if and only if* $\mathcal{U} \Vdash \Theta(\alpha)$ *implies* $\mathcal{U} \Vdash \Psi(\alpha)$,
(iv) $\mathcal{U} \Vdash \neg\Theta(\alpha)$ *if and only if* $\mathcal{U} \nVdash \Theta(\alpha)$,
(v) $\mathcal{U} \Vdash (\exists x \in {}^*S)\Theta(\alpha, x)$ *if and only if for some* $\beta : \mathcal{V} \to \mathcal{U}$ *and* $\delta \in {}^*S(\mathcal{V})$
$$\mathcal{V} \Vdash \Theta(\alpha \circ \beta, \delta),$$

(vi) $\mathcal{U} \Vdash (\forall x \in {}^*S)\Theta(\alpha, y)$ *if and only if for all* $\beta : \mathcal{V} \to \mathcal{U}$ *and* $\delta \in {}^*S(\mathcal{V})$

$$\mathcal{V} \Vdash \Theta(\alpha \circ \beta, \delta).$$

As is evident in the theorem above, the internal logic in $\mathrm{Sh}(\mathbb{U})$ is classical, i.e. the topos is Boolean.

The fundamental theorem for ultrasheaves is Moerdijk's theorem which also holds in $\mathrm{Sh}(\mathbb{U})$.

Theorem 3.3. *Let $\mathcal{U}$ be an ultrafilter, φ an L-formula and $\alpha \in {}^*S(\mathcal{U})$. Then*

$$\mathcal{U} \Vdash {}^*\varphi(\alpha) \text{ if and only if } (\exists U \in \mathcal{U})(\forall x \in U)\varphi(\alpha(x)).$$

From this we obtain the nice corollary:

Corollary 3.4. ${}^*S(\mathcal{U}) \models {}^*\varphi(\alpha) \iff \mathcal{U} \Vdash {}^*\varphi^{{}^*S}(\alpha)$.

From this corollary, the model theoretic characterization of the Rudin-Keisler ordering and the monotonicity and local character of the forcing relation we get a new proof of Blass' theorem. We also show that the corollary can not be strengthened to hold for formulas containing the standard predicate.

In general, one says that a topos $\mathcal{E}$ satisfies the axiom of choice if every object P in $\mathcal{E}$ is *projective*, i.e. for any epimorphism $\eta : X \to P$ there is a morphism $\sigma : P \to X$ such that $\eta \circ \sigma = 1$.

Theorem 3.5. *The topos theoretic axiom of choice does not hold in* $\mathrm{Sh}(\mathbb{U})$.

The paper concludes by showing that $\mathrm{Sh}(\mathbb{U})$ contains a model of Nelson's internal set theory (IST) [16]. IST consists of Zermelo-Fraenkel set theory expanded with a predicate $\mathrm{St}(x)$ and three additional axioms: transfer (T), idealization (I) and standardization (S). Let V be a set-theoretic universe and consider only formulas in the language consisting of ${}^*\!\in$, ${}^*\!=$ and $\mathrm{St}^{{}^*V}$. Below we give the formulation of the axioms for IST in ${}^*V$.

We will use the usual abbreviation $(\forall^{st} x \in {}^*S) \dots$ for $(\forall x \in {}^*S)\, \mathrm{St}(x) \to \dots$.

Theorem 3.6 (Transfer)**.** *Let $\varphi(\overline{x}, y)$ be an L-formula with $\overline{x}, y$ as its only free variables. Then the following is true in the internal logic of* $\mathrm{Sh}(\mathbb{U})$*:*

$$(\forall^{st} x_1 \in {}^*T_1) \dots (\forall^{st} x_n \in {}^*T_n)[(\forall^{st} y \in {}^*S)^*\varphi(\overline{x}, y) \to (\forall y \in {}^*S)^*\varphi(\overline{x}, y)].$$

Theorem 3.7 (Idealization)**.** *Let $\varphi(x, y, z)$ be an L-formula and ${}^*R$ a sheaf in* $\mathrm{Sh}(\mathbb{U})$. *Then the following is true for any $\mathcal{U} \in \mathbb{U}$: if, for all finite sets $S = \{s_1, \dots, s_n\} \subseteq R$ we have*

$$\mathcal{U} \Vdash (\exists x \in {}^*T)(\bigwedge_{i=1}^{n} {}^*\varphi(x, {}^*s_i, \alpha))$$

then

$$\mathcal{U} \Vdash (\exists x \in {}^*T)(\forall^{st} y \in {}^*R)\, {}^*\varphi(x, y, \alpha).$$

This idealization principle is proved by Palmgren [18] for $\mathrm{Sh}(\mathbb{F})$.

Theorem 3.8 (Standardization)**.** *Let $\varphi(x, y)$ be an L-formula and ${}^*S$ a sheaf in* $\mathrm{Sh}(\mathbb{U})$. *Then there exists a subsheaf ${}^*T$ of ${}^*S$ such that at any ultrafilter $\mathcal{U}$*

$$\mathcal{U} \Vdash \forall^{st} z\, (z \in {}^*T \leftrightarrow z \in {}^*S \wedge \varphi(z, \alpha)).$$

Standardization is a restricted version of a separation axiom. It is not full separation since the condition on the elements in ${}^*T$ only holds true for the standard elements in ${}^*T$.

4. Results in Paper II

In this paper we give an exact characterization of the relation between $\mathrm{Sh}(\mathbb{F})$ and $\mathrm{Sh}(\mathbb{U})$. Following Moerdijk's suggestion we prove that $\mathrm{Sh}(\mathbb{U})$ is the double negation subtopos of $\mathrm{Sh}(\mathbb{F})$.

Given an (intuitionistic) logic one can force it to become classical by adding the law of excluded middle to the assumptions. For a topos of sheaves there is a corresponding transformation, namely by adding the double negation topology to the underlying site. Not all of the original sheaves will be sheaves with respect to the new topology, but the internal logic in the resulting topos of sheaves will be classical.

The main theorem of the paper is

Theorem 4.1. $\mathrm{Sh}(\mathbb{U})$ *is equivalent to the double negation subtopos of* $\mathrm{Sh}(\mathbb{F})$.

In order to prove the theorem we introduce the topos $\mathrm{Sh}(\mathbb{F}, J_\infty)$ by defining a new topology J_∞ on $\mathbb{F}$.

Definition 4.2. A basis for the J_∞-*topology* are small families $\{\alpha_i : \mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ (for any set I) such that the induced morphism

$$\coprod_{i\in I} \mathcal{F}_i \to \mathcal{F}$$

is epic.

From Blass [3] we know that the category $\mathbb{F}$ has all coproducts. Now we can prove:

Theorem 4.3. $\mathrm{Sh}(\mathbb{U}) \cong \mathrm{Sh}(\mathbb{F}, J_\infty)$.

The second step in the proof of the main theorem is:

Theorem 4.4. *A presheaf F is in* $\mathrm{Sh}_{\neg\neg}(\mathbb{F}, J)$ *if and only if it is in* $\mathrm{Sh}(\mathbb{F}, J_\infty)$.

We close the paper by showing how the main theorem can be used to transfer the truth of formulas between the toposes $\mathrm{Sh}(\mathbb{U})$ of ultrasheaves and $\mathrm{Sh}(\mathbb{F})$ of sheaves on filters.

Theorem 4.5. *Let $\Theta(\alpha)$ be a first order formula with a free variable of a sort interpreted as an ultrasheaf F. Then, if $\Theta(\alpha)$ is true in* $\mathrm{Sh}(\mathbb{U})$*, its double negation translation $\Theta'(\alpha)$ is true in* $\mathrm{Sh}(\mathbb{F})$.

Theorem 4.6. *Let $\Theta(\alpha)$ be a first order formula with a free variable of a sort interpreted as an ultrasheaf F. Assume, moreover, that $\Theta(\alpha)$ is without universal quantifiers and has double negation stable predicates. If $\Theta(\alpha)$ is true in* $\mathrm{Sh}(\mathbb{F})$ *then $\Theta(\alpha)$ is also true in* $\mathrm{Sh}(\mathbb{U})$.

5. Results in Paper III

This paper contains some results on the relation between ultrasheaves and the corresponding ultrapowers. There are also some results on model theoretic aspects of ultrasheaves.

Two L-structures S and T give rise to two interpretations of the language L in the topos $\mathrm{Sh}(\mathbb{U})$, the sheaves ${}^*S$ and ${}^*T$. In the paper we show that the ${}^*(\,)$–transform preserves elementary equivalence with respect to L, L-homomorphisms, L-embeddings and L-isomorphisms.

We show that Łoś's theorem can be derived from Moerdijk's theorem. We do it by showing, without using Łoś's theorem, that the left- and righthand sides of the two theorems are equivalent.

In Paper I we gave an example that the equivalence

$$^*S(\mathcal{U}) \models {}^*\varphi(\alpha) \iff \mathcal{U} \Vdash {}^*\varphi^{{}^*S}(\alpha)$$

does not hold at the trivial ultrafilter 1 for all $L \cup \{\mathrm{St}\}$-formulas $\Theta(\alpha)$. In this paper we show that for any non-trivial ultrafilter there is a $L \cup \{\mathrm{St}\}$-formula $\Theta(x)$ such that the equivalence fails for $\Theta(x)$.

In general the collection of elements of the form $\alpha \in {}^*S(\mathcal{U})$ (that is the coproduct $\coprod_{\mathcal{U}\in\mathbb{U}} {}^*S(\mathcal{U})$) is not a small set. However, if you restrict yourself to distinct free variables of the sort $^*S$ in $\mathrm{Sh}(\mathbb{U})$ they do form a small set.

Definition 5.1. Given an ultrasheaf $^*S$, define an equivalence relation on the coproduct $\coprod_{\mathcal{U}\in\mathbb{U}} {}^*S(\mathcal{U})$ by saying that two elements $\alpha \in {}^*S(\mathcal{U})$ and $\beta \in {}^*S(\mathcal{V})$ are equivalent (we write $\alpha \sim \beta$) if there are $\gamma : \mathcal{W} \to \mathcal{U}$ and $\delta : \mathcal{W} \to \mathcal{V}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "\delta"] \arrow[d, "\gamma"'] & \mathcal{V} \arrow[d, "\beta"] \\
\mathcal{U} \arrow[r, "\alpha"'] & S
\end{tikzcd}$$

The relation to distinct free variables of the sort $^*S$ is this: for any formula $\Theta(x)$ with a free variable of sort $^*S$, and any ultrafilter $\mathcal{U}$ we have for $\alpha, \beta \in {}^*S(\mathcal{U})$ that, if $\alpha \sim \beta$ then

$$\mathcal{U} \Vdash \Theta(\alpha) \leftrightarrow \Theta(\beta).$$

Then we prove:

Theorem 5.2. *$^*S/\sim$ is a small set.*

The last result in the paper is that every ultrasheaf $^*S$ is saturated, i.e. that any set of formulas with a small set of free variables that is finitely satisfiable in $^*S$, is satisfiable in $^*S$.

Theorem 5.3. *Let $\{\varphi_i(x, \overline{y}) \mid i \in I\}$ be a set of L-formulas such that the set of free variables is small.*

Then the following is true for any $\mathcal{U} \in \mathbb{U}$: if, for every finite subset $I_0 \subseteq I$ and any $\overline{\alpha} \in {}^*S(\mathcal{U})$

$$\mathcal{U} \Vdash (\exists x \in {}^*S) \bigwedge_{i\in I_0} {}^*\varphi_i(x, \overline{\alpha})$$

then there is a cover $\beta : \mathcal{V} \to \mathcal{U}$ and an element $\delta \in {}^*S(\mathcal{V})$ such that for every $i \in I$

$$\mathcal{V} \Vdash {}^*\varphi_i(\delta, \overline{\alpha} \circ \beta).$$

6. Future research

Here we give some examples of different ways in which the study of the ultrasheaves could progress.

From the saturation of the ultrasheaves proved in Paper III one should study the possibility to prove something like the Isomorphism theorem for ultrapowers.

Because of the saturation it should also be profitable to study types of models S and their realizations in $\mathrm{Sh}(\mathbb{U})$.

The set of distinct free variables introduced in Paper III is in fact the colimit of the ultrasheaf $^*S$ over $\mathbb{U}$. This construction could be further explored, perhaps proving that a formula is satisfied in the colimit if and only if it is forced (in the topos) at some ultrafilter.

Looking at slices of $\mathrm{Sh}(\mathbb{F})$ over trivial filters $(I, \{I\})$, one can define reduced product-sheaves for families S_i, $i \in I$, and attempt to extend Moerdijk's theorem to this situation. One should then show that the reduced product-sheaf for a constant family of S_i is equal to a sheaf in $\mathrm{Sh}(\mathbb{F})$. One motivation for looking at this is to compare it with Ellerman's sheaves on Stone spaces. It seems that the exact relationship to Ellerman's work is that there is a connected covering of Ellerman's topos by the topos of reduced product-sheaves.

Another way of continuing the research would be to look for more new proofs of classic results on ultrapowers. In the thesis we have the new proofs of Łoś's and Blass' theorems, in Paper III and I respectively, as examples. The interest in this would lie in showing that viewing ultrapowers as sheaves is "natural" and that definitions and results fit naturally into the sheaf-theoretic language. Perhaps one could also hope to prove some new model-theoretic results this way.

As proved in Paper I, we know that $\mathrm{Sh}(\mathbb{U})$ models IST. But could we describe the theory of $\mathrm{Sh}(\mathbb{U})$ in more detail, perhaps even finding axioms for it? One of the problems of working in $\mathrm{Sh}(\mathbb{F})$ is that you do not have a "standard part"-map. Could such a map be found internally in $\mathrm{Sh}(\mathbb{U})$?

In my work so far, I have always started out in set theory (a set theory strong enough to prove the existence of ultrafilters). Perhaps this is not necessary. You could start working with filters over some more general category. Some related work has been done by Butz [4] based on the filter construction of Pitts [21].

In his original article, Moerdijk remarks that $\mathrm{Sh}(\mathbb{F})$ by Deligne's theorem has "enough" points, and asks for an explicit description of these. Perhaps this question could be answered more easily with the help of the double negation subtopos $\mathrm{Sh}(\mathbb{U})$?

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