Completeness and Categoricity

Part II: 20th Century Metalogic to 21st Century Semantics

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Abstract

This paper is the second in a two-part series in which we discuss several notions of completeness for systems of mathematical axioms, with special focus on their interrelations and historical origins in the development of the axiomatic method. We argue that, both from historical and logical points of view, higher-order logic is an appropriate framework for considering such notions, and we consider some open questions in higher-order axiomatics. In addition, we indicate how one can fruitfully extend the usual set-theoretic semantics so as to shed new light on the relevant strengths and limits of higher-order logic.

Introduction

This paper is the sequel to (Awodey and Reck 2002). In this two-part series we discuss several notions of completeness for systems of mathematical axioms, including the notion of categoricity. In Part I, we first distinguished these notions conceptually and then documented how each of them arose historically. This developement occurred in connection with that of the axiomatic method in late nineteenth and early twentieth century mathematics, in the works of, among others, Dedekind, Peano, Hilbert, Huntington, and Veblen. We also illustrated how the interrelations between these various notions were first discussed, systematically and in some detail, in metalogical and metamathematical works by Fraenkel and Carnap from the 1920s. Some of these discussions were continued in more well-known works by Hilbert, Gödel, Tarski, and others in the late 1920s and 1930s.

A number of the questions first formulated in these early metatheoretic investigations still remain mathematically interesting today. However, the now-standard restriction to first-order logic in connection with notions such as categoricity and completeness conflicts with how these topics were originally treated in the 1920s and 30s in the early works by Carnap, Hilbert, Gödel, and Tarski, as became clear in Part I. Indeed, some of the most interesting remaining questions in this connection only acquire their real significance in a broader framework, namely that of higher-order logic. Thus from a historical point of view, the restriction to first-order logic is unwarranted and inadequate. In the present paper, we intend to show further that this restriction is also technically ill-advised, insofar as some aspects of these questions are more naturally and fruitfully treated in higher-order logic.

For that purpose, we first present a concise review of this expanded logical framework. We then give partial answers to some of questions that were mentioned in Part I, but were left open since first being raised in the 1920s and 30s. Besides expanding the logical framework to that of higher-order logic, we also take a wider view of semantics than is customary in contemporary metalogic and metamathematics, or was even possible until quite recently. Namely, we extend the range of semantic notions from the standard set-theoretic ones to more general topological and category-theoretic semantics. This might seem even more radical than the move to higher-order logic, but we believe it is justified by the light it sheds on some previously obscure topics. It also allows us to establish some strengthenings of earlier results along lines hardly foreseeable by Carnap or Tarski, but not incompatible with their point of view. We conclude by indicating what we consider to be some promising directions for further work.

1 Higher-order axiomatics

1.1 Limitations of first-order logic

In what follows we are concerned with, among other things, the use of logic as a tool in formal axiomatics, as discussed in Part I. In this connection, we occasionally find fault with the standard framework of first-order logic and set-theoretic semantics, and we consider several alternatives. We wish to firmly acknowledge here at the outset that first-order logic and set-theoretic semantics are important and useful tools in formal axiomatics. Our proposal is not to reject or replace them, but to augment them for the specific purposes at hand. Moreover, it is clear that the proposed use of higher-order logic, particularly its semantics, involves philosophical presuppositions beyond those of first-order logic which some thinkers find questionable.[1] Thus let it also be clearly said that we do not intend this study as a contribution to the on-going discussion of the relative merits of higher-order logic from a philosophical point of view; our interests are mainly historical and mathematical.

That said, the evident difficulty involved in using first-order logic (hereafter FOL) in formal axiomatics is its inability to fully characterize structures with infinite models. The Löwenheim-Skolem theorems show that it is impossible to fully axiomatize an infinite mathematical structure, even up to isomorphism, using only FOL. It follows that FOL is not suitable for characterizing the basic objects of mathematics, like the natural, real, and complex numbers, and the Euclidean spaces.[2]

Moreover, many objects of mathematical study today are described generally by axioms that are not intended to be categorical, but are not of first order either. For example, rings with conditions on ideals, like Noetherian or principal ideal domain; structures on manifolds like vector bundles or tensor fields; the various kinds of spaces used in functional analysis like Hilbert and Banach spaces, and even classical mathematical objects like Euclidean and projective spaces are all determined axiomatically. It is hardly an exaggeration to say that the axiomatic method has succeeded, since its modern beginnings around 1900, in taking over mathematics. But, as the examples just mentioned illustrate, it is not only FOL that is being used in service of this method.

Of course, one can describe the *models* of such non-first-order axiomatic notions in terms of set theory. But this does not alter the fact that their axiomatic presentation is essentially higher-order. Nor will it do, in such cases, to treat higher-order logic as many-sorted first-order logic, as is occasionally suggested. For in specifying structures such as those just mentioned involving higher types of relations or functions, it is essential that these types be interpreted as such, and not as additional first-order structure, if the axiomatization is to serve its intended purpose. We thus believe that higher-order axiomatic theories are best recognized and studied on their own terms, rather than being converted into set theory or first-order logic.

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[1] See (Jané 1993) and (Corcoran 2002) for discussion.

[2] See (Tennant 2000) for another interesting weakness of FOL

1.2 Higher-order logic

We first present a simple and fairly standard extension of FOL which has the expressive capacity to formulate many of the axiomatic treatments of modern mathematics. Logical languages of this general kind, which are descendant from the type theory mentioned in Part I, Section 3.1, are usually called *higher-order logic* or *simple type theory*.[3]

Higher-order systems of logic are those having variables and quantification over "higher types" of relations or functions among the elements of "lower type". Thus, for example, one can extend the usual language $(A, +, \cdot, 0, 1)$ of ring theory by adding also variables $X, Y, \ldots$ ranging over subsets of the basic domain A. This allows one to axiomatize e.g. principal ideal domains by adding to the theory of commutative rings the familiar condition:

$$\forall I \subseteq A\,(\text{“}I\text{ is an ideal”} \rightarrow \exists x \in A(\ I = (x)\ )) \tag{1}$$

where the expressions "I is an ideal" and the principle ideal (x) are defined as usual. Of course, one also adds some logical vocabulary to express subset formation and membership.

We now give an informal description of a particular language of higher-order logic that is sufficient for the purposes of our further discussion. More details of related systems can be found e.g. in (Lambek and Scott 1986).

The language of HOL

The language of higher order logic (HOL) consists of type symbols, terms, and formulas. We write $\tau : X$ to indicate that the term τ has type X.

Types In addition to basic type symbols $A, B, \ldots$, and a type P of formulas, further types are built up inductively by the type-forming operations:

$$X \times Y,\ X \rightarrow Y,\ P(X)$$

Terms In addition to variables of each type $x_1, x_2, \ldots : X$, and possibly some basic, typed constant symbols, further terms are built up induc-

[3]Type theory is currently experiencing a sort of renaissance because of its applications in computer science. There are literally hundreds of different logical systems that can be called "higher-order logic" or "type theory".

tively by the term-forming operations:

$$\begin{aligned}&\langle \sigma, \tau \rangle,\ p_1(\tau),\ p_2(\tau)\\&\alpha(\tau),\ \lambda x : X.\sigma\\&\{x : X \mid \varphi\}\end{aligned}$$

Formulas In addition to equations $\sigma = \tau$ and atomic formulas $\tau \in \alpha$, further formulas are built up inductively by the usual logical operations:

$$\neg\varphi,\ \varphi \wedge \psi,\ \varphi \vee \psi,\ \varphi \Rightarrow \psi,\ \forall x : X(\varphi),\ \exists x : X(\varphi)$$

The type of a term is determined in the expected way by the types of the terms used in forming it, and these formations are subject to some obvious conditions for significance; e.g. $p_1(\tau) : A$ if $\tau : A \times B$. We make use of the usual conventions in writing formulas whenever convenient, such as writing $\langle x, y, z \rangle$ for $\langle x, \langle y, z \rangle \rangle$. Note that we have included the possibility of basic type and constant symbols, to be used as the basic language of an axiomatic theory. The theory of rings, for example, has one basic type symbol, say A, and the following basic constant symbols indicated with their types:

$$\begin{aligned}&0,\ 1 : A\\&+,\ \cdot : A \times A \to A\end{aligned}$$

Generally, we define a *theory* to consist of a basic *language* of type symbols and constants, together with a set of sentences in that language, called the *axioms*. We shall assume here that a theory has finitely many basic symbols and axioms, although there is no reason in principle why one cannot consider infinite theories. In these terms, e.g. the theory of rings thus consists of the language $(A, +, \cdot, 0, 1)$ and the usual handful of axioms for rings (with unit); and the theory of principal ideal domains results by adding the further axiom (1) above.

We emphasize that this use of HOL for presenting axiomatic theories, while familiar enough from everyday mathematical practice, is quite different from the original use intended by logicists like Frege and Russell, and also from that made of it by Carnap in his *Untersuchungen zur allgemeinen Axiomatik* (Carnap 1928), mentioned in Part I, Section 3.2. These pioneers had what has been called a "universal" conception of logic ((van Heijenoort 1967),(Goldfarb 1979)), according to which there is a single logical system with a single, fixed domain of quantification (namely, "everything"), and with fixed higher types consisting of "all" functions, concepts, propositional functions, etc. By contrast, the conception in use here has (possibly several) basic types, which can be interpreted in various ways, just as is common in the semantics of first-order logic. Indeed, the clearest way to understand the language of HOL presented here is as an extension of the usual language of FOL by adding the higher types $X \to Y$ and $P(X)$ and their associated terms $\langle\sigma,\tau\rangle$, $p_1(\tau)$, $p_2(\tau)$, $\alpha(\tau)$, $\lambda x : X.\sigma$, $\{x : X \mid \varphi\}$, and then building FOL formulas as usual from those terms and variables in equations and basic formulas $\tau \in \alpha$. In particular, any conventional theory in FOL is also a theory in HOL in the present sense.

When needed, a system of formal deduction can be specified in the usual way, as a formal system with logical axioms and rules of inference. One such system is outlined in the Appendix below, but we emphasize that there are many equivalent formulations.

1.3 Semantics

The semantics for HOL is essentially an extension of that for FOL, adjusted to take advantage of the simplifications resulting from the presence of additional types (see e.g. remark 1 below). We shall assume given a "semantic universe" with suitable structure for interpreting the language of HOL. Here we use sets and functions, but later we will generalize to other "universes" (suitable categories) with the required structure.

Rather than stating the formal definition of a model of a theory, we shall give a particular case of it which should be sufficient for the reader to infer the general notion.[4] Suppose we have a theory of the form (A, c, α), with one basic type symbol A, one constant symbol c, and one axiom α. For instance, it might be the theory of semi-groups, with c being $\cdot : A \times A \to A$ and α being the associativity law:

$$\forall x, y, z : A \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

An *interpretation* assigns to each type X a non-empty[5] set $[\![X]\!]$, in such a way that:

[4] See (Lambek and Scott 1986), (MacLane and Moerdijk 1992), (Awodey and Butz 2000) for details.

[5] This restriction merely simplifies the deductive calculus given in the Appendix.

$$
\begin{aligned}
[\![A \times B]\!] &= [\![A]\!] \times [\![B]\!] && \text{(the Cartesian product)}\\
[\![A \to B]\!] &= [\![B]\!]^{[\![A]\!]} && \text{(the set of functions)}\\
[\![P(A)]\!] &= \mathcal{P}([\![A]\!]) && \text{(the power set)}\\
[\![P]\!] &= \{\top, \bot\} && \text{(any two-element set)}
\end{aligned}
$$

A term $\tau(x) : Y$ containing a free variable $x : X$ is interpreted as a function $[\![\tau]\!] : [\![X]\!] \to [\![Y]\!]$, in such a way that:

$$
\begin{aligned}
[\![c]\!] &\in [\![X]\!] && \text{for a basic constant } c : X\\
[\![x]\!] &= 1_{[\![X]\!]} : [\![X]\!] \to [\![X]\!] && \text{(identity function)}\\
& && \text{for a variable } x : X\\
[\![\langle \sigma, \tau \rangle]\!] &= ([\![\sigma]\!], [\![\tau]\!]) && \text{(the ordered pair)}\\
[\![p_i(\tau)]\!] &= \pi_i([\![\tau]\!]) && \text{(the } i\text{-th projection)}\\
[\![\alpha(\tau)]\!] &= [\![\alpha]\!]([\![\tau]\!]) && \text{(functional application)}\\
[\![\lambda x : X.\sigma]\!] &= \text{the function } x \mapsto [\![\sigma]\!]\\
[\![\{x : X \mid \phi\}]\!] &= \text{the subset } \{x \epsilon [\![X]\!] \mid [\![\phi]\!] = \top\}\\
[\![\tau \in \alpha]\!] &= \top \text{ iff } [\![\tau]\!] \epsilon [\![\alpha]\!]\\
[\![\phi \wedge \psi]\!] &= [\![\phi]\!] \wedge [\![\psi]\!] && \text{and similarly for } \neg, \vee, \Rightarrow\\
[\![\forall x : X \phi]\!] &= \top \text{ iff for all } x \epsilon [\![X]\!],\ [\![\phi]\!] = \top\\
[\![\exists x : X \phi]\!] &= \top \text{ iff for some } x \epsilon [\![X]\!],\ [\![\phi]\!] = \top
\end{aligned}
$$

Note that we use the boolean operations $\wedge$, etc., on $\{\top, \bot\}$ to interpret the corresponding logical operations.

An interpretation $[\![-]\!]$ *satisfies* a sentence σ ("σ is *true* under $[\![-]\!]$") just if

$$[\![\sigma]\!] = \top$$

Of course, a *model* of a theory is an interpretation that satisfies the axioms. If M is a model, we also write $[\![-]\!]_M$ when considering it as just an interpretation, and we use the notation:

$$M \models \sigma \quad \text{for} \quad [\![\sigma]\!]_M = \top$$

Although it may look a bit unfamiliar at first sight, this definition agrees with the usual one for models of a first-order theory. For instance, in the example above of semi-groups, an interpretation in the present sense consists of a set $[\![A]\!]$ equipped with a binary operation $[\![\cdot]\!] : [\![A]\!] \times [\![A]\!] \to [\![A]\!]$. A model is such a structure for which

$$([\![A]\!], [\![\cdot]\!]) \models \forall x, y, z : A(x \cdot (y \cdot z) = (x \cdot y) \cdot z),$$

which is easily seen to mean just that the operation $[\![\cdot]\!]$ is associative.

According to our definition, the higher types of functions and relations are interpreted by the corresponding sets (in the conventional terminology, the models are "standard models" rather than "Henkin models"). Observe that such an interpretation is fully determined by the interpretation of the basic language. Thus in particular, an interpretation in this sense of a first-order language is just a first-order structure.

Remark 1. The "internal" notion of satisfaction used here may be unfamiliar; it differs from the more customary, "external" notion from elementary model theory in that *truth* is represented as an element of a set of truth values $\{\top, \bot\}$, and a formula $\varphi(x)$ where $x : X$ is represented as a function,

$$[\![\varphi(x)]\!] : [\![X]\!] \to \{\top, \bot\}.$$

Of course, $[\![\varphi(x)]\!]$ is just the characteristic function of the subset:

$$[\![\{x : X \,|\, \varphi(x)\}]\!] = \{a \in [\![X]\!] \mid [\![X]\!] \models \varphi(a)\} \subseteq [\![X]\!]$$

A sentence (closed formula) is therefore interpreted as one of the truth values $\top$ or $\bot$, with the "true" sentences ($= \top$) being exactly those that hold under the interpretation.

The reason for internalizing truth in this way is that, while it is equivalent to the external approach for set-theoretic semantics, this internal notion can easily be generalized to other semantic universes in a way that external semantics cannot. A similar procedure is sometimes used in connection with *boolean-valued models.*

We now use the semantics to define the notion of *semantic consequence* $\phi \models \psi$ between sentences in the usual way:

$$\phi \models \psi \quad \text{if for every interpretation, } [\![\phi]\!] \leq [\![\psi]\!]$$

where the ordering of truth-values is the usual one, $\bot \leq \top$. Semantic consequence between formulas is defined analogously, using the pointwise ordering of the interpreting functions. The notion of semantic consequence with respect to a theory is defined in the expected way, by considering only those interpretations that are models of the theory.

One sometimes hears it said that HOL is "stronger" than FOL, but this is only so with respect to its expressive capacity, not its semantic consequences. More precisely, the relation of higher-order semantic consequence is conservative over first-order semantic consequence. For let $\mathbb{T}$ be a first-order theory, regarded as a theory in HOL. The models of $\mathbb{T}$ in the HOL sense are then exactly the models in the usual FOL sense. Thus if the first-order sentence ϕ is true in every HOL model, then it is semantically valid in the sense of FOL.

Semantic consequence for HOL differs from that for FOL in several important respects: it is not compact; the usual Löwenheim-Skolem theorems do not hold; and its theorems are not recursively enumerable (see (Shapiro 1991) for further discussion).

1.4 Completeness and categoricity

We can now consider more precisely the question of how completeness and categoricity for an axiomatic theory are related in the context of HOL. As already mentioned in Part I, Section 3.2, the main early studies are notably (Fraenkel 1928), (Carnap 1928), and (Lindenbaum and Tarski 1935).[6] Briefly, the main positive results are that categoricity implies semantic completeness (in the sense of Part I, Section 1, Definition 1) generally, as for theories in FOL, while in certain cases the converse also holds, which is perhaps more surprising.

Proposition 2. *If a theory $\mathbb{T}$ is categorical, then it is semantically complete.*

Proof. (sketch) Given categorical $\mathbb{T}$, it suffices to show that if $M \models \sigma$ for some model M and sentence σ, then also $N \models \sigma$ for any other model N. But since $\mathbb{T}$ is categorical, there is an isomorphism of $\mathbb{T}$-models $i : M \cong N$, in the usual sense. Now it is easy to see that isomorphisms preserve satisfaction, just as in the first-order case. In more detail, one shows by structural induction that for any formula ϕ, one has $[\![\phi]\!]_M = [\![\phi]\!]_N \circ i^n$, as maps $M^n \to \{\top, \bot\}$, where there are n free variables in ϕ, and $i^n : M^n \cong N^n$ is the induced isomorphism on Cartesian products. Thus in particular, if $M \models \sigma$ for some sentence σ, then $\top = [\![\sigma]\!]_M = [\![\sigma]\!]_N$, and so also $N \models \sigma$. $\square$

The more interesting question in this connection is, under what conditions does the converse of proposition 2 hold? As already noted above, our

[6] Some recent authors who have also called attention to this topic are (Corcoran 1980), (Corcoran 1981), (Read 1997), and(Awodey and Carus 2001). This section addresses a question raised in (Corcoran 1981).

restriction to finite sets of axioms is essential here. Indeed, it is not hard to find non-isomorphic models that are logically equivalent (since the number of sets of sentences is bounded). The (infinite) "theory" of such a model would then be semantically complete but not categorical.

For the finite theories under consideration here, however, the situation is rather different. As already mentioned in Part I, Section 3.2, in (Carnap 1928) the implication from semantic completeness to categoricity was conjectured and an erroneous proof was offered. The following (correct) proof of a special case is due to Dana Scott:[7]

Proposition 3. *If a theory* $\mathbb{T}$ *has only one basic type and no basic constant symbols, then* $\mathbb{T}$ *is categorical if it is semantically complete.*

Proof. (sketch) Let σ be the conjunction of the finitely many axioms, and define the new sentence

$$\begin{aligned} \sigma_0 &=_{df} \sigma \wedge (\forall U : P(X))(\sigma^U \rightarrow U \cong X) \\ &= \text{“}X \text{ is the least subset of } X \text{ that satisfies } \sigma\text{”} \end{aligned}$$

in which X is the basic type, U is a variable of type $P(X)$, $U \cong X$ is expressed by the usual definition of isomorphism, and σ^U is a new sentence derived from σ by relativizing all types and quantifiers occurring in σ from X to U.

If σ is satisfied, then so is σ_0 (by the axiom of choice for sets). But if σ is also complete, then we claim that $\sigma \Leftrightarrow \sigma_0$. For if $M \models \sigma$, then we can take some $M' \subseteq M$ such that $M' \models \sigma_0$; since then also $M' \models \sigma$, we also have $M \models \sigma_0$, since σ is complete. But σ_0 is evidently categorical, so σ must also be categorical. □

While it is not difficult to extend this result to a few other cases, we do not know the extent to which it holds in general. Some easy sufficient conditions for the categoricity of a finite theory, given its semantic completeness, are having a definable model (Lindenbaum and Tarski 1935), having a model with no proper submodels, and being categorical in some power. The latter follows from the fact—easily inferred from the foregoing theorem—that all models of a semantically complete theory must have the same cardinality. We know of no counter-examples to the conjecture that semantic completeness of a finite theory implies categoricity in general.

[7] Scott produced this clever proof in response to a talk on Carnap's failed work by the first author. See also (Awodey and Carus 2001).

In sum, it seems that Carnap's conjecture remains undecided, with little indication as to which way it will go. This is surely to be counted as one of the leading open questions in higher-order axiomatics.

2 Topological semantics

In this section we consider an alternative to the usual set-theoretic semantics for HOL.[8] This is drawn from category theory, and is a special case of so-called "topos semantics", which we won't consider in general (see (Lambek and Scott 1986),(MacLane and Moerdijk 1992), and (Awodey and Butz 2000)). The topological semantics outlined here should however suffice to give the reader a general impression of what is involved in interpreting HOL in semantic "universes" other than that of sets.

We first briefly review the motivation for considering alternate semantics for HOL. The first and most obvious reason is that the set-theoretic semantic consequence relation is not deductively axiomatizable in any reasonable sense. Specifically, given a conventional deductive consequence relation $\varphi \vdash \psi$, the Gödel Incompleteness Theorem tells us that this relation cannot be *complete*, in the sense of Part I, Section 1, Definition 1, with respect to set-theoretic semantic consequence.

This does not necessarily mean that higher-order deduction is somehow defective, however. It is at least sound for set-valued semantics, in the sense that $\varphi \vdash \psi$ implies $\varphi \models \psi$. Moreover, it is conservative over first-order deduction, by a simple argument from the semantic conservativity mentioned in section 1.3, above. And as we shall see below, it is in fact complete with respect to the topological semantics to be considered here.

Another reason to broaden the scope of semantics for HOL is that, like completeness, this also affects the notion of *categoricity* for axiomatic theories, effectively making it a stronger condition. Indeed, since categoricity is a semantic notion, restricting semantics to sets makes it dependent on often nontrivial properties of sets, which can have peculiar, unwanted consequences; in Example 4 below, for instance, we indicate a simple theory that is categorical just in case the continuum hypothesis holds. The categoricity of certain axiomatic theories like the natural and real numbers seems to provide confirmation of their adequacy, independent of the more subtle properties of sets. Generalizing the range of semantics conforms better to this intention, as will be discussed further in section 3 below.

[8]This section draws on (Awodey 2000), which the reader can consult for more detail.

Finally, one simple reason for considering alternate semantics is that one is interested in the semantic objects themselves. The possibility of using logic to reason about structures on objects other than sets (as happens with e.g. topological groups) makes the systematic investigation of such objects useful in itself. This is indeed the case with the topological semantics considered below; the semantic objects employed (sheaves) are everyday mathematical objects.[9] This is not the case for the most familiar alternate semantics for HOL, the so-called "Henkin models". These are used only for proving deductive completeness, and have no independent mathematical interest.

The objects used in topological semantics are "continuously varying sets", in a sense made precise in 2.2 below. We first motivate this idea in 2.1 by considering an analogy to the ring of continuous, real-valued functions on a topological space. That example also shows how continuous variability can be used to violate some properties of constants, which is essentially what permits the completeness of higher-order deduction with respect to topological semantics, discussed in 2.3 below.

2.1 Ring of continuous functions

The real numbers $\mathbb{R}$ form a topological space, an abelian group, a commutative ring, a complete ordered field, and much more. Let us consider the properties expressed in just the *language of rings:*

$$0, 1, a+b, a \cdot b, -a$$

and first-order logic. For example, $\mathbb{R}$ is a field:

$$\mathbb{R} \models \forall x(x = 0 \vee \exists y \; x \cdot y = 1)$$

Now consider the *product ring* $\mathbb{R} \times \mathbb{R}$, with elements of the form

$$r = (r_1, r_2)$$

and the *product operations*:

$$\begin{aligned} 0 &= (0,0) \\ 1 &= (1,1) \\ (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ (x_1, x_2) \cdot (y_1, y_2) &= (x_1 \cdot y_1, x_2 \cdot y_2) \\ -(x_1, x_2) &= (-x_1, -x_2) \end{aligned}$$

[9]See e.g. (Hartshorne 1977), (Iversen 1986).

Since these operations are still associative, commutative, and distributive, $\mathbb{R} \times \mathbb{R}$ is still a ring.

But the element $(1,0) \neq 0$ cannot have an inverse, since $(1,0)^{-1}$ would have to be $(1^{-1}, 0^{-1})$. Therefore $\mathbb{R} \times \mathbb{R}$ *is not a field.*

In a similar way, one can form the more general product rings $\mathbb{R} \times \ldots \times \mathbb{R} = \mathbb{R}^n$, or $\mathbb{R}^I$ for any index-set I.

While not in general fields, product rings $\mathbb{R}^I$ are always (von Neumann) *regular*:

$$\mathbb{R}^I \models \forall x \exists y \ (x \cdot y \cdot x = x).$$

For, given x, we can take $y = (y_i)$ with:

$$y_i = \begin{cases} x_i^{-1}, & \text{if } x_i \neq 0 \\ 0, & \text{if } x_i = 0 \end{cases}$$

One can produce rings that violate even more properties of $\mathbb{R}$ by passing to "continuously varying reals". What is a "continuously varying real number"? Let X be a topological space; then a "real number r_x varying continuously over X" is just a continuous function:

$$r : X \to \mathbb{R}$$

We equip these functions with the pointwise operations:

$$(f+g)(x) = f(x) + g(x), \quad \text{etc.}$$

The set $\mathcal{C}(X)$ of all such functions then forms a *subring* of the product ring over the index set $|X|$ of points of the space X, that is, as rings: $\mathcal{C}(X) \subseteq \mathbb{R}^{|X|}$. But unlike the product ring, $\mathcal{C}(X)$ is not regular:

$$\mathcal{C}(X) \not\models \forall f \exists g \ (f \cdot g \cdot f = f)$$

For take e.g. $X = \mathbb{R}$ and $f(x) = x^2$, then we must have:

$$g(x) = \frac{1}{x^2}, \qquad \text{if } x \neq 0$$

but of course:

$$g(0) = \lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty$$

so there can be no *continuous* g satisfying $f \cdot g \cdot f = f$.

Thus the "continuously varying reals" $\mathcal{C}(X)$ have even fewer properties of the field of "constant" reals $\mathbb{R}$ than do the product rings $\mathbb{R}^I$. In this way, passing from constants to continuous variation "abstracts away" some properties of the constants.

2.2 Continuously variable sets

Just as the real numbers could be generalized to the "continuously variable reals" (continuous functions), we now generalize the notion of a set to that of a "continuously variable set," i.e. a *sheaf*.

As a first step, observe that the type-forming operations of product, powerset, equality, etc. can be interpreted in other "universes" of sets. Indeed, consider the universe of "pairs of sets," **Sets** $\times$ **Sets**. The objects have the form:

$$A = (A_1, A_2)$$

and the operations are defined componentwise:

$$\begin{aligned}(A_1, A_2) \times (B_1, B_2) &= (A_1 \times B_1, A_2 \times B_2) \\ \mathcal{P}(A_1, A_2) &= (\mathcal{P}(A_1), \mathcal{P}(A_2)) \\ P &= (P, P)\end{aligned}$$

Term-formation is similarly componentwise. Indeed, the logical operations can also be defined componentwise:

$$\begin{aligned}(a_1, a_2) \in (A_1, A_2) &= (a_1 \in A_1, a_2 \in A_2) \\ (\varphi_1, \varphi_2) \wedge (\psi_1, \psi_2) &= (\varphi_1 \wedge \psi_1, \varphi_2 \wedge \psi_2) \\ &\text{etc.}\end{aligned}$$

This interpretation of the logical language models HOL in the sense that the usual logical axioms and rules of inference (e.g. as given in the Appendix) are all validated. On the other hand, it does not satisfy *all* the properties of **Sets**. For example:

$$\mathbf{Sets} \models A \cong 0 \vee \exists x \ x \in A$$

But in **Sets**$\times$**Sets** we can take as A the object $(1, 0)$, which is not isomorphic to 0, and then $a \in (1, 0)$ means $a = (a_1, a_2)$ with $a_1 \in 1$ and $a_2 \in 0$, which is impossible.

Just as in the case of rings, we can also generalize to $\mathbf{Sets} \times \ldots \times \mathbf{Sets} = \mathbf{Sets}^n$, and indeed to $\mathbf{Sets}^I$ for any index set I, to get the "universe" of I-indexed families of sets.

All such "product universes" have some things in common, e.g. they all satisfy the axiom of choice (which, by the way, can be seen to be formally analogous to regularity for rings). To find even more general "universes" we consider even more general families of sets,

$$(F_x)_{x \in X}$$

varying continuously over an arbitrary topological space X. But what should a "continuously varying set" be? The problem is that we cannot simply take a "continuous set-valued function"

$$F : X \to \mathbf{Sets}$$

as we did for rings of real-valued functions, since **Sets** is not a topological space.

In modern mathematics, one often encounters continuously varying structures; let us recall how this is typically done, in order to find the notion we seek. A "continuously varying space" $(Y_x)_{x \in X}$ over a space X is called a *fiber bundle*. It consists of a space $Y = \sum_{x \in X} Y_x$ and a continuous "indexing" projection $\pi : Y \to X$, with $\pi^{-1}\{x\} = Y_x$, as indicated below.

$$\begin{array}{l} Y \;=\; \displaystyle\sum_{x \in X} Y_x \\ \Big\downarrow{\scriptstyle \pi} \\ X \end{array}$$

A "continuously varying group" $(A_x)_{x \in X}$ is a *sheaf of groups*. It consists essentially of a fiber bundle $\pi : A = \sum_{x \in X} A_x \to X$ satisfying the additional requirements:

1. each A_x is a group,

2. the operations in the fibers A_x "fit together continuously",

3. π is a local homeomorphism (see below).

We can then answer the question of what a "continuously varying set" should be by saying it is a *sheaf of sets*, i.e. a fiber bundle,

$$\begin{array}{l} F \;=\; \displaystyle\sum_{x \in X} F_x \\ \Big\downarrow{\scriptstyle \pi} \\ X \end{array}$$

such that π is a *local homeomorphism*, in the sense that each point $y \in F$ has some neighborhood U on which π is a homeomorphism $U \xrightarrow{\sim} \pi(U)$.

This ensures in particular that each fiber $F_x = \pi^{-1}(x)$ is discrete, and that the variation across the fibers is continuous, in a suitable sense.

To define the semantics of HOL in sheaves, one needs to interpret the basic type-forming and logical operations. Some of these can be defined pointwise, $(F \times G)_x \cong (F_x \times G_x)$. Others, however, cannot; for instance, the exponential G^F of sheaves F, G is the "sheaf-valued hom" $\mathrm{hom}(F, G)$, defined in terms of *germs of continuous maps* $F \to G$, for which $(G^F)_x \ncong G_x^{F_x}$. This is what makes topological semantics different from the product semantics of indexed families.

Like the product universes $\mathbf{Sets}^n$, the universe $\mathrm{sh}(X)$ of all sheaves on a given space X models HOL in the sense that the axioms are all true and the rules of inference are all sound. But in general, sheaves violate e.g. the axiom of choice. Indeed, one can find sheaf models of HOL that also violate many other properties of sets.

2.3 Topological completeness

If we think of sheaves as sets varying continuously in a parameter, the constant sets occur as the special case of no variation. The semantics given in 1.3 above then yield the topological semantics just outlined, with standard set-theoretic semantics as a special case. Some logical statements that are not true of variable sets in general are true of all constant sets, as a result of their special properties. In this sense, the logic of the constant sets is quite strong, while the logic of variable sets is much weaker. That is, fewer things are true of all variable sets than are true of constant ones. This is just like the difference between the field of real numbers and the ring of real-valued functions. Now one can ask, what is the logic of continuously varying sets? That is to say, which sentences of HOL are true in all sheaf models? The answer is given by the following theorem from (Awodey and Butz 2000).[10]

Theorem. *HOL is complete with respect to topological semantics.*

The completeness referred to is deductive completeness in the sense of our Definition 3, Part I, Section 1, with respect to the standard, classical deductive consequence relation, as specified in the Appendix. Thus if a sentence is true in all topological models, then it is provable.

The reader may wonder how this result is to be reconciled with the Gödel incompleteness of deductive higher-order logic. Roughly speaking, the situation is this: the sense in which a sentence is "true but unprovable"

[10] The proof uses recent results in topos theory (Butz and Moerdijk 1999) that are rooted in geometry.

in Gödel's theorem involves only "true of all *constant* sets," but not "true of all *variable* sets." Thus a "true but unprovable" Gödel-style sentence is only true of the constant sets, but it is violated by some variable ones (else it would be provable).

3 Notions of categoricity

Having now considered completeness and categoricity with respect to standard, set-theoretic semantics, and deductive completeness with respect to alternate semantics, we turn to possible alternate notions of categoricity. We have already mentioned that some axiomatic theories in higher-order logic are categorical in the usual sense that any two (standard) models are isomorphic just in case the sets used to model them are assumed to have certain properties, such as satisfying the continuum hypothesis or the axiom of choice. This is essentially because these properties of sets are expressible in HOL. For example, the following simple theory is categorical just if the continuum hypothesis holds.

Example 4. The theory $\mathbb{T}_0$ has one basic type symbol U, one relation symbol $R : PP(U)$, and two axioms expressing the conditions "U is countably infinite" and "$|U| \lneqq |R|$".

But the idea of categoricity as a basic criterion of adequacy for a system of axioms seems to presume that it is not sensitive to such questions as whether the continuum hypothesis holds. Indeed such issues seem irrelevant to the categoricity of descriptions of at least some classical mathematical notions, like the natural numbers. As was seen in Part I above, some version of categoricity was one of the main early conditions of adequacy for axiom systems, quite independently of a precise specification of the theory of sets, or any understanding of their more subtle properties.

In this section, we consider several strengthenings of the notion of categoricity that are not sensitive in this way to special properties of the semantics, although they do have their own peculiarities. The notions considered are called *unique*, *variable*, and *provable* categoricity. Some of the classical theories of greatest interest do indeed have these stronger properties. Finally, we consider the category-theoretic concept of *universality* and its relation to axiomatic descriptions.

3.1 Unique categoricity

This notion strengthens conventional categoricity by requiring that any two models M and N be isomorphic via a *unique* isomorphism $M \cong N$. This is clearly equivalent to saying that the theory at issue is categorical and, furthermore, that its models have no non-trivial automorphisms.[11]

The classical axiomatizations for the natural and real numbers do indeed have this stronger property (the complex numbers do, too, if one eliminates complex conjugation as an automorphism by adding a constant symbol for i). As will become more clear below, axiomatizations are sometimes categorical because there are some natural or canonical maps between models (as opposed to ones gotten, say, by the axiom of choice), and the axioms then suffice to make these canonical maps isomorphisms. The property of unique categoricity also seems to accompany some of the other strengthenings to be considered, and it is found in connection with the category theoretical notion of universality.

3.2 Variable categoricity

We have already considered the notion of a continuously varying model M over a space of parameters X, as made precise by the concept of a *sheaf of models*, which is a model in the "universe" $sh(X)$ of continuously variable sets (cf. 2.2 above). The notion of *variable categoricity* is simply the obvious generalization of categoricity to such variable models:

Definition 5. A theory $\mathbb{T}$ is called *variably categorical* if any two continuously variable models M, N over any space X are isomorphic.

This condition requires more than just that there is an isomorphism

$$h_x : M_x \xrightarrow{\sim} N_x \qquad \text{for each } x \in X.$$

In addition, the various h_x must fit together to form a single, continuous isomorphism

$$h : M \xrightarrow{\sim} N \qquad \text{over } X.$$

Thus, in effect, the h_x must also vary continuously with the parameter x.

Note that this notion does generalize conventional categoricity, since the conventional notion is the special case of variation over a one-point parameter space. In this sense, conventional categoricity is the limiting or trivial case of variable categoricity.

[11] Cf. (Tarski 1983), p. 313.

There is, of course, no sense to requiring that models over *different* spaces be isomorphic, since there is no notion of a map between such models (at least in the current situation).

There is an obvious *unique* version of this notion, obtained by requiring unique isomorphisms between models. For example, the classical theories of $\mathbb{N}$ and $\mathbb{R}$ have this property — they are *uniquely* variably categorical. The contrived theory $\mathbb{T}_0$ above does not have it, however (even assuming CH). The reason why is roughly that, in a given model M, a variable subset $R_M \subseteq P(U_M)$ might be *pointwise* isomorphic to $P(U_M)$, just for cardinality reasons, without there being a *continuous* isomorphism $R_M \xrightarrow{\sim} P(U_M)$ over the space of parameters X.

As suggested by the previous remark, the basic idea behind variable categoricity is that the strong requirement that the isomorphisms must also be continuously parametrized with the models tends to "break up" accidental or arbitrary choices of maps, and restrict to those that are somehow intrinsic to, or canonically associated with, the structure at issue. The following notion provides another, rather different, way of restricting the possible isomorphisms, namely by requiring them to be definable or provable.

3.3 Provable categoricity

We want to formulate the idea that the connecting isomorphism between any two models of a categorical theory is definable from the language of the theory, and is provably an isomorphism from the axioms of the theory.[12] To specify this notion, suppose our theory $\mathbb{T}$ is of the form:

$$U,\ f : T(U),\ \alpha(U, f)$$

where U is a basic type symbol, f a basic constant symbol of type $T(U)$, and $\alpha(U, f)$ a sentence in the language U, f and higher-order logic. Here we display U in the type symbol $T(U)$ to remind ourselves that the type of f may contain U as a parameter, e.g. if f represents a binary operation on U, then $T(U)$ is $U \times U \to U$. Similarly, the axiom $\alpha(U, f)$ likely contains the basic language U, f.

Now consider the new theory $\mathbb{T}^2$, which is essentially two copies of $\mathbb{T}$ written side-by-side. It has:

- basic types: $U_1,\ U_2$
- basic terms: $f_1 : T(U_1),\ f_2 : T(U_2)$

[12] Cf. (Tarski 1983), p. 310.

- axioms: $\alpha_1(U_1, f_1)$, $\alpha_2(U_2, f_2)$

where $T(U_1)$ is built from U_1 in the same way that $T(U)$ was built from U, e.g. if $f : U \times U \to U$, then $f_1 : U_1 \times U_1 \to U_1$, and similarly for $T(U_2)$. The axioms are similarly just the axiom α of $\mathbb{T}$ with the respective substitutions of (U_1, f_1) and (U_2, f_2) for (U, f).

Observe that a model of $\mathbb{T}^2$ is just a pair of models of $\mathbb{T}$,

$$\mathbf{Mod}(\mathbb{T}^2) = \mathbf{Mod}(\mathbb{T}) \times \mathbf{Mod}(\mathbb{T}).$$

Definition 6. $\mathbb{T}$ is called *provably categorical* if:

$$\mathbb{T}^2 \vdash \exists h : U_1 \to U_2 \text{ “}h \text{ is a } \mathbb{T}\text{-model isomorphism”}$$

where the formula “h is a $\mathbb{T}$-model isomorphism” is to be spelled out in higher-order logic in the obvious way.

The idea behind provable categoricity is that the theory $\mathbb{T}$ has enough “logical strength” on its own to ensure that any two $\mathbb{T}$-models are isomorphic. This notion is plainly dependent on the logical consequence relation represented by $\vdash$. Here we are assuming the classical, syntactic consequence relation in higher-order logic (as given in the Appendix). A different (weaker) notion results if we take instead e.g. semantic consequence for classical **Set**-valued semantics. That notion is clearly equivalent to conventional categoricity. Of course, any theory that is provably categorical is also categorical.

A stronger condition will result from a weaker notion of logical consequence $\vdash$. For example, using intuitionistic provability instead of classical by omitting the law of excluded middle makes it more difficult for a theory to be provably categorical. It is not hard to make up theories that are provably categorical classically, but not so intuitionistically.

The familiar theories of natural and real numbers are provably categorical (even intuitionistically). The contrived theory $\mathbb{T}_0$ which depends on the continuum hypothesis is evidently not (else one could prove CH in higher-order logic).

As these remarks make clear, there is a connection between provable categoricity and semantic considerations like completeness. Indeed the completeness of the higher-order deductive consequence relation with respect to topological semantics is used in the proof of the following:

Theorem 7. *A theory is provably categorical if and only if it is variably categorical.*

The even stronger notion of intuitionistically provable categoricity mentioned above is equivalent to a certain semantic notion that is phrased in terms of arbitrary *toposes*, but we have chosen not to go into that here.

3.4 Universality

Category theory provides a notion of "unique specification" that is related to categoricity in an interesting way, which remains to be clarified. Although this is not the place for a thorough discussion, it seems at least worth mentioning the basic connection and a couple of examples.

The basic concept we have in mind is that of a *universal mapping property*, which can be used to characterize a particular mathematical structure. The connection with the present topic results from the fact that universal mapping properties are unique characterizations up to isomorphism; any two structures that satisfy a universal mapping property are necessarily isomorphic. Indeed, such structures are *uniquely* isomorphic; so universal mapping properties may be compared with uniquely categorical theories.

The two notions do not seem to be equivalent, however. While some concepts can be formulated both in terms of a categorical, axiomatic theory and a universal mapping property, some concepts seem to be given most naturally in one way or the other, as the following examples illustrate.

Examples. 1. The natural numbers are characterized by the universal mapping property called "natural numbers object", due to Lawvere (Lawvere 1969). In any category with a terminal object 1, consider arbitrary structures of the form:

$$1 \xrightarrow{a} U \xrightarrow{f} U$$

(no conditions on f). A *natural numbers object* is a *universal* structure of this type. That is, one (N, o, s) such that given any such (U, a, f) there is a unique homomorphism $h : (N, o, s) \to (U, a, f)$, i.e. a map $h : N \to U$ such that $ho = a$ and $hs = fh$, as indicated in the commutative diagram below.

$$\begin{tikzcd} 1 \arrow[r, "o"] \arrow[d, equal] & N \arrow[r, "s"] \arrow[d, dotted, "h"'] & N \arrow[d, dotted, "h"] \\ 1 \arrow[r, "a"] & U \arrow[r, "f"] & U \end{tikzcd}$$

This characterization is equivalent to the familiar Peano axioms when both are interpreted in categories like **Sets**. It is worth mentioning that it also applies in much more general categories than **Sets**, where the Peano axioms cannot be interpreted.

2. A notion that can be given by a universal mapping property, but not by any familiar axioms, is that of the *free group* on a set of generators. Consider the case of two generators: the *free group* $F(x, y)$ on the elements x, y has the property that for any group G and elements $g, g' \in G$, there is a unique homomorphism $h : F(x, y) \to G$ with $h(x) = g$ and $h(y) = g'$. The concept of a *polynomial ring* is defined by a similar universal mapping property.

3. The real numbers $\mathbb{R}$ provide an example of a structure characterized by a (uniquely) categorical theory that is not determined by any known universal mapping property.

Of course, it may well be that one can find higher-order axioms for free groups, polynomial rings, etc., or even for any particular universal mapping property. Conversely, the real numbers can perhaps be characterized by a suitable universal mapping property.[13] We don't know whether either of these is the case, but simply mention the connection between categoricity and universality as a direction for possible further research. Indeed, this line of thought seems to be quite closely related to Carnap's work on extremal axioms and Hilbert's Axiom of Line Completeness, mentioned in Part I, Section 3.2.

Appendix: Deduction for higher-order logic

The *deductive consequence relation* $\varphi \vdash \psi$ between formulas is specified by a deductive calculus in the usual way. The following rules of inference could be reduced considerably by defining some logical operations in terms of others. See (Lambek and Scott 1986) for some alternatives.

1. *Order*

   (a) $\varphi \vdash \varphi$

   (b) $\varphi \vdash \psi$ and $\psi \vdash \vartheta$ implies $\varphi \vdash \vartheta$

   (c) $\varphi \vdash \psi$ implies $\varphi[\tau/x] \vdash \psi[\tau/x]$

[13] See (Pavlovic and Pratt 1999), (Escardo and Simpson 2001) for some recent attempts.

2. *Equality*

   (a) $\top \vdash \tau = \tau$
   (b) $\tau = \tau' \vdash \varphi[\tau/x] \Rightarrow \varphi[\tau'/x]$
   (c) $\vartheta \vdash \varphi \Rightarrow \psi$ and $\vartheta \vdash \psi \Rightarrow \varphi$ implies $\vartheta \vdash \varphi = \psi$
   (d) $\forall x.(\alpha(x) = \beta(x)) \vdash \alpha = \beta$

3. *Products*

   (a) $\top \vdash \langle p_1\tau, p_2\tau\rangle = \tau$
   (b) $\top \vdash p_i\langle \tau_1, \tau_2\rangle = \tau_i, \quad i = 1, 2$

4. *Exponents*

   (a) $\top \vdash (\lambda x.\tau)(x) = \tau$
   (b) $\top \vdash \lambda x.\alpha(x) = \alpha \quad$ (x not free in α)

5. *Elementary logic*

   (a) $\bot \vdash \varphi$
   (b) $\varphi \vdash \top$
   (c) $\neg\neg\varphi \vdash \varphi$
   (d) $\vartheta \vdash \neg\varphi$ iff $\vartheta \wedge \varphi \vdash \bot$
   (e) $\vartheta \vdash \varphi$ and $\vartheta \vdash \psi$ iff $\vartheta \vdash \varphi \wedge \psi$
   (f) $\vartheta \vee \varphi \vdash \psi$ iff $\vartheta \vdash \psi$ and $\varphi \vdash \psi$
   (g) $\vartheta \wedge \varphi \vdash \psi$ iff $\vartheta \vdash \varphi \Rightarrow \psi$
   (h) $\vartheta \vdash \varphi(x)$ iff $\vartheta \vdash \forall x \varphi(x)$ (x not free in ϑ)
   (i) $\exists x \varphi(x) \vdash \vartheta$ iff $\varphi(x) \vdash \vartheta$ (x not free in ϑ)

The τ's are any terms; φ, ψ, ϑ are any formulas; α, β are any terms of the same exponential type. Substitution $\varphi[\tau/y]$ must include a convention to avoid binding free variables in τ. The type $P(X)$ and the associated terms $\tau \in \alpha$ and $\{x : X|\varphi\}$ are treated as alternate notation for $X \to P$, $\alpha(\tau)$, and $\lambda x : X.\varphi$, respectively.

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References

Awodey, S. (2000). Continuity and logical completeness. Technical Report CMU-PHIL-116, Carnegie Mellon University.

Awodey, S. and C. Butz (2000). Topological completeness for higher-order logic. *Journal of Symbolic Logic 65*, 1168–82.

Awodey, S. and A. Carus (2001). Carnap, completeness and categoricity: The *Gabelbarkeitssatz* of 1928. *Erkenntnis 54*, 145–72.

Awodey, S. and E. Reck (2002). Completeness and categoricity, part I: 19th century axiomatics to 20th century metalogic. *This journal*.

Butz, C. and I. Moerdijk (1999). Topological representation of sheaf cohomology of sites. *Compositio Mathematica 118*, 217–33.

Carnap, R. (1928). Untersuchungen zur Allgemeinen Axiomatik. Manuscript, Carnap Nachlaß, University of Pittsburgh.

Corcoran, J. (1980). Categoricity. *History and Philosophy of Logic 1*, 187–207.

Corcoran, J. (1981). From categoricity to completeness. *History and Philosophy of Logic 2*, 113–19.

Corcoran, J. (2002). Second-order logic. In C. A. Anderson (Ed.), *Church Memorial Volume*. Dondrecht: Kluwer. To appear.

Escardo, M. and A. Simpson (2001). A universal characterization of the closed interval. In *Proceedings of the 16th Annual Symposium on Logic in Computer Science (LICS'01)*. To appear.

Fraenkel, A. (1928). *Einleitung in die Mengenlehre* (third, revised ed.). Berlin: Springer.

Goldfarb, W. (1979). Logic in the twenties: the nature of the quantifier. *Journal of Symbolic Logic 44*, 351–68.

Hartshorne, R. (1977). *Algebraic Geometry.* New York: Springer.

Iversen, B. (1986). *Cohomology of Sheaves.* New York: Springer.

Jané, I. (1993). A critical appraisal of second-order logic. *History and Philosophy of Logic 14*, 67–86.

Lambek, J. and P. J. Scott (1986). *Introduction to Higher-Order Categorical Logic.* Cambridge: Cambridge University Press.

Lawvere, F. (1969). Adjointness in foundations. *Dialectica 23*, 281–96.

Lindenbaum, A. and A. Tarski (1934–1935). Über die Beschränkheit der Ausdrucksmittel deduktiver Theorien. In *Ergebnisse eines Mathematischen Kolloquiums, Fascicule 7*, pp. 15–22.

MacLane, S. and I. Moerdijk (1992). *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* New York: Springer.

Pavlovic, D. and V. Pratt (1999). On coalgebra of real numbers. *Electronic Notes in Theoretical Computer Science 19.*

Read, S. (1997). Completeness and categoricity: Frege, Gödel and model theory. *History and Philosophy of Logic 18*, 79–93.

Shapiro, S. (1991). *Foundations Without Foundationalism: A Case for Second-Order Logic.* Oxford: Oxford University Press.

Tarski, A. (1983). *Logic, Semantics, Metamathematics* (second ed.). Indianapolis: Hackett. Edited, with introduction, by J. Corcoran. J. H. Woodger, trans. First edition published by Oxford University Press, Oxford, 1956.

Tennant, N. (2000). Deductive versus expressive power: A pre-Gödelian predicament. *Journal of Philosophy 47*, 257–77.

van Heijenoort, J. (1967). Logic as calculus and logic as language. *Boston Studies in Philosophy of Science 3*, 440–46.