# TOPOLOGICAL COMPLETENESS OF FIRST-ORDER MODAL LOGIC

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As McKinsey and Tarski [19] showed, the Stone representation theorem for Boolean algebras extends to algebras with operators to give topological semantics for (classical) propositional modal logic, in which the "necessity" operation is modeled by taking the interior of an arbitrary subset of a topological space. The topological interpretation was extended by Awodey and Kishida [3] in a natural way to arbitrary theories of full first-order logic. This paper proves the resulting system of first-order S4 modal logic to be complete with respect to such topological semantics.

## 1. Topological Semantics for First-Order S4

This section reviews the topological semantics for first-order modal logic given in [3]. Since it combines topological semantics for propositional modal logic and the usual semantics for first-order logic as formulated denotationally, it is helpful to first review these.

### 1.1. **Topological Semantics for Propositional S4.**

A language of propositional modal logic results by adding the sentential operator $\Box$, called a *modal operator*, to a language of classical propositional logic. While $\Box$ usually means "necessity," another modal operator $\Diamond$, for "possibility," can be had by defining $\Diamond\varphi := \neg\Box\neg\varphi$. The system S4 of propositional modal logic consists of the axioms and rule listed below, in addition to all axioms and rules of classical propositional logic. Here, $\top$ stands for truth, or any theorem of propositional logic.

$$\Box\varphi \vdash \varphi$$

$$\Box\varphi \vdash \Box\Box\varphi$$

$$\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$$

$$\top \vdash \Box\top$$

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$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}$$

We use a calculus of binary sequents $\varphi \vdash \psi$, whose basic rules are reflexivity and transitivity of $\vdash$.[1]

McKinsey and Tarski [19] showed that these rules are exactly those of the interior operation in topological spaces.[2] More precisely, given a language $\mathcal{L}$ of propositional modal logic, an *interpretation* of $\mathcal{L}$ is a map $[\![\cdot]\!]$ from the set of sentences of $\mathcal{L}$ to a topological space $X$ such that $[\![p]\!]$ is an arbitrary subset of $X$ for each atomic sentence $p$, and which satisfies the following conditions for sentential connectives and propositional constants.

$$[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$$

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$

$$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$$

$$[\![\top]\!] = X$$

$$[\![\bot]\!] = \varnothing$$

$$[\![\Box\varphi]\!] = \mathbf{int}([\![\varphi]\!]).$$

An interpretation $(X, [\![\cdot]\!])$ models a sentence $\varphi$ if $\varphi$ is "true" under $[\![\cdot]\!]$, i.e.:

$$(X, [\![\cdot]\!]) \models \varphi \iff [\![\varphi]\!] = X.$$

Then the correspondence between the rules of the interior operator and the S4 rules, as well as between the rules of the Boolean set operations and the rules of sentential operators, immediately implies that (propositional) S4 is sound with respect to this topological semantics.

**Theorem 1.** *For every pair of sentences $\varphi$, $\psi$ of a propositional modal language $\mathcal{L}$,*

$$\varphi \vdash \psi \textit{ is provable in S4} \implies \textit{every topological interpretation } (X, [\![\cdot]\!]) \textit{ has } [\![\varphi]\!] \subseteq [\![\psi]\!].$$

*In particular,*

$$\vdash \varphi \textit{ is provable in S4} \implies \textit{every topological interpretation } (X, [\![\cdot]\!]) \textit{ has } (X, [\![\cdot]\!]) \models \varphi.$$

[1] Cf. e.g. [1], pp. 138f.

[2] Actually they showed the dual result for the closure operation.

S4 is also complete with respect to the topological semantics, in the following strong form. The proof of this result will be reviewed later in Subsection 2.1.

**Theorem 2.** *For any consistent theory $\mathbb{T}$ containing S4 in $\mathcal{L}$, there exist a topological space $X$ and a topological interpretation $[\![\cdot]\!]$ such that, for every pair of sentences $\varphi$, $\psi$ of $\mathcal{L}$,*

$$\varphi \vdash \psi \text{ is provable in } \mathbb{T} \iff [\![\varphi]\!] \subseteq [\![\psi]\!].$$

1.2. **Denotational Semantics for (Classical) First-Order Logic.** In extending the topological semantics to first-order modal logic, it will be useful to extend the notation $[\![\varphi]\!]$ to first-order formulas with free variables. In this subsection we review the semantics of first-order logic in this "denotational" formulation.

As usual, a *structure* $M$ for a first-order language $\mathcal{L}$ of signature $\{R_i, f_j, c_k\}_{i\in I, j\in J, k\in K}$ is a tuple $\langle |M|, R_i{}^M, f_j{}^M, c_k{}^M\rangle_{i\in I, j\in J, k\in K}$ consisting of a set $|M|$ and interpretations for basic symbols in $\mathcal{L}$. Thus $M$ interprets relation symbols $R$ of arity $n$, function symbols $f$ of arity $m$, and constant symbols $c$ in $\mathcal{L}$ by assigning $R^M \subseteq |M|^n$, $f^M : |M|^m \to |M|$, and $c^M \in |M|$. Based on this interpretation, the satisfaction or modeling relation is recursively defined as usual. For a formula $\varphi(\bar{x})$ of $\mathcal{L}$ in the context of variables $\bar{x} = (x_1, \ldots, x_n)$ and a sequence $\bar{a} = (a_1, \ldots, a_n)$ of elements of $|M|$, we write

$$M \vDash \varphi[a_1, \ldots, a_n]$$

to mean "$M$ is a model of $\varphi$ with $a_1, \ldots, a_n$ in place of the variables $x_1, \ldots, x_n$." Here, by putting $\varphi$ in the context $x_1, \ldots, x_n$, it is presupposed that $x_1, \ldots, x_n$ are all distinct, and that $\varphi$ has no free variables except $x_1, \ldots, x_n$. (We write $M \vDash \sigma$ when $n = 0$.) Then we say $\varphi$ *holds in* $M$, written (with a slight abuse of notation) $M \vDash \varphi$, to mean that $M \vDash \varphi[\bar{a}]$ for any suitable $\bar{x}$ and every $\bar{a} \in |M|^n$. Given any theory $\mathbb{T}$ in $\mathcal{L}$, we say $M$ is a *model of* $\mathbb{T}$, written $M \vDash \mathbb{T}$, if $M \vDash \varphi$ for every theorem $\varphi$ of $\mathbb{T}$, i.e., every formula $\varphi$ such that $\mathbb{T}$ proves the sequent $\vdash \varphi$.

Using this relation $\vDash$, we extend the notion of *interpretation* to formulas. In a structure $M$ for $\mathcal{L}$, the interpretation of a formula $\varphi$ of $\mathcal{L}$ in the context $\bar{x} = (x_1, \ldots, x_n)$ is the set

$$[\![\, \bar{x} \mid \varphi \,]\!]^M := \{\, \bar{a} \in |M|^n \mid M \vDash \varphi[\bar{a}]\,\}.$$

In short, the interpretation of $\varphi$ is the *(logically) definable relation* on $M$ defined by $\varphi$. In the case of $n = 0$, we have $|M|^0 \cong \{\text{true}\}$ and then, for a closed formula $\sigma$,

$$[\![\sigma]\!]^M = \{ * \in \{\text{true}\} \mid M \vDash \sigma \} = \begin{cases} \{\text{true}\} & \text{if } M \vDash \sigma, \\ \varnothing & \text{if } M \nvDash \sigma. \end{cases}$$

Also, the interpretations for terms $t(\bar{x})$ are given by definable maps $[\![\, \bar{x} \mid t \,]\!] : |M|^n \to |M|$ in the obvious way. It follows that, for a formula $\varphi$, we have

$$M \vDash \varphi \iff [\![\, \bar{x} \mid \varphi \,]\!]^M = |M|^n \tag{1}$$

for any suitable $\bar{x}$. It is worth noting that, instead of the definition in terms of $\vDash$, we could recursively define $[\![\, \bar{x} \mid \varphi \,]\!]$ and $[\![\, \bar{x} \mid t \,]\!]$ directly from $R^M$, $f^M$, $c^M$ with operations on sets and images of maps. In that case (1) would be the *definition* of $\vDash$.

In terms of $[\![\cdot]\!]$, the soundness and completeness of first-order logic are expressed as follows.

**Theorem 3.** *Given a language $\mathcal{L}$ of first-order logic, the following holds for any sentence $\sigma$ of $\mathcal{L}$:*

$$\sigma \textit{ is a theorem of first-order logic} \iff \textit{every structure } M \textit{ has } M \vDash \sigma.$$

*Or, in terms of $[\![\cdot]\!]$, for any pair of formulas $\varphi$, $\psi$ of $\mathcal{L}$ with no free variables except $\bar{x}$,*

$$\varphi \vdash \psi \textit{ is provable in first-order logic} \iff \textit{every interpretation } [\![\cdot]\!] \textit{ has } [\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!].$$

1.3. **Sheaves over a Space.** In unifying the two sorts of semantics reviewed so far, we will use the notion of sheaf over a topological space, which has both topological and set-theoretic aspects.

**Definition 1.** A *sheaf* over a topological space $X$ consists of a topological space $F$ and a *local homeomorphism* $\pi : F \to X$, meaning that every point $a$ of $F$ has some open neighborhood $U$ (i.e., $a \in U$) such that $\pi(U)$ is open and the restriction $\pi|_U : U \to \pi(U)$ of $\pi$ to $U$ is a homeomorphism. $X$ is called the *base space*, $F$ the *total space*, and $\pi$ the *projection* from $F$ to $X$.[3]

[3] The notion of a sheaf is sometimes defined as a certain kind of functor, in which case the notion used here is called an étale space. The functorial notion is equivalent to the notion here (in the categorical sense). This paper only considers sheaves over topological spaces; but the definition in terms of functors enables one to define sheaves more generally over various categories (see e.g. [17] for details) and obtain more general models of modal logic.

One of the important properties of sheaves is that any local homeomorphism $\pi : F \to X$ is not only continuous but also is an *open map*, i.e., $\pi(U) \subseteq X$ is open in $X$ for every $U$ open in $F$.

We can also look at sheaves from the following viewpoint. Given a sheaf $(F, \pi)$ over $X$, define for any point $p \in X$ the "stalk" $F_p = \pi^{-1}(\{p\}) \subseteq F$, also called the *fiber* over $p$. Because fibers do not intersect each other, $F$ is partitioned into fibers, so that the underlying set $|F|$ of the space $F$ can be recovered by taking the disjoint union of all the fibers. That is, we can write

$$|F| = \sum_{p \in X} F_p,$$

where $\sum$ indicates that the union is disjoint. Note that each fiber $F_p$ forms a discrete subspace of $F$.

We require a few more notions. A map from a sheaf $(F, \pi_F)$ over $X$ to another $(G, \pi_G)$ over $X$ is a continuous map $f : F \to G$ over $X$, i.e., one that respects the fibers with $\pi_G \circ f = \pi_F$.

$$\begin{array}{ccc} F & \xrightarrow{\;f\;} & G \\ & {\scriptstyle \pi_F} \searrow \quad = \quad \swarrow {\scriptstyle \pi_G} & \\ & X & \end{array}$$

Thus the underlying map $f$ can be written as a bundle of maps from fibers to fibers:

$$f = \sum_{p \in X} f_p : \sum_{p \in X} F_p \longrightarrow \sum_{p \in X} G_p, \qquad f_p : F_p \to G_p \text{ for each } p \in X.$$

It follows that such continuous maps are local homeomorphisms, and hence are open maps.

The *product* of two sheaves $(F, \pi_F)$ and $(G, \pi_G)$ over $X$ is the usual "fibered product"

$$\begin{array}{ccc} F \times_X G & \longrightarrow & G \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \pi_G} \\ F & \xrightarrow[\pi_F]{} & X \end{array}$$

given by the underlying set

$$|F \times_X G| = \{ (a, b) \in F \times G \mid \pi_F(a) = \pi_G(b) \} = \sum_{p \in X} F_p \times G_p$$

with its subspace topology of $F \times G$, paired with the projection sending $(a, b)$ to $\pi_F(a) = \pi_G(b)$.

Lastly, for a sheaf $(F, \pi)$ over $X$, the *diagonal map* $\Delta : F \to F \times_X F$ defined as $a \mapsto (a, a)$ is a map of sheaves, and hence is an open map. Therefore, in particular, the image

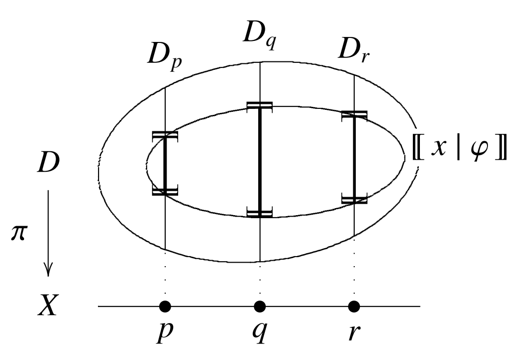
$$\Delta(F) = \{ (a, a) \in F \times_X F \mid a \in F \} \subseteq F \times_X F$$

of $F$ is an open subset of $F \times_X F$. We note that, for any topological space $F$ in general, if $\pi : F \to X$ is an open continuous map with the diagonal map $\Delta : F \to F \times_X F$ an open map as well, then $(F, \pi)$ is a sheaf over $X$.

1.4. **Topological Semantics for First-Order S4.** Now the two semantics reviewed in Subsections 1.1 and 1.2 are combined using sheaves in the following manner. Consider a first-order modal language $\mathcal{L}$, and a sheaf $\pi : D \to X$ over a topological space $X$ as our "domain" of quantification. Letting $D^n$ be the $n$-fold "fibered" product of $D$ over $X$, choose an arbitrary subset (not necessarily open) $[\![R]\!] \subseteq D^n$ for each $n$-ary relation symbol $R$ of $\mathcal{L}$. We also choose arbitrary maps of sheaves $[\![f]\!] : D^n \to D$ for $n$-ary function symbols $f$ of $\mathcal{L}$, including "global sections" $[\![c]\!] : X \to D$ for constants $c$ considered as 0-ary function symbols, noting that $X = D^0$. Then, restricted to each fiber $D_p$, we have $[\![R]\!]_p \subseteq {D_p}^n$, $[\![f]\!]_p : {D_p}^n \to D_p$, $[\![c]\!]_p = [\![c]\!](p) \in D_p$ so that

$$\langle D_p, [\![R]\!]_p, [\![f]\!]_p, [\![c]\!]_p, \dots \rangle = \mathfrak{D}_p$$

is a structure for $\mathcal{L}$. We use the usual first-order semantics in each fiber and take a disjoint union to interpret the first-order operations in $\mathcal{L}$.




$$\begin{array}{ccc} [\![ x \mid \varphi ]\!] & \subseteq & |D| \\ \| & & \| \\ \sum_{p \in X} [\![ x \mid \varphi ]\!]_p & & \sum_{p \in X} D_p \\ [\![ x \mid \varphi ]\!]_p & \subseteq & D_p \end{array}$$

On the other hand, $\Box$ is interpreted by the interior operations on $X$ and $D$. That is,

$$[\![\Box\sigma]\!] = \mathbf{int}_X([\![\sigma]\!]) \subseteq X,$$

$$[\![ \bar{x} \mid \Box\varphi ]\!] = \mathbf{int}_{D^n}([\![ \bar{x} \mid \varphi ]\!]) \subseteq D^n.$$

for sentences $\sigma$ and formulas $\varphi$ (with no more than $n$ variables). Let us record the essential features of such models in the following.

**Definition 2.** A *topological interpretation* $(D, [\![\cdot]\!])$ for a first-order language $\mathcal{L}$ with modal operator $\Box$ consists of the following data:

(i) a topological space $X$;

(ii) a space $D$ with a local homeomorphism $\pi : D \to X$ (interpreting the domain of "individuals");

(iii) for each relation symbol $R$ of $\mathcal{L}$ of arity $n$, a subset $[\![R]\!] \subseteq D^n$, where $D^n$ is the $n$-fold product of $D$ over $X$.

(iv) for each function symbol $f$ of $\mathcal{L}$ of arity $n$ (including constant symbols with $n = 0$), a continuous map $[\![f]\!] : D^n \to D$ over $X$.

and interprets $\mathcal{L}$ in the following manner:

(v) first-order operations in $\mathcal{L}$ are interpreted by the usual operations on sets and maps;

(vi) the modal operator $\Box$ is interpreted by topological interior on the spaces $X$ and $D^n$.

For a formula $\varphi$ of $\mathcal{L}$, we again write $(D, [\![\cdot]\!]) \models \varphi$ for $[\![\, \bar{x} \mid \varphi \,]\!] = D^n$ in a suitable context of variables $\bar{x}$. For a theory $\mathbb{T}$ in $\mathcal{L}$, we say $(D, [\![\cdot]\!])$ is a *topological model of* $\mathbb{T}$, written $(D, [\![\cdot]\!]) \models \mathbb{T}$, if $(D, [\![\cdot]\!]) \models \varphi$ for every theorem $\varphi$ of $\mathbb{T}$. The class of such interpretations provides semantics for the following system of first-order modal logic.

**Definition 3.** The system FOS4 of first-order modal logic is obtained by simply joining the following two sets of axioms and rules:

1. Usual rules and axioms of classical first-order logic. Their application is indifferent to whether formulas contain $\Box$ or not.
2. Propositional S4 axioms and rule for all formulas:

$$\Box\varphi \vdash \varphi$$

$$\Box\varphi \vdash \Box\Box\varphi$$

$$\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$$

$$\top \vdash \Box\top$$

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}$$

**Theorem 4.** *Every topological interpretation* $(D, [\![\cdot]\!])$ *satisfies all axioms and rules of FOS4.*

*Proof.* The classical rules are satisfied by corresponding operations of sets, and the S4 rule and axioms are satisfied by the topological interior operation. $\square$

Let us take some examples of consequence of FOS4. Not only $\Box\exists y\Box\varphi \vdash \exists y\Box\varphi$ but also the following is provable in FOS4.

$$\dfrac{\dfrac{\dfrac{\Box\varphi \vdash \exists y\Box\varphi}{\Box\Box\varphi \vdash \Box\exists y\Box\varphi}}{\Box\varphi \vdash \Box\exists y\Box\varphi}}{\exists y\Box\varphi \vdash \Box\exists y\Box\varphi}$$

In terms of the topological interpretation, this means that the direct image $[\![\, \bar{x} \mid \exists y\Box\varphi \,]\!]$ of the open set $[\![\, \bar{x}, y \mid \Box\varphi \,]\!]$ under $1_{D^n} \times \pi$ is fixed under **int**, i.e.

$$\mathbf{int}([\![\, \bar{x} \mid \exists y\Box\varphi \,]\!]) = [\![\, \bar{x} \mid \Box\exists y\Box\varphi \,]\!] = [\![\, \bar{x} \mid \exists y\Box\varphi \,]\!],$$

and hence is open. Also, by substituting $\Box x = z$ for $\varphi(z)$ in $x = y \vdash \varphi(x) \to \varphi(y)$, we have

$$x = y \vdash \Box x = x \to \Box x = y,$$

where $\vdash \Box x = x$ follows from $\vdash x = x$ with an S4 rule; thus FOS4 proves $x = y \dashv\vdash \Box x = y$. Topologically, this means that the diagonal

$$[\![\, x, y \mid x = y \,]\!] = \{\, (a, a) \in D \times_X D \mid a \in D \,\} = \Delta[D] \subseteq D \times_X D$$

is open. Therefore $\pi : D \to X$ must be a sheaf for FOS4 to be sound. Indeed, the identity of formulas $\Box(\varphi[t/x])$ and $(\Box\varphi)[t/x]$ requires $\mathbf{int}([\![\, \bar{y} \mid t \,]\!]^{-1}([\![\, x \mid \varphi \,]\!])) = [\![\, \bar{y} \mid t \,]\!]^{-1}(\mathbf{int}([\![\, x \mid \varphi \,]\!]))$, which in turn requires $D$ to be a sheaf. In short, even the well-definedness of the semantics requires sheaves.

1.5. **Example of the Topological Interpretation.** Consider $\mathbb{R}^+ = \{a \in \mathbb{R} \mid 0 < a\}$, the positive reals with the usual topology, and the projection $\pi : \mathbb{R}^+ \to S^1$ onto the unit circle in the plane $\mathbb{R}\times\mathbb{R}$ such that $\pi(a) = (\cos 2\pi a, \sin 2\pi a)$. Thus the circle $S^1$ has an infinite spiral $\mathbb{R}^+$ above it, but with an open, downward end at 0 (see the figure below). Let $\mathfrak{M} = (\mathbb{R}^+, [\![\cdot]\!])$ interpret the binary relation symbol $\leqslant$ by the "no-greater-than" relation of real numbers on this sheaf, as follows:

$$[\![\, x, y \mid x \leqslant y \,]\!]_p = \{\, (a, b) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid a \leqslant b \text{ and } \pi(a) = \pi(b) = p \,\}.$$

I.e., in each fiber $\mathbb{R}^+{}_p$, the order is just the usual one on the reals.

Now consider the truth of the following sentences under this interpretation:

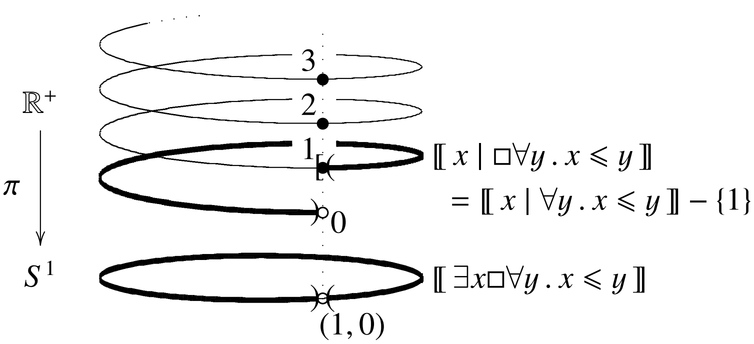
(2) $\exists x \forall y . x \leqslant y$ "There exists $x$ such that $x$ is the least."

(3) $\exists x \Box \forall y . x \leqslant y$ "There exists $x$ such that $x$ is necessarily the least."

$[\![\, x \mid \forall y . x \leqslant y \,]\!] = (0, 1]$ is the set of points of $\mathbb{R}^+$ that are the least in their own fibers. Thus we have $[\![\, \exists x \forall y . x \leqslant y \,]\!] = \pi\,((0, 1]) = S^1$ and so (2) is true in $\mathfrak{M}$. On the other hand,

$$[\![\, x \mid \Box\forall y . x \leqslant y \,]\!] = \mathbf{int}([\![\, x \mid \forall y . x \leqslant y \,]\!]) = \mathbf{int}((0, 1]) = (0, 1).$$

So $[\![\, \exists x \Box \forall y . x \leqslant y \,]\!] = \pi\,((0, 1)) = S^1 - \{(1, 0)\} \neq S^1$, i.e., (3) is *not* true.




In this way, $1 \in \mathbb{R}^+$ is "actually the least" in its fiber (or "possible world") $\mathbb{R}^+{}_{(1,0)} = \{1, 2, 3, \ldots\}$, but not "necessarily the least." Intuitively speaking, 1 is the least in the world $\mathbb{R}^+{}_{(1,0)}$, but any neighborhood of this world, no matter how small a one we take, contains some world ($\{\varepsilon, 1+\varepsilon, 2+\varepsilon, 3+\varepsilon, \ldots\}$ for $\varepsilon > 0$) in which 1 is no longer the least. Note that here we used the notion "1 in worlds near by" for explanation. Even though 1 only exists in $\mathbb{R}^+{}_{(1,0)}$, this notion still makes sense because the local homeomorphism property of the sheaf allows us to find an associated point in any other world in a sufficiently small neighborhood. In contrast, for

(4) $\quad \Box\exists x\forall y.x \leqslant y \quad$ "There necessarily exists $x$ such that $x$ is the least."

we have $[\![\Box\exists x\forall y.x \leqslant y]\!] = \mathbf{int}([\![\exists x\forall y.x \leqslant y]\!]) = \mathbf{int}(S^1) = S^1$, and so (4) is true in $\mathfrak{M}$. In general, $\Box$ expresses "continuous truth."

Finally, note that

$$\mathfrak{M} \nvDash \Box\exists x\forall y.x \leqslant y \to \exists x\Box\forall y.x \leqslant y,$$

thus providing a counter-model for a "Barcan formula" of the form "$\Box\exists \to \exists\Box$". Note also that

$$[\![\, x, y \mid x \leqslant y \,]\!] = \Delta\,(\mathbb{R}^+) \cup (\{\,(a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\,\} \cap \mathbb{R}^+ \times_{S^1} \mathbb{R}^+)$$

is open. It follows that $[\![\, x, y \mid \Box x \leqslant y \,]\!] = \mathbf{int}([\![\, x, y \mid x \leqslant y \,]\!]) = [\![\, x, y \mid x \leqslant y \,]\!]$, and therefore $[\![\exists x\forall y.\Box x \leqslant y]\!] = [\![\exists x\forall y.x \leqslant y]\!] = S^1$, which means

$$\mathfrak{M} \nvDash \exists x\forall y\Box x \leqslant y \to \exists x\Box\forall y.x \leqslant y.$$

I.e., $\mathfrak{M}$ is also a counter-model for the Barcan formula of the form "$\forall\Box \to \Box\forall$". (In contrast, "converse Barcan" "$\Box\forall \to \forall\Box$" and "$\exists\Box \to \Box\exists$" are provable in FOS4, and hence valid in the topological semantics.)

## 2. Completeness

We say a theory $\mathbb{T}$ is FOS4 if it satisfies all the axioms and rules of FOS4.

**Theorem 5.** *For any consistent FOS4 theory $\mathbb{T}$ in a first-order modal language $\mathcal{L}$, there exist a topological space $X$, a sheaf $\pi : D \to X$, and a topological interpretation $(D, [\![\cdot]\!])$ such that*

$$\mathbb{T} \textit{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!]$$

*for every pair of formulas $\varphi$, $\psi$ of $\mathcal{L}$.*

### 2.1. Completeness Proof for Propositional S4.

Before giving our proof for Theorem 5, we first review a proof for the completeness of propositional S4 with respect to its topological semantics, since the key idea of this proof will be extended in our proof for FOS4.

**Theorem 2.** *For any consistent theory* $\mathbb{T}$ *containing S4 in a propositional modal language* $\mathcal{L}$ *with* $\Box$*, there exist a topological space X and a topological interpretation* $[\![\cdot]\!]$ *such that*

$$\mathbb{T} \textit{ proves } \varphi \vdash \psi \iff [\![\varphi]\!] \subseteq [\![\psi]\!]$$

*for every pair of sentences* $\varphi$*,* $\psi$ *of* $\mathcal{L}$*.*

*Proof (sketch).* Consider the Lindenbaum algebra $B$ of $\mathbb{T}$, which is a Boolean algebra equipped with the operation $b : [\varphi] \mapsto [\Box\varphi]$. Take the set $\mathcal{U}$ of ultrafilters in $B$. In other words, every $u \in \mathcal{U}$ is a (two-valued) model of $\mathbb{T}$ and $\mathcal{U}$ is the collection of all such models. Then take the Stone representation $\widehat{\cdot} : B \to \mathcal{P}(\mathcal{U})$, i.e., for every sentence $\varphi$ of $\mathcal{L}$,

$$\widehat{[\varphi]} = \{ u \in \mathcal{U} \mid [\varphi] \in u \}.$$

In other words, $\widehat{[\varphi]}$ is the collection of (two-valued) models in which $\varphi$ is true. We interpret $\varphi$ by $\widehat{[\varphi]}$, viz. $[\![\varphi]\!] = \widehat{[\varphi]}$.

Now topologize the set $\mathcal{U}$ with basic open sets

$$V_\varphi = \widehat{[\Box\varphi]} \tag{5}$$

for all formulas $\varphi$. Then the interior operation **int** for this topology satisfies $\widehat{[\Box\varphi]} = \mathbf{int}(\widehat{[\varphi]})$. Because the Boolean structure is preserved by the Boolean homomorphism $\widehat{\cdot}$, e.g. $\widehat{[\varphi \wedge \psi]} = \widehat{[\varphi]} \cap \widehat{[\psi]}$, interpreting $\varphi$ with $[\![\varphi]\!] = \widehat{[\varphi]}$ gives a topological interpretation as defined in Subsection 1.1. Therefore $\widehat{\cdot}$ being injective implies

$$\mathbb{T} \text{ proves } \varphi \vdash \psi \iff [\varphi] \leqslant [\psi] \iff [\![\varphi]\!] \subseteq [\![\psi]\!]. \qquad \Box$$

This proof can be summarized as follows: We take the collection of models of an S4 theory in question regarded as a non-modal propositional theory and topologize it with the interpretations of $\Box$ sentences as basic open sets. This idea is one of the two keys for our proof. For FOS4, we will take the collection of models, each with a domain of individuals, of a FOS4 theory in question and topologize it with the interpretations of $\Box$ formulas as basic open sets. To implement this idea, we need the other key, i.e., the main lemma in the next subsection.

2.2. **Sufficient Set of Models with All Names.** This subsection prepares the underlying sets of a topological space $X$ and a sheaf $\pi : D \to X$ as in the statement of Theorem 5. They have to be prepared in such a way that the projection $\pi$ will be a local homeomorphism with the desired topology on $X$ and $F$, which we ensure with Property (v) in the lemma.

Before stating the main lemma, let us introduce the following notation: Given any first-order modal language $\mathcal{L}$, we write $\overline{\mathcal{L}} := \mathcal{L} \cup \{\Box\varphi\}$ for the first-order *non-modal* language gained by adding to $\mathcal{L}$ an $n$-ary basic relation symbol $\Box\varphi$ for every formula $\varphi$ of $\mathcal{L}$ with exactly $n$ free variables but no function or constant symbols. Note that $\overline{\mathcal{L}}$ yields exactly the same set of formulas as $\mathcal{L}$ does, and therefore any consistent theory $\mathbb{T}$ in $\mathcal{L}$ is a consistent theory in $\overline{\mathcal{L}}$ as well.

**Main Lemma.** *Given a first-order modal language $\mathcal{L}$ and a consistent FOS4 theory $\mathbb{T}$ in $\mathcal{L}$, there exist $\mathcal{L}^*$, $\mathbb{T}^*$ and $\mathfrak{M}$ such that*

(i) *$\mathcal{L}^*$ is an extension of $\mathcal{L}$ gained by adding new constant symbols.*

(ii) *$\mathbb{T}^*$ is a FOS4 theory in $\mathcal{L}^*$.*

(iii) *$\mathbb{T}^*$ is a conservative extension of $\mathbb{T}$ over $\mathcal{L}$, i.e., every pair of formulas $\varphi$, $\psi$ of $\mathcal{L}$ has*

$$\mathbb{T} \text{ proves } \varphi \vdash \psi \iff \mathbb{T}^* \text{ proves } \varphi \vdash \psi.$$

(iv) *$\mathfrak{M}$ is a set of structures for $\overline{\mathcal{L}^*}$, and moreover a sufficient set of models of $\mathbb{T}^*$, i.e.,*

$$\mathbb{T}^* \text{ proves } \vdash \varphi \iff \text{each } M \in \mathfrak{M} \text{ and every } \bar{a} \in |M|^n \text{ have } M \vDash \varphi[\bar{a}]$$

*for any formula $\varphi$ of $\mathcal{L}^*$ in a context of variables $\bar{x} = (x_1, \ldots, x_n)$. Note that this with* (ii) *entails the following for any such $\varphi$ of $\mathcal{L}$:*

$$\mathbb{T} \text{ proves } \vdash \varphi \iff \text{each } M \in \mathfrak{M} \text{ and every } \bar{a} \in |M|^n \text{ have } M \vDash \varphi[\bar{a}].$$

(v) *$\mathcal{L}^*$ labels $\mathfrak{M}$, i.e., for each $M \in \mathfrak{M}$ and every $a \in |M|$, there is a constant symbol $c$ in $\mathcal{L}^*$ such that $c^M = a$.*

*Proof.* Since $\mathbb{T}$ is a consistent theory in $\overline{\mathcal{L}}$, Gödel's completeness theorem of first-order logic implies that there is a class $\mathbf{M} \neq \varnothing$ of structures for $\overline{\mathcal{L}}$ such that

$$\mathbb{T} \text{ proves } \vdash \varphi \iff M \vDash \varphi \text{ for all } M \in \mathbf{M},$$

which means $\mathbf{M}$ satisfies (iv) above (for $\mathbf{M}$, $\mathcal{L}$, $\mathbb{T}$ in place of $\mathfrak{M}$, $\mathcal{L}^*$ $\mathbb{T}^*$, respectively) if $\mathbf{M}$ is a set. While $\mathbf{M}$ may well be too large to be a set, the Löwenheim-Skolem theorem implies that there is a cardinal number $\lambda$ such that the set $\mathfrak{M}_0 = \{ M \mid \|M\| \leqslant \lambda, M \vDash \mathbb{T} \} \subseteq \mathbf{M}$ satisfies (iv) above (for $\mathfrak{M}_0$, $\mathcal{L}$, $\mathbb{T}$ in place of $\mathfrak{M}$, $\mathcal{L}^*$, $\mathbb{T}^*$). Note that, because $\mathbb{T}$ is FOS4, $M \vDash \mathbb{T}$ implies the following for every context $\bar{x} = (x_1, \dots, x_n)$:

$$\llbracket \bar{x} \mid \varphi \rrbracket^M \subseteq \llbracket \bar{x} \mid \psi \rrbracket^M \implies \llbracket \bar{x} \mid \Box\varphi \rrbracket^M \subseteq \llbracket \bar{x} \mid \Box\psi \rrbracket^M,$$

$$\llbracket \bar{x} \mid \Box\bar{x} = \bar{x} \rrbracket^M = |M|^n,$$

$$\llbracket \bar{x} \mid \Box(\varphi \wedge \psi) \rrbracket^M = \llbracket \bar{x} \mid \Box\varphi \wedge \Box\psi \rrbracket^M,$$

$$\llbracket \bar{x} \mid \Box\varphi \rrbracket^M \subseteq \llbracket \bar{x} \mid \varphi \rrbracket^M,$$

$$\llbracket \bar{x} \mid \Box\varphi \rrbracket^M = \llbracket \bar{x} \mid \Box\Box\varphi \rrbracket^M,$$

$$\llbracket \bar{x}, \bar{y} \mid \bar{x} = \bar{y} \rrbracket^M = \llbracket \bar{x}, \bar{y} \mid \Box\bar{x} = \bar{y} \rrbracket^M.$$

Although (iv) as well as trivially (i)-(iii) hold for $\mathfrak{M}_0$, $\mathcal{L}$, $\mathbb{T}$ in place of $\mathfrak{M}$, $\mathcal{L}^*$, $\mathbb{T}^*$, (v) does not necessarily hold. To ensure (v), we use a technique which may be called "lazy Henkinization," which is to take

$$\mathcal{L}^* := \mathcal{L} \cup \{ c_i \mid i < \lambda \} \text{ by adding new constant symbols, and}$$

$$\mathfrak{M} := \{ M_f \mid M \in \mathfrak{M}_0 \text{ and } f : \lambda \twoheadrightarrow |M| \text{ is a surjection} \},$$

where $M_f$ is the expansion of $M$ to $\overline{\mathcal{L}^*} := \mathcal{L}^* \cup \{\Box\varphi\}$ with $c_i{}^{M_f} = f(i)$ for all $i < \lambda$. And take the theory $\mathbb{T}^*$ of $\mathfrak{M}$, i.e.,

$$\mathbb{T}^* := \{ \varphi \mid \varphi \text{ is a formula of } \mathcal{L}^* \text{ such that } M_f \vDash \varphi[\bar{a}] \text{ for all } M_f \in \mathfrak{M} \text{ and } \bar{a} \in |M_f|^n = |M|^n \},$$

to show that (i)-(v) hold for $\mathfrak{M}$, $\mathcal{L}^*$, $\mathbb{T}^*$.

Properties (i), (iv) and (v) are ensured by definition. (iii) follows because we have the following for every pair of formulas $\varphi$, $\psi$ of $\mathcal{L}$ in a context of at most $n$ free variables:

$$\begin{aligned}
\mathbb{T}^* \text{ proves } \varphi \vdash \psi &\iff \mathbb{T}^* \text{ proves } \vdash \varphi \to \psi \\
&\iff M_f \vDash \varphi[\bar{a}] \to \psi[\bar{a}] \text{ for all } M \in \mathfrak{M}_0,\ f : \lambda \twoheadrightarrow |M|, \text{ and } \bar{a} \in |M|^n \\
&\iff M \vDash \varphi[\bar{a}] \to \psi[\bar{a}] \text{ for all } M \in \mathfrak{M}_0 \text{ and } \bar{a} \in |M|^n \\
&\iff \mathbb{T} \text{ proves } \vdash \varphi \to \psi \\
&\iff \mathbb{T} \text{ proves } \varphi \vdash \psi.
\end{aligned}$$

To show that (ii) holds, viz. that $\mathbb{T}^*$ is FOS4, observe the following facts.

(a) Axioms and rules of first-order logic all hold in $\mathbb{T}^*$; e.g., $\mathbb{T}^*$ proving $\vdash \forall y\, \varphi(y, \bar{x})$ entails its proving $\vdash \varphi(c_i, \bar{x})$ for every $i < \lambda$. This is because each $M_f \in \mathfrak{M}$ is a model of first-order logic.

(b) For every formula $\varphi(\bar{y}, \bar{x})$ of $\mathcal{L}$ with $\bar{y}$ of length $m$ and constant symbols $\bar{c} = (c_{i_1}, \ldots, c_{i_m})$ with $i_1, \ldots, i_m < \lambda$, we have

$$\mathbb{T}^* \text{ proves } \vdash \varphi(\bar{c}, \bar{x}) \iff \mathbb{T} \text{ proves } \vdash \varphi(\bar{y}, \bar{x}).$$

The "$\Leftarrow$" direction follows from (ii) and (a). To show "$\Rightarrow$", suppose $\mathbb{T}^*$ proves $\vdash \varphi(\bar{c}, \bar{x})$ and fix $M \in \mathfrak{M}_0$ and any $\bar{a} = (a_1, \ldots, a_m) \in |M|^m$. There is $f : \lambda \twoheadrightarrow |M|$ such that $f(i_1) = a_1, \ldots, f(i_m) = a_m$, i.e. $[\![\bar{c}]\!]^{M_f} = \bar{a}$. For every $\bar{b} \in |M|^n$, then, $\mathbb{T}^*$ proving $\vdash \varphi(\bar{c}, \bar{x})$ implies $M_f \vDash \varphi(\bar{c}, \bar{b})$ and hence $M_f \vDash \varphi[\bar{a}, \bar{b}]$, which means $M \vDash \varphi[\bar{a}, \bar{b}]$ because $\varphi(\bar{y}, \bar{x})$ is in $\mathcal{L}$. Thus $\mathbb{T}$ proves $\vdash \varphi(\bar{y}, \bar{x})$, since $\mathfrak{M}_0$ is sufficient.

Now fix any S4 axiom in $\mathcal{L}^*$ of the form $\varphi(\bar{c}) \vdash \psi(\bar{c})$. Because $\varphi(\bar{x}) \vdash \psi(\bar{x})$ is an axiom of $\mathbb{T}$, it proves $\vdash \varphi(\bar{x}) \to \psi(\bar{x})$ and hence (b) above implies that $\mathbb{T}^*$ proves $\vdash \varphi(\bar{c}) \to \psi(\bar{c})$. Thus $\mathbb{T}^*$ proves $\varphi(\bar{c}) \vdash \psi(\bar{c})$. $\mathbb{T}^*$ also satisfies the S4 rule in $\mathcal{L}^*$, viz. $\varphi(\bar{c}) \vdash \psi(\bar{c})$ entailing $\Box\varphi(\bar{c}) \vdash \Box\psi(\bar{c})$, because

$$\begin{aligned}
\mathbb{T}^* \text{ proves } \vdash \varphi(\bar{c}) \to \psi(\bar{c}) &\implies \mathbb{T} \text{ proves } \vdash \varphi(\bar{x}) \to \psi(\bar{x}) && \text{by (b)} \\
&\implies \mathbb{T} \text{ proves } \vdash \Box\varphi(\bar{x}) \to \Box\psi(\bar{x}) && \text{because } \mathbb{T} \text{ is FOS4} \\
&\implies \mathbb{T}^* \text{ proves } \vdash \Box\varphi(\bar{c}) \to \Box\psi(\bar{c}) && \text{by (b).}
\end{aligned}$$

Thus $\mathbb{T}^*$ is FOS4. □

The reader may wonder why we used "lazy Henkinization" rather than the usual method of adding Henkin constants to attain (v). That method does not serve our purpose for the following reason. Suppose we add to $\mathcal{L}$ a constant $c_\varphi$ for a formula $\varphi$ of $\mathcal{L}$, together with the corresponding Henkin axiom $\exists x\varphi(x) \vdash \varphi(c_\varphi)$. Then (ii) implies that the extended theory $\mathbb{T}^*$ proves $\Box\exists x\varphi(x) \vdash \Box\varphi(c_\varphi)$, while it also proves $\Box\varphi(c_\varphi) \vdash \exists x\Box\varphi(x)$. Therefore the "Barcan formula" $\Box\exists x\varphi(x) \vdash \exists x\Box\varphi(x)$ of $\mathcal{L}$ is provable in $\mathbb{T}^*$, and hence in $\mathbb{T}$ as well by (iii), although it is not sound in the topological semantics, as observed in Subsection 1.5.

2.3. **Space of Models with Logical Topology.** Given a first-order modal language $\mathcal{L}$ and a consistent FOS4 theory $\mathbb{T}$ in $\mathcal{L}$, take $\mathcal{L}^*$, $\mathbb{T}^*$ and $\mathfrak{M}$ as in the main lemma. We topologize $\mathfrak{M}$ to be the base space of the desired sheaf, by taking as basic open sets the subsets of $\mathfrak{M}$ logically definable in $\mathcal{L}^*$.

**Definition 4.** For a closed formula $\sigma$ of $\mathcal{L}^*$, define

$$V_\sigma := \{\, M \in \mathfrak{M} \mid M \vDash \Box\sigma \,\} \subseteq \mathfrak{M}.$$

Note $\Box$ in the definition, which generalizes (5).

**Remark 1.** The set of $V_\sigma$ for all closed formulas $\sigma$ of $\mathcal{L}^*$ forms a basis for a topology on $\mathfrak{M}$, because

- $\mathfrak{M} = V_\top$ since $\mathbb{T}^*$ proves $\vdash \Box\top$ for a closed theorem $\top$ of first-order logic.
- $V_\sigma \cap V_\rho = V_{\sigma\wedge\rho}$ since $\mathbb{T}^*$ proves $\Box\sigma \wedge \Box\rho \dashv\vdash \Box(\sigma \wedge \rho)$.

**Definition 5.** Let $\mathfrak{M}$ have the topology generated by the $V_\sigma$ for all closed formulas $\sigma$ of $\mathcal{L}^*$.

2.4. **Sheaves with Logical Topology.** Now that we have a topological space $\mathfrak{M}$, we construct a topological fiber bundle over $\mathfrak{M}$, i.e. a space $\widetilde{\mathfrak{M}}$ with a continuous projection $\pi : \widetilde{\mathfrak{M}} \to \mathfrak{M}$. We also construct all of its finite powers $\widetilde{\mathfrak{M}}^n$ simultaneously. First we define the underlying sets.

**Definition 6.** Define

$$\widetilde{\mathfrak{M}} := \sum_{M\in\mathfrak{M}} |M| = \{\, \langle M, a\rangle \mid M \in \mathfrak{M} \text{ and } a \in |M| \,\},$$

and the projection $\pi : \widetilde{\mathfrak{M}} \to \mathfrak{M}$ by $\langle M, a\rangle \mapsto M$. Similarly, define

$$\widetilde{\mathfrak{M}}^n := \sum_{M\in\mathfrak{M}} |M|^n = \{\, \langle M, \bar{a}\rangle \mid M \in \mathfrak{M} \text{ and } \bar{a} \in |M|^n \,\},$$

and the projection $\pi^n : \widetilde{\mathfrak{M}}^n \to \mathfrak{M}$ by $\langle M, \bar{a}\rangle \mapsto M$.

It is helpful to introduce a notation for logically definable subsets and maps.

**Definition 7.** For each formula $\varphi(\bar{x})$ of $\mathcal{L}^*$ in the context of variables $\bar{x} = (x_1, \dots, x_n)$, define

$$[\![\, \bar{x} \mid \varphi \,]\!]^{\mathfrak{M}} := \sum_{M\in\mathfrak{M}} [\![\, \bar{x} \mid \varphi \,]\!]^M = \left\{ \langle M, \bar{a}\rangle \in \widetilde{\mathfrak{M}}^n \mid M \vDash \varphi[\bar{a}] \right\} \subseteq \widetilde{\mathfrak{M}}^n.$$

Also, for each term $t$ of $\mathcal{L}^*$ in the context of variables $\bar{x}$, define

$$\begin{aligned} [\![\, \bar{x} \mid t \,]\!]^{\mathfrak{M}} := \sum_{M\in\mathfrak{M}} [\![\, \bar{x} \mid t \,]\!]^M : \widetilde{\mathfrak{M}}^n = \sum_{M\in\mathfrak{M}} |M|^n &\longrightarrow \sum_{M\in\mathfrak{M}} |M| = \widetilde{\mathfrak{M}} \\ \langle M, \bar{a}\rangle &\longmapsto \langle M, [\![\, \bar{x} \mid t \,]\!]^M(\bar{a})\rangle. \end{aligned}$$

**Lemma 1.** *For each pair of formulas $\varphi$, $\psi$ of $\mathcal{L}^*$ in a context of variables $\bar{x}$,*

$$[\![\, \bar{x} \mid \varphi \,]\!]^{\mathfrak{M}} \subseteq [\![\, \bar{x} \mid \psi \,]\!]^{\mathfrak{M}} \iff \mathbb{T}^* \text{ proves } \varphi \vdash \psi.$$

*Proof.* Because $\mathfrak{M}$ is a sufficient set of models of $\mathbb{T}^*$ as in (iv) of the main lemma. □

To define the desired topology on the sets $\widetilde{\mathfrak{M}}^n$, we use subsets logically definable in $\mathcal{L}^*$.

**Definition 8.** For a formula $\varphi(\bar{x})$ of $\mathcal{L}^*$ in the context of variables $\bar{x} = (x_1, \ldots, x_n)$, define

$$U_\varphi := [\![\, \bar{x} \mid \Box\varphi \,]\!]^{\mathfrak{M}} = \left\{ \langle M, \bar{a} \rangle \in \widetilde{\mathfrak{M}}^n \mid M \vDash \Box\varphi[\bar{a}] \right\} \subseteq \widetilde{\mathfrak{M}}^n.$$

Note □ in the definition, which generalizes (5).

**Remark 2.** The sets $U_\varphi$ form a basis for a topology on $\widetilde{\mathfrak{M}}^n$, because

- $\widetilde{\mathfrak{M}}^n = U_\top$ since $\mathbb{T}^*$ proves $\vdash \Box\top$ with $\top$ in the context $\bar{x}$.
- $U_\varphi \cap U_\psi = U_{\varphi\wedge\psi}$ since $\mathbb{T}^*$ proves $\Box\varphi(\bar{x}) \wedge \Box\psi(\bar{x}) \dashv\vdash \Box(\varphi(\bar{x}) \wedge \psi(\bar{x}))$.

**Definition 9.** For each $n$, let $\widetilde{\mathfrak{M}}^n$ have the topology generated by $U_\varphi$ for all formulas $\varphi(\bar{x})$ of $\mathcal{L}^*$ in the context of variables $\bar{x} = (x_1, \ldots, x_n)$.

**Lemma 2.** *Each projection $\pi^n : \widetilde{\mathfrak{M}}^n \to \mathfrak{M}$ is a local homeomorphism.*

*Proof.* For each $n$, it is enough to show the following:

(i) $\pi^n$ is continuous.

(ii) Every point of $\widetilde{\mathfrak{M}}^n$ lies in a global section, i.e., each $p \in \widetilde{\mathfrak{M}}^n$ has a continuous map $g : \mathfrak{M} \to \widetilde{\mathfrak{M}}^n$ such that $\pi_n \circ g$ is the identity, $g(\mathfrak{M})$ is open in $\widetilde{\mathfrak{M}}^n$, and $p \in g(\mathfrak{M})$.

(i) Every basic open set $V_\sigma$ in $\mathfrak{M}$ has

$$\begin{aligned}(\pi^n)^{-1}(V_\sigma) &= \left\{ \langle M, \bar{a} \rangle \in \widetilde{\mathfrak{M}}^n \mid M \vDash \Box\sigma \right\} \\ &= \left\{ \langle M, \bar{a} \rangle \in \widetilde{\mathfrak{M}}^n \mid M \vDash \Box\sigma[\bar{a}] \right\} = U_\sigma,\end{aligned}$$

i.e., the inverse image by $\pi^n$ of basic open $V_\sigma$ is open. Thus $\pi^n$ is continuous.

(ii) Fix $p \in \widetilde{\mathfrak{M}}^n$. By the definition of $\mathfrak{M}$, $p = \langle M_f, \bar{a} \rangle$ with a surjection $f : \lambda \twoheadrightarrow |M_f|$. Therefore there are $i_1, \ldots, i_n < \lambda$ such that $\bar{a} = (f(i_1), \ldots, f(i_n)) = ({c_{i_1}}^{M_f}, \ldots, {c_{i_n}}^{M_f})$. Then the map $g : \mathfrak{M} \to \widetilde{\mathfrak{M}}^n$ defined by $g(N) = \langle N, {c_{i_1}}^N, \ldots, {c_{i_n}}^N \rangle$ is continuous because

$$g^{-1}\left(U_\varphi\right) = \left\{ N \in \mathfrak{M} \mid \langle N, {c_{i_1}}^N, \ldots, {c_{i_n}}^N \rangle \in U_\varphi \right\}$$

$$= \left\{ N \in \mathfrak{M} \mid N \models \Box\varphi[{c_{i_1}}^N, \dots, {c_{i_n}}^N] \right\}$$
$$= \{ N \in \mathfrak{M} \mid N \models \Box\varphi(c_{i_1}, \dots, c_{i_n}) \} = V_{\varphi(c_{i_1},\dots,c_{i_n})}.$$

While $\pi_n \circ g(N) = N$ and $p = \langle M_f, \bar{a} \rangle = g(M_f) \in g(\mathfrak{M})$, $g(\mathfrak{M})$ is open because

$$\begin{aligned} g(\mathfrak{M}) &= \left\{ \langle N, b_1, \dots, b_n \rangle \in \widetilde{\mathfrak{M}}^n \mid b_1 = {c_{i_1}}^N, \dots, b_n = {c_{i_n}}^N \right\} \\ &= \left\{ \langle N, b_1, \dots, b_n \rangle \in \widetilde{\mathfrak{M}}^n \mid N \models b_1 = c_{i_1} \wedge \cdots \wedge b_n = c_{i_n} \right\} \\ &= \left\{ \langle N, b_1, \dots, b_n \rangle \in \widetilde{\mathfrak{M}}^n \mid N \models \Box b_1 = c_{i_1} \wedge \cdots \wedge \Box b_n = c_{i_n} \right\} \\ &= \left\{ \langle N, b_1, \dots, b_n \rangle \in \widetilde{\mathfrak{M}}^n \mid N \models \Box(b_1 = c_{i_1} \wedge \cdots \wedge b_n = c_{i_n}) \right\} = U_{x_1 = c_{i_1} \wedge \cdots \wedge x_n = c_{i_n}}. \end{aligned}$$ □

2.5. **These Sheaves Form a Topological Interpretation.** We now have topological spaces $\mathfrak{M}$, $\widetilde{\mathfrak{M}}$ and a local homeomorphism $\pi : \widetilde{\mathfrak{M}} \to \mathfrak{M}$, providing (i) and (ii) of Definition 2. This subsection finishes our proof by showing (iii)–(vi) for the obvious topological interpretation $(\widetilde{\mathfrak{M}}, [\![\cdot]\!]^{\mathfrak{M}})$ defined with

$$[\![R]\!]^{\mathfrak{M}} = [\![\, \bar{x} \mid R(\bar{x}) \,]\!]^{\mathfrak{M}} \subseteq \widetilde{\mathfrak{M}}^n,$$
$$[\![f]\!]^{\mathfrak{M}} = [\![\, \bar{x} \mid f(\bar{x}) \,]\!]^{\mathfrak{M}} : \widetilde{\mathfrak{M}}^n \to \widetilde{\mathfrak{M}}$$

for basic relation symbols $R$ and function symbols $f$ of arity $n$ of $\mathcal{L}$. While (iii) is immediate, (v) is summarized as Lemma 3, which is obvious since $[\![\cdot]\!]^{\mathfrak{M}}$ is just the disjoint union of $[\![\cdot]\!]^M$.

**Lemma 3.** *The operations on $[\![\cdot]\!]^M$ interpreting first-order operations in $\mathcal{L}$ extends to $[\![\cdot]\!]^{\mathfrak{M}}$; e.g.,*

$$\begin{aligned} [\![\, \bar{x} \mid \varphi(t(\bar{x})) \,]\!]^{\mathfrak{M}} &= \sum_{M \in \mathfrak{M}} [\![\, \bar{x} \mid \varphi(t(\bar{x})) \,]\!]^M \\ &= \sum_{M \in \mathfrak{M}} \left([\![\, \bar{x} \mid t \,]\!]^M\right)^{-1} \left([\![\, y \mid \varphi(y) \,]\!]^M\right) \\ &= \left(\sum_{M \in \mathfrak{M}} [\![\, \bar{x} \mid t \,]\!]^M\right)^{-1} \left(\sum_{M \in \mathfrak{M}} [\![\, y \mid \varphi(y) \,]\!]^M\right) \\ &= \left([\![\, \bar{x} \mid t \,]\!]^{\mathfrak{M}}\right)^{-1} \left([\![\, y \mid \varphi(y) \,]\!]^{\mathfrak{M}}\right). \end{aligned}$$

Next, (iv) follows from:

**Lemma 4.** *For each function symbol $f$ of arity $n$ of $\mathcal{L}$, $[\![f]\!]^{\mathfrak{M}} : \widetilde{\mathfrak{M}}^n \to \widetilde{\mathfrak{M}}$ is a continuous map over $\mathfrak{M}$.*

*Proof.* $[\![f]\!]^{\mathfrak{M}}$ is over $\mathfrak{M}$ by definition, and is continuous because every formula $\varphi$ of $\mathcal{L}$ in the context of variable $y$ has $\left([\![f]\!]^{\mathfrak{M}}\right)^{-1}\left(U^1_\varphi\right)$ open in $\widetilde{\mathfrak{M}}^n$, since Lemma 3 implies

$$\left([\![f]\!]^{\mathfrak{M}}\right)^{-1}\left(U^1_\varphi\right) = \left([\![\, \bar{x} \mid f \,]\!]^{\mathfrak{M}}\right)^{-1}\left([\![\, y \mid \Box\varphi(y) \,]\!]^{\mathfrak{M}}\right)$$
$$= [\![\, \bar{x} \mid \Box\varphi(f(\bar{x})) \,]\!]^{\mathfrak{M}} = U_{\varphi(f(\bar{x}))}. \qquad \Box$$

Finally, (vi) follows from Lemmas 5 and 6.

**Lemma 5.** *Each* $(\widetilde{\mathfrak{M}}^n, \pi^n)$ *is the n-ary fibered product of* $(\widetilde{\mathfrak{M}}, \pi)$ *over* $\mathfrak{M}$.

*Proof.* Because $\widetilde{\mathfrak{M}}^0$ and $\mathfrak{M}$ have the same underlying sets $\left|\widetilde{\mathfrak{M}}^0\right| = \sum_{M\in\mathfrak{M}}\{\text{true}\} = |\mathfrak{M}|$ and the same bases, $\pi^0 : \widetilde{\mathfrak{M}}^0 \to \mathfrak{M}$ is a homeomorphism and hence $(\widetilde{\mathfrak{M}}^0, \pi^0)$ is the 0-ary product of $(\widetilde{\mathfrak{M}}, \pi)$. And $(\widetilde{\mathfrak{M}}^1, \pi^1) = (\widetilde{\mathfrak{M}}, \pi)$ is trivially the 1-ary product of $(\widetilde{\mathfrak{M}}, \pi)$. Suppose $(\widetilde{\mathfrak{M}}^n, \pi^n)$ and $(\widetilde{\mathfrak{M}}^m, \pi^m)$ are the $n$-ary and $m$-ary products of $(\widetilde{\mathfrak{M}}, \pi)$, to show their product to be the $(n+m)$-ary product $(\widetilde{\mathfrak{M}}^{n+m}, \pi^{n+m})$ of $(\widetilde{\mathfrak{M}}, \pi)$. While the underlying sets and maps of these two products are identical, we need to show $\mathcal{O}(\widetilde{\mathfrak{M}}^n \times_{\mathfrak{M}} \widetilde{\mathfrak{M}}^m) = \mathcal{O}(\widetilde{\mathfrak{M}}^{n+m})$, where $\mathcal{O}(\widetilde{\mathfrak{M}}^n \times_{\mathfrak{M}} \widetilde{\mathfrak{M}}^m)$ is the topology of the product of $\widetilde{\mathfrak{M}}^n$ and $\widetilde{\mathfrak{M}}^m$ over $\mathfrak{M}$, and $\mathcal{O}(\widetilde{\mathfrak{M}}^{n+m})$ the topology generated by logically definable subsets $U_\varphi$. It will be convenient to indicate the dimension of a basic open set by writing $U^n_\varphi = [\![\, \bar{x} \mid \varphi \,]\!]^{\mathfrak{M}} \subseteq \widetilde{\mathfrak{M}}^n$.

Let $p_1$ and $p_2$ be the projection from $\widetilde{\mathfrak{M}}^n \times \widetilde{\mathfrak{M}}^m$ onto $\widetilde{\mathfrak{M}}^n$ and $\widetilde{\mathfrak{M}}^m$, respectively. Then the sets

$$p_1{}^{-1}\left(U^n_\varphi\right) \cap p_2{}^{-1}\left(U^m_\psi\right) \cap \left|\widetilde{\mathfrak{M}}^n \times_{\mathfrak{M}} \widetilde{\mathfrak{M}}^m\right|$$

for all formulas $\varphi(\bar{x})$ and $\psi(\bar{y})$ of $\mathcal{L}^*$, with $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_m)$, form a basis for $\mathcal{O}(\widetilde{\mathfrak{M}}^n \times_{\mathfrak{M}} \widetilde{\mathfrak{M}}^m)$. Since $\mathbb{T}^*$ proves $\Box\varphi \wedge \Box\psi \dashv\vdash \Box(\varphi \wedge \psi)$, we have

$$\begin{aligned}
p_1{}^{-1}\left(U^n_\varphi\right) \cap p_2{}^{-1}\left(U^m_\psi\right) \cap \left|\widetilde{\mathfrak{M}}^n \times_{\mathfrak{M}} \widetilde{\mathfrak{M}}^m\right| &= \left\{ \langle M, \bar{a}, \bar{b}\rangle \in \widetilde{\mathfrak{M}}^{n+m} \mid M \vDash \Box\varphi[\bar{a}] \text{ and } M \vDash \Box\psi[\bar{b}] \right\} \\
&= \left\{ \langle M, \bar{a}, \bar{b}\rangle \in \widetilde{\mathfrak{M}}^{n+m} \mid M \vDash \Box\varphi[\bar{a}] \wedge \Box\psi[\bar{b}] \right\} \\
&= \left\{ \langle M, \bar{a}, \bar{b}\rangle \in \widetilde{\mathfrak{M}}^{n+m} \mid M \vDash \Box(\varphi[\bar{a}] \wedge \psi[\bar{b}]) \right\} \\
&= U^{n+m}_{\varphi\wedge\psi}.
\end{aligned}$$

Thus $\mathcal{O}(\widetilde{\mathfrak{M}}^n \times_{\mathfrak{M}} \widetilde{\mathfrak{M}}^m) \subseteq \mathcal{O}(\widetilde{\mathfrak{M}}^{n+m})$.

Fix any $\langle M, \bar{a}, \bar{b}\rangle \in U^{n+m}_\varphi$ for any $\varphi(\bar{x}, \bar{y})$ of $\mathcal{L}^*$ with $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_m)$. By (v) in the main lemma, we can pick $i_1, \ldots, i_n, j_1, \ldots, j_m < \lambda$ such that $\bar{c}_i = (c_{i_1}, \ldots, c_{i_n})$, $\bar{c}_j = (c_{j_1}, \ldots, c_{j_m})$ and $\bar{a} = \bar{c}^M_i$, $\bar{b} = \bar{c}^M_j$. Then $\langle M, \bar{a}, \bar{b}\rangle \in U^{n+m}_\varphi$ means $M \vDash \Box\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{a} = \bar{c}_i \wedge \bar{b} = \bar{c}_j$.

Next note that $\mathbb{T}^*$ proves the following to be equivalent:

$$\Box\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{x} = \bar{c}_i \wedge \bar{y} = \bar{c}_j$$

$$\Box\varphi(\bar{c}_i, \bar{c}_j) \wedge \Box\bar{x} = \bar{c}_i \wedge \Box\bar{y} = \bar{c}_j$$

$$\Box(\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{x} = \bar{c}_i) \wedge \Box(\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{y} = \bar{c}_j)$$

$$\Box(\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{x} = \bar{c}_i \wedge \bar{y} = \bar{c}_j).$$

Therefore we have $M \vDash \Box(\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{a} = \bar{c}_i)$ and $M \vDash \Box(\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{b} = \bar{c}_j)$, and hence

$$\begin{aligned}
\langle M, \bar{a}, \bar{b}\rangle \in p_1{}^{-1}\left(U^n_{\varphi(\bar{c}_i,\bar{c}_j)\wedge\bar{x}=\bar{c}_i}\right) \cap p_2{}^{-1}\left(U^m_{\varphi(\bar{c}_i,\bar{c}_j)\wedge\bar{y}=\bar{c}_j}\right) \cap \left|\widetilde{\mathfrak{M}}^n \times_{\mathfrak{M}} \widetilde{\mathfrak{M}}^m\right| \\
= \Big\{ \langle M, \bar{a}, \bar{b}\rangle \in \widetilde{\mathfrak{M}}^{n+m} \mid M \vDash \Box(\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{a} = \bar{c}_i) \\
\text{and } M \vDash \Box(\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{b} = \bar{c}_j) \Big\} \\
= \Big\{ \langle M, \bar{a}, \bar{b}\rangle \in \widetilde{\mathfrak{M}}^{n+m} \mid M \vDash \Box(\varphi(\bar{c}_i, \bar{c}_j) \wedge \bar{a} = \bar{c}_i \wedge \bar{b} = \bar{c}_j) \Big\} \\
\subseteq \Big\{ \langle M, \bar{a}, \bar{b}\rangle \in \widetilde{\mathfrak{M}}^{n+m} \mid M \vDash \Box\varphi[\bar{a}, \bar{b}] \Big\} = U^{n+m}_{\varphi}.
\end{aligned}$$

Thus $\mathcal{O}(\widetilde{\mathfrak{M}}^{n+m}) \subseteq \mathcal{O}(\widetilde{\mathfrak{M}}^n \times_{\mathfrak{M}} \widetilde{\mathfrak{M}}^m)$. □

**Lemma 6.** $[\![\, \bar{x} \mid \Box\varphi \,]\!]^{\mathfrak{M}} = \mathbf{int}_{\widetilde{\mathfrak{M}}^n}([\![\, \bar{x} \mid \varphi \,]\!]^{\mathfrak{M}})$ *for every formula $\varphi$ of $\mathcal{L}$ in the context of variables $\bar{x}$.*

*Proof.* Because $[\![\, \bar{x} \mid \Box\varphi \,]\!]^{\mathfrak{M}} \subseteq [\![\, \bar{x} \mid \varphi \,]\!]^{\mathfrak{M}}$ is basic open $U_\varphi$ in $\widetilde{\mathfrak{M}}^n$, it is enough to show that every basic open $U_\psi = [\![\, \bar{x} \mid \Box\psi \,]\!]^{\mathfrak{M}}$ such that $[\![\, \bar{x} \mid \Box\psi \,]\!]^{\mathfrak{M}} \subseteq [\![\, \bar{x} \mid \varphi \,]\!]^{\mathfrak{M}}$ has $[\![\, \bar{x} \mid \Box\psi \,]\!]^{\mathfrak{M}} \subseteq [\![\, \bar{x} \mid \Box\varphi \,]\!]^{\mathfrak{M}}$. If $[\![\, \bar{x} \mid \Box\psi \,]\!]^{\mathfrak{M}} \subseteq [\![\, \bar{x} \mid \varphi \,]\!]^{\mathfrak{M}}$, it means by Lemma 1 that $\mathbb{T}^*$ proves

$$\Box\psi \vdash \varphi$$

and hence

$$\Box\Box\psi \vdash \Box\varphi$$

and

$$\Box\psi \vdash \Box\varphi,$$

i.e., $[\![\, \bar{x} \mid \Box\psi \,]\!]^{\mathfrak{M}} \subseteq [\![\, \bar{x} \mid \Box\varphi \,]\!]^{\mathfrak{M}}$ again by Lemma 1. □

**Corollary 1.** $(\widetilde{\mathfrak{M}}, [\![\cdot]\!]^{\mathfrak{M}})$ *is a topological interpretation.*

**Theorem 5.** *For any consistent FOS4 theory $\mathbb{T}$ in a first-order modal language $\mathcal{L}$, there exists a topological interpretation $(D, [\![\cdot]\!])$ such that*

$$\mathbb{T} \textit{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!]$$

*for every pair of formulas $\varphi$, $\psi$ of $\mathcal{L}$.*

*Proof.* $(\widetilde{\mathfrak{M}}, [\![\cdot]\!]^{\mathfrak{M}})$ of Corollary 1 is a topological interpretation such that every pair of formulas $\varphi$, $\psi$ of $\mathcal{L}$ has

$$\mathbb{T} \text{ proves } \varphi \vdash \psi \iff \mathbb{T}^* \text{ proves } \varphi \vdash \psi \iff [\![\, \bar{x} \mid \varphi \,]\!]^{\mathfrak{M}} \subseteq [\![\, \bar{x} \mid \psi \,]\!]^{\mathfrak{M}}$$

by (ii) of the main lemma and Lemma 1. □

2.6. **Compactness.** As is the case in usual first-order logic, the usual compactness theorem for FOS4 follows as a corollary to the completeness theorem.

**Theorem 3.** *If every finite subset of a set* $\Gamma$ *of sentences of* $\mathcal{L}$ *has a (topological) model, then* $\Gamma$ *does too.*

In this case, the theorem has the following topological interpretation as well: the space $\mathfrak{M}$ is a *compact topological space*, as are all the spaces $\widetilde{\mathfrak{M}}^n$.

*Proof.* To prove this for the case of $\widetilde{\mathfrak{M}}^n$, first observe that for any set $S$ of formulas of $\mathcal{L}$ in the context $\bar{x} = (x_1, \ldots, x_n)$, with $S_{\neg\Box} := \{\, \neg\Box\varphi \mid \varphi \in S \,\}$, and for any $\langle M, \bar{a} \rangle \in \widetilde{\mathfrak{M}}^n$ we have

$$\begin{aligned}
& \langle M, \bar{a} \rangle \models S_{\neg\Box} & \\
\iff\; & M \models \neg\Box\varphi[\bar{a}] & \text{for all } \varphi \in S \\
\iff\; & M \not\models \Box\varphi[\bar{a}] & \text{for all } \varphi \in S \\
\iff\; & \langle M, \bar{a} \rangle \notin U_\varphi & \text{for all } \varphi \in S \\
\iff\; & \langle M, \bar{a} \rangle \in \widetilde{\mathfrak{M}}^n \setminus U_\varphi & \text{for all } \varphi \in S \\
\iff\; & \langle M, \bar{a} \rangle \in \bigcap_{\varphi \in S} (\widetilde{\mathfrak{M}}^n \setminus U_\varphi) = \widetilde{\mathfrak{M}}^n \setminus \bigcup_{\varphi \in S} U_\varphi. &
\end{aligned}$$

Now suppose $\widetilde{\mathfrak{M}}^n = \bigcup_{\varphi \in S} U_\varphi$ for some set $S$ of formulas of $\mathcal{L}$ in the context $\bar{x} = (x_1, \ldots, x_n)$. Then $S_{\neg\Box}$ has no model, because $\widetilde{\mathfrak{M}}^n \setminus \bigcup_{\varphi \in S} U_\varphi = \varnothing$. Hence compactness of first-order logic yields some finite subset $S^* \subseteq S$ such that finite $S^*_{\neg\Box} \subseteq S_{\neg\Box}$ has no model. This means $\widetilde{\mathfrak{M}}^n \setminus \bigcup_{\varphi \in S^*} U_\varphi = \varnothing$. Thus $\bigcup_{\varphi \in S^*} U_\varphi = \widetilde{\mathfrak{M}}^n$. This also establishes the case of $\mathfrak{M} \cong \widetilde{\mathfrak{M}}^0$ with $V_\sigma$ corresponding to $U_\sigma$. □

## Appendix A. Topos-theoretic Formulation

Our semantics is a special case of an interpretation in topos theory, which was also used to arrive at the original proof before extracting the elementary description given here. We skecth the topos-theoretic content for the interested reader.

A.1. **Topological Semantics for Propositional S4.** The topological interpretation of propositional modal logic, specifically with an S4 modal operator, is given by a Boolean algebra $B$ equipped with a monotone operation $b : B \to B$ which is a "Cartesian comonad", i.e. which satisfies:

(i) $x \leqslant y \Rightarrow bx \leqslant by$. (I.e., $b$ is monotone.)

(ii) $b1 = 1$ and $b(x \wedge y) = bx \wedge by$. (I.e., $b$ preserves finite products.)

(iii) $bx \leqslant x$ and $bx \leqslant b^2x$. (I.e., $b$ is a comonad.)

Such a $B$ is called a *topological Boolean algebra*. Because (iii) implies that $b$ is idempotent, $b$ has the fixed points $B_b = \{ bx \mid x \in B \}$. Consider the decomposition of $b$ into $r : B \to B_b$ and $i : B_b \to B$ such that $r(x) = bx$ and $i(x) = x$.

$$\begin{array}{ccccc} B & & \xrightarrow{\quad b \quad} & & B \\ & \searrow^{r} & \| & \nearrow^{i} & \\ & & B_b & & \end{array}$$

Then, for every pair $x \in B_b$ and $y \in B$, we have $ix = x \leqslant y$ in $B$ iff $x \leqslant ry = by$ in $B_b$, as follows: For $x \leqslant y$ entails $bx \leqslant by$ by (i) where $bx = x$ since $x \in B_b$ is a fix point of $b$, and $x \leqslant by$ entails $x \leqslant y$ because $by \leqslant y$ as in (iii). Thus we have adjunction $i \dashv r$, with coalgebras $B_b$.

In the special case where $B$ is a powerset $\mathcal{P}(X)$, $b$ corresponds to the interior operation for a topology $\mathcal{O}(X) = B_b$ on $X$, with $i = \textbf{inc} \dashv r = \textbf{int}$:

$$\mathcal{P}(X) \underset{\textbf{int}}{\overset{\textbf{inc}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{O}(X),$$

which is to say the following, for any subset $Y$ and open subset $U$ of $X$:

$$U \subseteq Y \iff U \subseteq \textbf{int}(Y).$$

A.2. **Comonad on Sets/|X|.** To generalize the topological semantics in the previous section to interpret first-order S4, we use sheaves on a space rather than just open subsets. Let $X$ be a topological space. Then the scheme of our generalization will be:

$$\begin{array}{rcl} \text{Propositional} & \rightsquigarrow & \text{First-order} \\ \mathcal{P}(X) & \rightsquigarrow & \mathbf{Sets}/|X| \\ \mathcal{O}(X) & \rightsquigarrow & \mathrm{Sh}(X) \end{array}$$

Here $|X|$ is the underlying set, or set of points, of $X$, $\mathbf{Sets}/|X|$ the topos of sets over $|X|$, and $\mathrm{Sh}(X)$ the topos of all (functorial) sheaves for $X$. It is helpful to note $\mathrm{Sh}(X)$ is equivalent to the topos $\mathbf{LH}/X$ of local homeomorphisms (or étale spaces) over $X$, whose objects are local homeomorphisms $\pi_F : F \to X$ and whose arrows from $\pi_F : F \to X$ to $\pi_G : G \to X$ are maps of sheaves $f : F \to G$.

Now, to set up the counterpart of the propositional case $b : B \to B$ as in

$$b \circlearrowright B \underset{r}{\overset{i}{\underset{\longrightarrow}{\overset{\hookleftarrow}{\bot}}}} B_b,$$

consider the following functors:

$$\mathbf{Sets}/|X| \underset{\mathbf{R}}{\overset{\mathbf{U}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Sh}(X) \simeq \mathbf{LH}/X$$

The simplest description of $\mathbf{U}$ and $\mathbf{R}$ is that they are the geometric morphism induced by the continuous map $\mathrm{id} : |X| \to X$. (Note that $\mathbf{Sets}/|X| \simeq \mathrm{Sh}(|X|)$ for the discrete space $|X|$.) More concretely,

- For a sheaf $F$, or its associated local homeomorphism $\pi_F : F \to X$, $\mathbf{U}$ forgets the topological structure of $\pi_F$, i.e., $\mathbf{U}F = |\pi_F| : |F| \to |X|$ is just a function. (Here the total space $F = \coprod_{x \in X} P_x$ for stalks $P_x$ of germs at $x$.)
- $\mathbf{R}(f : Y \to |X|) = \mathrm{Hom}_{\mathbf{Sets}/|X|}(-, f) \in \mathrm{Sh}(X)$ restricted to the open sets of $X$. That is, for any open $U \subseteq X$, $\mathbf{R}f(U) = \mathrm{Hom}_{\mathbf{Sets}/|X|}(U, f)$, whose typical element is an arrow $s$ which makes the diagram below commute:

$$\begin{array}{ccc} U & \xrightarrow{s} & Y \\ & \searrow \;\; = \;\; \swarrow{\scriptstyle f} & \\ & |X| & \end{array}$$

  In other words, $\mathbf{R}f$ is the sheaf of sections over open subsets of the discrete bundle $f$.

Now consider the comonad $\mathbf{B} : \mathbf{Sets}/|X| \to \mathbf{Sets}/|X|$ defined by $\mathbf{B} = \mathbf{U} \circ \mathbf{R}$. Clearly, $\mathbf{B}$ is cartesian, i.e., preserves finite limits, since both $\mathbf{U}$ and $\mathbf{R}$ are so.

The following two natural transformations will appear in the proof of Lemma 7.

- Counit $\epsilon : \mathbf{B} \to 1$, which makes all instances of the following diagram in **Sets** commute:

$$\begin{array}{ccc} \mathbf{B}A & \xrightarrow{\epsilon_A} & A \\ & \searrow \; = \; \swarrow & \\ & |X| & \end{array}$$

- Unit $\eta : 1 \to \mathbf{RU}$. The isomorphism for the adjunction $\mathbf{U} \dashv \mathbf{R}$ being natural implies the following diagram in $\mathbf{Sets}/|X|$ commutes for all sheaves $F \to X$.

$$\begin{array}{ccccc} \mathbf{U}F & \xrightarrow{\mathbf{U}\eta_F} & \mathbf{URU}F = \mathbf{BU}F & \xrightarrow{\epsilon_{\mathbf{U}F}} & \mathbf{U}F \\ & & = & & \\ & & \underset{1_{\mathbf{U}F}}{\longrightarrow} & & \end{array}$$

A.3. **Boolean Algebra of Subobjects.** For a sheaf $F$, consider the Boolean algebra $\mathrm{Sub}_{\mathbf{Sets}/|X|}(\mathbf{U}F)$; note that $\mathrm{Sub}_{\mathbf{Sets}/|X|}(\mathbf{U}F) \cong \mathcal{P}(|F|)$ for domain $|F|$ of $\mathbf{U}F = |\pi_F| : |F| \to |X|$. Then we can "restrict" the comonad $\mathbf{B}$ to an operation on $\mathrm{Sub}(\mathbf{U}F)$ as in the following lemma:

**Lemma 7.** *We can define a monotone and Cartesian operation* $b : \mathrm{Sub}(\mathbf{U}F) \to \mathrm{Sub}(\mathbf{U}F)$ *by the pullback* $b(A) = (\mathbf{U}\eta)^*\mathbf{B}A$ *as in:*

$$\begin{array}{ccc} bA & \longrightarrow & \mathbf{B}A \\ \downarrow & & \downarrow \\ \mathbf{U}F & \xrightarrow{\mathbf{U}\eta_F} & \mathbf{BU}F \end{array}$$

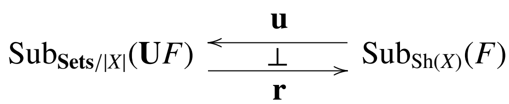
*Proof.* The diagram below commutes:

(6)
$$\begin{array}{ccccc} bA & \longrightarrow & \mathbf{B}A & \xrightarrow{\epsilon_A} & A \\ \downarrow & & \downarrow & = & \downarrow \\ \mathbf{U}F & \xrightarrow{\mathbf{U}\eta_F} & \mathbf{BU}F & \xrightarrow{\epsilon_{\mathbf{U}F}} & \mathbf{U}F \\ & & = & & \\ & & \underset{1_{\mathbf{U}F}}{\longrightarrow} & & \end{array}$$

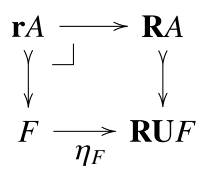
$b$ is Cartesian because pullbacks preserve finite products. $\square$

We now study this operator $b$. It can be described in two more ways, each of which will be of use.

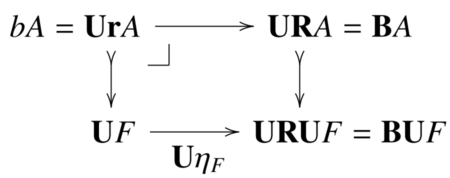
**Lemma 8.** *$b$ can be decomposed as $b = \mathbf{u} \circ \mathbf{r}$ for the following $\mathbf{u} \dashv \mathbf{r}$:*



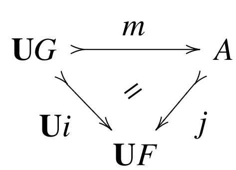

- *For $G \in \mathrm{Sub}_{\mathrm{Sh}(X)}(F)$, $\mathbf{u}G = \mathbf{U}G \in \mathrm{Sub}_{\mathbf{Sets}/|X|}(\mathbf{U}F)$.*
- *For $A \in \mathrm{Sub}_{\mathbf{Sets}/|X|}(\mathbf{U}F)$, $\mathbf{r}A \in \mathrm{Sub}_{\mathrm{Sh}(X)}(F)$ is given by the pullback as follows:*



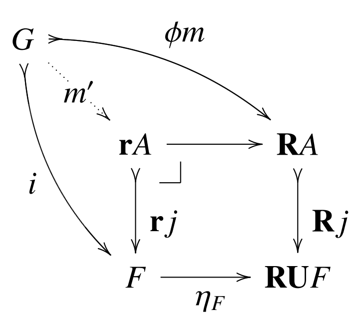

*Proof.* $b = \mathbf{u} \circ \mathbf{r}$ because hitting the diagram with $\mathbf{u}$ gives the left square in (6), as follows:




To show $\mathbf{u} \dashv \mathbf{r}$, let the isomorphisms for the adjunction $\mathbf{U} \dashv \mathbf{R}$ be $\phi : \mathrm{Hom}_{\mathbf{Sets}/|X|}(\mathbf{U}G, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sh}(X)}(G, \mathbf{R}A)$. Suppose $\mathbf{u}G \leqslant A$ in $\mathrm{Sub}_{\mathbf{Sets}/|X|}(\mathbf{U}F)$, i.e. there is $m : \mathbf{u}G = \mathbf{U}G \to A$ as in:

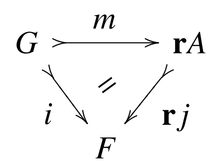



Then $\mathbf{R}j \circ \phi m = \phi(j \circ m) = \phi(\mathbf{U}i) = \eta_F \circ i$ by the naturality of $\phi$. Hence there is a unique arrow $m'$ making the diagram below commute:

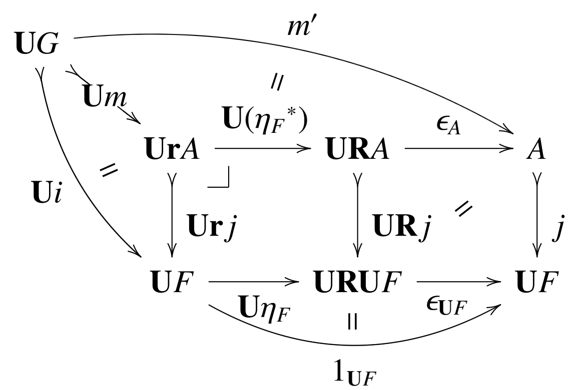



Thus $\mathbf{r}j \circ m' = i$, and therefore $G \leqslant \mathbf{r}A$ in $\mathrm{Sub}_{\mathrm{Sh}(X)}(F)$.

Suppose $G \leqslant \mathbf{r}A$ in $\mathrm{Sub}_{\mathrm{Sh}(X)}(F)$, i.e. there is $m$ as in:



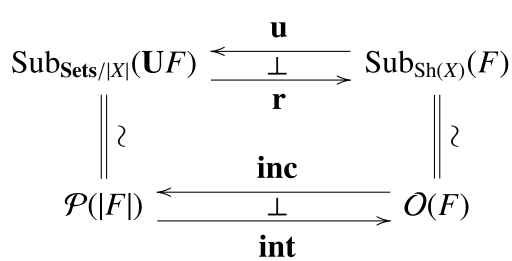
Then, letting $m' = \epsilon_A \circ \mathbf{U}(\eta_F{}^*) \circ \mathbf{U}m$, we have $j \circ m' = \mathbf{U}\eta_F \circ \epsilon_{\mathbf{U}F} \circ \mathbf{U}i = \mathbf{U}i$.



Therefore $\mathbf{u}G \leqslant A$ in $\mathrm{Sub}_{\mathbf{Sets}/|X|}(\mathbf{U}F)$. Thus $\mathbf{u} \dashv \mathbf{r}$. □

**Lemma 9.** $\mathbf{u}$ *and* $\mathbf{r}$ *coincide with the usual inclusion and interior operations, respectively, for the topology on the total space* $F$.

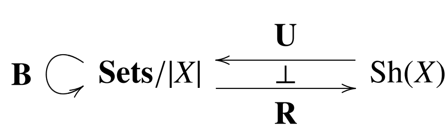
*Proof.* $\mathbf{u}$ and $\mathbf{r}$ commute respectively with **inc** and **int** in the diagram below.



(Here $|F| = \mathrm{dom}(\mathbf{U}F)$, the set of points of the total space $F$.) □

**Corollary 2.** *For any sheaf* $F$, $(\mathcal{P}(|F|), b)$ *is a topological Boolean algebra in the sense of Subsection A.1 above, with* $b = \mathbf{int}$ *the interior operation for the total space* $F$.

**Theorem 4.** *For any space* $X$*, consider the canonical geometric morphism* $\mathbf{U} \dashv \mathbf{R}$ *to sheaves on* $X$ *and the composition* $\mathbf{B} = \mathbf{UR}$ *as in:*



*Then for any sheaf $F$ on $X$, the usual interpretation of first-order logic over* $\mathbf{U}F$ *in* $\mathbf{Sets}/|X|$ *also satisfies the rules of FOS4, when basic terms are interpreted by maps of sheaves and the* $\Box$ *symbol is interpreted by the modal operator* $b : \mathrm{Sub}_{\mathbf{Sets}/|X|}(\mathbf{U}F) \to \mathrm{Sub}_{\mathbf{Sets}/|X|}(\mathbf{U}F)$*, which is induced by* $\mathbf{B}$ *as in Lemma 7; moreover,* $bA = \mathbf{int}(A)$ *is the interior operation in the total space* $F$*.*

A.4. **More General Cases.** Our semantics for first-order S4 apply not only to categories $\mathrm{Sh}(X)$ of sheaves for topological spaces $X$, but also for *any* (surjective) geometric morphism $C^* \dashv C_* : \mathcal{E} \twoheadrightarrow \mathcal{F}$ of topoi, with associated comonad $C = C^* \circ C_* : \mathcal{E} \to \mathcal{E}$. This follows from the formulation of Lemma 7. (The interpretation as "interior" is of course only available in the special topological case.) The completeness of S4 with respect to topological models then implies completeness with respect to the larger class of topos models.

Note that since we are assuming classical logic, we should restrict the general models $\mathcal{E} \to \mathcal{F}$ to those with $\mathcal{E}$ a *Boolean* topos in order to have soundness. The more general situation of arbitrary $\mathcal{E}$ is required to model *intuitionistic* first-order S4, which is treated in this same spirit in [18].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