INTRODUCTION TO THE UNIVALENT FOUNDATIONS OF MATHEMATICS

CONSTRUCTIVE TYPE THEORY AND HOMOTOPY

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The purpose of this survey talk is to introduce a new and rather surprising connection between Logic, Geometry, and Algebra which has recently come to light in the form of an interpretation of the constructive type theory of Per Martin-Löf into homotopy theory, resulting in new examples of certain algebraic structures which are important in topology. This connection was discovered quite recently, and various aspects of it are now under active investigation, in particular in connection with the Univalent Foundations program.

1. Type Theory and Homotopy

1.1. **Type theory.** Recall that **Martin-Löf constructive type theory** is a formal system consisting of the following data:

$$\begin{aligned}
\text{Types:}\quad & X,\ Y, \ldots, A \times B,\ A \to B, \ldots \\
\text{Terms:}\quad & x : A,\ b : B,\ \langle a, b\rangle,\ \lambda x.b(x) \\
\text{Dependent Types:}\quad & x : A \vdash B(x) \\
& x : A, y : B(x) \vdash C(x, y) \\
& x : A \vdash \sum_{y:B(x)} C(x, y) \\
& x : A \vdash \prod_{y:B(x)} C(x, y)
\end{aligned}$$

The dependent types are regarded as **indexed families of types**. There are simple type forming operations $A \times B$ and $A \to B$, as well as operations on dependent types including sum $\sum_{x:A} B(x)$ and product $\prod_{x:A} B(x)$ types. There are also dependent terms and term-forming operations. Finally, there are equations $s = t : A$ between terms of the same type, as in any algebraic theory.

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1.2. **Curry-Howard.** The system has a curious dual interpretation:

- once as **mathematical** objects: types or sets and their terms or elements, which are being constructed,
- once as **logical** objects: formulas or propositions and their proofs or verifications, which are being derived.

This is known as the **Curry-Howard correspondence** or the **Propositions-as-Types** interpretation. This is what is responsible for the "constructive" character of the system: a proof that something or a certain kind exists, for instance, carries with it an instance of such a thing.

According to the second, logical, interpretation, we therefore have first **propositional**, and then **predicate logic**, with **quantifiers** $\forall$ and $\exists$. It's now natural from the logical point of view to add a primitive relation of **equality** between terms of the same type:

$$x, y : A \vdash \mathtt{Id}_A(x, y)$$

This can be read as the **type of proofs** that $x = y$.

The rules for this type are such that terms a and b that are **definitially equal** $a = b$ are also **propositionally equal**, in the sense that their identity type $\mathtt{Id}_A(a, b)$ is inhabited by a term:

$$a = b \quad \Longrightarrow \quad t : \mathtt{Id}_A(a, b) \quad \text{(for some } t\text{)}$$

But the converse is generally not true — this is known as **intensionality**. Forcing the converse by adding a rule of **extentionality** spoils some of the computational virtues of the system, such as the decidability of type-checking. Intensionality gives rise to a structure within type theory of great complexity and interest. Indeed, it is a structure that has arisen independently elsewhere in mathematics *twice*: namely in topology and in higher category theory.

1.3. **The homotopy interpretation (I).** Suppose we have terms of ascending identity types:

$$\begin{aligned} a,\ b &: A \\ p,\ q &: \mathtt{Id}_A(a, b) \\ \alpha,\ \beta &: \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p, q) \\ \ldots &: \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_{\ldots}}}(\ldots) \end{aligned}$$

This suggests the following interpretation:

$$
\begin{array}{rcl}
\text{Types} & \rightsquigarrow & \text{Spaces} \\
\text{Terms} & \rightsquigarrow & \text{Maps} \\
a : A & \rightsquigarrow & \text{Points } a \in A \\
p : \mathtt{Id}_A(a,b) & \rightsquigarrow & \text{Paths } p : a \to b \\
\alpha : \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p,q) & \rightsquigarrow & \text{Homotopies } \alpha : p \Rightarrow q \\
\vdots & &
\end{array}
$$

$$
\begin{tikzcd}
A & a \bullet \arrow[r, bend left, "p"] \arrow[r, bend right, "q"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet\, b
\end{tikzcd}
$$

This **homotopy interpretation** turns out to work very well — indeed, so well that the type theory seems to capture a certain **logic of homotopy** that was not previously formalized or even recognized. It allows one to use the system to reason formally about homotopy in a way that, moreover, can even be implemented on a computer using *existing* proof assistants right out of the box.

Let us see how to interpret the rest of the type theory, namely dependent types and identity types. First, we have the following:

1.3.1. *Rules for identity types.*

$$\frac{A : \mathtt{type}}{x, y : A \ \vdash\ \mathtt{Id}_A(x,y) : \mathtt{type}}\ \mathtt{Id}\text{ formation}$$

$$\frac{a : A}{\mathtt{r}(a) : \mathtt{Id}_A(a,a)}\ \mathtt{Id}\text{ introduction}$$

$$\frac{\begin{array}{c} x, y : A, z : \mathtt{Id}_A(x,y) \ \vdash\ B(x,y,z) : \mathtt{type} \\ x : A \ \vdash\ b(x) : B\big(x, x, \mathtt{r}(x)\big) \end{array}}{x, y : A, z : \mathtt{Id}_A(x,y) \ \vdash\ \mathtt{J}(b,x,y,z) : B(x,y,z)}\ \mathtt{Id}\text{ elimination}$$

$$\frac{a : A}{\mathtt{J}\big(b, a, a, \mathtt{r}(a)\big) \ =\ b(a) : B\big(a, a, \mathtt{r}(a)\big)}\ \mathtt{Id}\text{ conversion}$$

The introduction rule provides a witness $\mathtt{r}(a)$ that a is identical to itself, called the *reflexivity term*. The distinctive elimination rule can be recognized as Lawvere's adjoint rule; it has the form:

$$\frac{x = y \qquad B(x,x)}{B(x,y)}$$

The final conversion rule is a sort of book-keeping for proof terms.

These rules force the dependent types to have the following **lifting property**:

$$\cfrac{\cfrac{p : \mathtt{Id}_A(a,b) \qquad \cfrac{x : A \vdash B(x)}{x : A \vdash \lambda y.y : B(x) \to B(x)}}{p_* : B(a) \to B(b)} \qquad \overline{a} : B(a)}{p_*(\overline{a}) : B(b)}$$

$$\begin{tikzcd} B \arrow[d] & \overline{a} \arrow[r, dotted] & p_*(\overline{a}) \\ A & a \arrow[r, "p"'] & b \end{tikzcd}$$

And with a bit more fuss, one can even lift the whole "path" $p : a \to b$. This essentially forces the interpretation of dependent types as fibrations:

$$\text{Dependent types } x : A \vdash B(x) \quad \rightsquigarrow \quad \text{Fibrations} \quad \begin{tikzcd} B \arrow[d] \\ A \end{tikzcd}$$

To complete the interpretation, how shall we interpret the dependent Identity type? We require a fibration of the form:

$$\begin{tikzcd} \mathtt{Id}_A \arrow[d] \\ A \times A \end{tikzcd}$$

Since the fiber $\mathtt{Id}_A(a,b)$ over a pair (a,b) is to be the space of paths from a to b, we take the total fibration to be the **path space** fibration,

$$\mathtt{Id}_A = A^I \longrightarrow A \times A,$$

which classifies homotopies between maps into A. Thus terms of type A^I are homotopies between their two projections.

1.4. **The homotopy interpretation (II).** That's the basic idea, except that we don't really use concrete spaces and homotopy, of course, but rather their abstract description in terms of **Quillen model categories**. This has two benefits worth mentioning: first, there is the usual advantage of axiomatics, namely that we thereby subsume a range of different models. Secondly, it allows a completeness theorem that I'll discuss further below.

In fact, all that is really required is half of the Quillen model category structure, namely a *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{C}$:

$$\mathcal{L} \hookrightarrow \mathcal{C} \hookleftarrow \mathcal{R}$$

which consists of two collections $\mathcal{L}$ (the "left-class", regarded as trivial cofibrations) and $\mathcal{R}$ (the "right-class", regarded as fibrations) of maps in $\mathcal{C}$ such that:

(1) Every map $f : A \to B$ has a factorization as $f = p \circ i$, where $i \in \mathcal{L}$ and $p \in \mathcal{R}$.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & C \arrow[d, "p"] \\ & B, \end{tikzcd}$$

(2) given maps $f : A \to B$ in $\mathcal{L}$ and $g : C \to D$ in $\mathcal{R}$ in a commutative square,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted, "j"] & D \end{tikzcd}$$

there's a diagonal filler j as indicated.

Now we can give the type theory a **Fibrations as Types** interpretation:

$$\begin{array}{rcl} \text{Closed Types } A & \rightsquigarrow & \mathcal{R}\text{-objects } A \longrightarrow 1 \\ \text{Dependent Types } x : A \vdash B(x) & \rightsquigarrow & \mathcal{R}\text{-Maps } B \longrightarrow A \\ \text{Dependent Terms } x : A \vdash b(x) : B(x) & \rightsquigarrow & \text{Sections } \begin{tikzcd} A \arrow[r, "b"] \arrow[dr, "1"'] & B \arrow[d] \\ & A \end{tikzcd} \end{array}$$

The `Id`-type is interpreted by factoring the diagonal Δ into a "path object" according to axiom (1):

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[dr, "\Delta"'] & A^I \arrow[d, "p"] \\ & A \times A, \end{tikzcd}$$

The formation and introduction rules are thereby satisfied.

For the elimination rule, we have the following set-up: the premises describe a fibration $q : B \to A^I$ together with a map $b : A \to B$ such that $q \circ b = r$. This yields the following (outer) commutative square:

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "r"'] & B \arrow[d, "q"] \\ A^I \arrow[r, "="'] \arrow[ur, dotted, "j"] & A^I \end{tikzcd}$$

Because q is a right map and r is, by definition, a left map, by axiom (2) there exists a diagonal filler j, which we choose as the interpretation of the elimination term:

$$x, y : A, z : \mathtt{Id}_A(x, y) \vdash \mathtt{J}(b, x, y, z) : B(x, y, z).$$

Commutativity of the bottom triangle is just the conclusion of the elimination rule, namely j is a section of q. And the commutativity of the top triangle is the required conversion rule:

$$x : A \vdash \mathtt{J}(b, x, x, \mathtt{r}(x)) = b(x) : B(x, x, \mathtt{r}(x)).$$

We've now basically shown the following:

Theorem 1.1 (Awodey & Warrren 2008)**.** *Martin-Löf type theory has a* ***sound*** *interpretation into any Quillen model category (even into any weak factorization system).*

To introduce some logician's terminology, the notion of **soundness** means that a formally provable statement is always true under the specified interpretation:

$$\text{provable} \quad \overset{\textit{sound}}{\Longrightarrow} \quad \text{true in all models}$$

Thus here, if a certain type can be shown to have a term, then the corresponding interpretation must always have a point.

The converse notion is **completeness**: a statement is provable if its interpretation is always true:

$$\text{provable} \quad \overset{\textit{complete}}{\Longleftarrow} \quad \text{true in all models}$$

Theorem 1.2 (Garner & Gambino 2009)**.** *The homotopy interpretation of Martin-Löf type theory is essentially* ***complete*** *with respect to models in weak factorization systems.*

There is a technical issue of selecting path objects A^I and diagonal fillers j as interpretations of $\mathtt{Id}_A$-types and $\mathtt{J}$-terms in a "coherent" way, i.e. respecting substitution of terms for variables; various solutions are available. Being able to prove this fundamental result is the second reason for using *abstract* homotopy for our semantics.

Summing up this first half of the lecture, we have now seen that:

Martin-Löf type theory provides a "logic of homotopy".

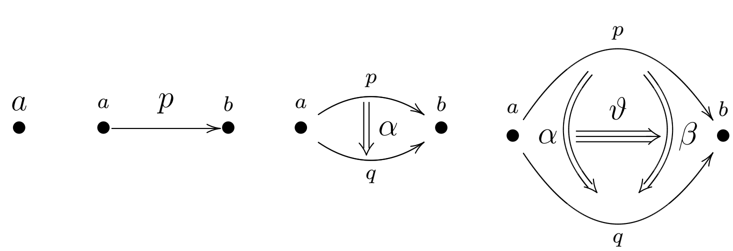
2. Type Theory and Higher Categories

It's now reasonable to ask, how **expressive** is the logical system as a language for homotopy theory? What homotopically relevant facts, properties, and constructions are logically expressible?

2.1. **Fundamental groupoids.** Let's return to the system of identity terms of various orders:

$$\begin{aligned} a,\ b &: A \\ p,\ q &: \mathtt{Id}_A(a,b) \\ \alpha,\ \beta &: \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p,q) \\ \vartheta &: \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_{\dots}}}(\alpha,\beta) \\ &\vdots \end{aligned}$$

These can be represented suggestively by the following figure.



Just as in homotopy, the terms of the lowest orders bear the structure of a groupoid: the **fundamental groupoid** of the type A. Indeed, the laws of identity provable in the system correspond to the groupoid structure:

$$\begin{array}{rll} r : \mathtt{Id}(a,a) & \text{reflexivity} & a \to a \\ s : \mathtt{Id}(a,b) \to \mathtt{Id}(b,a) & \text{symmetry} & a \leftrightarrows b \\ t : \mathtt{Id}(a,b) \times \mathtt{Id}(b,c) \to \mathtt{Id}(a,c) & \text{transitivity} & a \to b \to c \end{array}$$

But also just as in homotopy, the **groupoid laws** of associativity, inverse, and unit only hold up to the existence of a term of the next higher order — i.e. "up to homotopy".

Indeed, the entire system of terms of all orders forms a globular set:

$$A \leftleftarrows \mathtt{Id}_A \leftleftarrows \mathtt{Id}_{\mathtt{Id}_A} \leftleftarrows \dots$$

which bears the structure of a **weak** infinite-dimensional groupoid:

Theorem 2.1 (Lumsdaine, Garner & van den Berg, 2009)**.** *The system of all identity terms of over any fixed type is a weak ω-groupoid.*

What we are seeing here is that the fundamental groupoid of a *space* is really a construction in what I called the **logic of homotopy**, since it can be constructed directly in the logical system and then interpreted in any particular space. The topological fact that points, paths, and (higher) homotopies do not form an actual groupoid, but only a weak, higher-dimensional groupoid, is not merely *analogous* to the type-theoretic case; it's an instance of the same phenomenon.

2.2. **Martin-Löf complexes.** Next we ask, how *special* are the weak ω-groupoids arising in type theory? In light of the soundness and completeness results of the first part, it stands to reason that they are **quite general**. Now, the so-called "Homotopy Hypothesis" of Grothendieck states that arbitrary weak ω-groupoids classify homotopy types of spaces. We can formulate an analogous "**Type Theory Hypothesis**" – at least as a motivating conjecture – stating that the logical groupoids constructed in type theory are essentially the same as the arbitrary ones. I will conclude by sketching a way of making this conjecture more precise.

Given any globular set:

$$A_\bullet = A_0 \leftleftarrows A_1 \leftleftarrows A_2 \leftleftarrows \dots$$

We make a type theory $\mathbb{T}(A_\bullet)$ over one basic type A, with the elements of the various A_n as primitive constants:

$$\begin{aligned} A_0 &\ni a, a', \ldots : A \\ A_1 &\ni p, p', \ldots : \mathtt{Id}_A(s(p), t(p)) \\ &\vdots \end{aligned}$$

where $s, t : A_n \rightrightarrows A_{n-1}$ are the (n-dimensional) source and target maps.

Now we "turn the crank" and generate a weak ω-groupoid consisting of terms of Id-types of all orders, derived from these basic terms. The resulting groupoid,

$$G(A_\bullet),$$

is the **logical groupoid generated by the globular set $A_\bullet$**.

This is a neat way to make weak ω-groupoids, by the way, because it is in a sense mechanical. Given any ∞-graph $A_\bullet$, a computer can be programmed to systematically generate the cells of the weak ω-groupoid $G(A_\bullet)$ and calculate their relations.

Now this construction is functorial in $A_\bullet$,

$$G : \mathrm{GSets} \longrightarrow \mathrm{GSets},$$

and it determines a monad on the globular sets. The algebras for this "free logical groupoid monad" – called **Martin-Löf complexes** – are the logical groupoids generated from systems of type theory with primitive terms of various identity types and equations among terms.

Now we can formulate our conjecture more precisely:

Conjecture 2.2.

(1) *The free Martin-Löf complex on a globular set is equivalent to the free weak ω-groupoid.*
(2) *The category of all Martin-Löf complexes is equivalent to that of all weak ω-groupoids.*

Of course, there is still quite a bit of slack in the word **equivalent**. An even more precise formulation can already be *proved* when we restrict to the 1-dimensional case. For such truncated structures, we have namely:

Theorem 2.3 (Awodey, Hofstra & Warren, 2010)**.**

(1) *The free 1-dimensional Martin-Löf complex on a graph is equivalent to the free groupoid.*
(2) *The category of all 1-dimensional Martin-Löf complexes admits a Quillen model structure, and is Quillen-equivalent to the category of groupoids.*

In this precise sense, for the case of 1-dimensional structures at least, the type-theoretically generated groupoids essentially agree with the algebraic ones. Obviously, one should now proceed to compare the type theories truncated at higher dimensions and the corresponding higher groupoids.

To conclude, we have seen that the homotopy interpretation of Martin-Löf type theory captures the important fundamental (weak ω-) groupoid construction, and therewith a clearly significant amount of homotopy theory.

3. Further Topics

3.1. The entire classifying category $\mathcal{C}(\mathbb{T})$ of a type theory $\mathbb{T}$ is an $(\infty, 1)$-category (Lumsdaine 2010).

3.2. One can also show that (the nerve of) $\mathcal{C}(\mathbb{T})$ is a quasi-category. This should allow one to use the work of Joyal and Lurie to analyse the other type-theoretic operations $\Sigma, \Pi, \mathtt{W}, \mathtt{U}$ in terms of related categorical concepts, as in theory of an ∞-topos.

3.3. This dovetails with Voevodsky's Univalent Foundations program.